Math. Proc. Camb. Phil. Soc. (1986), **100**, 65
Printed in Great Britain

Vogt's theorem on categories of homotopy coherent diagrams

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(*Received* 14 *March* 1985; *revised* 31 *October* 1985)

Let *Top* be the category of compactly generated topological spaces and continuous maps. The category, *Top*, can be given the structure of a simplicially enriched category (or **S**-category, **S** being the category of simplicial sets). For **A** a small category, Vogt (in [**22**]) constructed a category, Coh (**A**, *Top*), of homotopy coherent **A**-indexed diagrams in *Top* and homotopy classes of homotopy coherent maps, and proved a theorem identifying this as being equivalent to Ho ($Top^{\mathbf{A}}$), the category obtained from the category of commutative **A**-indexed diagrams by localizing with respect to the level homotopy equivalences. Thus one of the important consequences of Vogt's result is that it provides concrete coherent models for the formal composites of maps and formal inverses of level homotopy equivalences which are the maps in Ho ($Top^{\mathbf{A}}$). The usefulness of such models and in general of Vogt's results is shown in the series of notes [**14**–**17**] by the second author in which these results are applied to give an obstruction theory applicable in prohomotopy theory.

Vogt's proof is quite difficult to follow as it relies very heavily on results from his work with Boardman [**1**]. In this paper, we give a simplification of Vogt's proof. Our proof replaces Vogt's *Top*-enriched categorical methods by simplicially enriched methods. This allows us not only to generalize Vogt's theorem to handle other **S**-categories than *Top* but also to distinguish between two parts of the proof. Each distinct part uses different properties of the **S**-category being considered. (It should be noted that whilst Vogt uses the monoid multiplication $t_1 * t_2 = t_1 t_2$ on [0, 1], our results being simplicial use the alternative form $t_1 * t_2 = \sup(t_1, t_2)$.)

From the first author's description of Coh (**A**, *Top*) (given in [**6**]), one obtains a generalization Coh ($\mathbf{A}, \mathbf{B}_S$) for any **S**-category $\mathbf{B}_S$. This category Coh ($\mathbf{A}, \mathbf{B}_S$) is defined to be the category associated to the simplicial class **S**-*Cat* ($\mathbf{S}(\mathbf{A}), \mathbf{B}_S$) where **S**(**A**) is a certain comonad simplicial resolution of **A**. There is a natural functor

$$\gamma \colon \mathbf{B}^{\mathbf{A}} \to \mathrm{Coh}\,(\mathbf{A}, \mathbf{B}_S)$$

sending an actually commutative diagram to itself considered as a homotopy coherent diagram. (More details are given in the first section.)

The first part shows that γ *inverts level homotopy equivalences in* $\mathbf{B}^{\mathbf{A}}$. By a *level homotopy equivalence*, we mean a map $f\colon X \to Y$ in $\mathbf{B}^{\mathbf{A}}$ so that, for each object i in **A**, the map $f(i)\colon X(i) \to Y(i)$ is a homotopy equivalence. Thus each $f(i)$ will have a homotopy inverse $g(i)$ but these $(g(i)\colon\ Y(i) \to X(i))$ will usually not be natural, i.e. they will not give us a map $g\colon Y \to X$ in $\mathbf{B}^{\mathbf{A}}$. We prove that the $g(i)$ *do* give one a coherent map from X to Y, in fact we prove more (see Theorem 1·1 for a detailed statement): *if $f\colon X \to Y$ is a coherent map between coherent diagrams X and Y of type* $\mathbf{A}$ (*that is a diagram of form* $\mathbf{A} \times [1]$ *agreeing with X and with Y on* $\mathbf{A} \times \{0\}$ *and* $\mathbf{A} \times \{1\}$ *respectively*) *and if each $f(i)\colon X(i) \to Y(i)$ is a homotopy equivalence, then there is a coherent map, $g\colon Y \to X$, homotopy inverse to f in* Coh $(\mathbf{A}, \mathbf{B}_S)$.

Our proof depends on filling arguments and so it needs the assumption that $\mathbf{B}_S$ is 'locally Kan', i.e. that each hom simplicial set $\mathbf{B}_S(X, Y)$ is a Kan complex. This means that this part of the result does not apply to $\mathbf{S}$ itself but does apply to categories of spaces, Kan complexes, simplicial modules, chain complexes, crossed complexes and probably also to Cat^n (Groups). It will also apply to categories of commutative diagrams in the above cases.

The second part uses an adaptation of a construction of Graeme Segal to 'rigidify' a coherent diagram; that is to say, given a homotopy coherent $\mathbf{A}$-indexed diagram X, we construct a commutative diagram $\hat{X}$ indexed by $\mathbf{A}$ and a coherent map

$$X \to \hat{X}$$

such that for each object A of $\mathbf{A}$, the corresponding

$$X(A) \to \hat{X}(A)$$

is a homotopy equivalence. The construction requires that $\mathbf{B}_S$ be complete or co-complete, but in fact in the useful case of $\mathbf{B}_S = Kan$, the category of Kan complexes, a short additional argument shows that if X is a coherent diagram of Kan complexes, so is $\hat{X}$, and hence the rigidification works in this case even though Kan is not complete.

Using this construction together with the results of the previous section one shows that the *induced functor*

$$\gamma*\colon \mathrm{Ho}(\mathbf{B}^{\mathbf{A}}) \to \mathrm{Coh}\,(\mathbf{A}, \mathbf{B}_S)$$

is an equivalence of categories (thus generalizing Vogt's result for the case $\mathbf{B}_S = Top$).

The rigidification construction is a type of homotopy coherent Kan extension. Such constructions have been mentioned in the literature, e.g. by Heller [**24**], but we were unable to find a reference which handled our particular case in a detailed enough way. We therefore include the details for completeness. In fact the theory of coherent Kan extensions (as opposed to homotopy Kan extensions) is little represented in the literature. A detailed treatment of their general theory is in preparation.

Finally we should briefly mention a recent preprint of Dwyer and Kan [**23**] received after the first version of this paper was written. They prove, amongst other things, results linking localized categories Ho $(\mathbf{S}^{\mathbf{A}})$ and Ho $(\mathbf{S}^{\mathbf{S}(\mathbf{A})})$ (in our notation), but where localization is with respect to level weak equivalences, not level homotopy equivalences. Their results therefore do not give them information on homotopy coherent diagrams, although it may be possible to deduce special cases of our generalized form of Vogt's theorem from their results.

Throughout the paper we will use the language of enriched category theory. A useful reference for this theory is Kelly [**11**].

We would like to thank the referee for helpful advice which improved the presentation of our results.

1. *Descriptions of homotopy coherence*

We start with a brief introduction to homotopy coherence, as this is quite hard to find in the literature.

The intuitive idea of a homotopy coherent diagram is best illustrated by a sequence of interrelated examples, namely the n-simplices for different n. Recall that the category $[n]$ has as objects the integers $\{0, 1, \ldots, n\}$ and has a unique map, (ij), from i to j if $i \leqslant j$. Thus we get:

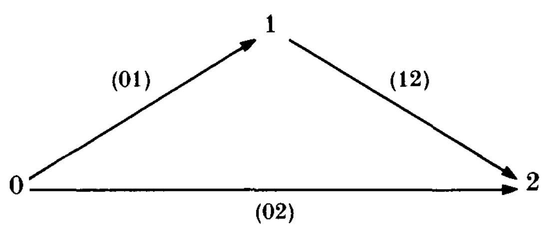
[0] has a single morphism.

[1] looks like

$$0 \rightarrow 1$$

together with, of course, identities at 0 and 1.

[2] is likewise



where the composite of the morphism (01) with (12) is (02) and so on.

Diagrams in *Top* of type [0] and [1] are, of course, respectively spaces and maps. At $n = 2$, the first homotopy phenomenon can occur; a diagram of type [2] in *Top*

$$F\colon [2] \rightarrow Top$$

consists of three spaces $F(0)$, $F(1)$ and $F(2)$ and maps $F(01)$, $F(12)$, and $F(02)$ such that $F(02) = F(12)\,F(01)$.

A homotopy commutative diagram of type [2] has the same data as this but one merely requires that $F(12)\,F(01)$ is homotopic to $F(02)$; a homotopy coherent diagram of type [2] in addition specifies the homotopy

$$F(012)\colon F(0) \times I \rightarrow F(2)$$

such that

$$F(012)\,(x, 1) = F(12)\,F(01)\,(x)$$

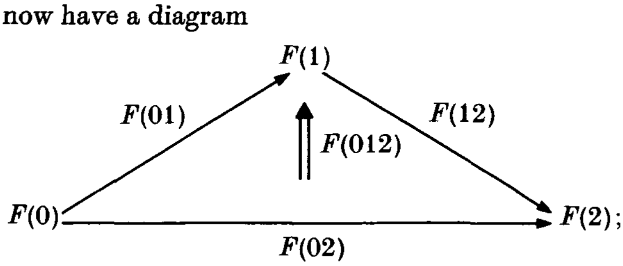
and

$$F(012)\,(x, 0) = F(02)\,(x)$$

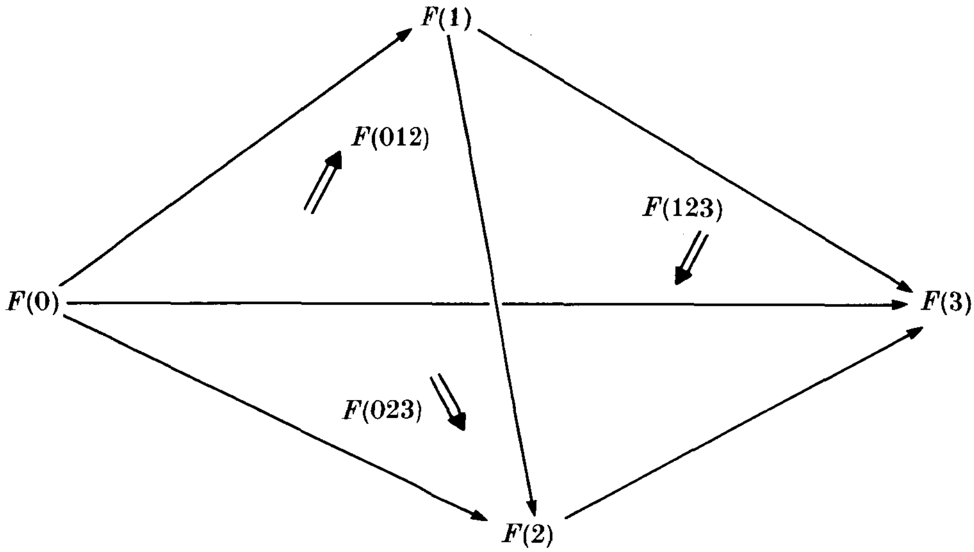
for all $x \in F(0)$, $I = [0, 1]$ being the unit interval.

The full significance of the difference between homotopy commutative and homotopy coherent is evident when one considers $n = 3$.

For $n = 2$, we now have a diagram



for $n = 3$ we thus already have a diagram



plus a homotopy $F(013)$ on the back face of the simplex. More precisely, we have homotopies

$$F(012)\colon F(0)\times I\to F(2),\quad F(012)\colon F(02)\simeq F(12)F(01),$$
$$F(013)\colon F(0)\times I\to F(3),\quad F(013)\colon F(03)\simeq F(13)F(01),$$
$$F(023)\colon F(0)\times I\to F(3),\quad F(023)\colon F(03)\simeq F(23)F(02),$$
$$F(123)\colon F(1)\times I\to F(3),\quad F(123)\colon F(13)\simeq F(23)F(12).$$

These are composable as follows:

$$F(03)\overset{F(023)}{\simeq}F(23)F(02)\overset{F(23)F(012)}{\simeq}F(23)F(12)F(01)$$

and

$$F(03)\overset{F(013)}{\simeq}F(13)F(01)\overset{F(123)(F(01)\times I)}{\simeq}F(23)F(12)F(01).$$

These compositions may be conveniently schematized in a face diagram square

For homotopy coherence, we need a homotopy between the composed homotopies, i.e. a map

$$F(0123)\colon F(0)\times I^2\to F(3)$$

'filling in' the square.

For $n = 4$, the actual diagram is more difficult to make precise, but the face diagram (a cube with one commuting face) can still be drawn. Apart from its faces all of which compose well, one needs an additional

$$F(01234): F(0) \times I^3 \rightarrow F(4)$$

linking these composites – and so on.

Clearly one might attempt some sort of inductive definition using the face diagrams of higher dimensional simplices; however, this would seem to be technically very difficult and it is better to approach the general case in one go. This is what Vogt does in [**21**].

There is a formal similarity between the chains of composable maps used by Vogt and the bar construction of the comonad free simplicial group resolution used in group cohomology. This leads one to consider the analogue for (small) categories (which are, in any case, 'merely' monoids with many objects).

From the category *Cat* of small categories, there is an obvious forgetful functor to the category of directed graphs with distinguished loops at the vertices. The free category construction gives a left adjoint to this forgetful functor and together they induce a comonad on *Cat* (cf. Mac Lane [**13**] for the basic ideas behind monads and comonads). This in turn yields an **S**-category $\mathbf{S}(\mathbf{A})$ together with an augmentation functor

$$\mathbf{S}(\mathbf{A}) \xrightarrow{K} \mathbf{A}$$

which for each pair (A, B) of objects of **A** gives a homotopy equivalence of simplicial sets

$$\mathbf{S}(\mathbf{A})(A, B) \rightarrow K(\mathbf{A}(A, B), 0)$$

where $K(\mathbf{A}(A, B), 0)$ is the simplicial set with $K(\mathbf{A}(A, B), 0)_0 = \mathbf{A}(A, B)$ and all n-simplexes with $n > 0$ degenerate.

In [**6**] one finds a proof that coherent diagrams of type **A** à la Vogt correspond precisely to **S**-functors

$$\mathbf{S}(\mathbf{A}) \rightarrow Top_S.$$

The 'secret' behind the success of $\mathbf{S}(\mathbf{A})$ in capturing coherence phenomena may be seen as follows:

$\mathbf{S}(\mathbf{A})_0$ consists of composable chains of maps in **A**, none of which is the identity.

$\mathbf{S}(\mathbf{A})_1$ consists of composable chains of maps in $\mathbf{S}(\mathbf{A})_0$, none of which is the identity, and hence can be considered as consisting of chains of maps in **A** together with a choice of bracketing.

To see this let us examine $\mathbf{A} = [3]$. $\mathbf{S}([3])(0, 3)_0$ for instance is the set

$$\{(03), (01, 13), (02, 23), (01, 12, 23)\}$$

whilst

$$\mathbf{S}([3])(0, 3)_1 = \{((01, 13)), ((01), (12, 23)), ((02, 23)), ((01, 12)), (23)), ((01, 12, 23))\}$$

plus degenerate simplices. (Think of words in the generators of a group giving a free group and then words in these words giving the next level of a resolution, and so on.)

It is an instructive exercise to calculate $\mathbf{S}(\mathbf{A})$ for $\mathbf{A} = [n]$, $n = 2, 3, 4$ say, and to check that this does tally with our 'intuitive' description. The case $\mathbf{A} = [3]$ is handled in some detail in our notes [**7**].

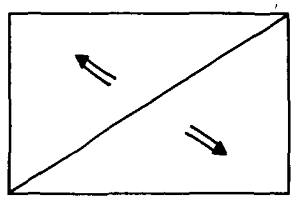
Since we now have a simplicial description of homotopy coherence, it seems a good idea to use this as a basis for a general definition.

Let $\mathbf{B}_S$ be an **S**-category and **A** a small category. A *homotopy coherent diagram of type* **A** *in* $\mathbf{B}_S$ is an **S**-functor from **S(A)** to $\mathbf{B}_S$. (This is equivalent to specifying a simplicial map from Ner (**A**), the nerve of **A**, to $\mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$, the homotopy coherent nerve of $\mathbf{B}_S$ (see [**6**] and Section 2 below for details on $\mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$.)

Remark. It is worth remarking that in general the naive description, even for $\mathbf{A} = [3]$, may not quite agree with that given by this definition. The difficulty is to know what a homotopy between homotopies should be relative to the **S**-category structure of $\mathbf{B}_S$. If one takes it to be a map

$$I \times I = \Delta[1] \times \Delta[1] \rightarrow \mathbf{B}_S(F(0), F(3))$$

all turns out well. The slight difficulty occurs because in the square face diagram the higher simplicial homotopies go from the middle to the sides,



and unless reverse homotopies can be defined in $\mathbf{B}_S$ (e.g. if $\mathbf{B}_S$ is locally Kan) some difficulty in reconciling an 'exterior' view of the higher homotopies with the 'correct' interior view may be experienced.

The following 'bare hands' description of a homotopy coherent diagram generalizes that of Vogt to the case of an arbitrary **S**-category (see [**6**]). We start with some notation; we work simplicially throughout.

As before $I = \Delta[1]$. We let $m: I^2 \rightarrow I$ be the multiplicative structure on I making it into a simplicial monoid. It is given by

$$m(0,0) = 0,$$
$$m(0,1) = m(1,0) = m(1,1) = 1.$$

For any n, we write I^n for the n-cube, the product of n copies of I. Finally, given two simplicial maps

$$f: K_1 \rightarrow \mathbf{B}_S(x,y),$$
$$g: K_2 \rightarrow \mathbf{B}_S(y,z),$$

we will denote the composition

$$K_1 \times K_2 \xrightarrow{f \times g} \mathbf{B}_S(x,y) \times \mathbf{B}_S(y,z) \xrightarrow{c} \mathbf{B}_S(x,z)$$

simply by gf.

Now we can give the data for a homotopy coherent diagram

$$F: \mathbf{S}(\mathbf{A}) \rightarrow \mathbf{B}_S$$

as follows: to each object A of **A** it assigns an object $F(A)$ of $\mathbf{B}_S$; for each

$$\sigma = (f_0, \ldots, f_n) \in \mathbf{A}^{n+1}(A, B)$$

a simplicial map

$$F(\sigma): I^n \rightarrow \mathbf{B}_S(F(A), F(B))$$

such that

(i) if $f_0 = \mathrm{id}$, $F(\sigma) = F(\partial_0 \sigma)\,(\mathrm{proj} \times I^{n-1})$, where $\mathrm{proj}: I \rightarrow \Delta[0]$ is the unique map to $\Delta[0]$;

(ii) if $f_i = \mathrm{id}$, $0 < i < n$,

$$F(\sigma) = F(\partial_i \sigma).(I^i \times m \times I^{n-i});$$

(iii) if $f_n = \mathrm{id}$, $F(\sigma) = F(\partial_n \sigma)(I^{n-1} \times \mathrm{proj});$

$(\mathrm{iv})_i$ $F(\sigma)|(I^{i-1} \times \{0\} \times I^{n-i}) = F(\partial_i \sigma)$, $1 \leqslant i \leqslant n-1;$

$(\mathrm{v})_i$ $F(\sigma)|(I^{i-1} \times \{1\} \times I^{n-i}) = F(\sigma_i') F(\sigma_i)$,

where $\sigma_i = (f_0, \ldots, f_{i-1})$, and $\sigma_i' = (f_i, \ldots, f_n)$, $1 \leqslant i \leqslant n-1$.

(Here we are using ∂_i for the face operators in the nerve of **A**.)

We will need this detailed description later when discussing a homotopy coherent analogue of the nerve construction.

Given a small category **A** and two homotopy coherent diagrams, F and G, of type **A** in an S-category $\mathbf{B}_S$, we want to define a coherent map, f from F to G.

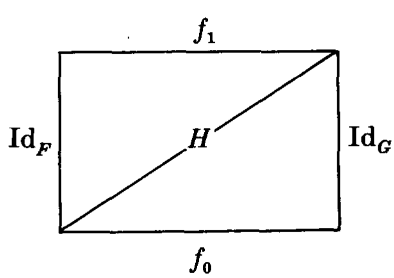
Given ordinary diagrams F, G: $\mathbf{A} \to \mathbf{B}$ a natural transformation from F to G can be considered to be a functor from $\mathbf{A} \times [1]$ to **B** taking the value F on $\mathbf{A} \times \{(0)\}$ and the value of G on $\mathbf{A} \times \{(1)\}$. Thus there is an obvious way of defining a coherent map in our more general situation.

Given coherent diagrams F, G of type **A** in $\mathbf{B}_S$, a *coherent map* f: $F \to G$ is a coherent diagram of type $\mathbf{A} \times [1]$ in $\mathbf{B}_S$ agreeing with F on $\mathbf{A} \times \{(0)\}$ and with G on $\mathbf{A} \times \{(1)\}$.

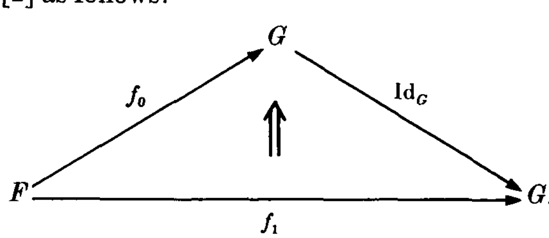
We say two coherent maps f_0, f_1: $F \to G$ are *homotopic* if there is a coherent diagram H of type $\mathbf{A} \times [1] \times [1]$ in $\mathbf{B}_S$ agreeing with f_0 on $\mathbf{A} \times [1] \times \{(0)\}$ and with f_1 on

$$\mathbf{A} \times [1] \times \{(1)\}$$

and such that $H|\mathbf{A} \times \{(0)\} \times [1]$ gives the identity map on F whilst $H|\mathbf{A} \times \{(1)\} \times [1]$ gives the identity map on G:



Remark. If $\mathbf{B}_S$ is locally Kan, this is equivalent to saying that there is a coherent diagram of type $\mathbf{A} \times [2]$ as follows:



As a map is a coherent diagram of type $\mathbf{A} \times [1]$ in $\mathbf{B}_S$, f can be described by a simplicial map

$$f\colon \mathrm{Ner}(\mathbf{A} \times [1]) \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S),$$

or since

$$\begin{aligned} \mathrm{Ner}(\mathbf{A} \times [1]) &\cong \mathrm{Ner}(\mathbf{A}) \times \mathrm{Ner}([1]) \\ &\cong \mathrm{Ner}(\mathbf{A}) \times \Delta[1] \end{aligned}$$

by a simplicial map

$$f\colon \mathrm{Ner}(\mathbf{A}) \times \Delta[1] \to \mathrm{Ner}_{\text{h.c.}}(\mathbf{B}_S)$$

with agreement of f on the ends with F and G respectively. Similarly homotopies of coherent maps can be defined to be maps

$$H\colon \mathrm{Ner}(\mathbf{A}) \times \Delta[1] \times \Delta[1] \to \mathrm{Ner}_{\text{h.c.}}(\mathbf{B}_S)$$

with the obvious agreement on the sides.

Now suppose $\mathbf{B}_S$ is locally Kan, and that

$$f\colon F_0 \to F_1$$

and

$$g\colon F_1 \to F_2$$

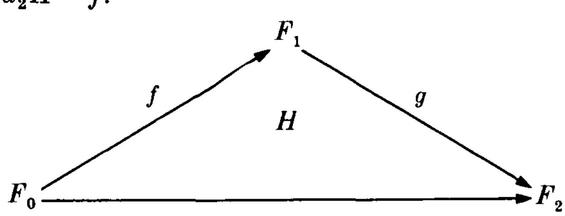
are coherent maps. Then one clearly has a simplicial map

$$\mathrm{Ner}(\mathbf{A}) \times \Lambda^1[2] \to \mathrm{Ner}_{\text{h.c.}}(\mathbf{B}_S)$$

which, since $\mathrm{Ner}_{\text{h.c.}}(\mathbf{B}_S)$ is weakly Kan (see Section 2 for a discussion of this), can be extended to give

$$H\colon \mathrm{Ner}(\mathbf{A}) \times \Delta[2] \to \mathrm{Ner}_{\text{h.c.}}(\mathbf{B}_S),$$

say with $d_0 H = g$, $d_2 H = f$:



We would like to take $d_1 H$ as the composite of f and g; however, this would not be well defined, as there may be more than one extension possible. If on the other hand we pass to homotopy classes, one easily sees that the weak Kan condition satisfied by $\mathrm{Ner}_{\text{h.c.}}(\mathbf{B}_S)$ implies that

$$[g][f] = [d_1 H]$$

gives a well defined composition, as any two choices of extension differ by a homotopy. Also, this composition is associative, has identities, etc., and hence one has a category which we will denote $\mathrm{Coh}(\mathbf{A}, \mathbf{B}_S)$, of coherent diagrams of type $\mathbf{A}$ and homotopy classes of coherent maps between them.

Remark. We can view this category in a slightly different way. Recall that, given any simplicial set $K.$, one can construct an associated (small) category by taking chains of 1-simplices and then adding in the relations coming from the 2-simplices. (If $K.$ is a weak Kan complex all chains of 1-simplices can be replaced by single 1-simplices.) Now $\mathrm{Ner}_{\text{h.c.}}(\mathbf{B}_S)$ is a simplicial class; if we form a new simplicial class $\mathbf{S}(\mathrm{Ner}(\mathbf{A}), \mathrm{Ner}_{\text{h.c.}}(\mathbf{B}_S))$ of all simplicial maps from $\mathrm{Ner}(\mathbf{A})$ to $\mathrm{Ner}_{\text{h.c.}}(\mathbf{B}_S)$ and then form its associated (large) category then we get exactly the category $\mathrm{Coh}(\mathbf{A}, \mathbf{B}_S)$ defined above.

Any actual commutative diagram

$$\mathbf{A} \xrightarrow{F} \mathbf{B}$$

can be considered as an **S**-functor (with the trivial simplicial enrichment on the category **A**)

$$\mathbf{A} \xrightarrow{F} \mathbf{B}_S$$

and hence it gives a composite **S**-functor

$$\mathbf{S}(\mathbf{A}) \longrightarrow \mathbf{A} \xrightarrow{F} \mathbf{B}_S = \mathbf{S}(\mathbf{A}) \xrightarrow{F_s} \mathbf{B}_S.$$

Any natural transformation $f\colon F \to G$ in $\mathbf{B}^{\mathbf{A}} = \mathrm{Func}(\mathbf{A}, \mathbf{B})$, the category of functors from **A** to **B**, similarly gives rise to a coherent map $f_s\colon F_s \to G_s$ (since it can be considered to be a functor

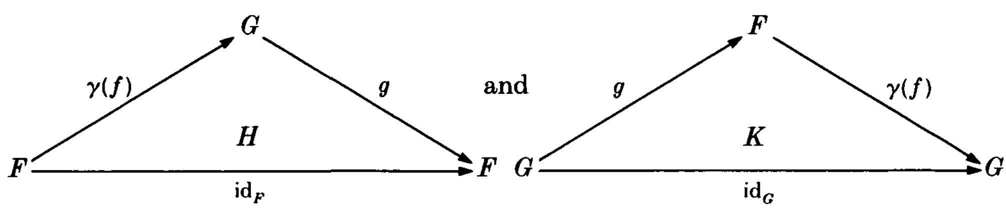
$$\mathbf{A} \times [1] \to B).$$

Thus we get a functor

$$\gamma\colon \mathbf{B}^{\mathbf{A}} \to \mathrm{Coh}\,(\mathbf{A}, \mathbf{B}_S).$$

Our task will be to analyse this functor γ.

Firstly we notice that if f is such that $\gamma(f)$ is invertible (in $\mathrm{Coh}\,(\mathbf{A}, \mathbf{B}_S)$ then as there will be diagrams



(i.e. of type $\mathbf{A} \times \Delta[2]$), f must satisfy the following condition:

for each object i of **A**, $f(i)\colon F(i) \to G(i)$ is a homotopy equivalence.

Definition. We will say in general, that a morphism $f\colon F \to G$ is a *level homotopy equivalence* if for each i in **A** the morphism $f(i)\colon F(i) \to G(i)$ is a homotopy equivalence (thus adapting the terminology of Edwards–Hastings [**8**]).

What has just been observed is thus that if f is such that $\gamma(f)$ is invertible then f is a level homotopy equivalence. What is less obvious is that the converse of this result holds, i.e. any level homotopy equivalence is inverted by γ. Because of this, writing Σ for the class of level homotopy equivalences in $\mathbf{B}^{\mathbf{A}}$ and $\mathrm{Ho}\,(\mathbf{B}^{\mathbf{A}})$ for the category obtained by formally inverting the maps in Σ, one gets an induced functor

$$\bar{\gamma}\colon \mathrm{Ho}\,(\mathbf{B}^{\mathbf{A}}) \to \mathrm{Coh}\,(\mathbf{A}, \mathbf{B}_S).$$

We will study this functor, producing conditions on $\mathbf{B}_S$ which will imply that it is an equivalence.

Our main object in section 3 will be to prove that level homotopy equivalences are inverted by γ. In fact we will prove more.

THEOREM 1·1. *Let* **A** *be a small category,* $\mathbf{B}_S$ *a locally Kan* **S**-*category, and* $f\colon F \to G$ *a coherent map between coherent diagrams of type* **A** *in* $\mathbf{B}_S$. *If for each object* i *in* **A**, *the map* $f(i)\colon F(i) \to G(i)$ *is a homotopy equivalence, then* $[f]$ *is an isomorphism in* $\mathrm{Coh}\,(\mathbf{A}, \mathbf{B}_S)$.

Before giving the proof of this theorem, we will need to study the homotopy coherent nerve of an **S**-category, introduced by the first author in [**6**]. In particular we will need detailed information on the extent to which the Kan conditions are satisfied.

2. *The homotopy coherent nerve*

Let $\mathbf{A}$ be a small category. Then a well known construction yields a simplicial set called the nerve of $\mathbf{A}$, explicitly

$$\mathrm{Ner}(\mathbf{A})_n = \mathrm{Cat}([n], \mathbf{A}).$$

Let $\mathbf{B}_S$ be an $\mathbf{S}$-category. We define, by analogy with the above, a homotopy coherent nerve of $\mathbf{B}_S$ to be a simplicial class $\mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$ specified by

$$\mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)_n = \mathbf{S}\text{-}\mathrm{Cat}(\mathbf{S}[n], \mathbf{B}_S).$$

The following result is fairly easy to check:

There is a one-to-one correspondence between $\mathbf{S}$-functors

$$F\colon \mathbf{S}(\mathbf{A}) \to \mathbf{B}_S$$

and simplicial maps

$$F\colon \mathrm{Ner}(\mathbf{A}) \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S).$$

This result has as an immediate consequence the necessity of studying $\mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$ in detail as it clearly contains more or less all the information necessary for the study of coherent diagrams. We start by studying the extent to which it satisfies the various Kan conditions. To explain to some extent the resulting set of theorems and to fix notation, we shall briefly recall the statement of the analogous results for $\mathrm{Ner}(\mathbf{A})$.

Firstly the notation: if $n \geqslant 0$ then $\Delta[n]$ denotes the standard n-simplex in $\mathbf{S}$, which it is convenient to view as

$$\Delta[n] = \mathrm{hom}(-, [n]).$$

For each $0 \leqslant i \leqslant n$, $\Delta^i[n]$ denotes the subobject of $\Delta[n]$ given by

$$\Lambda^i[n]_m = \begin{cases} \Delta[n]_m & \text{for} \quad 0 \leqslant m < n-1 \\ \Delta[n] - \{d_i\} & \text{if} \quad m = n-1 \\ \text{degenerate simplices if} & m \geqslant n, \end{cases}$$

where $d_i\colon [n-1] \to [n]$ is the increasing map missing out the element i. Of course $\Lambda^i[n]$ is an n-simplex with its centre and ith face missing.

An (n, i) box in a simplicial set $K.$ is a simplicial map

$$F\colon \Lambda^i[n] \to K..$$

A filler for F is a simplicial map,

$$\bar{F}\colon \Delta[n] \to K.$$

which restricts to F on $\Lambda^i[n]$. We say that $K.$ satisfies the (n, i) *extension condition* (of Kan) if any (n, i)-box has a filler. $K.$ is a *weak Kan complex* if it satisfies the (n, i) extension condition for all pairs (n, i) with $0 < i < n$ and it is a *Kan complex* if, in addition, it satisfies all $(n, 0)$ and (n, n)-extension conditions. Thus $K.$ is Kan if all boxes have fillers.

Example. If $\mathbf{A}$ is a category, $\mathrm{Ner}(\mathbf{A})$ is a weak Kan complex. If all morphisms in $\mathbf{A}$ are isomorphisms (i.e. if $\mathbf{A}$ is a groupoid) then $\mathrm{Ner}(\mathbf{A})$ is a Kan complex and conversely. In general, a box

$$f\colon \Lambda^0[n] \to \mathrm{Ner}(\mathbf{A})$$

has a filler if $f(01)$ is an isomorphism and similarly any (n, n)-box with $f(n-1, n)$ an isomorphism has a filler.

As suggested above, the point of this example is that a similar phenomenon occurs with $\mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$ if $\mathbf{B}_S$ is locally Kan. (In fact the following results specialize down to that of the above example when $\mathbf{B}_S$ is a simplicial category with trivial hom simplicial sets.)

THEOREM 2·1. *If $\mathbf{B}_S$ is a locally Kan* **S***-category and*

$$F: \Lambda^i[n] \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$$

is a (n, i)-box in $\mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$ for $0 < i < n$, then it has a filler

$$\bar{F}: \Delta[n] \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S).$$

(In other words if $\mathbf{B}_S$ is locally Kan, $\mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$ is weakly Kan.)

Proof. For any n, $1 \leqslant i \leqslant n$ and $\epsilon = 0$ or 1, we will denote by $J^n_{i,\epsilon}$ the subcomplex of I^n given by

$$J^n_{i,\epsilon} = \text{‘}I^n - \operatorname{int} I^n - I^{i-1} \times \{\epsilon\} \times I^{n-i}\text{’}$$

or more precisely

$$J^n_{i,\epsilon} = \partial I^n - I^{i-1} \times \{\epsilon\} \times I^{n-i},$$

where ∂I^n is the simplicial boundary of I^n.

The map F specifies n different coherent diagrams of type $[n-1]$ in $\mathbf{B}_S$ agreeing on suitable faces. Each of these is completely determined by what F does on its top dimensional simplices. Thus it is as if we knew $F(\partial_j \sigma)$ for all $j \neq i$, and hence that we had a map ϕ_F defined on the union of the faces $\{I^{j-1} \times \{0\} \times I^{(n-1)-j}\}$ of I^{n-1}. We also have all the information on the $F(\sigma'_k) F(\sigma_k)$, as both σ_k and σ'_k are simplices of $\Lambda^i[n]$; i.e. we know ϕ_F also on $I^{j-1} \times \{1\} \times I^{(n-1)-j}$ for all j.

The map F thus gives us already a simplicial map

$$\phi_F: J^{n-1}_{1,0} \to \mathbf{B}_S(F(0), F(n)).$$

$J^{n-1}_{1,0} \hookrightarrow I^{n-1}$ is a simplicial cofibration, and $\mathbf{B}_S(F(0), F(n))$ is a Kan complex, so there exists an extension

$$\bar{F}(\sigma): I^{n-1} \to \mathbf{B}_S(F(0), F(n)).$$

If one now puts $\bar{F}(\partial_i \sigma) = \bar{F}(\sigma) | I^{i-1} \times \{0\} \times I^{n-i-1}$, then the identities (i), ..., $(\mathrm{v})_i$ are all satisfied, thus completing the proof.

If $\mathbf{B}_S = \mathit{Top}_S$, the fillers in $\mathbf{B}_S(F(0), F(n))$ can be specified by choosing specific retractions

$$|\Delta[n]| \to |\Lambda^i[n]|,$$

which will then allow one to write down a specific $\bar{F}$ given F. Of course a change in the choice of fillers, will change $\bar{F}$.

PROPOSITION 2·2. *If $\mathbf{B}_S$ is locally Kan and*

$$F: \Lambda^0[n] \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$$

is a $(n, 0)$-box such that $F(0) = F(1)$ and $F(01)$ is the identity on $F(0)$, then there is a filler

$$\bar{F}: \Delta[n] \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S).$$

Proof. F gives us a box

$$\phi_F: J^{n-1}_{1,1} \to \mathbf{B}_S(F(0), F(n))$$

which extends, as before, to a map

$$\bar{F}\colon I^{n-1} \to \mathbf{B}_S(F(0), F(n)),$$

and one takes

$$\bar{F}(\partial_0\sigma) = \bar{F}(\sigma)\,|\,\{1\}\times I^{n-1},$$

but as $\bar{F}(01) = F(01)$ is the identity on $F(0)$, we are all right. (However, this does show that, in general, one should not hope to get away so lightly!)

('Dualizing', we get a similar result for (n, n)-boxes, if $F(n-1) = F(n)$ and $F(n-1, n)$ is the relevant identity.)

The question immediately arises: *is there a filler for a* $(n, 0)$*-box if* $F(0, 1)$ *is merely a homotopy equivalence?*

To attack this question we first look at a result which is, in some way, a converse of it.

Let us suppose that f is a morphism in $\mathbf{B}_S$, which has the property that for each $(n, 0)$-box

$$F\colon \Lambda^0[n] \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$$

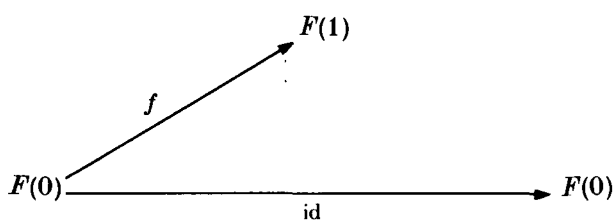
having $F(0, 1) = f$ (and let us say $\mathbf{B}_S$ locally Kan as well), there is a filler

$$F\colon \Delta[n] \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S).$$

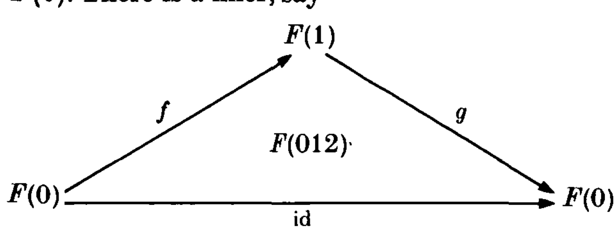
What are the properties of f?

First look at $n = 2$:

We consider the $(2, 0)$-box



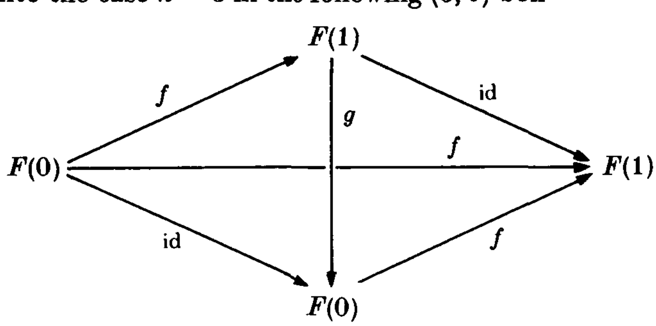
i.e. with $F(2) = F(0)$. There is a filler, say



so f has a one-sided homotopy inverse with homotopy

$$F(012)\colon \mathrm{id} \simeq gf.$$

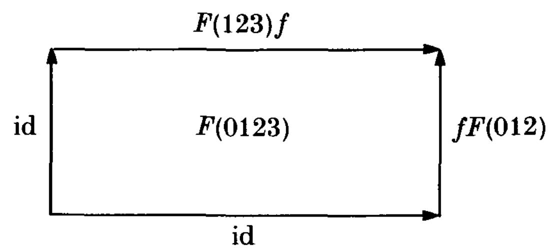
We feed this into the case $n = 3$ in the following $(3, 0)$-box



with $F(2) = F(0)$, $F(3) = F(1)$, $F(012)$ the homotopy already found, and the $F(023)$ and $F(013)$ homotopies trivial. The existence of a filler gives a homotopy

$$F(123)\colon \mathrm{id} \simeq fg,$$

so f is a homotopy equivalence. Our face diagram gives



so the homotopies $F(123)f$ and $fF(012)$ are homotopic via the homotopies involved in $F(0123)$. Using $n = 4$, one can show that $F(012)g$ and $gF(123)$ are similarly homotopic. Thus we have a strongly coherent set of data $(f, g, F(012), F(123))$ for this homotopy equivalence f.

Such 'strong homotopy equivalences' seem first to have been considered by Lashof [**12**]. Vogt in [**21**] proved that in the topological case, any homotopy equivalence f is part of such a strong homotopy equivalence (f, g, H, K). Spencer [**20**] has considered this in an abstract setting (proposition 2·3, p. 416). It is clear from these proofs that the analogous result should work in any locally Kan **S**-category as it only needs results about the 2-categorical structure of *Top* (given by spaces, maps and homotopy classes of homotopies).

We start with the definition of homotopy equivalence in an arbitrary **S**-category $\mathbf{B}_S$. Although it should be clear what the definition is, it is still useful to have it explicitly stated.

$f \in \mathbf{B}_S(\mathrm{A}, \mathrm{B})_0$ is a *homotopy equivalence* if there is a $g \in \mathbf{B}_S(B, A)_0$ and $H \in \mathbf{B}_S(A, A)_1$, $K \in \mathbf{B}_S(B, B)_1$ such that

$$d_0 H = gf, \quad d_1 H = \mathrm{id}_A,$$
$$d_0 K = fg, \quad d_1 K = \mathrm{id}_B.$$

A *strong homotopy equivalence* is a quadruple (f, g, H, K) with two higher homotopies joining fH and Kf, and Hg and gK respectively. The neatest way to make this explicit is the following:

there are two simplicial maps

$$F\colon I \times I \to \mathbf{B}_S(A, B),$$
$$G\colon I \times I \to \mathbf{B}_S(B, A),$$

which are such that for $\sigma_1 \in \Delta[1] = I$, the non-degenerate 1-simplex,

$$F(\{0\} \times \sigma_1) = s_0(f)$$
$$F(\{1\} \times \sigma_1) = fH$$
$$F(\sigma_1 \times \{0\}) = s_0(f)$$
$$F(\sigma_1 \times \{1\}) = Kf,$$

similarly for G with the roles of f and g, and of H and K, reversed.

PROPOSITION 2·3. *Let $\mathbf{B}_S$ be a locally Kan* $\mathbf{S}$*-category. Then, if $f \in \mathbf{B}_S(A,B)_0$ is a homotopy equivalence with g as its homotopy inverse and $H \in \mathbf{B}_S(A,A)_1$ with $d_0 H = gf$, $d_1 H = \mathrm{id}_A$, there is a $K \in \mathbf{B}_S(B,B)_1$ such that (f,g,H,K) is a strong homotopy equivalence.*

Proof. It is well known (or see Gabriel and Zisman [9]) that, if X is a simplicial set, one can form its fundamental groupoid ΠX by considering the free groupoid on the 1-skeleton and then dividing out by the relations coming from the 2-simplices. Also it is clear that, if X is a Kan complex, each element in the set $\Pi X(x,y)$ for $x,y \in X_0$ (i.e. each path from x to y in X) is the image of a 1-simplex (i.e. an edge) in X from x to y, the proof being by induction on the length of the path.

The functor Π preserves products, so from the compositions

$$c\colon \mathbf{B}_S(A,B) \times \mathbf{B}_S(B,C) \to \mathbf{B}_S(A,C),$$

we derive compositions

$$c^*\colon \Pi\mathbf{B}_S(A,B) \times \Pi\mathbf{B}_S(B,C) \to \Pi\mathbf{B}_S(A,C).$$

Thus we have a 2-category $\Pi\mathbf{B}_S$ with, for each pair A,B of objects in $\mathbf{B}_S$, a groupoid $\Pi\mathbf{B}_s(A,B)$. (Of course the objects and maps of $\Pi\mathbf{B}_S(A,B)$ can be interpreted as being maps and homotopy classes of homotopies between them – provided that $\mathbf{B}_S$ is locally Kan.)

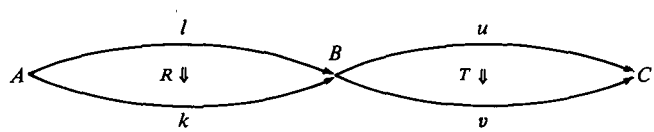
We follow Vogt [20] in stating two lemmas.

LEMMA 1. *Suppose A, B, C are objects in $\mathbf{B}_S$, l, $k \in \mathbf{B}_S(B,C)_0$, u, $v \in \mathbf{B}_S(B,C)_0$ and $R \in \mathbf{B}_S(A,B)_1$, $T \in \mathbf{B}_S(B,C)_1$, with $d_0 R = l$, $d_1 R = k$, $d_0 T = u$, $d_1 T = v$, then*

$$[uR + Tl] = [Tk + vR],$$

where $+$ denotes composition in $\Pi\mathbf{B}_S(A,C)$ and $[\ ]$ denotes the equivalence class of the enclosed element in $\Pi\mathbf{B}_S(A,C)$.

Proof. Diagrammatically one has



This is thus just a statement of the Godement interchange law in this case and it states that composition is a functor in $\Pi\mathbf{B}_S$.

LEMMA 2. *If f, g are as in the proposition, then the induced functors*

$$f^*\colon \Pi\mathbf{B}_S(B,C) \to \Pi\mathbf{B}_S(\mathbf{A},C)$$

and

$$g^*\colon \Pi\mathbf{B}_S(A,C) \to \Pi\mathbf{B}_S(B,C)$$

are bijective on hom-sets.

Proof. We refer the reader to Vogt's paper [21]. The necessary changes are minor and can be easily made.

Thus as long as $\mathbf{B}_S$ is locally Kan, we can embed any homotopy equivalence data in strong homotopy equivalence data. We next turn to apply this in our study of $\mathrm{Ner}_{\text{h.c.}}(\mathbf{B}_S)$ when $\mathbf{B}_S$ is locally Kan.

PROPOSITION 2·4. *If* $\mathbf{B}_S$ *is locally Kan and*

$$F: \Lambda^0[n] \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$$

is such that $F(0,1)$ *is a homotopy equivalence, then* F *has a filler,*

$$\bar{F}: \Delta[n] \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S).$$

Proof. The strong homotopy equivalence data for $F(0,1)$ will be denoted

$$(F(01), g, H, K)$$

as before. (In [7] we have given the case $n = 3$ in detail by way of illustration.)

We have already a box

$$\phi_F: J^{n-1}_{1,1} \to \mathbf{B}_S(F(0), F(n))$$

for which we take a filler

$$G(\sigma): I^{n-1} \to \mathbf{B}_S(F(0), F(n)),$$

and set $G = G(\sigma)|\{1\} \times I^{n-2}$. We also have a $(n-1)$-cube

$$GH: I^{n-1} \to \mathbf{B}_S(F(0), F(n)).$$

Now let $G_{\epsilon,i} = G|\{1\} \times I^{i-2} \times \{\epsilon\} \times I^{n-i-1}$ for each $\epsilon = 0, 1$, then

$$\left.\begin{aligned} G_{0,i} &= F(1, \ldots, i-1, i+1, \ldots, n)\, F(01) \\ G_{1,i} &= F(i, \ldots, n)\, F(1, \ldots, i)\, F(01) \end{aligned}\right\} \quad \text{for} \quad 2 \leqslant i \leqslant n-1.$$

Let

$$\left.\begin{aligned} H_{0,i} &= F(1, \ldots, i-1, i+1, \ldots, n) \\ H_{1,i} &= F(i, \ldots, n)\, F(1, \ldots, i) \end{aligned}\right\} \quad \text{again for} \quad 2 \leqslant i \leqslant n-1,$$

so that $G_{\epsilon,i} = H_{\epsilon,i} F(01)$. As before, let L be the square linking $KF(01)$ with $F(01)H$ and consider the $(n-1)$-cube $H_{\epsilon,i}L$. We have that

$$\begin{aligned} H_{0,i}L|I \times \{0\} \times I^{n-3} &= H_{0i} F(01) \\ &= F(1, \ldots, i-1, i+1, \ldots, n)\, F(01) \end{aligned}$$

and

$$F(0, 1, \ldots, i-1, i+1, \ldots, n): I^{n-2} \to \mathbf{B}_S(F(0), F(n))$$

have a common face, similarly for $H_{0,i}L|\{0\} \times I \times I^{n-3}$. Using the hyperprisms which result, and the Kan condition on $\mathbf{B}_S(F(0), F(n))$, one obtains some $(n-1)$-cubes, $S_{0,i}$, such that

$$\begin{aligned} S_{0,i}|I \times \{0\} \times I^{n-3} &= S_{0,i}|\{0\} \times I \times I^{n-3} \\ &= F(0, 1, \ldots, 0-1, i-1, \ldots, n), \\ S_{0,i}|I \times \{1\} \times I^{n-3} &= G_{0,i}H \end{aligned}$$

and

$$S_{0,i}|\{1\} \times I \times I^{n-3} = H_{0,i}KF(0,1).$$

One now repeats this construction for each side, adapting the details for $S_{1,i}$ in the obvious way.

Fitting the sides $H_{\epsilon,i}K$ and Gg together, one has a $(n-1)$-box in $\mathbf{B}_S(F(1), F(n))$ and thus a filler M. We take

$$\bar{F}(1, 2, \ldots, n) = M|\{0\} \times I^{n-2}.$$

Finally one has an n-box in $\mathbf{B}_S(F(0), F(n))$, using $MF(0,1)$, GH, $G(\sigma)$ and the faces $S_{\epsilon,i}$. The remaining face of the filler gives us $\bar{F}(0, 1, \ldots, n)$ with the required properties.

Clearly the above proof can be easily adapted to give a proof of the following 'dual' proposition.

PROPOSITION 2·4*. *If* $\mathbf{B}_S$ *is locally Kan and*

$$F: \Lambda[n] \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$$

is such that $F(n-1, n)$ *is a homotopy equivalence, then* F *has a filler*

$$F: \Delta[n] \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S).$$

3. *The invertibility of level homotopy equivalences in* Coh $(\mathbf{A}, \mathbf{B}_S)$

In this section we give the proof of Theorem 1·1. For the convenience of the reader, we recall the statement of that result:

THEOREM 1·1. *Let* $\mathbf{A}$ *be a small category,* $\mathbf{B}_S$ *a locally Kan* $\mathbf{S}$*-category, and* $f: F \to G$ *a coherent map between coherent diagrams of type* $\mathbf{A}$ *in* $\mathbf{B}_S$. *If for each object* i *in* $\mathbf{A}$, *the map* $f(i): F(i) \to G(i)$ *is a homotopy equivalence, then* $[f]$ *is an isomorphism in* Coh $(\mathbf{A}, \mathbf{B}_S)$.

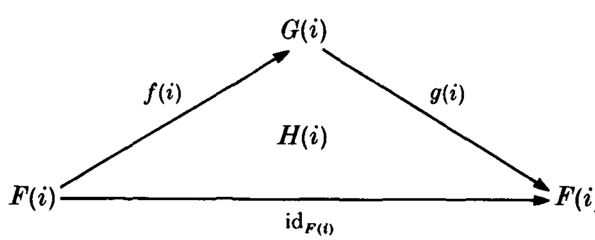
Proof. We start by reformulating the result with the aim of reducing it to special cases of $\mathbf{A}$.

We have that f may be described by

$$f: \mathrm{Ner}(\mathbf{A}) \times \Delta[1] \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S).$$

Similarly, specifying that $(f(i), g(i), H(i), K(i))$ for each i in $\mathbf{A}$ be a strong homotopy equivalence implies that we have a simplicial map

$$\mathrm{Ner}(|\mathbf{A}|) \times \Delta[2] \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$$



for each $i \in |\mathbf{A}|$ (here $|\mathbf{A}|$ denotes the set (or discrete category) of objects of $\mathbf{A}$). Thus we obtain a simplicial map

$$\phi: (\mathrm{Ner}(\mathbf{A}) \times \Lambda^0[2]) \cup (\mathrm{Ner}(|\mathbf{A}|) \times \Delta[2]) \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$$

by defining ϕ on $\mathrm{Ner}(\mathbf{A}) \times d_1\Delta[2]$ by the identity on F.

Using the $K(i)$'s, we similarly can define

$$\psi: (\mathrm{Ner}(\mathbf{A}) \times \Lambda^2[2]) \cup (\mathrm{Ner}(|\mathbf{A}|) \times \Delta[2]) \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S).$$

Suppose, for the moment, that ϕ and ψ can both be extended over $\mathrm{Ner}(\mathbf{A}) \times \Delta[2]$, giving say $\bar{\phi}$ and $\bar{\psi}$. Then $\bar{\phi}$ gives a map $[g_1]$ in Coh $(\mathbf{A}, \mathbf{B}_S)$ such that $[g_1][f] = [\mathrm{id}_F]$ whilst $\bar{\psi}$ gives a $[g_2]$ with $[f][g_2] = [\mathrm{id}_G]$. Associativity of composition in Coh $(\mathbf{A}, \mathbf{B}_S)$ then implies $[g_1] = [g_2]$, so $[f]$ is invertible in Coh $(\mathbf{A}, \mathbf{B}_S)$. Thus it is sufficient to prove that ϕ and ψ do extend as supposed.

Now $\bar{\phi}$ and $\bar{\psi}$ will be determined once their values on all $\sigma_n \times \Delta[2]$ are given, for all

$n \in \mathbf{N}$, $\sigma_n \in \mathrm{Ner}(\mathbf{A})_n$. Thus it sufficies to prove the result for the case $\mathbf{A} = [n]$ as the general case will then follow by the usual glueing/colimit type argument. We are therefore reduced to proving the following proposition and its 'dual' with $\Lambda^0[2]$ replaced by $\Lambda^2[2]$.

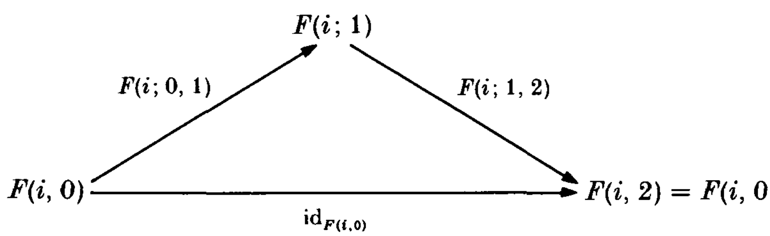
PROPOSITION 3·2. *Let $\mathbf{B}_S$ be a locally Kan simplicial category and n be a non-negative integer. Suppose*

$$F_1 \colon \Delta[n] \times \Lambda^0[2] \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$$
$$F_2 \colon \{0, 1, \ldots, n\} \times \Delta[2] \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$$

are two simplicial morphisms such that

(*i*) *$F_2 | \{i\} \times \Delta[2]$ is part of a strong homotopy,*

$$(F(i;\, 0, 1),\, F(i;\, 1, 2),\, F(i;\, 0, 1, 2),\, K)$$



(ii) $F_1 | \{0, 1, \ldots, n\} \times \Lambda^0[2] = F_2 | \{0, 1, \ldots, n\} \times \Lambda^0[2]$.

Then there is an extension

$$F \colon \Delta[n] \times \Delta[2] \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$$

of both F_1 and F_2.

Proof. If $n = 0$, there is nothing to prove as $|[0]| = [0]$, so suppose that we have the result for every $k < n$. One can thus extend F_1 and F_2, which together give us a map

$$(\Delta[n] \times \Lambda^0[2]) \cup (sk_0 \Delta[n] \times \Delta[2]) \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S),$$

to one

$$F' \colon (\Delta[n] \times \Lambda^0[2]) \cup (sk_{n-1} \Delta[n] \times \Delta[2]) \to \mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S).$$

where sk_l indicates the l-skeleton functor. It therefore remains for us to extend F' over the remaining simplices of dimensions $n+1$ and $n+2$ of $\Delta[n] \times \Delta[2]$.

Using our knowledge of $\mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$ (Propositions 2·1 and 2·4), we see that we can extend if one has an $(n+1, i)$-box ($i \neq 0, n+1$), an $(n+2, i)$-box ($i \neq 0, n+2$), or an $(n+1, 0)$ or $(n+2, 0)$-box, if the first map is part of a homotopy equivalence.

We first enumerate the $(n+2)$-simplices of $\Delta[n] \times [2]$ in a way which will enable us to fill them in succession:

The non-degenerate $(n+2)$-simplices are of the form

$$\sigma_{i,j} = (0, 1, \ldots, i, i, \ldots, j, j, \ldots, n) \times (0, \ldots, \underset{\substack{\uparrow \\ i}}{0}, 1, \ldots, \underset{\substack{\uparrow \\ j+1}}{1}, 2, \ldots, 2)$$

for $i \leqslant j$, and one thus has the relations

$$\partial_{j+1} \sigma_{i,j} = \partial_{j+1} \sigma_{i,j-1}$$

and

$$\partial_i \sigma_{i,j} = \partial_i \sigma_{i-1,j}$$

and no others.

Firstly let us look at $\sigma_{n,n} = (0,1,\dots,n,n,n)\times(0,\dots,0,1,2)$ and examine its faces. We already know the value of F' on the following faces:

for $i = 0,\dots,n-1$, $\partial_i\sigma_{n,n}\in sk_{n-1}(\Delta[n])\times\Delta[2])$, known by hypothesis;

for $i = n+1, n+2$, $\partial_i\sigma_{n,n}\in\Delta[n]\times\Lambda^0[2]$, therefore in the domain of F_1.

We thus have a $\Lambda^n[n+2]$ and we can therefore fill its image in $\mathrm{Ner}_{\mathrm{h.c.}}(\mathbf{B}_S)$ and have, as a pay-off, a knowledge of the image of $\partial_n\sigma_{n,n}$, which is the same as $\partial_n\sigma_{n-1,n}$ (this giving us a face of $\sigma_{n-1,n}$).

Looking at $\sigma_{n-1,n}$, we already know $\partial_i\sigma_{n-1,n}$ for $i=0,\dots,n-2$ and for $i=n+2$, and we have just received as a gift from $\sigma_{n,n}$, the necessary information on $\partial_n\sigma_{n-1,n}$. We are ignorant about ∂_{n-1} and ∂_{n+1}. We turn our attention to $\partial_{n+1}\sigma_{n-1,n}$. On this $(n+1)$-simplex we already know $\partial_i\partial_{n+1}\sigma_{n-1,n}$ for $i=0,\dots,n-2,n,n+1$, so we know a $\Lambda^{n-1}[n+1]$ which we can thus fill. Returning to $\sigma_{n-1,n}$, we now have knowledge of a $\Lambda^{n-1}[n+2]$ and we can fill, getting a gift of $\partial_{n-1}\sigma_{n-1,n} = \partial_{n-1}\sigma_{n-2,n}$. We also have the information on $\partial_{n+1}\sigma_{n-1,n} = \partial_{n+1}\sigma_{n-1,n-1}$, which will be useful later.

Suppose now that we have defined our extension on each

$$\sigma_{n-m,n},\ m = 0,1,\dots,k-1 < n,$$

and thus we have knowledge of

$$\partial_{n-m}\sigma_{n-m-1,n} = \partial_{n-m}\sigma_{n-m,n}$$

and also of

$$\partial_{n+1}\sigma_{n-m,n} = \partial_{n+1}\sigma_{n-m,n-1}.$$

For $\sigma_{n-k,n}$, we already have $\partial_i\sigma_{n-k,n}$ for $i=0,\dots,n-k-1,\ n-k+2,\dots,n$ and $n+2$, also ∂_{n-k+1} as a gift from $\sigma_{n-k+1,n}$. We lack ∂_{n+1} and ∂_{n-k}. As before we attack ∂_{n+1} for which we know ∂_i for $i=0,\dots,n-k-1,n-k+1,\dots,n,n+1$, that is, we know our extended map on a $\Lambda^{n-k}[n+1]$ and we can fill it, noting that

$$\partial_{n-1}\sigma_{n-k,n} = \partial_{n+1}\sigma_{n-k,n-1}$$

so this face will be useful later on.

We should note that if $k=n$, our first morphism in $F(\partial_{n+1}\sigma_{0,n})$ is a homotopy equivalence since

$$\sigma_{0,n} = (0,0,1,\dots,n,n)\times(0,1,\dots,1,2).$$

Thus we have $F(\sigma_{l,n})$ for each $l=0,1,\dots,n$. We continue the process with $F(\sigma_{n-1,n-1})$. On $\sigma_{n-1,n-1}$, we know ∂_i for $i=0,\dots,n-2,n,n+2$, but also for ∂_{n+1}, a gift from $\sigma_{n-1,n}$. We therefore can extend.

The pattern repeats. One can easily formalize this into an inductive proof in which we take as hypothesis the existence of an extension F on $\sigma_{i,j}$,

$$j = n, n-1,\dots,n-l,\quad i = n-l, n-l-1,\dots,n-k+1,\quad \text{etc.}$$

At each stage, one must hypothesize the 'gifts' from earlier stages and of course one has to take some slight care, as above for $\sigma_{0,n}$, when one gets to the $(0,n-k)$ stage each time, as one then needs the fact that at this stage $F(\sigma_{0,n-k})$ has a homotopy equivalence as first morphism so an extension exists for the $(n+2,0)$-box that is known.

(A slight difficulty arises each time when filling $\sigma_{0,n-k}$ as we do not seem to know four faces, d_0, d_1, d_{n-k+1} and d_{n-k+2}, but we note the following:

$$d_1\sigma_{0,n-k} = d_1\sigma_{1,n-k}$$

and so is already known,

$$d_{n-k+2}\sigma_{0,n-k} = d_{n-k+2}\sigma_{0,n-k+1}$$

and so is also already known, and in the faces of $d_{n-k+1}\sigma_{0,n-k}$, we have

$$\begin{aligned} d_1 d_{n-k+1}\sigma_{0,n-k} &= d_{n-k} d_1 \sigma_{0,n-k} \\ &= d_{n-k} d_1 \sigma_{1,n-k} \\ &= d_1 d_{n-k+1}\sigma_{1,n-k} \end{aligned}$$

and so this is all right as well.)

This induction completes the proof.

The previous results, and in particular Theorem 1·1, show that the natural functor

$$\gamma\colon \mathbf{B}^{\mathbf{A}} \to \mathrm{Coh}\,(\mathbf{A}, \mathbf{B}_S)$$

factors through a localization

$$p\colon \mathbf{B}^{\mathbf{A}} \to Ho\,(\mathbf{B}^{\mathbf{A}}) = \mathbf{B}^{\mathbf{A}}(\Sigma^{-1})$$

where Σ is the class of level homotopy equivalences in $\mathbf{B}^{\mathbf{A}}$. We will write

$$\bar{\gamma}\colon \mathrm{Ho}\,(\mathbf{B}^{\mathbf{A}}) \to \mathrm{Coh}\,(\mathbf{A}, \mathbf{B}_S)$$

for the unique functor such that $\gamma = \bar{\gamma}p$.

The remainder of this paper analyses this functor $\bar{\gamma}$, finding conditions under which it is an equivalence.

4. *Rigidification of a coherent functor*

In order to show that $\bar{\gamma}$ is an equivalence, we have to show that it is full and faithful and that, given any coherent diagram F in $\mathbf{B}_S$, there is a commutative diagram isomorphic to F in Coh $(\mathbf{A}, \mathbf{B}_S)$. We start with the second of these.

Precisely this sort of problem has been tackled in the topological case by Segal [**17**] in his Appendix B. Analysis of his result gives the skeleton of a proof for the non-topological and general case. Although he does not mention this, his $\pi * F$ construction is an example of a particular type of Kan-extension. As with many of the results in this theory, the 2-categorical and bicategorical analogues have already been studied, e.g. by Street [**19**], Giraud [**10**], and Bozapalides [**5**]. Combining these two special cases, one arrives at the following idea.

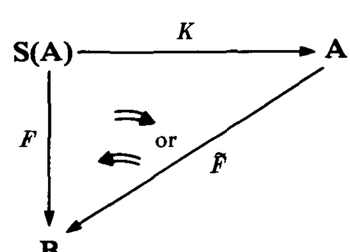
There is a canonical augmentation

$$\mathbf{S}(\mathbf{A}) \xrightarrow{K} \mathbf{A}$$

and we have a functor

$$F\colon \mathbf{S}(\mathbf{A}) \to \mathbf{B}_S;$$

clearly if we want the best approximation to F by an 'actual' functor from $\mathbf{A}$ to $\mathbf{B}$, we need to form a right or left Kan extension



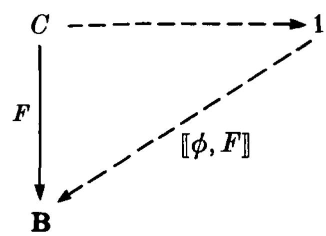
The problem is that there is no reason to suppose that F is isomorphic to $\tilde{F}K$. The reason is soon clear. $\tilde{F}$ will be defined using an S-limit or colimit, but we are dealing with homotopy phenomena so we should form an $\tilde{F}$ via a coherent limit or colimit, that is a homotopy limit or colimit.

We will briefly recall this theory here as it helps motivate the construction of a suitable $\hat{F}$ later ('suitable' in the sense that it works!). For the details of this theory, we refer the reader to the paper of Bourn and the first author [**3**]. For details on general indexed limits see Kelly [**11**], or the original Borceux-Kelly [**2**].

Let C be an $\mathbf{S}$-category. The *projective ϕ-indexed cone functor over F* (denoted $[\![\phi, F]\!]: \mathbf{B}^{op} \to \mathbf{S}$) is given by the formula

$$[\![\phi, F]\!](B) = \int_A \mathbf{S}(\phi(A), \mathbf{B}_S(B, FA)).$$

Thus $[\![\phi, F]\!]$ is the right Kan extension of F along the distributor (or profunctor) ϕ:



The simplicial functor F admits a *projective ϕ-indexed limit* if $[\![\phi, F]\!]$ is representable, i.e. if there is an object in $\mathbf{B}_S$, denoted by ϕ-lim F, such that

$$[\![\phi, F]\!](B) = \mathbf{B}_S(B, \phi\text{-lim}\, F).$$

The notion of *ϕ-indexed colimit* is dual.

The advantage of indexed limits is that one can choose the indexation to capture coherence phenomena. The simplest example of this is the Bousfield–Kan homotopy limit [**4**] which has the following description:

if $\mathbf{I}$ is a category, $F: \mathbf{I} \to \mathbf{S}$ a functor, then

$$\operatorname{ho\,lim} F = \int_i \mathbf{S}(\operatorname{Ner}(\mathbf{I}/i), F(i))$$

and for B is $\mathbf{S}$

$$\mathbf{S}(B, \operatorname{ho\,lim} F) = [\![\operatorname{Ner}(I/-), F]\!](B),$$

so the Bousfield–Kan homotopy limit is $\operatorname{Ner}(I/-)$-indexed.

We have to make two generalizations from this example. Firstly we need a generalization replacing $\mathbf{S}$ by an arbitrary simplicial category $\mathbf{B}_S$, and then we need to replace the actual functor by a coherent functor, i.e. a simplicial functor $\mathbf{S}(\mathbf{A}) \to \mathbf{B}_S$.

Let $\mathbf{B}_S$ be as usual a simplicial category and let B be an object of $\mathbf{B}_S$. B defines functors

$$\mathbf{B}_S(B, -): \mathbf{B}_S \to \mathbf{S}$$

and

$$\mathbf{B}_S(-, B): \mathbf{B}_S^{op} \to \mathbf{S}.$$

We say that $\mathbf{B}_S$ is *tensored* if $\mathbf{B}_S(B, -)$ has a left adjoint (which is denoted $-\,\overline{\otimes}\, B$), whilst $\mathbf{B}_S$ is *cotensored* if $\mathbf{B}_S(-, B)$ has a right adjoint, which is denoted $\overline{\mathbf{B}}_S(-, B)$.

Thus, if $\mathbf{B}_s$ is tensored and K is a simplicial set, there is a natural isomorphism

$$\mathbf{S}(K, \mathbf{B}_S(B, -)) \cong \mathbf{B}_S(K \,\overline{\otimes}\, B, -),$$

and if $\mathbf{B}_S$ is cotensored then

$$\mathbf{S}(K, \mathbf{B}_S(-, B)) \cong \mathbf{B}_S(-, \bar{\mathbf{B}}_S(K, B)).$$

Remark. (i) One can think of $K \bar{\otimes} B$ as being a sort of product of the simplicial set K with B, while $\bar{\mathbf{B}}_S(K, B)$ is an 'object of maps' from K to B.

(ii) Although we shall use co-tensors and tensors in conjunction with indexed limits and colimits, they are themselves simple examples of such.

Next we turn to an indexation which is suitable for non-trivially simplicial categories and yet which reduces to the Bousfield–Kan Ner $(I/-)$-indexation in the case that the category has trivial simplicial structure. We shall give the obvious generalization for the case of $\mathbf{S}(\mathbf{A})$, referring the reader to [**3**] for the general theory.

Let $\mathbf{A}$ be a category and $\mathbf{S}(\mathbf{A})$ the resolved simplicial category. If C is an object of $\mathbf{A}$ (and hence also of $\mathbf{S}(\mathbf{A})$), we define a simplicial object in $\mathbf{S}$, $\mathbf{S}(\mathbf{A})/C$, by

$$\coprod_{\mathbf{A}} \mathbf{S}(\mathbf{A})(A, C) \substack{\rightarrow\\ \leftarrow\\ \rightarrow} \coprod_{A_0, A_1} \mathbf{S}(\mathbf{A})(A_0, A_1) \times \mathbf{S}(\mathbf{A})(A_1, C) \substack{\leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow} \cdots$$

$$\coprod_{A_0, \ldots, A_n} \mathbf{S}(\mathbf{A})(A_0, A_1) \times \ldots \times \mathbf{S}(\mathbf{A})(A_{n-1}, A_n) \times \mathbf{S}(\mathbf{A})(A_n, C)$$

with faces and degeneracies given by the following formulae:

$$(\mathbf{S}(\mathbf{A})/C)_{n-1} \underset{s_i}{\overset{d_i}{\rightleftarrows}} (\mathbf{S}(\mathbf{A})/C)_n,$$

$$d_0(z_1, \ldots, z_n, g) = (z_2, \ldots, z_n, g),$$

$$d_i(z_1, \ldots, z_i, z_{i+1}, \ldots, g) = (z_1, \ldots, z_{i+1} \cdot z_i, \ldots, g) \quad (0 < i < n),$$

$$d_n(z_1, \ldots, z_n, g) = (z_1, \ldots, g \cdot z_n),$$

$$s_i(z_1, \ldots, z_i, z_{i+1}, \ldots, z_{n+1}, g) = (z_1, \ldots, z_i, 1_{A_i}, z_{i+1}, \ldots, z_{n-1}, g).$$

This construction thus gives a bisimplicial set $\mathbf{S}(\mathbf{A})/C$.

To reduce to a simplicial set, we take the diagonal and define

$$H_{\mathbf{S}(\mathbf{A})}(C) = \mathrm{Diag}\,(\mathbf{S}(\mathbf{A})/C).$$

We define for $F\colon \mathbf{S}(\mathbf{A}) \to \mathbf{B}_S$, $\operatorname{ho\,lim} F = H_{\mathbf{A}} - \lim F$, that is

$$\operatorname{ho\,lim} F = \int_C \bar{\mathbf{B}}_S(H_{\mathbf{A}}(C), FC)$$

(of course assuming that $\mathbf{B}_S$ is a complete $\mathbf{S}$-category).

This functor reduces to Bousfield and Kan's homotopy limit if F factors via $\mathbf{S}(\mathbf{A}) \xrightarrow{K} \mathbf{A}$.

It is now easy to suggest a definition of the coherent right Kan extension of F along $K\colon \mathbf{S}(\mathbf{A}) \to \mathbf{A}$, provided $\mathbf{B}_S$ is cotensored.

We set, for X in $\mathbf{A}$,

$$\begin{aligned}\hat{F}(X) &= \operatorname{ho\,lim}\,(X \downarrow K \xrightarrow{\delta_X} \mathbf{S}(\mathbf{A}) \xrightarrow{F} \mathbf{B}_S)\\ &= \int_{f: X \to A} \mathbf{B}_S(H(X \downarrow K)(f, A),\ F\delta_X(f, A)),\end{aligned} \tag{1}$$

where $X \downarrow K$ is now a simplicial comma category. (The objects of $X \downarrow K$ are pairs $(f\colon X \to KA, A)$, and for (f, A), (g, A') in $X \downarrow K$ the simplicial set of maps from (f, A) to (g, A') is given by

$$(X \downarrow K)((f,A),(g,A'))_n = \{\sigma \in \mathbf{S}(\mathbf{A})(A,A') \mid g = K(\sigma)f\};$$

δ_X is the simplicial functor, $\delta_X(f, A) = A$.)

Our objects in defining $\hat{F}$ were (i) that it be a 'true' functor from $\mathbf{A}$ to $\mathbf{B}$ (and this it clearly is by construction) and (ii) to have that the functor $K\colon \mathbf{S}(\mathbf{A}) \to \mathbf{A}$ induces a natural transformation

$$\partial^*\colon \hat{F}K \to F$$

which on any object X is a homotopy equivalence (as this will ensure that $\hat{F}K$ and F are isomorphic in Coh $(\mathbf{A}, \mathbf{B}_S)$.)

In order to compare $\hat{F}$ and F, it will help to have an end formula for $F(X)$. This is given by the Yoneda type result

$$\hat{F}(X) = \int_A \bar{\mathbf{B}}_S(\mathbf{S}(\mathbf{A})(X,A), F(A)). \tag{2}$$

We next need to make (1) look more like (2), and this we do by simplifying (1) as follows.

We have

$$H_{(X\downarrow K)}(f,A) = \text{Diag}\,((X \downarrow K)/(f,A)),$$

where $(X \downarrow K)/(f, A)$ is the bisimplicial set with

$$\coprod_{(f_0,A_0),\ldots,(f_n,A_n)} (X\downarrow K)((f_0,A_0),(f_1,A_1)) \times \ldots \times (X \downarrow K)((f_n,A_n),(f,A))$$

in dimension n.

Given the description of $X \downarrow K$, it is clear that

$$\coprod_{A_0,\ldots,A_n} \mathbf{A}(X,A_0) \times \mathbf{S}(\mathbf{A})(A_0,A_1) \times \ldots \times \mathbf{S}(\mathbf{A})(A_n,A) = \psi(X,A)_n,$$

say, is the disjoint union of the $((X \downarrow K)/(f,A))_{n,0}$ over the various maps $f\colon X \to A$. As all of these have the same codomain A, we have

$$\hat{F}(X) \cong \int_A \bar{\mathbf{B}}\,(\text{Diag}\,\psi(X,A), F(A))$$

with $\psi(X, A)$ having face and degeneracies induced from those given for $H_{(X\downarrow K)}(f, A)$. Clearly the induced map from $\hat{F}(X)$ to $F(X)$ is given by the 'augmentation' map

$$\partial(X,A)\colon \psi(X,A) \to \mathbf{S}(\mathbf{A})(X,A),$$

defined by composing the disjoint union of the various augmentation maps

$$\mathbf{S}(\mathbf{A})(A_i, A_{i+1}) \to \mathbf{A}(A_i, A_{i+1}) \quad (0 \leqslant i < n)$$

with the map

$$d_0\colon \coprod_{A_0,\ldots,A_n} \mathbf{A}(X,A_0) \times \ldots \times \mathbf{A}(A_{n-1},A_n) \times \mathbf{S}(\mathbf{A})(A_n,A) \to \mathbf{S}(\mathbf{A})(X,A)$$

given by

$$d_0(f_0,\ldots,f_n,\sigma) = (f_n \cdots f_0)^*(\sigma).$$

Our best hope of proving ∂^* to be a homotopy equivalence would seem to be to prove that $\partial(X, A)$ is a homotopy equivalence naturally in X and A and then to prove some general lemmas about natural homotopy equivalences between indexing functors.

Now $\mathbf{S}(\mathbf{A})$ is defined by a comonad generated by a free-forget pair going to directed graphs with 'identity' loops. Thus, as a simplicial resolution of $\mathbf{A}$, it will be split at the underlying directed graph level. This implies that the augmented map,

$$\mathbf{S}(\mathbf{A})(A_i, A_{i+1}) \to \mathbf{A}(A_i, A_{i+1}) \quad (0 \leqslant i < n),$$

is a homotopy equivalence of simplicial sets, but that the homotopy inverse need not be compatible with composition. We can thus use these maps safely in the required range $(0 \leqslant i < n)$, since our only operation on these A_i's is disjoint union (no 'integration' is done with them).

Thus $\psi(X, A)$ is naturally homotopy equivalent to a bisimplicial set $\psi'(X, A)$ with

$$\psi'(X, A)_{n,p} = \coprod_{A_0, \ldots, A_n} \mathbf{A}(X, A_0) \times \ldots \times \mathbf{A}(A_{n-1}, A_n) \times \mathbf{S}(\mathbf{A})(A_n, A)_p.$$

We now show that the augmentation

$$d: \psi'(X, A) \to \mathbf{S}(\mathbf{A})(X, A)$$

given above has a homotopy inverse. For this we define a 'contraction' or an 'extra degeneracy' s_{-1} by

$$s_{-1}(f_0, \ldots, f_n, \sigma) = (1_X, f_0, \ldots, f_n, \sigma).$$

It is easy to check that s_{-1} satisfies the simplicial identities, so

$$d_0: \psi'(X, A) \to \mathbf{S}(\mathbf{A})(X, A)$$

is a homotopy equivalence of bisimplicial sets. Again this is natural in X and A. Composing, we get the map

$$\partial(X, A): \psi(X, A) \to \mathbf{S}(\mathbf{A})(X, A)$$

to be a homotopy equivalence of bisimplicial sets (here as above we are considering $\mathbf{S}(\mathbf{A})(X, A)$ as a bisimplicial set, constant in the second direction; when we need to stress this point, as we will later, we will write $K(\mathbf{S}(\mathbf{A})(X, A), 0)$ for this bisimplicial set.)

We will need to make this notion of 'natural homotopy' more precise.

Definition. Let F, G be two $\mathbf{S}$-functors from $\mathbf{S}(\mathbf{A})^{\mathrm{op}} \times \mathbf{S}(\mathbf{A})$ to $\mathbf{B}_S$ where $\mathbf{B}_S$ is cotensored. Suppose $f_0, f_1: F \to G$ are two coherent natural maps. We shall say that f_0, f_1 are *naturally homotopic* if there is an $\mathbf{S}$-natural transformation

$$h: F \to \bar{\mathbf{B}}_S(\Delta[1], G)$$

such that composition of h with d_0^* and d_1^* gives respectively f_0 and f_1 (where

$$d_i: \Delta[0] \to \Delta[1]$$

are the two end maps and we have identified $\bar{\mathbf{B}}_S(\Delta[0], G)$ with G).

LEMMA 4·1. *Given two* $\mathbf{S}$*-functors*

$$F, G: \mathbf{S}(\mathbf{A})^{\mathrm{op}} \times \mathbf{S}(\mathbf{A}) \to \mathbf{B}_S$$

with $\mathbf{B}_S$ *cotensored, and two naturally homotopic maps*

$$f_0, f_1 \colon F \to G,$$

then the induced maps

$$\int_A f_0, \int_A f_1 \colon \int_A F(A,A) \to \int_A G(A,A)$$

are homotopic in $\mathbf{B}_S$.

Proof. There is some natural

$$h \colon F \to \overline{\mathbf{B}}_S(\Delta[1], G),$$

hence a map

$$\int_A h \colon \int_A F(A,A) \to \int_A \overline{\mathbf{B}}_S(\Delta[1], G(A,A)),$$

but since $\overline{\mathbf{B}}_S$ commutes with ends (as it is a right adjoint), we have

$$\int_A \overline{\mathbf{B}}_S(\Delta[1], G(A,A)) \cong \overline{\mathbf{B}}_S\left(\Delta[1], \int_A G(A,A)\right),$$

so $\int_A h$ gives the required homotopy between $\int f_0$ and $\int f_1$. (Note: here we are implicitly using the useful fact that if $\mathbf{B}_S$ is cotensored,

$$\begin{aligned} \mathbf{B}_S(X, Y)_1 &\cong \mathbf{S}(\Delta[1], \mathbf{B}_S(X, Y)) \\ &\cong \mathbf{B}_S(X, \overline{\mathbf{B}}_S(\Delta[1], Y)), \end{aligned}$$

so $\overline{\mathbf{B}}_S(\Delta[1], -)$ acts like a cocylinder functor and can be used to give an explicit description of homotopies.)

COROLLARY 4·2. *If* $f \colon F \to G$, $g \colon G \to F$ *are homotopy inverse to each other, the homotopies being natural, then* $\int F(A,A)$ *and* $\int G(A,A)$ *are homotopically equivalent.*

LEMMA 4·3. *Let* $K, L \colon \mathbf{S}(\mathbf{A}) \to \mathbf{S}$ *be two* $\mathbf{S}$*-functors and* $F \colon \mathbf{S}(\mathbf{A}) \to \mathbf{B}_S$ *a coherent diagram in* $\mathbf{B}_S$. *Suppose*

$$h \colon K \to \mathbf{S}(\Delta[1], L)$$

is a natural homotopy between $f_0 = hd_0^*$ *and* $f_1 = hd_1^*$; *then the two induced maps*

$$f_i^* \colon \overline{\mathbf{B}}_S(L, F) \to \overline{\mathbf{B}}_S(K, F)$$

are naturally homotopic.

Proof. h induces a map $\bar{h} \colon K \times \Delta[1] \to L$ and hence

$$\bar{h}^* \colon \overline{\mathbf{B}}_S(L, F) \to \overline{\mathbf{B}}_S(K \times \Delta[1], F) \cong \mathbf{B}_S(\Delta[1], \overline{\mathbf{B}}_S(K, F))$$

which gives the required natural homotopy.

We have already noted that

$$\partial(X, A) \colon \psi(X, C) \to K(\mathbf{S}(\mathbf{A})(X, C), 0)$$

is a homotopy equivalence, natural in C. Taking diagonals, we get

$$\operatorname{Diag} \psi(X, C) \to \mathbf{S}(\mathbf{A})(X, C) = \operatorname{Diag} K(\mathbf{S}(\mathbf{A})(X, C),\ 0)$$

is a homotopy equivalence natural in C, since $\operatorname{Diag}_{\cdot,\cdot}$ is given by a coend

$$(\operatorname{Diag} K_{\cdot,\cdot})_{n'} = \int^{[n]} \Delta[n] \times K_{n,n'}$$

and so the dual of 4·2 applies to give the desired natural homotopy equivalence. (Note: this actually uses C as a 'dummy' variable; it also uses the tensoring on $\mathbf{S}$.)

Now applying 4·2 again, but this time with respect to 'integration' over C, we obtain a homotopy equivalence

$$\int_C \bar{\mathbf{B}}_S(\mathbf{S}(\mathbf{A})(X,C),F(C)) \to \int_C \bar{\mathbf{B}}_S(\mathrm{Diag}\,\psi(X,C),F(C))$$

but, as mentioned before,

$$\int_C \bar{\mathbf{B}}_S(\mathbf{S}(\mathbf{A})(X,C),F(C)) \cong F(X)$$

by the enriched version of the Yoneda lemma, and

$$\int_C \bar{\mathbf{B}}_S(\mathrm{Diag}\,\psi(X,C),F(C)) = \hat{F}(X)$$

by definition.

Corollary 4·4. *For any* $F\colon \mathbf{S}(\mathbf{A}) \to \mathbf{B}_S$, *with* $\mathbf{B}_S$ *cotensored, one has for any* X *in* A,

$$F(X) \to \hat{F}(X)$$

is a homotopy equivalence.

As $\hat{F}$ is an actual commutative diagram, this gives us the required rigidification.

Corollary 4·5. *Given any* F *in* $\mathrm{Coh}(\mathbf{A}, \mathbf{B}_S)$, *there is a commutative diagram* $\hat{F}$ *and a natural isomorphism*

$$F \to \hat{F}$$

in $\mathrm{Coh}(\mathbf{A}, \mathbf{B}_S)$.

Applying 4·5 to the indexing category $\mathbf{A} \times [1]$, we get

Corollary 4·6. *Given coherent diagrams* F, G *of type* $\mathbf{A}$ *in a complete* $\mathbf{S}$-*category* $\mathbf{B}_S$ *and a coherent map* $f\colon F \to G$, *there is a commutative diagram in* $\mathrm{Coh}(\mathbf{A}, \mathbf{B}_S)$

$$\begin{array}{ccc} F & \xrightarrow{\quad f \quad} & G \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \hat{F} & \xrightarrow{\quad \hat{f} \quad} & \hat{G} \end{array}$$

in which $F \xrightarrow{f} G$ *is within the image of the canonical functor*

$$\bar{\gamma}\colon \mathrm{Ho}(\mathbf{B}^{\mathbf{A}}) \to \mathrm{Coh}(\mathbf{A}, \mathbf{B}_S).$$

Together with 1·1, these results imply the following Theorem which is the formal statement of our generalized version of Vogt's theorem.

Theorem 4·7. *Let* $\mathbf{A}$ *be a small category and* $\mathbf{B}_S$ *a locally Kan complete* $\mathbf{S}$-*category. Then there is an equivalence of categories*

$$\mathrm{Ho}(\mathbf{B}^{\mathbf{A}}) \xrightarrow{\bar{\gamma}} \mathrm{Coh}(\mathbf{A}, \mathbf{B}_S).$$

This theorem is also true if one replaces 'complete' by 'co-complete', as the construction of Segal [18] gives an $\check{F}$ defined as a left coherent Kan extension and thus as a homotopy colimit.

Remark. In the introduction, we mentioned that the above generalized form of Vogt's theorem also applied to the case $\mathbf{B}_S = Kan$ even though *Kan* is not complete. We sketch below the steps necessary to prove this.

The essential point is to show that if $F\colon \mathbf{S}(\mathbf{A}) \to \mathbf{S}$ is a simplicial functor with $F(A)$ a Kan complex for each object A, then $\hat{F}(A)$ is also a Kan complex.

First one uses the cosimplicial replacement technique given by Bourn and the first named author [**3**] to produce a cosimplicial simplicial set Y, say with

$$F(A) = \mathrm{Tot}\,(Y),$$

where Tot is the Bousfield–Kan total complex functor. From the analysis of the Tot functor given in [**4**], one can deduce that $\mathrm{Tot}\,(Y)$ is Kan if Y is a fibrant cosimplicial simplicial set. The explicit description of Y coming from [**3**] allows one to check that this is indeed the case if each FA is Kan. A detailed proof of a generalization of this result is included in a forthcoming paper by the authors on coherent ends and coherent Kan extensions.

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