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Quillen closed model structures for sheaves

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Abstract

In this paper I give a general procedure of transferring closed model structures along adjoint functor pairs. As applications I derive from a global closed model structure on the category of simplicial sheaves closed model structures on the category of sheaves of 2-groupoids, the category of bisimplicial sheaves and the category of simplicial sheaves of groupoids. Subsequently, the homotopy theories of these categories are related to the homotopy theory of simplicial sheaves.

1. Introduction

There are two ways of trying to generalize the well-known closed model structure on the category of simplicial sets to the category of simplicial objects in a Grothendieck topos. One way is to concentrate on the local aspect, and to use the Kan-fibrations as a starting point. In [14] Heller showed that for simplicial presheaves there is a local (there called *right*) closed model structure. In [2] Brown showed that for a topological space X the category of "locally fibrant" sheaves of spectra on X is a *category of fibrant objects*, which is something a little bit weaker than a closed model structure. This has been extended to simplicial objects in an arbitrary Grothendieck topos by Jardine in [15].

The other way, which is the one that I will make use of here, is to concentrate on the global aspect, and to use the cofibrations being monomorphisms as a starting point. For the category of simplicial sheaves on a topological space X a global closed model structure has been given, with some assumptions on X, by Brown and Gersten in [3]. For simplicial presheaves a global (there called *left*) closed model structure has been given by Heller in [14]. These results have been extended to arbitrary Grothendieck

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topoi by Joyal in [18], where it is shown that the category of simplicial objects in an arbitrary Grothendieck topos enjoys a closed model structure where the cofibrations are exactly the monomorphisms.

The local and the global extensions are *different*: they *do* have the same weak equivalences, and therefore describe the same homotopy category, but the describing structures are different, and also the classes of fibrations are not the same (in fact, a global fibration is always a local one, but not conversely). A good review on the relation between the local and global theory can be found in [16].

The category of 2-groupoids enjoys a closed model structure, as shown by Moerdijk and Svensson in [28]. It is related to the closed model structure on the category of simplicial sets via the 2-categorical nerve functor (see [32]) and its left adjoint. This last relation induces an adjointness at the homotopy level, which gives a categorical description of homotopy 2-types: every topological space with trivial homotopy groups in dimensions ≥ 3 is homotopy equivalent to the classifying space of a 2-groupoid. Thus, 2-groupoids are an alternative to other structures classifying 2-types such as Whitehead's crossed modules (see [34]), Ellis' crossed squares (see [7] and [8]), and Loday's Cat^1-groups (see [23] and [4]).

The category of bisimplicial sets also enjoys a closed model structure, as shown by Moerdijk in [27]. The diagonal $\delta : \Delta \to \Delta \times \Delta$ induces an adjointness relation between the category of simplicial sets and the category of bisimplicial sets. This adjointness becomes an equivalence on the level of homotopy, and is extensively used in K-theory (see [12] and [31], for example).

Finally there is a closed model structure on the category of simplicial groupoids, as will be shown in [5], which is related to the closed model structure on the category of bisimplicial sets above by the dimensionwise nerve and fundamental groupoid functors. Simplicial groupoids are the necessary generalization for sheaf purposes of simplicially enriched groupoids, which, according to Dwyer and Kan in [6], model all homotopy types, and extensive work on simplicial sheaves of groupoids has been done by Joyal and Tierney in [19] and [21].

These three examples of closed model structures share a common ground which makes it possible to extend them to sheaves on a site $(\mathbb{C}, J)$: for all of them there is an adjointness relation with the category of simplicial sets with certain properties concerning weak equivalences. These adjunctions can be lifted to adjunctions between the corresponding categories of sheaves on $(\mathbb{C}, J)$, and I use the properties just mentioned to transfer the global closed model structure on the category of simplicial sheaves, to obtain global closed model structures on the category of sheaves of 2-groupoids on $(\mathbb{C}, J)$, the category of bisimplicial sheaves on $(\mathbb{C}, J)$, and on the category of simplicial sheaves of groupoids on $(\mathbb{C}, J)$. This closed model structure on the category of simplicial sheaves of groupoids appears to be different from the closed model structures in [21], but they do have the same weak equivalences and hence classify the same homotopy.

For the transfer I define the weak equivalences using the right adjoint, and I show that the left adjoint preserves certain weak equivalences by considering a boolean cover for the topos of sheaves under consideration. This argument is a so called Barr-cover argument (see [24]). I prove the factorization axioms "as in simplicial sheaves", that is, by a so called small object argument (see [10] or [29], and also the Appendix).

A special case of the transfer occurs in [11], where the closed model structure on the category of simplicial groups is transferred (there called *lifted*) to the category of n-hypergroupoids of groups. This observation does not make the proof given there easier, though.

In all the above cases the adjointness relation used to define the closed model structure induces an adjointness relation between the homotopy categories. Thus, studying homotopy of sheaves of 2-groupoids and of simplicial sheaves of groupoids gives information about, and a better understanding of, the homotopy of simplicial sheaves. This hopefully can be used in the Grothendieck programme [13] of finding a useful homotopy analogue of a sheaf, where gluing is also possible if the matching is only "up to homotopy".

This paper is organized as follows. Section 2 is a short review on Quillen closed model structures. In Section 3 I describe the general transfer, and give the conditions needed to make the argument work. Section 4 contains the proof of this transfer theorem. Section 5 gives a short review of Joyal's closed model structure on the category of simplicial sheaves in [18]. In Section 6 I apply the general theorem to the situation for sheaves of 2-groupoids. This gives a closed model structure on this category, which for presheaves is different from the closed model structure on the category of presheaves of 2-groupoids given in [28] since that is a local one. In Section 7 I apply the general theorem to the situation for bisimplicial sheaves, yielding a closed model structure on this category. In Section 8 I apply the general theorem to the situation for simplicial sheaves of groupoids, which sheds new light on the results on this category in [19] and [21]. All applications are accompanied by a characterization of the homotopy that is classified.

There is an appendix on the notion of small sheaf.

2. Quillen closed model structures

2.1. Axioms

Recall (for example from [30]) that a Quillen closed model structure on a category $\mathbb{C}$ consists of three classes of arrows: weak equivalences, fibrations and cofibrations, such that the following axioms are satisfied (an arrow which is a fibration and a weak equivalence will be called a trivial fibration, and an arrow which is a cofibration and a weak equivalence will be called a trivial cofibration):

- CM1 $\mathbb{C}$ has all finite limits and colimits;
- CM2 For any pair of composable arrows f and g, if two of the three f, g, $g \circ f$ are weak equivalences, so is the third;
- CM3 The classes of weak equivalences, fibrations and cofibrations are closed under retracts;

CM4 (factorization axioms) Any arrow can be factored as a cofibration followed by a trivial fibration, and as a trivial cofibration followed by a fibration;

CM5 (lifting axioms) For any diagram

$$\begin{array}{ccc} C & \xrightarrow{\zeta} & D \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C' & \xrightarrow{\vartheta} & D' \end{array}$$

with f a cofibration and g a trivial fibration or f a trivial cofibration and g a fibration there exists an arrow $\xi : C' \to D$ such that $\xi \circ f = \zeta$ and $g \circ \xi = \vartheta$ (fibrations are said to have the *right lifting property* with respect to trivial cofibrations, etc.).

Such a category $\mathbb{C}$ with a closed model structure on it can be localized with respect to its weak equivalences. The resulting category is called the *homotopy category of* $\mathbb{C}$, and is denoted by $\mathrm{Ho}(\mathbb{C})$. The homotopy category has the property that an arrow in $\mathbb{C}$ is a weak equivalence iff its image in $\mathrm{Ho}(\mathbb{C})$ is an isomorphism. For details about this the reader is referred to [29].

2.2. Comparing homotopy categories

Homotopy categories can be compared directly, that is, by considering arrows up to weak equivalence. In general this is not easy, and various methods have been developed to get around the problem. One of these, which is especially for categories with closed model structures, appears in [29]. Since it will be used repeatedly in the sequel I will recall it here:

Lemma 2.1. *Let $\mathbb{C}$ and $\mathbb{D}$ be categories equipped with a closed model structure, and let*

$$\mathbb{C} \underset{L}{\overset{R}{\rightleftarrows}} \mathbb{D}$$

be a pair of adjoint functors, L being the left and R the right adjoint functor. Suppose that L preserves cofibrations and that L carries weak equivalences between cofibrant objects in $\mathbb{C}$ into weak equivalences in $\mathbb{D}$. Also suppose that R preserves fibrations and that R carries weak equivalences between fibrant objects in $\mathbb{D}$ into weak equivalences in $\mathbb{C}$. Then there is a canonical adjunction

$$\mathrm{Ho}(\mathbb{C}) \underset{\underline{L}}{\overset{\underline{R}}{\rightleftarrows}} \mathrm{Ho}(\mathbb{D}).$$

Suppose in addition for C a cofibrant object of $\mathbb{C}$ and D a fibrant object of $\mathbb{D}$ that a map $L(C) \to D$ is a weak equivalence iff the associated map $C \to R(D)$ is a weak equivalence. Then the adjunction morphisms $\mathrm{id} \to \underline{L} \circ \underline{R}$ *and* $\underline{R} \circ \underline{L} \to \mathrm{id}$ *are isomorphisms so the categories* $\mathrm{Ho}(\mathbb{C})$ *and* $\mathrm{Ho}(\mathbb{D})$ *are equivalent.* □

3. The transfer

Consider categories $\mathbb{C}$ and $\mathbb{D}$ and functors as in

$$\mathbb{C} \underset{L}{\overset{R}{\rightleftarrows}} \mathbb{D},$$

with L left adjoint to R, and the category $\mathbb{C}$ comes equipped with a closed model structure. The idea is to use the adjoint functors to equip $\mathbb{D}$ with three classes of arrows: weak equivalences, fibrations and cofibrations, and to show that under certain assumptions on the closed model structure on $\mathbb{C}$, on the category $\mathbb{D}$, and on the adjunction $L \dashv R$ this defines a closed model structure on $\mathbb{D}$.

Definition 3.1. In the situation above, define the following classes of arrows in $\mathbb{D}$:
- *weak equivalences*: an arrow d in $\mathbb{D}$ is a weak equivalence if $R(d)$ is a weak equivalence in $\mathbb{C}$.
- *fibrations*: an arrow d in $\mathbb{D}$ is a fibration if $R(d)$ is a fibration in $\mathbb{C}$, equivalently, if d has the right lifting property with respect to all arrows $L(c)$ with c a trivial cofibration in $\mathbb{C}$.
- *cofibrations*: an arrow d in $\mathbb{D}$ is a cofibration if d has the left lifting property with respect to all trivial fibrations in $\mathbb{D}$.

This definition agrees with the one given in [11], and also with the definition in [33], where the case of the adjunction $c\,Sd^2 \dashv Ex^2 N$ between simplicial sets and categories is treated.

I will prove the factorization axioms for the closed model structure on $\mathbb{D}$ by a small object argument, using that the (trivial) cofibrations in $\mathbb{C}$ are "generated" by (trivial) cofibrations between λ-small objects (see the Appendix), and that L preserves "enough" of this generating smallness structure. More precisely, the following will be assumed about the closed model structure on $\mathbb{C}$:

Definition 3.2. Let λ be an infinite regular cardinal. A closed model structure on a category $\mathbb{C}$ is *λ-generated* if $\mathbb{C}$ has all colimits, and every (trivial) cofibration in $\mathbb{C}$ is a transfinite composition of pushouts of coproducts of (trivial) cofibrations between λ-small objects. The closed model structure is *generated* if it is λ-generated for some infinite regular cardinal λ.

The important example of a generated closed model structure, that will be used in the applications here, is the closed model structure on the category of simplicial sheaves as given in [18]. I will recall it briefly in Section 5.

Next it will be necessary that $\mathbb{D}$ has all λ-filtered colimits too, and that L preserves the notion of λ-smallness. This last condition holds in almost all practical cases because its right adjoint R is almost always λ'-accessible (see [26]) for some $\lambda' \geq \lambda$, which suffices since then for a λ'-filtered category $\mathbb{I}$ and an $\mathbb{I}$-indexed diagram D one has the following sequence of isomorphisms:

$$\mathbb{D}(L(C), \lim_{\to \mathbb{I}} D_I) \cong \mathbb{C}(C, R(\lim_{\to \mathbb{I}} D_I)) \cong \mathbb{C}(C, \lim_{\to \mathbb{I}} R(D_I))$$
$$\cong \lim_{\to \mathbb{I}} \mathbb{C}(C, R(D_I)) \cong \lim_{\to \mathbb{I}} \mathbb{D}(L(C), D_I).$$

The condition that is crucial to the argument is a statement about what L does to trivial cofibrations, in particular, whether a certain colimit of these is a weak equivalence in $\mathbb{D}$. In the applications this condition will be proven by lifting the statement to a boolean cover of the topos of sheaves, noting that this cover is a model of classical set theory, and using the closed model structure on the category of 2-groupoids, for example, to show that this colimit is in fact a trivial cofibration.

Theorem 3.3. *Let λ be an infinite regular cardinal. Let $\mathbb{C}$ be a category equipped with a λ-generated closed model structure, let $\mathbb{D}$ be a category having finite limits and all colimits, and let*

$$\mathbb{C} \underset{L}{\overset{R}{\rightleftarrows}} \mathbb{D}$$

be such that $L \dashv R$, with L preserving λ-smallness. Suppose that for every arrow d in $\mathbb{D}$ which is a transfinite composition of pushouts of coproducts of arrows $L(c)$ with c a trivial cofibration in $\mathbb{C}$ the arrow $R(d)$ is a weak equivalence in $\mathbb{C}$. Then Definition 3.1 defines a closed model structure on $\mathbb{D}$.

4. Proof of Theorem 3.3

Axiom CM1 holds by assumption. Axiom CM2 and two parts of axiom CM3 follow directly from the corresponding facts in $\mathbb{C}$ using the adjunction. The remaining part of axiom CM3 follows by a standard formal argument.

The first factorization axiom states that every arrow in $\mathbb{D}$ can be factored as a cofibration followed by a trivial fibration. I will prove that every arrow in $\mathbb{D}$ can be factored as a transfinite composition of pushouts of arrows $L(c)$ with c a coproduct of cofibrations between λ-small objects in $\mathbb{C}$ followed by an arrow in $\mathbb{D}$ having the right lifting property with respect to all arrows $L(c)$ with c a cofibration in $\mathbb{C}$. The first part of the factorization to be constructed will be a cofibration since by a formal adjunction argument the functor L preserves cofibrations and

Lemma 4.1. *The class of cofibrations in $\mathbb{D}$ is closed under coproducts, pushouts and transfinite composition.*

Proof. Formal lifting arguments. □

The second part of this factorization is a trivial fibration since

Lemma 4.2. *An arrow d in $\mathbb{D}$ having the right lifting property with respect to all arrows $L(c)$ with c a cofibration in $\mathbb{C}$ is a trivial fibration.*

Proof. By the adjunction $R(d)$ has the right lifting property with respect to all cofibrations in $\mathbb{C}$, hence is a trivial fibration in $\mathbb{C}$. $\square$

To construct the above factorization of an arrow $d : D \to D'$, define with transfinite induction for each ordinal $\kappa \in \lambda$ an object D^κ of $\mathbb{D}$, an arrow $\iota^{\kappa+1} : D^\kappa \to D^{\kappa+1}$ and an arrow $d^\kappa : D^\kappa \to D'$, as follows. $D^0 = D$, $d^0 = d$. Having obtained D^κ and d^κ, consider the set Σ^κ of all diagrams

$$\begin{array}{ccc} L(C_\sigma) & \longrightarrow & D^\kappa \\ {\scriptstyle L(c_\sigma)}\downarrow & & \downarrow{\scriptstyle d^\kappa} \\ L(C'_\sigma) & \longrightarrow & D' \end{array}$$

with c_σ a cofibration between λ-small objects in $\mathbb{C}$, which is a set since there is only a set of λ-small objects of $\mathbb{C}$, the functor L preserves λ-smallness and there is only a set of arrows between two λ-small objects. Define $D^{\kappa+1}$, $\iota^{\kappa+1}$ and $d^{\kappa+1}$ by the diagram

$$\begin{array}{ccccc} L\left(\coprod_{\sigma\in\Sigma^\kappa} C_\sigma\right) & \longrightarrow & D^\kappa & & \\ \downarrow & & \downarrow{\scriptstyle \iota^{\kappa+1}} & \searrow{\scriptstyle d^\kappa} & \\ L\left(\coprod_{\sigma\in\Sigma^\kappa} C'_\sigma\right) & \longrightarrow & D^{\kappa+1} & & \\ & \searrow & & \searrow{\scriptstyle d^{\kappa+1}} & \\ & & & & D' \end{array}$$

in which the rectangle is a pushout in $\mathbb{D}$.

For a limit ordinal $\kappa \le \lambda$, having obtained $D^{\kappa'}$ and $d^{\kappa'}$ for all $\kappa' \in \kappa$, define $D^\kappa = \varinjlim_\kappa D^{\kappa'}$, $\iota^\kappa : D \to D^\kappa$ the canonical arrow, and $d^\kappa : D^\kappa \to D'$ the arrow induced by the $d^{\kappa'}$'s. By definition $\iota^\lambda : D \to D^\lambda$ is a transfinite composition of pushouts of arrows $L(c)$ with c a coproduct of small cofibrations in $\mathbb{C}$. d^λ has the right lifting property with respect to all arrows $L(c)$ with c a cofibration in $\mathbb{C}$ since it follows from the cofibrations in $\mathbb{C}$ being generated that this lifting problem reduces to c a cofibration between λ-small objects in $\mathbb{C}$, and then given a diagram

$$\begin{array}{ccc} L(C) & \longrightarrow & D^\lambda \\ {\scriptstyle L(c)}\downarrow & & \downarrow{\scriptstyle d^\lambda} \\ L(C') & \longrightarrow & D' \end{array}$$

with c a cofibration between λ-small objects in $\mathbb{C}$, λ-smallness of $L(C)$ implies that ζ must factor through D^κ for some $\kappa \in \lambda$, which gives a lifting to $D^{\kappa+1}$ in the $(\kappa+1)$-th step above. So $d^\lambda \circ \iota^\lambda$ is the required factorization of d.

The second factorization axiom states that every arrow in $\mathbb{D}$ can be factored as a trivial cofibration followed by a fibration. I will prove something more, namely that every arrow in $\mathbb{D}$ can be factored as a transfinite composition of pushouts of arrows $L(c)$ with c a coproduct of trivial cofibrations between λ-small objects in $\mathbb{C}$ followed by a fibration. By the last assumption in the statement of the theorem the first part of this factorization is indeed a *trivial* cofibration.

To construct this factorization one proceeds analogously to the former factorization, again using that L preserves λ-smallness. The fact that the trivial cofibrations in $\mathbb{C}$ are generated ensures that the last part of the factorization is a fibration.

The first lifting axiom holds by definition of the cofibrations in $\mathbb{D}$.

The second lifting axiom states that it is possible to lift in every diagram

$$\begin{array}{ccc} D & \xrightarrow{\zeta} & E \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle e} \\ D' & \xrightarrow{\vartheta} & E' \end{array}$$

in $\mathbb{D}$ with d a trivial cofibration and e a fibration. To find such a lifting I will use a standard argument (see e.g. [27]), adapted to this situation. Use the proof of the second factorization axiom to factor d as a transfinite composition of pushouts of arrows $L(c)$ with c a coproduct of trivial cofibrations between λ-small objects in $\mathbb{C}$ followed by a fibration, say $d = p \circ i$. Now p is even a weak equivalence by axiom CM2, which implies that $R(p)$ is a trivial fibration in $\mathbb{C}$, in other words, p has the right lifting property with respect to all arrows $L(c)$ with c a cofibration in $\mathbb{C}$. So, using lemma 4.1, there is a lifting in

$$\begin{array}{ccc} D & \xrightarrow{i} & \bullet \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle p} \\ D' & = & D' \end{array}$$

by the first lifting axiom, and in

$$\begin{array}{ccc} D & \xrightarrow{\qquad\zeta\qquad} & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle e} \\ \bullet & \xrightarrow{p} D' \xrightarrow{\vartheta} & E' \end{array}$$

since it can be done for i of the form $L(c)$ with c a trivial cofibration in $\mathbb{C}$, which suffices by formal lifting arguments. Composing both gives the required lifting.

All axioms for a closed model structure have been checked, which finishes the proof of Theorem 3.3.

5. Simplicial sheaves

In [18] Joyal showed that the category of simplicial sheaves enjoys a generated closed model structure. Because this closed model structure will be used for the applications later on I will recall it briefly.

Let $\mathcal{E}$ be the topos of sheaves on a site $(\mathbb{C}, J)$, which will be fixed from now on. Denote the topos of simplicial sheaves on $(\mathbb{C}, J)$, in other words, of simplicial objects in

$\mathcal{E}$, by $s\mathcal{E}$. For X an object of $s\mathcal{E}$ its homotopy sheaves are defined as follows. $\pi_0(X)$ is the coequalizer in $\mathcal{E}$ of the pair $X_1 \rightrightarrows X_0$. To describe $\pi_n(X)$, consider X as a simplicial sheaf over X_0, i.e., consider X in the base extension $\mathcal{E}/X_0 \to \mathcal{E}$, so that X has a generic basepoint. Now Kan's construction (see [22]) can be applied internally in the topos $\mathcal{E}/X_0$ since it only uses finite limits and colimits, yielding $\pi_n(X)$ as an object over X_0.

Definition 5.1. Define the following classes of arrows in $s\mathcal{E}$:

weak equivalences: an arrow $f : X \to Y$ in $s\mathcal{E}$ is a weak equivalence if

(1) $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is an isomorphism,

(2) for all $n \geq 1$, the square

$$\begin{array}{ccc} \pi_n(X) & \longrightarrow & \pi_n(Y) \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

is a pullback.

cofibrations: an arrow $f : X \to Y$ in $s\mathcal{E}$ is a cofibration if it is a monomorphism.

fibrations: an arrow $f : X \to Y$ in $s\mathcal{E}$ is a fibration if it has the right lifting property with respect to all trivial cofibrations in $s\mathcal{E}$.

In [18] it is proven that

Theorem 5.2. *Definition* 5.1 *defines a closed model structure on* $s\mathcal{E}$. □

Being a topos over **Sets** the category $s\mathcal{E}$ has all (set-indexed) colimits. In [18] it is shown that there exists an infinite regular cardinal λ such that every trivial cofibration in $s\mathcal{E}$ is a transfinite composition of pushouts of trivial cofibrations between λ-small objects. Using the same method it can be shown that also every cofibration in $s\mathcal{E}$ is a transfinite composition of pushouts of cofibrations between λ-small objects, for the same λ. In short, the closed model structure on $s\mathcal{E}$ is λ-generated for some large enough λ.

Another important property of the above closed model structure on $s\mathcal{E}$ is the following. Take a boolean cover of the topos $\mathcal{E}$, that is, a surjective geometric morphism $f : \mathcal{F} \to \mathcal{E}$ where $\mathcal{F}$ is a boolean topos satisfying the axiom of choice, for example, the Barr-cover of $\mathcal{E}$ (see [17]). In general, constructions with finite limits and colimits are preserved under the inverse image of a geometric morphism. When the geometric morphism is surjective these so-called geometric constructions are also reflected. The same holds for geometric properties, i.e., properties which are equivalent to the invertibility of a geometrically constructed arrow. Thus, in order to verify that a certain geometrically constructed arrow in $\mathcal{E}$ has a certain geometric property it suffices to verify the property in $\mathcal{F}$. Now $\mathcal{F}$ is a boolean topos, which is a model of classical set theory. Thus in checking statements in $\mathcal{F}$ one can pretend to be working in **Sets**. The property of being a weak equivalence in $s\mathcal{E}$ is geometrical (see [18]), which thus implies that a geometrically constructed arrow in $s\mathcal{E}$ is a weak equivalence iff the same construction in s**Sets** yields a weak equivalence. When I make use of this property of weak equivalences in $s\mathcal{E}$ I will say that I apply the *Barr-cover argument.*

6. Application 1: Homotopy 2-types of sheaves

The first application I will give is the transferring of the closed model structure on $s\mathcal{E}$ to the category $2\mathrm{Grpd}(\mathcal{E})$ of sheaves of 2-groupoids on $(\mathbb{C}, J)$, in other words, of 2-groupoids in $\mathcal{E}$, using the closed model structure on the category **2Grpd** of 2-groupoids in [28]. The category of (sheaves of) 2-groupoids only serves as a sample: once a closed model structure with the same properties is obtained on a category classifying a certain finite homotopy type, there is also a closed model structure on the corresponding category of sheaves.

6.1. 2-Groupoids

It is not necessary to describe the closed model structure on **2Grpd** in full detail for two reasons. First, because full detail is already supplied by [28], and second because I will only use a very specific part.

There is an adjoint functor pair

$$s\mathbf{Sets} \underset{W}{\overset{\mathbb{N}}{\rightleftarrows}} \mathbf{2Grpd}$$

where $\mathbb{N}$ is the 2-categorical nerve (see [32]):

$$\mathbb{N}(\mathcal{G}) = G_0 \leftleftarrows G_1 \leftleftarrows G_2 \times_{G_1} (G_1 \times_{G_0} G_1) \leftleftarrows \cdots, \tag{1}$$

and W is the Whitehead 2-groupoid described in [28]:

$$W(X) = X_0 \leftleftarrows F(X_1) \leftleftarrows F(X_2)/X_3, \tag{2}$$

where F describes a free construction, i.e., $F(X_i)$ consists of finite formal composites of i-cells of X and their formal inverses.

The closed model structure on **2Grpd** has the following properties (see [28]) that are relevant here:

Lemma 6.1. *An arrow g in* **2Grpd** *is a weak equivalence iff $\mathbb{N}(g)$ is a weak equivalence in s***Sets**. □

Lemma 6.2. *For every simplicial set X the unit $\eta : X \to \mathbb{N}W(X)$ is a weak equivalence iff $\pi_n(X) = 0$ for any $n > 2$.* □

Lemma 6.3. *The functor W : s***Sets** $\to$ **2Grpd** *preserves weak equivalences and cofibrations.* □

So it will not be necessary to know exactly what the trivial cofibrations in **2Grpd** are, but only that they are part of a closed model structure, and that they are preserved by W.

6.2. Applying Theorem 3.3

I will now show that for sheaves of 2-groupoids on $(\mathbb{C}, J)$ Theorem 3.3 can be applied. The category 2Grpd($\mathcal{E}$) has all (set-indexed) colimits, calculated by taking the pointwise colimit, which exists because **2Grpd** has all colimits, followed by sheafification. Finite limits in 2Grpd($\mathcal{E}$) can be calculated pointwise, and then for objects and arrows separately.

Because of their geometric nature equations (1) and (2) also define an adjoint functor pair

$$s\mathcal{E} \underset{W}{\overset{\mathbb{N}}{\rightleftarrows}} \text{2Grpd}(\mathcal{E})$$

(the ambiguity in notation introduced here will do no harm because it will always be clear from the context which version of $\mathbb{N}$ and W is meant, and it avoids the need of writing lots of subscripts $_{\mathcal{E}}$). The functor $\mathbb{N}$: 2Grpd($\mathcal{E}$) $\to s\mathcal{E}$ is expressed, in Eq. (1), by finite limits, hence $\mathbb{N}$ preserves filtered colimits, which implies, as remarked in section 3, that W preserves λ-smallness.

It remains to be shown that every arrow in 2Grpd($\mathcal{E}$) which is a transfinite composition of pushouts of coproducts of arrows $W(f)$ with f a trivial cofibration in $s\mathcal{E}$ is a weak equivalence in 2Grpd($\mathcal{E}$). Because the construction of the functors W and $\mathbb{N}$ is geometrical the Barr-cover argument can be applied to the present situation: an arrow in $s\mathcal{E}$ constructed using finite limits and colimits and the functors W and $\mathbb{N}$ is a weak equivalence in $s\mathcal{E}$ if the corresponding construction in *s***Sets** yields a weak equivalence. Thus I only need to show that every arrow in *s***Sets** which is the $\mathbb{N}$-image of a transfinite composition of pushouts of coproducts of arrows $W(f)$ with f a trivial cofibration in *s***Sets** is a weak equivalence.

Starting with a trivial cofibration in *s***Sets**, W sends it to a trivial cofibration in **2Grpd** by Lemma 6.3. Every transfinite composition of pushouts of coproducts of such arrows is then again a trivial cofibration in **2Grpd** since by the closed model structure the class of trivial cofibrations is closed under these operations. Lemma 6.1 then gives that the $\mathbb{N}$-image of this arrow is a weak equivalence in *s***Sets**, as was needed.

Concluding:

Theorem 6.4. *There is a closed model structure on the category of sheaves of* 2-*groupoids on* $(\mathbb{C}, J)$, *where the weak equivalences are arrows for which the* 2-*categorical nerve is a weak equivalence of simplicial sheaves.*

6.3. A Mac Lane–Whitehead result

One of the first results on classification of finite homotopy types occurred in [25], where it was shown that homotopy 2-types can be classified by what is there called algebraic 3-types. As a consequence of Theorem 6.4 I will give the sheaf analogon of this result in terms of 2-groupoids: sheaves of 2-groupoids classify homotopy 2-types of sheaves.

Theorem 6.5. *The functors W and $\mathbb{N}$ induce adjoint functors*

$$\mathrm{Ho}(s\mathcal{E}) \underset{\underline{W}}{\overset{\underline{\mathbb{N}}}{\rightleftarrows}} \mathrm{Ho}(2\mathrm{Grpd}(\mathcal{E}))\,.$$

$\underline{W}$ and $\underline{\mathbb{N}}$ induce an equivalence of categories between the full subcategory of $\mathrm{Ho}(s\mathcal{E})$ *given by those simplicial sheaves X for which $\pi_n(X) = 0$ for every $n > 2$, and* $\mathrm{Ho}(2\mathrm{Grpd}(\mathcal{E}))$.

Proof. The first statement is a straightforward application of Lemma 2.1: W preserves cofibrations by a formal lifting argument and weak equivalences by Lemma 6.3 and the Barr-cover argument, and $\mathbb{N}$ preserves fibrations and weak equivalences by definition. For the second part, note that $\mathbb{N}$ lands in the full subcategory of $s\mathcal{E}$ given by those simplicial sheaves X for which $\pi_n(X) = 0$ for every $n > 2$ since for $\mathcal{G}$ a 2-groupoid $\mathbb{N}(\mathcal{G})$ is coskeletal, hence has trivial homotopy, above dimension 2. When the adjunction is restricted to this subcategory the unit becomes a weak equivalence by Lemma 6.2 and the Barr-cover argument, and the counit is then also a weak equivalence: apply axiom CM2 to one of the triangular identities. So on the level of homotopy categories the restricted adjunction becomes an equivalence of categories. □

7. Application 2: Bisimplicial sheaves

The second application I will give is the transferring of the closed model structure on $s\mathcal{E}$ to the category $bis\mathcal{E}$ of bisimplicial sheaves on $(\mathbb{C}, J)$, in other words, of bisimplicial objects in $\mathcal{E}$, using the closed model structure on the category of bisimplicial sets in [27].

7.1. Bisimplicial sets

I will recall some results from [27].

The diagonal $\delta : \boldsymbol{\Delta} \to \boldsymbol{\Delta} \times \boldsymbol{\Delta}$ induces an adjoint functor pair

$$s\mathbf{Sets} \underset{\delta_!}{\overset{\delta^*}{\rightleftarrows}} bis\mathbf{Sets}$$

where δ^* is given by composition with δ. The left adjoint functor $\delta_!$ is completely determined by the images of the standard simplices $\Delta[n]$, which are given by

$$\delta_!(\Delta[n]) = \Delta[n,n] \overset{\mathrm{def}}{=} (\boldsymbol{\Delta} \times \boldsymbol{\Delta})(-,([n],[n]))\,. \tag{3}$$

In [27] a closed model structure on $bis\mathbf{Sets}$ is defined essentially via Theorem 3.3. Indeed, weak equivalences, fibrations and cofibrations are defined as in Definition 3.1, and it is proven that

Lemma 7.1. *$\delta_!$ sends $\Lambda^k[n] \hookrightarrow \Delta[n]$ to a trivial cofibration in bis***Sets**. □

7.2. Applying Theorem 3.3

I will now show that for bisimplicial sheaves on $(\mathbb{C}, J)$ Theorem 3.3 can be applied. Being a topos over **Sets** the category $bis\mathcal{E}$ has finite limits and all colimits.

The diagonal $\delta : \Delta \to \Delta \times \Delta$ induces for *every* Grothendieck topos $\mathcal{E}$ an adjoint functor pair

$$s\mathcal{E} \underset{\delta_!}{\overset{\delta^*}{\rightleftarrows}} bis\mathcal{E},$$

and a further right adjoint δ_* to δ^*, hence δ^* preserves all colimits, which implies that $\delta_!$ preserves λ-smallness.

It remains to be shown that every arrow in $s\mathcal{E}$ which is the δ^*-image of a transfinite composition of pushouts of coproducts of arrows $\delta_!(f)$ with f a trivial cofibration in $s\mathcal{E}$ is a weak equivalence in $s\mathcal{E}$. The constructions of the functors $\delta_!$ and δ^* are geometrical: $\delta^*(\mathbb{X})_n = X_{n,n}$ and $\delta_!(X) = \int^n \Delta[n,n] \times X_n = (\coprod_{n\geq 0} \Delta[n,n] \times X_n)/\sim$ for a suitable equivalence relation $\sim$. Applying the Barr-cover argument again, an arrow in $s\mathcal{E}$ constructed using finite limits and colimits and the functors $\delta_!$ and δ^* is a weak equivalence in $s\mathcal{E}$ if the corresponding construction in s**Sets** yields one. Thus it suffices to show that every arrow in s**Sets** which is the δ^*-image of a transfinite composition of pushouts of coproducts of arrows $\delta_!(f)$ with f a trivial cofibration in s**Sets** is a weak equivalence.

Starting with a trivial cofibration in s**Sets**, it can be written as a transfinite composition of pushouts of coproducts of horn-inclusions since the horn-inclusions generate the class of trivial cofibrations. Lemma 7.1 gives that $\delta_!$ of such a horn-inclusion is a trivial cofibration in *bis***Sets**. By the closed model structure on *bis***Sets** the class of trivial cofibrations is closed under coproducts, pushouts and transfinite composition, which not only implies that $\delta_!$ of the original arrow is a trivial cofibration in *bis***Sets**, but also, using that $\delta_!$ preserves colimits, that every transfinite composition of pushouts of coproducts of such arrows is again a trivial cofibration. By definition of the weak equivalences in *bis***Sets** the δ^*-images of these arrows are weak equivalences in s**Sets**, as was needed.

Concluding:

Theorem 7.2. *There is a closed model structure on the category of bisimplicial sheaves on $(\mathbb{C}, J)$, where the weak equivalences are arrows for which the diagonal is a weak equivalence of simplicial sheaves.*

7.3. Classifying homotopy with bisimplicial sheaves

To compare the homotopy category of bisimplicial sheaves with the homotopy category of simplicial sheaves one more property of the functor $\delta_!$ will be needed.

Lemma 7.3. *The functor $\delta_! : s\mathcal{E} \to bis\mathcal{E}$ preserves weak equivalences between cofibrant objects.*

Proof. Consider a weak equivalence $X \to Y$ between cofibrant objects in $s\mathcal{E}$. Factor it as $p \circ i$ where i is a trivial cofibration and p a fibration. Because of the closed model structure on $bis\mathcal{E}$ and the preservation of fibrations by δ^* the functor $\delta_!$ preserves trivial cofibrations, so $\delta_!(i)$ is a weak equivalence. To see that $\delta_!(p)$ is also a weak equivalence, find a dotted lifting in

$$\begin{array}{ccc} 0 & \longrightarrow & \bullet \\ \downarrow & \overset{p'}{\nearrow} & \downarrow p \\ Y & = & Y \end{array}$$

which exists because Y is cofibrant and p is a trivial fibration by axiom CM2. Since p' is a right inverse it is a monomorphism, i.e., a cofibration in $s\mathcal{E}$, and it is a weak equivalence by axiom CM2. Therefore, $\delta_!(p')$ is a weak equivalence, and since $\delta_!(p) \circ \delta_!(p') = \mathrm{id}$ axiom CM2 again implies that $\delta_!(p)$ is a weak equivalence as well. □

Theorem 7.4. *The functors $\delta_!$ and δ^* induce an equivalence of homotopy categories*

$$\mathrm{Ho}(s\mathcal{E}) \underset{\underline{\delta_!}}{\overset{\underline{\delta^*}}{\rightleftarrows}} \mathrm{Ho}(bis\mathcal{E}).$$

Proof. $\delta_!$ preserves cofibrations by the same formal lifting argument as in the proof of Theorem 6.5, and weak equivalences between cofibrant objects by Lemma 7.3. δ^* preserves fibrations and weak equivalences by definition. Applying Lemma 2.1 gives that $\delta_!$ and δ^* induce adjoint functors $\underline{\delta_!}$ and $\underline{\delta^*}$ between the homotopy categories.

In [27] Moerdijk has shown that δ^* and δ_* induce an equivalence of homotopy categories. Because adjoints are unique up to isomorphism, $\underline{\delta_!}$ is isomorphic to the functor induces by δ_*, which implies that $\underline{\delta_!}$ and $\underline{\delta^*}$ also give an equivalence of homotopy categories. □

8. Application 3: Simplicial sheaves of groupoids

The third application I will give is the transferring of the closed model structure on $s\mathcal{E}$ to the category $s\mathrm{Grpd}(\mathcal{E})$ of simplicial sheaves of groupoids on $(\mathbb{C}, J)$, in other words, of simplicial groupoids in $\mathcal{E}$, via $bis\mathcal{E}$. This is much more practical than trying to transfer the closed model structure on $bis\mathcal{E}$ directly, since then one has to prove for example that this closed model structure is generated, which is not necessary in the present approach. Note that there is *no* assumption on the simplicial sheaf of objects of the groupoid, contrary to [6], where it is assumed that the simplicial set of objects of any simplicial groupoid is discrete, thus is in fact a simplicially enriched groupoid.

8.1. Simplicial groupoids

I will say *nothing* here about the closed model structure on the category s**Grpd** of simplicial groupoids since it will not be used in the sequel. Instead, I will collect a few things that will be used in this section, among which a brief recall of two other closed model structures on bisimplicial sets, and of one on presheaves of groupoids. The reader is referred to the forthcoming paper [5] for details about the closed model structure on simplicial groupoids.

There are adjoint functor pairs

$$s\mathbf{Sets} \underset{\delta_!}{\overset{\delta^*}{\rightleftarrows}} bis\mathbf{Sets} \underset{\Pi}{\overset{N}{\rightleftarrows}} s\mathbf{Grpd}$$

where δ^* and $\delta_!$ are as in the previous section, and N and Π are the dimensionwise (say vertical) nerve and fundamental groupoid, respectively. The composite $\delta^* \circ N$ is the "classifying space" functor, and will be denoted by B. The composite $\Pi \circ \delta_!$ will be denoted by P.

The closed model structure on simplicial sheaves of Section 5 also gives a closed model structure on simplicial presheaves, of course. In particular, this gives two closed model structures on bisimplicial sets, a "vertical" and a "horizontal" one, by considering bisimplicial sets as simplicial objects in the presheaf category of simplicial sets, in the two possible directions. Since global and local weak equivalences coincide, an arrow in *bis***Sets** is a weak equivalence in the vertical (resp. horizontal) structure iff it is for every horizontal (resp. vertical) dimension a weak equivalence of simplicial sets. In both structures the cofibrations are monomorphisms of bisimplicial sets.

In [1] it is shown that there is a remarkable relation between the weak equivalences in the horizontal and vertical structures, and the weak equivalences in the "diagonal" structure in Section 7:

Lemma 8.1. *If $f : X \to Y$ is an arrow in bis***Sets** *such that $f_{n,\bullet} : X_{n,\bullet} \to Y_{n,\bullet}$ is a weak equivalence in s***Sets** *for each n, then $\delta^*(f) : \delta^*(X) \to \delta^*(Y)$ is also a weak equivalence.* □

The category of (pre)sheaves of groupoids enjoys a closed model structure with internal categorical equivalences of groupoids as weak equivalences, as shown in [20]. The nerve functor $N : \mathrm{Grpd}(\mathcal{E}) \to s\mathcal{E}$ sends these weak equivalences to weak equivalences of simplicial (pre)sheaves. The cofibrations in this structure are the arrows whose object part is a monomorphism, which together with the full & faithfulness of weak equivalences implies that trivial cofibrations are always monomorphisms.

8.2. Applying Theorem 3.3

I will now show that for simplicial sheaves of groupoids on $(\mathbb{C}, J)$ Theorem 3.3 can be applied. The category $s\mathrm{Grpd}(\mathcal{E})$ has all colimits and finite limits since the category of sheaves of groupoids has them and colimits and finite limits of simplicial objects are pointwise.

As before, the geometric nature of the adjoint functors P and B gives an adjoint functor pair

$$s\mathcal{E} \underset{P}{\overset{B}{\rightleftarrows}} s\mathrm{Grpd}(\mathcal{E}).$$

As in Section 6 the nerve functor can be expressed by finite limits, which implies that N preserves filtered colimits. Together with the preservation of colimits by δ^* as shown in Section 7 this gives that P preserves λ-smallness.

It remains to be shown that every arrow in $s\mathrm{Grpd}(\mathcal{E})$ which is a transfinite composition of pushouts of coproducts of arrows $P(f)$ with f a trivial cofibration in $s\mathcal{E}$ is a weak equivalence in $s\mathrm{Grpd}(\mathcal{E})$. By arguments as before this can be done using the Barr-cover argument: weak equivalences are geometrical, and P and B are constructed geometrically. Hence it suffices to show that every arrow in s**Sets** which is the B-image of a transfinite composition of pushouts of coproducts of arrows $P(f)$ with f a trivial cofibration in s**Sets** is a weak equivalence. To do this I will use an argument from [5] which exploits the closed model structures recalled above.

Starting with a trivial cofibration in s**Sets**, it can be written as a transfinite composition of pushouts of coproducts of horn-inclusions. It therefore suffices, since P preserves colimits and since a pushout of a transfinite composition is a transfinite composition of pushouts, to prove the above for arrows f which are horn-inclusions. Denote by T_n the antidiscrete groupoid on n points, i.e., with exactly one arrow between any two points. From the description in [27] of $\delta_!(\Lambda^k[n])$ it follows that for fixed horizontal dimension l the monomorphism $P(\Lambda^k[n])_l \hookrightarrow P(\Delta[n])_l$ of groupoids is given by

$$\coprod_{\substack{\alpha:[l]\to[n] \\ \mathrm{im}(\alpha)\cup\{k\}\neq[n]}} T_{n_\alpha} \hookrightarrow \coprod_{\alpha:[l]\to[n]} T_{n+1},$$

where

$$n_\alpha = \begin{cases} n & \text{if } \#(\mathrm{im}(\alpha)\cup k) = n, \\ n+1 & \text{if } \#(\mathrm{im}(\alpha)\cup k) < n. \end{cases}$$

Factor $P(\Lambda^k[n]) \hookrightarrow P(\Delta[n])$ as

$$\coprod_{\alpha\in\Lambda^k[n]} T_{n_\alpha} \hookrightarrow \coprod_{\alpha\in\Lambda^k[n]} T_{n+1} \hookrightarrow \coprod_{\alpha\in\Delta[n]} T_{n+1}.$$

The first part of this factorization is a trivial cofibration of internal groupoids in s**Sets** because T_{n_α} is a deformation retract of T_{n+1} and a coproduct of trivial cofibrations is again one. It therefore has the property, by the closed model structure on Grpd(s**Sets**), that every pushout of it is a trivial cofibration of internal groupoids as well. Since the nerve functor N : Grpd(s**Sets**) $\to$ s(s**Sets**), which is in fact nothing but the dimensionwise nerve, preserves monomorphisms and weak equivalences, N of such a pushout is a trivial cofibration in s(s**Sets**), i.e., is a trivial cofibration in the *vertical* structure on bisimplicial sets.

N of the second part of the factorization above can also be written as

$$\coprod_{x\in N(T_{n+1})} \Lambda^k[n] \hookrightarrow \coprod_{x\in N(T_{n+1})} \Delta[n]$$

which is a trivial cofibration of simplicial objects in s**Sets**. Considering a pushout of the second part, this pushout is preserved by N since N commutes with coproducts. Therefore, by the closed model structure on $s(s\mathbf{Sets})$, N of this pushout is a trivial cofibration in $s(s\mathbf{Sets})$, i.e., is a trivial cofibration in the *horizontal* structure on bisimplicial sets.

So N sends any pushout of the monomorphism $P(\Lambda^k[n]) \hookrightarrow P(\Delta[n])$ to an arrow in *bis*Sets which is a composite of a vertical and a horizontal trivial cofibration of bisimplicial sets. Considering a transfinite composition of those, δ^* sends it, since δ^* preserves colimits and monomorphisms and sends vertical and horizontal weak equivalences of bisimplicial sets to weak equivalences of simplicial sets by Lemma 8.1, to a transfinite composition of trivial cofibrations of simplicial sets. By the closed model structure on *s*Sets this is a trivial cofibration as well, in particular, it is a weak equivalence in *s*Sets, as was needed.

Concluding:

Theorem 8.2. *There is a closed model structure on the category of simplicial sheaves of groupoids on* $(\mathbb{C}, J)$, *where the weak equivalences are arrows for which the diagonal of the dimensionwise nerve is a weak equivalence of simplicial sheaves.*

8.3. Classifying homotopy with simplicial sheaves of groupoids

As in the previous cases, the homotopy category of simplicial sheaves of groupoids can be compared with the homotopy category of simplicial sheaves.

Theorem 8.3. *The functors P and B induce an equivalence of homotopy categories*

$$\mathrm{Ho}(s\mathcal{E}) \underset{P}{\overset{B}{\rightleftarrows}} \mathrm{Ho}(s\mathrm{Grpd}(\mathcal{E})).$$

Proof. By arguments as before P preserves cofibrations, and weak equivalences between cofibrant objects because the proof of Lemma 7.3 goes through without change. B preserves fibrations and weak equivalences by definition. Applying Lemma 2.1 gives that P and B induce adjoint functors between the homotopy categories. This adjointness is an equivalence of categories if both unit and counit are weak equivalences.

By definition, $\varepsilon_{\mathcal{G}} : PB(\mathcal{G}) \to \mathcal{G}$ is a weak equivalence if $B(\varepsilon_{\mathcal{G}}) : BPB(\mathcal{G}) \to B(\mathcal{G})$ is. By one of the triangular identities this map is left inverse to $\eta_{B(\mathcal{G})} : B(\mathcal{G}) \to BPB(\mathcal{G})$, so axiom CM2 gives that if the unit is a weak equivalence, the counit is too.

To show that the unit $\eta_X : X \to BP(X)$ is a weak equivalence, observe that $X = Bdis(X)$, where $dis : s\mathcal{E} \to s\mathrm{Grpd}(\mathcal{E})$ sends a simplicial object to the discrete simplicial groupoid on it, and consider the map $\varepsilon_{dis(X)} : P(X) = PBdis(X) \to dis(X)$. B of this map is left inverse to $\eta_{Bdis(X)} = \eta_X$ by the same triangular identity. So by axiom CM2 it now suffices to show that $\varepsilon_{dis(X)} : P(X) \to dis(X)$ is a weak equivalence in *s*Grpd$(\mathcal{E})$. But for every $Y \in s\mathcal{E}$ one has $s\mathrm{Grpd}(\mathcal{E})(P(X), dis(Y)) \cong s\mathcal{E}(X, Bdis(Y)) = s\mathcal{E}(X,Y) \cong s\mathrm{Grpd}(\mathcal{E})(dis(X), dis(Y))$, and because every object of *s*Grpd$(\mathcal{E})$ is weakly equivalent to a discrete one (see [20]), this implies that $\varepsilon_{dis(X)} : P(X) \to dis(X)$ induces an isomorphism in the homotopy category, i.e., that it is a weak equivalence, as required. □

The last argument in this proof, namely that $P(X) \to dis(X)$ is a weak equivalence, also implies that P and dis induce isomorphic functors between the homotopy categories. Thus, the above is an alternative proof of the result in [21] that dis and $(-)_0$ (which is taking the simplicial object of objects of a simplicial groupoid) induce an equivalence of homotopy categories.

Appendix A. λ-small sheaves

There are three important facts that make Quillen's small object argument (see [29, Lemma 3, Section II.3]) work. The first one is that the category of topological spaces has countable colimits of chains. The second one is that the (trivial) fibrations are characterized by the right lifting property with respect to a *set* of maps $\{C_\sigma \to C'_\sigma \mid \sigma \in \Sigma\}$ where each C_σ is "small". The third one is that those small objects C_σ have the property that $\mathbb{C}(C_\sigma, -)$ preserves countable colimits of chains.

To apply such a small object argument when working with a category of sheaves some adaptations must be made. For example, in the category of simplicial sets small objects are finitely generated, but what is the correct analogue of finiteness in the category of simplicial sheaves in this case? This also implies that countable colimits of chains might not be sufficient. And to obtain a characterization of the (trivial) fibrations as above some geometric (see Section 6) construction of small objects must yield a small object.

In [18] these problems are solved by defining λ-smallness essentially as λ-sequentially small (in fact, as λ-presentable, see [26]), taking some large enough infinite regular cardinal λ and showing that the class of λ-small objects satisfies the needed closure properties. I will define λ-smallness via cardinality, which immediately implies some of the properties, and then show that for large enough infinite regular λ the λ-small objects are λ-sequentially small and satisfy also the remaining properties.

A.1. λ-smallness

Definition A.1. Let $\mathbb{C}$ be a category and J a basis for a Grothendieck topology on $\mathbb{C}$. Let λ be an infinite regular cardinal. A sheaf X on the site $(\mathbb{C}, J)$ is *λ-small* if there exists a set $S = \{y \mid y \in X(C_y)\}$ of cardinality less than λ such that the presheaf $Y : \mathbb{C} \to \mathbf{Sets}$ which is given on objects by $Y(C) = \{x \in X(C) \mid x = y \restriction_C \text{ for some } y \in S,\ c : C \to C_y\}$ is a dense subpresheaf of X.

The reason for defining λ-smallness this way instead of as in [18] is that this definition is the direct generalization to sheaves of the notion of a set of cardinality less than λ. The disadvantage of this definition is that it defines λ-smallness in terms of the site, rather than in terms of the topos, as is done in [18]. For large enough λ, however, both definitions will appear to be the same (Theorem A.8). Note that this large enough λ may be smaller than the λ chosen in [18].

Now fix a site $(\mathbb{C}, J)$ and an infinite regular cardinal λ. A first property of the set of λ-small sheaves on $(\mathbb{C}, J)$ is the following (cf. [9, 6.2 Satz]):

Lemma A.2. *A colimit of a collection of cardinality less than λ consisting of λ-small sheaves is again λ-small.*

Proof. Let $\{X_I\}_{I\in\mathbb{I}}$ an $\mathbb{I}$-indexed collection of λ-small sheaves on $(\mathbb{C}, J)$, with $\mathbb{I}$ a category with less than λ many objects. Since every X_I is λ-small there exist sets $S_I = \{y \mid y \in X_I(C_y)\}$ of cardinality less than λ such that for each $I \in \mathbb{I}$ the presheaf $Y_I(C) = \{x \in X_I(C) \mid x = y \restriction_C \text{ for some } y \in S_I,\ c : C \to C_y\}$ is a dense subpresheaf of X_I. Take $S'_I = \{i_*(y) \mid i : I' \to I,\ y \in S_{I'}(C_y)\}$, which not necessarily has cardinality less than λ, and Y'_I likewise. This way each Y'_I is still a dense subpresheaf of X, and moreover $\{Y'_I\}_{I\in\mathbb{I}}$ is an $\mathbb{I}$-indexed collection of presheaves on $\mathbb{C}$. Because sheafification, denoted by $\underline{a}$, commutes with colimits $X = \varinjlim_{\mathbb{I}} X_I = \varinjlim_{\mathbb{I}} \underline{a}(Y'_I) = \underline{a}(\varinjlim_{\mathbb{I}} Y'_I)$, so $\varinjlim_{\mathbb{I}} Y'_I$ is a dense subpresheaf of X. Now take $S = \{[y] \mid y \in S_I \text{ for some } I \in \mathbb{I}\}$, which does have cardinality less than λ by the assumption on $\mathbb{I}$ and by infinite regularity of λ. Then $(\varinjlim_{\mathbb{I}} Y'_I)(C) = \varinjlim_{\mathbb{I}}(Y'_I(C)) = \{[y] \mid y \in Y'_I(C) \text{ for some } I \in \mathbb{I}\} = \{[(i_*(y))\restriction_C] \mid c : C \to C_y,\ [y] \in S_I \text{ for some } I \in \mathbb{I}\} = \{[y]\restriction_C \mid c : C \to C_y,\ [y] \in S\}$, which shows that X is again λ-small. □

The λ-small sheaves generate the topos $\mathrm{Sh}(\mathbb{C}, J)$, in the sense that:

Lemma A.3. *Every sheaf on $(\mathbb{C}, J)$ is λ-filtered union of its λ-small subsheaves.*

Proof. Let X be a sheaf on $(\mathbb{C}, J)$, and let $\{X_I\}_{I\in\mathbb{I}}$ be the collection of λ-small subsheaves of X. λ-filteredness if $\mathbb{I}$ is immediate from lemma A.2, so it remains to show $X = \varinjlim_{\mathbb{I}} X_I$. To this end, consider for each $x \in X(C_x)$ the λ-small subsheaf Z_x of X with $S = \{x\}$ and $Y_x(C) = \{x \restriction_C \mid c : C \to C_x\}$. Now $\bigcup\{Z_x \mid x \in X(C_x)\} = \bigcup\{\underline{a}(Y_x) \mid x \in X(C_x)\} = \underline{a}(\bigcup_{\text{presh.}}\{Y_x \mid x \in X(C_x)\}) = \underline{a}(X) = X$, from which it follows that X is the λ-filtered union of its λ-small subsheaves. □

In the terminology of [26], this lemma implies that the category $\mathrm{Sh}(\mathbb{C}, J)$ is λ-accessible for every λ.

A.2. Condition on λ

From now on I will assume that λ is such that for every object C of $\mathbb{C}$ the set $\coprod_{C'\in\mathbb{C}} \mathbb{C}(C', C)$ has cardinality less than λ. This assumption on λ is somewhat stronger than needed for each one of the following extra properties of λ-small sheaves, but since I will need all of them in order to apply a small object argument and since λ can always be chosen larger than necessary this causes no problems.

With this extra assumption λ-smallness can be characterized as follows:

Lemma A.4. *Let λ be such that for every object C of $\mathbb{C}$ the set $\coprod_{C' \in \mathbb{C}} \mathbb{C}(C', C)$ has cardinality less than λ. Then a sheaf X on $(\mathbb{C}, J)$ is λ-small iff X has a dense subpresheaf Y such that the cardinality of each of the sets $Y(C)$ is less than λ, and non-empty for only less than λ many $C \in \mathbb{C}$.*

Proof. If X satisfies the right-hand side, take $S = \coprod_{C \in \mathbb{C}} Y(C)$. Conversely, if X is λ-small with S and Y showing this, then because the site is of local weight less than λ the set S can be extended with all restrictions of elements of S without raising its cardinality. But then $S = \coprod_{C \in \mathbb{C}} Y(C)$, and the given condition on Y follows. □

A.3. Properties

In general, a subsheaf of a λ-small sheaf need not be λ-small, as the following example shows. Let $\mathbb{C}$ be the category with as objects the ordinals less than or equal to λ, and for every $\kappa < \lambda$ an arrow $\kappa \to \lambda$. Let J be the minimal topology, and X the sheaf on $(\mathbb{C}, J)$ defined by $X(C) = \{*\}$ for every $C \in \mathbb{C}$. Then clearly X is λ-small (take $S = \{* \mid * \in X(\lambda)\}$), but the subsheaf $Y(C) = \{* \mid C < \lambda\}$ is not.

With the extra assumption on λ, however, the set of λ-small sheaves does have that property:

Lemma A.5. *Assume that λ satisfies the extra assumption. Then every subsheaf of a λ-small sheaf is again λ-small.*

Proof. Let W be a subsheaf of the λ-small sheaf X on $(\mathbb{C}, J)$. By Lemma A.4 X has a dense subpresheaf Y with only less than λ many non-empty $Y(C)$, each of which has cardinality less than λ. Now $W \cap Y$ is a dense subpresheaf of W since $\underline{a}(W \cap Y) = \underline{a}(W) \cap \underline{a}(Y) = W \cap X = W$, and because $(W \cap Y)(C) \subseteq Y(C)$ for every $C \in \mathbb{C}$ the sets $(W \cap Y)(C)$ have cardinality less than λ, and are non-empty for only less than λ many C. Lemma A.4 again implies that W is indeed λ-small. □

In general, the product of two λ-small sheaves need not be λ-small, as the following example shows. Let $\mathbb{C}$ be the category with two objects 0 and 1, and λ many arrows from 1 to 0. Let J be the minimal topology, X the sheaf on $(\mathbb{C}, J)$ defined by $X(1) = \{*\}$, $X(0) = \lambda$, and X' the sheaf on $(\mathbb{C}, J)$ defined by $X(1) = \emptyset$, $X(0) = \{0\}$. Clearly X and X' are λ-small (take $S = \{* \mid * \in X(1)\}$ and $S' = \{0 \mid 0 \in X'(0)\}$), but $X \times X'$ is not.

With the extra assumption on λ, however, the set of λ-small sheaves does have that property:

Lemma A.6. *Assume that λ satisfies the extra assumption. Then the product of two λ-small sheaves on $(\mathbb{C}, J)$ is again λ-small.*

Proof. Let X and X' be λ-small sheaves on $(\mathbb{C}, J)$, thus, by Lemma A.4, having dense subpresheaves Y and Y' respectively such that $Y(C)$ and $Y'(C)$ are non-empty for less than λ many C and have cardinality less than λ. Then $X \times X'$ has a dense subpresheaf

$Y \times Y'$, there are only less than λ many non-empty sets $Y(C) \times Y'(C)$, and each one of those has cardinality less than $\lambda^2 = \lambda$. $\square$

A.4. Characterization

Before comparing λ-small sheaves as defined here with λ-smallness as defined in [18] the following is needed about λ-filtered colimits of sheaves.

Lemma A.7. *Assume that λ satisfies the extra assumption. Then λ-filtered colimits in* $\mathrm{Sh}(\mathbb{C}, J)$ *can be calculated pointwise.*

Proof. Standard sheaf arguments. $\square$

With this lemma:

Theorem A.8. *Let $(\mathbb{C}, J)$ be a site, and let λ be an infinite regular cardinal such that for every object C of $\mathbb{C}$ the set $\coprod_{C' \in \mathbb{C}} \mathbb{C}(C', C)$ has cardinality less than λ. Then a sheaf X on $(\mathbb{C}, J)$ is λ-small iff $\mathrm{Sh}(\mathbb{C}, J)(X, -)$ preserves λ-filtered colimits.*

Proof. Suppose X is λ-small with Y as in Lemma A.4, and let $\{Z_I\}_{I \in \mathbb{I}}$ be an $\mathbb{I}$-indexed collection of sheaves on $(\mathbb{C}, J)$, with $\mathbb{I}$ a λ-filtered category, and with colimit Z. There is a canonical map $\varphi : \varinjlim_{\mathbb{I}} \mathrm{Sh}(\mathbb{C}, J)(X, Z_I) \to \mathrm{Sh}(\mathbb{C}, J)(X, Z)$ given by $\varphi([f_I : X \to Z_I]) = p_I \circ f_I$, where p_I is the colimit injection $Z_I \to Z$.

To prove φ to be surjective, let $f : X \to Z$ be a sheaf morphism. For $y \in Y(C_y)$, suppose $f_{C_y}(y) = [z_y]$ with $z_y \in Z_{I_y}(C_y)$, using the description of Lemma A.7 for the λ-filtered colimit Z. Since there are less than λ many y's to be considered there is, by λ-filteredness of $\mathbb{I}$, an I_Y majoring all I_y's. Now every $[z_y]$ can be represented by an element of $Z_{I_Y}(C_y)$, which implies that the presheaf map $f \restriction_Y : Y \to Z$ factors through Z_{I_Y}. But X is the associated sheaf of Y and Z_{I_Y} is a sheaf so f also factors through Z_{I_Y}.

To prove φ to be injective, let $[f_I : X \to Z_I]$ and $[f_{I'} : X \to Z_{I'}]$ both have φ-image $f : X \to Z$. This means, again using Lemma A.7, that for all $x \in X(C_x)$ the elements of Z represented by $(f_I)_{C_x}(x)$ and $(f_{I'})_{C_x}(x)$ are equal, which by filteredness of $\mathbb{I}$ comes down to the existence of a diagram

$$\begin{array}{ccc} I & & I' \\ {\scriptstyle i}\searrow & & \swarrow{\scriptstyle i'} \\ & I_x & \end{array}$$

in $\mathbb{I}$ such that $(Z_i)_{C_x}((f_I)_{C_x}(x)) = (Z_{i'})_{C_x}((f_{I'})_{C_x}(x))$. In particular, this holds for every $y \in Y(C_y)$. Since there are less than λ many y's to be considered there exists, by λ-filteredness of $\mathbb{I}$, a cocone I_Y for the diagram consisting of I, I' and all I_y, say with maps $i_Y : I \to I_Y$ and $i'_Y : I' \to I_Y$ in $\mathbb{I}$. Now $(Z_{i_Y} \circ f_I) \restriction_Y = (Z_{i'_Y} \circ f_{I'}) \restriction_Y$, and because X is the associated sheaf of Y and Z_{I_Y} is a sheaf also $Z_{i_Y} \circ f_I = Z_{i'_Y} \circ f_{I'}$, in other words, $[f_I] = [f_{I'}]$.

Conversely, suppose X satisfies the right-hand side of the statement in the proposition. By Lemma A.3, $X = \varinjlim_{\mathbb{I}} X_I$ with $\{X_I\}_{I\in\mathbb{I}}$ the collection of λ-small subsheaves of X. Now $\mathrm{Sh}(\mathbb{C},J)(X,X) = \mathrm{Sh}(\mathbb{C},J)(X,\varinjlim_{\mathbb{I}} X_I) = \varinjlim_{\mathbb{I}} \mathrm{Sh}(\mathbb{C},J)(X,X_I)$. In particular, id : $X \to X$ factors through a certain X_I, which implies, because X_I is a subsheaf of X, that $X \cong X_I$, in other words, that X is λ-small. □

This proposition gives that for λ large enough the definition of λ-smallness given here agrees with the one in [18], which means that for such λ the λ-small objects are λ-sequentially small, which can be used to do a "λ-small object argument". Because of this observation there is no harm in confusing λ-smallness and λ-sequential smallness, especially when there is no sheaf in sight, as in Sections 3 and 4.

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References

[1] A. Bousfield and D. Kan, Homotopy Limits, Completions and Localisations, Lecture Notes in Mathematics, Vol. 304 (Springer, New York, 1972).
[2] K.S. Brown, Abstract homotopy theory and generalized sheaf cohomology, Trans. Amer. Math. Soc. 186 (1973) 419–458.
[3] K.S. Brown and S.M. Gersten, Algebraic K-theory as generalized sheaf cohomology, in: Algebraic K-theory I: Higher K-theories, Lecture Notes in Mathematics, Vol. 341 (Springer, Berlin, 1973) 266–292.
[4] M. Bullejos, A. Cegarra and J. Duskin, On cat^n-groups and homotopy types, J. Pure Appl. Algebra 86 (1993) 135–154.
[5] S.E. Crans and I. Moerdijk, On simplicial groupoids, in preparation.
[6] W. Dwyer and D.M. Kan, Homotopy theory and simplicial groupoids, Proc. Koninklijke Academie van Wetenschappen, A 87 (4) (1984) 379–385.
[7] G. Ellis, Crossed modules and their higher dimensional analogue, Ph.D. Thesis, University of Wales, 1984.
[8] G. Ellis and R. Steiner, Higher dimensional crossed modules and the homotopy groups of $(n+1)$-ads, J. Pure Appl. Algebra 46 (1987) 117–136.
[9] P. Gabriel and F. Ulmer, Lokal Präsentierbare Kategorien, Lecture Notes in Mathematics, Vol. 221 (Springer, Berlin, 1971).
[10] P. Gabriel and M. Zisman, Calculus of Fractions and Homotopy Theory, (Springer, Berlin, 1967).
[11] J. Garcia Cabello and A.R. Garzón, Closed model structures for algebraic models of n-types, Preprint, University of Granada, 1992.
[12] H. Gillet and D.R. Grayson, On the loop space of the q-construction, Illinois J. Math. 31 (1987) 574–597.
[13] A. Grothendieck, Pursuing stacks, letter to D.G. Quillen, 1986.
[14] A. Heller, Homotopy theories, Mem. Amer. Math. Soc. 71 (383) (1988).
[15] J. Jardine, Simplicial objects in a Grothendieck topos, in: Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory, Part I, Contemporary Mathematics, Vol. 55 (American Mathematical Society, Providence, RI, 1986) 193–239.

[16] J. Jardine, Simplicial presheaves, J. Pure Appl. Algebra 47 (1987) 35–87.
[17] P. Johnstone, Topos Theory, London Mathematical Society Monographs (Academic Press, London, 1977).
[18] A. Joyal, letter to A. Grothendieck, 1984.
[19] A. Joyal and M. Tierney, Classifying spaces for sheaves of simplicial groupoids, J. Pure Appl. Algebra 89 (1993) 135–161.
[20] A. Joyal and M. Tierney, Strong stacks and classifying spaces, in: Category Theory, Lecture Notes in Mathematics, Vol. 1488 (Springer, Berlin, 1991) 213–236.
[21] A. Joyal and M. Tierney, On the homotopy theory of sheaves of simplicial groupoids, preprint, 1993.
[22] D.M. Kan, A combinatorial definition of homotopy groups, Ann. of Math. 67 (1958) 282–312.
[23] J.-L. Loday, Spaces with finitely many non-trivial homotopy groups, J. Pure Appl. Algebra 24 (1982) 179–202.
[24] S. Mac Lane and I. Moerdijk, Sheaves in Geometry and Logic (Springer, Berlin, 1992).
[25] S. Mac Lane and J. Whitehead, On the 3-type of a complex, Proc. Nat. Acad. Sci. 36 (1) (1950) 41–48.
[26] M. Makkai and R. Paré, Accessible Categories: The Foundations of Categorical Model Theory, Contemporary Mathematics, Vol. 104 (American Mathematical Society, Providence, RI, 1989).
[27] I. Moerdijk, Bisimplicial sets and the group completion theorem, in: Algebraic K-Theory: Connections with Geometry and Topology (Kluwer, Dordrecht, 1989) 225–240.
[28] I. Moerdijk and J. Svensson, Algebraic classification of equivariant homotopy 2-types, part I, J. Pure Appl. Algebra 89 (1993) 187–216.
[29] D.G. Quillen, Homotopical Algebra, Lecture Notes in Mathematics, Vol. 43 (Springer, Berlin, 1967).
[30] D.G. Quillen, Rational homotopy theory, Ann. of Math. 90 (2) (1969) 205–295.
[31] D.G. Quillen, Higher algebraic K-theory: I, in: Algebraic K-theory I: Higher K-theories, Lecture Notes in Mathematics, Vol. 341 (Springer, Berlin, 1973) 84–147.
[32] R. Street, The algebra of oriented simplexes, J. Pure Appl. Algebra 49 (1987) 283–335.
[33] R.W. Thomason, *Cat* as a closed model category, Cahiers Topologie Géometrie Differentielle XXI (3) (1980) 305–324.
[34] J. Whitehead, Combinatorial homotopy II, Bull. Amer. Math. Soc. 55 (1949) 453–496.