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We construct Galois extensions of the $T(n)$–local sphere, lifting all finite abelian Galois extensions of the $K(n)$–local sphere. This is achieved by realizing them as higher semiadditive analogues of cyclotomic extensions. Combining this with a general form of Kummer theory, we lift certain elements from the $K(n)$–local Picard group to the $T(n)$–local Picard group.

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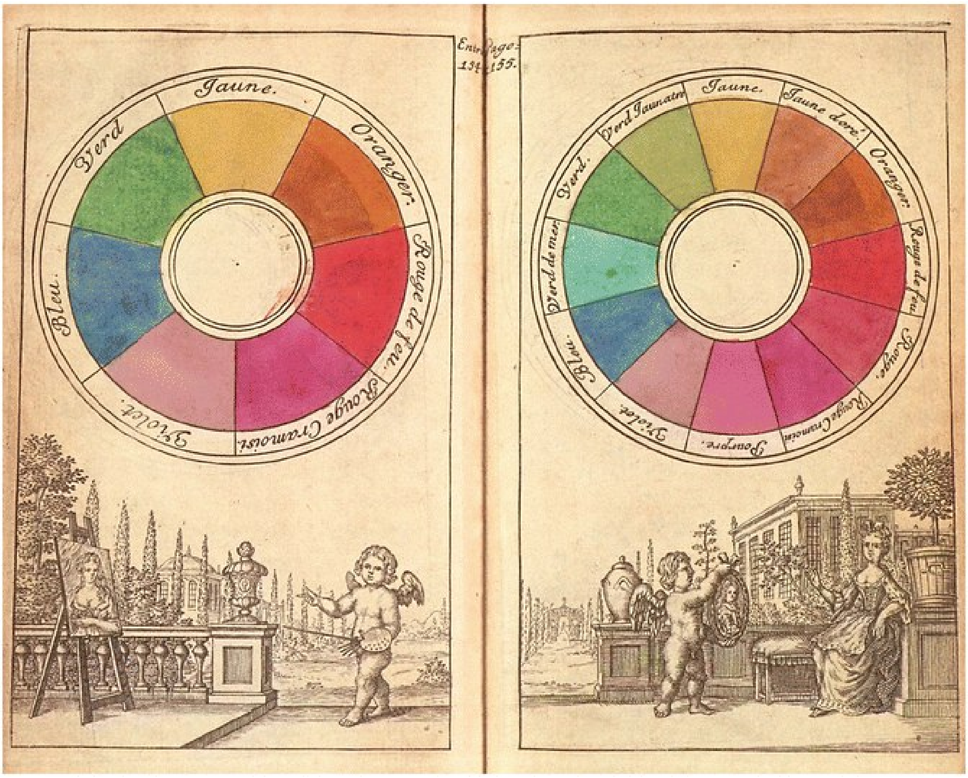

Color circles from the enlarged 1708 edition of the *Treatise on miniature painting*, Claude Boutet.

1 Introduction

1.1 Overview

1.1.1 Background and main results Chromatic homotopy theory, as a general approach, proposes to study the ∞–category $\mathrm{Sp}_{(p)}$, of p–local spectra, via the "chromatic height filtration". In practice, there are two prominent candidates for the "monochromatic layers" for such a filtration. The first are the $K(n)$–localizations $L_{K(n)}\colon \mathrm{Sp}_{(p)} \to \mathrm{Sp}_{K(n)}$, where $K(n)$ is the Morava K–theory spectrum of height n. The second are the telescopic localizations $L_{T(n)}\colon \mathrm{Sp}_{(p)} \to \mathrm{Sp}_{T(n)}$, where $T(n)$ is obtained by inverting a v_n–self map of a finite spectrum of type n. The two candidates are related by the inclusion $\mathrm{Sp}_{K(n)} \subseteq \mathrm{Sp}_{T(n)}$, which is known to be an equivalence for $n = 0, 1$ by the works of Miller [41] and Mahowald [37]. Whether this inclusion is an equivalence for all n is an open question and is the subject of the celebrated "telescope conjecture" of Ravenel. On the one hand, the ∞–categories $\mathrm{Sp}_{T(n)}$ are fundamental from a structural standpoint, as they arise via the thick subcategory theorem [30, Theorem 7]. They also admit a close connection to unstable homotopy theory (see Heuts [26]), and figure in the "redshift" phenomena for algebraic K-theory (see Ausoni and Rognes [2], Hahn and Wilson [24], Land, Mathew, Meier and Tamme [32] and Clausen, Mathew, Naumann and Noel [15]). However, they are hard to access computationally. On the other hand, the ∞–categories $\mathrm{Sp}_{K(n)}$, which a priori might contain somewhat less information, still exert a large control over $\mathrm{Sp}_{(p)}$, due to the nilpotence theorem of Hopkins, Devinatz and Smith [30, Corollary 5] and the chromatic convergence theorem of Hopkins and Ravenel [46, Theorem 7.5.7]. Moreover, they possess deep connections to the algebraic geometry of formal groups, and are consequently much more amenable to computations.

One of the key instances of the relationship between the theory of formal groups and $\mathrm{Sp}_{K(n)}$, is the construction[1] of the Lubin–Tate $\mathbb{E}_\infty$–ring spectrum E_n; see Goerss and Hopkins [23], or alternatively Lurie [36, Construction 5.1.1]. Simply put, E_n provides a faithful and relatively computable (highly structured) multiplicative (co)homology theory for $K(n)$–local spectra. Moreover, the cohomology operations of E_n can be understood in terms of the Morava stabilizer group $\mathbb{G}_n = \widehat{\mathbb{Z}} \ltimes \mathrm{Aut}(\overline{\Gamma})$, where Γ is a formal group law of height n over $\mathbb{F}_p$, and $\overline{\Gamma}$ is its base-change to $\overline{\mathbb{F}}_p$. From a more conceptual perspective, by the work of Devinatz and Hopkins [18], Rognes [49], Baker and Richter [5] and Mathew [38], E_n can be viewed as an "algebraic closure" of the $K(n)$–local sphere $\mathbb{S}_{K(n)}$ in $\mathrm{Sp}_{K(n)}$, with $\mathbb{G}_n$ as its Galois group. Hence, as in ordinary commutative algebra, one can apply "Galois descent" to study the ∞–category $\mathrm{Sp}_{K(n)}$ in terms of the far more tractable ∞–category of $K(n)$–local E_n–modules.

In light of that, it seems beneficial to study Galois extensions of $\mathbb{S}_{T(n)}$ in $\mathrm{Sp}_{T(n)}$ as well. In this regard, we have the following result.

Theorem A (Theorem 5.31) *Let G be a finite abelian group. For every G–Galois extension R of $\mathbb{S}_{K(n)}$ in $\mathrm{Sp}_{K(n)}$, there exists a G–Galois extension R^f of $\mathbb{S}_{T(n)}$ in $\mathrm{Sp}_{T(n)}$, such that $L_{K(n)}R^f \simeq R$.*

In particular, all of the Galois extensions of $\mathbb{S}_{K(n)}$, that are classified by finite quotients of the determinant map $\det\colon \mathbb{G}_n \to \mathbb{Z}_p^\times$, can be lifted to Galois extensions of $\mathbb{S}_{T(n)}$ in $\mathrm{Sp}_{T(n)}$. In fact, the lifting of the various abelian Galois extensions can be done in a compatible way. In the language of [38], the localization functor $L_{T(n)}\colon \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{K(n)}$ induces a map on the weak Galois groups (in the opposite direction), and we show that after abelianization this map admits a retract. The proof of Theorem A relies on the ∞–semiadditivity of the ∞–categories $\mathrm{Sp}_{T(n)}$ (see Carmeli, Schlank and Yanovski [14, Theorem A]), and the theory of "higher cyclotomic extensions", which we develop in this paper. The latter builds on the theory of semiadditive height and semisimplicity; see Carmeli, Schlank and Yanovski [13, Theorem D].

A related structural invariant, which is better understood for $\mathrm{Sp}_{K(n)}$ than for $\mathrm{Sp}_{T(n)}$, is the *Picard group*. Recall that for a symmetric monoidal ∞–category $\mathscr{C}$, the Picard group $\mathrm{Pic}(\mathscr{C})$ is the abelian group of isomorphism classes of invertible objects in $\mathscr{C}$ under tensor product. While $\mathrm{Pic}(\mathrm{Sp}_{K(n)})$ was intensively

[1] In this paper, we use the version of E_n whose coefficients satisfy $\pi_0 E_n \simeq W(\overline{\mathbb{F}}_p)[[u_1, \dots, u_{n-1}]]$.

studied (for instance in Hopkins, Mahowald and Sadofsky [28], Lader [31], Goerss, Henn, Mahowald and Rezk [22] and Heard [25]), very little is known about $\mathrm{Pic}(\mathrm{Sp}_{T(n)})$. Our second main result concerns the construction of nontrivial elements in $\mathrm{Pic}(\mathrm{Sp}_{T(n)})$.

Theorem B (Theorem 5.32) *For an odd prime p, the group $\mathrm{Pic}(\mathrm{Sp}_{T(n)})$ admits a subgroup isomorphic to $\mathbb{Z}/(p-1)$.*

Moreover, under $K(n)$–localization, this subgroup is mapped isomorphically onto the subgroup of $\mathrm{Pic}(\mathrm{Sp}_{K(n)})$, consisting of objects which are $(p-1)$–torsion and of symmetric monoidal dimension (a.k.a. Euler characteristic) 1. We also construct some nontrivial elements in $\mathrm{Pic}(\mathrm{Sp}_{T(n)})$ for $p=2$, and describe their image in the algebraic Picard group (Theorem 5.33). We deduce Theorem B from Theorem A by a generalized Kummer theory, which we develop in this paper.

1.1.2 Higher cyclotomic extensions We shall now outline our approach to Theorem A. As mentioned above, the Galois extensions of $\mathbb{S}_{K(n)}$ are governed by the Lubin–Tate spectrum E_n, whose construction relies on the theory of complex orientations. In the absence of an analogue of this construction in the $T(n)$–local world, we take our cue from classical algebra, where we have a natural source of abelian Galois extensions — the *cyclotomic extensions*. Namely, for a commutative ring R, we have the m^{th} cyclotomic extension $R[\omega_m] := R[t]/\Phi_m(t)$, where $\Phi_m(t)$ is the m^{th} cyclotomic polynomial, whose roots are the primitive m^{th} roots of unity. If m is invertible in R, this extension is Galois (though not necessarily connected) with respect to the natural action of $(\mathbb{Z}/m)^\times$.

As a concrete example, consider $R=\mathbb{Q}_p$. Starting integrally, for every $d\in\mathbb{N}$, the cyclotomic extension $\mathbb{Z}_p[\omega_{p^d-1}]$ splits into a product of copies of the $\mathbb{Z}/d$–Galois extension $W(\mathbb{F}_{p^d})$, exhibiting the latter as a subextension of a cyclotomic one. After inverting p, we get $\mathbb{Q}_p(\omega_{p^d-1}) = W(\mathbb{F}_{p^d})[p^{-1}]$, which assemble into the maximal unramified extension[2] $\mathbb{Q}_p^{\mathrm{ur}} := \bigcup_d \mathbb{Q}_p(\omega_{p^d-1})$ of $\mathbb{Q}_p$. However, as p is now invertible, we have also the cyclotomic extensions of p–power order, which assemble into $\mathbb{Q}_p(\omega_{p^\infty}) := \bigcup_r \mathbb{Q}_p[\omega_{p^r}]$. It is a classical theorem in number theory that all *abelian* Galois extensions of $\mathbb{Q}_p$ can be obtained in this way.

Theorem (Kronecker–Weber) *We have that $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{\mathrm{ab}} \simeq \hat{\mathbb{Z}}\times\mathbb{Z}_p^\times$, where $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\to\hat{\mathbb{Z}}$ classifies $\mathbb{Q}_p^{\mathrm{ur}}$, and the p–adic cyclotomic character $\chi\colon \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\to\mathbb{Z}_p^\times$ classifies $\mathbb{Q}_p(\omega_{p^\infty})$.*

Incidentally, for $1\le n<\infty$, we also have[3] $\mathbb{G}_n^{\mathrm{ab}}\simeq\hat{\mathbb{Z}}\times\mathbb{Z}_p^\times$. The finite Galois extensions of $\mathbb{S}_{K(n)}$ that are classified by the map $\mathbb{G}_n\to\hat{\mathbb{Z}}$, are the $K(n)$–localizations of the *spherical Witt vectors* $\mathbb{S}W(\mathbb{F}_{p^d})$; see Lurie [36, Example 5.2.7]. Hence, just as for $\mathbb{Q}_p$, they can be obtained from cyclotomic extensions of

[2]We denote by $\mathbb{Q}_p(\omega_m)$ the splitting field of $\Phi_m(t)$ over $\mathbb{Q}_p$, as opposed to $\mathbb{Q}_p[\omega_m] := \mathbb{Q}_p[t]/\Phi_m(t)$ which may be not connected, but rather a product of copies of $\mathbb{Q}_p(\omega_m)$.

[3]For height $n=1$, this similarity was also discussed in [49, Section 5.5].

order prime to p; see Proposition 5.13 and Corollary 5.15. Similarly, the $T(n)$–localizations of $\mathbb{S}W(\mathbb{F}_{p^d})$ constitute a lift of the said Galois extensions of $\mathbb{S}_{K(n)}$ to Galois extensions of $\mathbb{S}_{T(n)}$. However, unlike for $\mathbb{Q}_p$, the element p is not invertible in $\mathrm{Sp}_{K(n)}$, and in fact, a $K(n)$–local commutative algebra *cannot* admit primitive p–power roots of unity; see Devalapurkar [16, Theorem 1.3]. Nevertheless, and it is the main insight leading to the results of this paper, the higher semiadditivity of the ∞–categories $\mathrm{Sp}_{K(n)}$ (in the sense of Hopkins and Lurie [27]) allows one to view the Galois extensions classified by $\det\colon \mathbb{G}_n \to \mathbb{Z}_p^\times$, as a "higher analogue" of the classical cyclotomic extensions of p–power order. Furthermore, the higher semiadditivity of the ∞–categories $\mathrm{Sp}_{T(n)}$ (see Carmeli, Schlank and Yanovski [14, Theorem A]), is what allows us to construct their $T(n)$–local lifts. We shall now explain these ideas in more detail.

We begin by reformulating the construction of the (ordinary) cyclotomic extensions in a way which lends itself to ∞–categorical generalizations. Recall that a $(p^r)^{\text{th}}$ root of unity in a commutative ring R is a homomorphism $C_{p^r} \to R^\times$, and it is called *primitive*, if it is nowhere (in the algebrogeometric sense) of order p^{r-1}; see Definition 3.3. For a given R, the functor which assigns to every R–algebra, the set of its $(p^r)^{\text{th}}$ roots of unity, is corepresented by the group algebra $R[C_{p^r}]$. Consider now the short exact sequence of abelian groups

$$0 \to C_p \to C_{p^r} \to C_{p^{r-1}} \to 0,$$

the associated R–algebra homomorphism $f\colon R[C_{p^r}] \to R[C_{p^{r-1}}]$, and the set map $\iota\colon C_p \hookrightarrow R[C_{p^r}]$. If p is invertible in R, we can define the idempotent

$$\varepsilon := \frac{1}{p} \sum_{g \in C_p} \iota(g) \;\in\; R[C_{p^r}],$$

which splits f. That is, inverting ε and $1-\varepsilon$, respectively, yields a decomposition

$$R[C_{p^r}] \simeq R[C_{p^{r-1}}] \times R[\omega_{p^r}].$$

In particular, we get that the cyclotomic extension $R[\omega_{p^r}]$ corepresents the set of *primitive* $(p^r)^{\text{th}}$ roots of unity. Furthermore, the natural action of the group $(\mathbb{Z}/p^r)^\times$ on C_{p^r} induces an action on the group algebra $R[C_{p^r}]$, which then restricts to the cyclotomic extension $R[\omega_{p^r}]$ by the invariance of ε, making it a $(\mathbb{Z}/p^r)^\times$–Galois extension of R. Reformulated in this way, the construction of the cyclotomic extensions can be carried out for a commutative algebra object R in any additive symmetric monoidal ∞–category, provided that p is an invertible element in the ring $\pi_0(R)$; see Schwänzl, Vogt and Waldhausen [50, Theorem 3]. An extension of these ideas, which allows adjoining roots of any invertible element, was studied in Lawson [33].

To define *higher* cyclotomic extensions, we first observe that for a commutative algebra R in a symmetric monoidal ∞–category $\mathscr{C}$, the grouplike commutative monoid (or equivalently, the connective spectrum) of units $R^\times$, need not be discrete in general. Taking advantage of that, we define a *height n root of unity* of R, to be a morphism of the form $C_{p^r} \to \Omega^n R^\times$. For such a higher root of unity, we also have a corresponding notion of primitivity; see Definition 4.2. The functor assigning to each R–algebra the space of its height n roots of unity is corepresented by the higher group algebra $R[B^n C_{p^r}]$. By analogy with the above, we consider the fiber sequence

$$B^n C_p \to B^n C_{p^r} \to B^n C_{p^{r-1}},$$

the associated morphism of commutative R–algebras $f\colon R[B^n C_{p^r}] \to R[B^n C_{p^{r-1}}]$, and map of spaces

$$\iota\colon B^n C_p \to \mathrm{Map}(\mathbb{1}_{\mathscr{C}}, R[B^n C_{p^r}]).$$

To proceed, assume now that $\mathscr{C}$ is stable and n–semiadditive. This allows one to integrate families of morphisms in $\mathscr{C}$ indexed by n–finite spaces. In particular, we can consider the *cardinality* of the n–finite space $B^n C_p$. This is given by integrating over $B^n C_p$, the unit map $\mathbb{1}_{\mathscr{C}} \xrightarrow{1_R} R$, and is denoted by

$$|B^n C_p| := \int_{B^n C_p} 1_R \;\in\; \pi_0(R) = \pi_0 \mathrm{Map}(\mathbb{1}_{\mathscr{C}}, R).$$

Recall from Carmeli, Schlank and Yanovski [13, Definition 3.1.6] that R is said to be of height $\le n$ if the element $|B^n C_p|$ is invertible in $\pi_0(R)$, in which case we can define

$$\varepsilon := \frac{1}{|B^n C_p|} \int_{x \in B^n C_p} \iota(x) \;\in\; \pi_0(R[B^n C_{p^r}]).$$

Note that for $n = 0$ we have $|C_p| = p$, hence, R is of height 0 precisely when p is invertible in the ring $\pi_0(R)$. We then show that, as in the case $n = 0$, the element ε is idempotent and induces a $(\mathbb{Z}/p^r)^\times$–equivariant decomposition

$$R[B^n C_{p^r}] \simeq R[B^n C_{p^{r-1}}] \times R[\omega^{(n)}_{p^r}],$$

such that the projection onto the first factor coincides with f (Proposition 4.5). As a result, the commutative R–algebra $R[\omega^{(n)}_{p^r}]$ corepresents the space of *primitive* $(p^r)^{\text{th}}$ roots of unity of height n (Proposition 4.8). We call $R[\omega^{(n)}_{p^r}]$ the *height n cyclotomic extension* of R of order p^r.

To apply the abstract construction of higher cyclotomic extensions to the chromatic world, we recall from [13, Section 4.4] that the semiadditive height generalizes the chromatic height. Namely, all objects of $\mathrm{Sp}_{T(n)}$, and hence also of $\mathrm{Sp}_{K(n)}$, are of semiadditive height exactly n. We then prove that the resulting $(\mathbb{Z}/p^r)^\times$–equivariant algebras $\mathbb{S}_{K(n)}[\omega^{(n)}_{p^r}]$ are Galois (Proposition 5.2). Furthermore, by comparing the infinite cyclotomic extension $\mathbb{S}_{K(n)}[\omega^{(n)}_{p^\infty}]$, with Westerland's R_n [51], we deduce that it is classified by[4] $\det\colon \mathbb{G}_n \to \mathbb{Z}_p^\times$; see Theorem 5.8 and the following discussion. Thus, the determinant map can be viewed as the higher chromatic analogue of the p–adic cyclotomic character. In the same spirit, the realization of all the abelian Galois extensions of $\mathbb{S}_{K(n)}$ in terms of (ordinary and higher) cyclotomic extensions can be viewed as the higher chromatic analogue of the Kronecker–Weber theorem. Finally, we deduce Theorem A from the above, by showing that the $T(n)$–local higher cyclotomic extensions $\mathbb{S}_{T(n)}[\omega^{(n)}_{p^r}]$ are Galois as well (Proposition 5.2), using the nilpotence theorem in the guise of "nil-conservativity"; see [14, Section 4.4].

[4]This requires one to choose a normalizable formal group law in the sense of Hopkins and Lurie [27, Definition 5.3.1]. Westerland in [51] uses the Honda formal group law, in which case one has to replace det with $\det_\pm$.

1.1.3 Kummer theory We now outline the relationship between abelian Galois extensions and the Picard spectrum, which allows us to deduce Theorem B from Theorem A. Classically, given a field k which admits a primitive m^{th} root of unity, and a finite abelian group A which is m–torsion, Kummer theory identifies the set of isomorphism classes of A–Galois extensions of k, with $\mathrm{Ext}^1_{\mathbb{Z}}(A^*, k^\times)$, where $A^* = \hom(A, \mathbb{Q}/\mathbb{Z})$ is the Pontryagin dual[5] of A. One way to construct this identification is to observe that for every A–Galois extension L/k we can simultaneously diagonalize the action of all the elements of A on L, producing an eigenspace decomposition $L \simeq \bigoplus_{\varphi \in A^*} L_\varphi$ as k–vector spaces. The L_φ turn out to be all 1–dimensional, and the multiplication of L restricts to give isomorphisms $L_\varphi \otimes L_\psi \xrightarrow{\sim} L_{\varphi+\psi}$. As a result, a choice of basis elements $0 \neq x_\varphi \in L_\varphi$ provides a 1–cocycle representative of a class in the group $\mathrm{Ext}^1_{\mathbb{Z}}(A^*, k^\times)$, which can then be shown to depend only on L and to completely characterize it.

For a more general commutative ring R, which admits a primitive m^{th} root of unity (so, in particular, $m \in R^\times$), and an A–Galois extension S of R, one can still produce a decomposition $S \simeq \bigoplus_{\varphi \in A^*} R_\varphi$ and isomorphisms $R_\varphi \otimes R_\psi \xrightarrow{\sim} R_{\varphi+\psi}$ as before. However, this only implies that the R_φ are *invertible* R–modules, rather than that $R_\varphi \simeq R$. This leads to a classification of A–Galois extensions of R, which involves both the Picard group $\mathrm{Pic}(R)$ and the group of units $R^\times$. These groups can be recognized as the π_0 and π_1 respectively, of the *Picard spectrum* of R, which we denote by $\mathrm{pic}(R)$.

Generalizing this, we show that for every additive presentable symmetric monoidal ∞–category $\mathscr{C}$, such that the commutative ring $\pi_0 \mathbb{1}$ admits a primitive m^{th} root of unity, there is a homotopy equivalence of spaces (Theorem 3.18)

$$\mathrm{CAlg}^{A\text{-gal}}(\mathscr{C}) \simeq \mathrm{Map}_{\mathrm{Sp}^{\mathrm{cn}}}(A^*, \mathrm{pic}(\mathscr{C})),$$

where on the left-hand side we have the full subcategory (which turns out to be an ∞–groupoid) of $\mathrm{CAlg}(\mathscr{C})^{BA}$ consisting of A–Galois extensions of the unit $\mathbb{1}_{\mathscr{C}}$. This can be considered as a general form of "Kummer theory" in the context of ∞–categories. The main difficulty in establishing the above homotopy equivalence is to handle the multiplicativity of the eigenspace decomposition coherently. To this end, we realize the eigenspace decomposition, under the above assumptions, as a symmetric monoidal equivalence (Proposition 3.13)

$$\mathfrak{F}\colon \mathrm{Fun}(BA, \mathscr{C})_{\mathrm{Ptw}} \xrightarrow{\sim} \mathrm{Fun}(A^*, \mathscr{C})_{\mathrm{Day}},$$

where the subscript "Ptw" indicates the usual pointwise symmetric monoidal structure, while the subscript "Day" indicates the Day-convolution symmetric monoidal structure. This can be viewed as a general form of the discrete Fourier transform.

Applying the above to $A = \mathbb{Z}/m$, and taking π_0, gives rise to a (noncanonically) split short exact sequence of abelian groups (Proposition 3.23)

$$0 \to (\pi_0 \mathbb{1}^\times)/(\pi_0 \mathbb{1}^\times)^m \to \pi_0 \,\mathrm{CAlg}^{\mathbb{Z}/m\text{-gal}}(\mathscr{C}) \to \mathrm{Pic}^{\mathrm{ev}}(\mathscr{C})[m] \to 0,$$

[5] For instance, for $A = \mathbb{Z}/m$, we have $\mathrm{Ext}^1_{\mathbb{Z}}((\mathbb{Z}/m)^*, k^\times) = (k^\times)/(k^\times)^m$, so that cyclic Galois extensions of order m are classified by invertible elements of the base field up to m^{th} powers; see Birch [10].

where $\mathrm{Pic}^{\mathrm{ev}}(\mathscr{C}) \le \mathrm{Pic}(\mathscr{C})$ is the subgroup of invertible objects of monoidal dimension 1 (Definition 3.22), and $\mathrm{Pic}^{\mathrm{ev}}(\mathscr{C})[m]$ is its m–torsion subgroup. We note that when $\mathscr{C}$ is p–complete for some odd prime p, the $\mathbb{Z}_p$–algebra $\pi_0\mathbb{1}$ always admits primitive $(p-1)^{\mathrm{st}}$ roots of unity. Thus, to every $\mathbb{Z}/(p-1)$–Galois extension of $\mathbb{1}$, corresponds a (possibly trivial) $(p-1)$–torsion element of $\mathrm{Pic}^{\mathrm{ev}}(\mathscr{C})$.

Specializing to the chromatic world, we show that the $K(n)$–local Picard object Z_n, which corresponds to the higher cyclotomic $\mathbb{Z}/(p-1)$–Galois extension $\mathbb{S}_{K(n)}[\omega_p^{(n)}]$, generates the group (Proposition 5.23)

$$\mathrm{Pic}^{\mathrm{ev}}(\mathrm{Sp}_{K(n)})[p-1] \simeq \mathbb{Z}/(p-1).$$

We deduce that $\mathbb{S}_{T(n)}[\omega_p^{(n)}]$ corresponds to a $T(n)$–local Picard object $Z_n^f \in \mathrm{Pic}^{\mathrm{ev}}(\mathrm{Sp}_{T(n)})[p-1]$, which lifts Z_n, implying Theorem B. We use a variation of the above method to produce nontrivial $T(n)$–local Picard objects in the case $p=2$ as well, using the three $\mathbb{Z}/2$–subextensions of $\mathbb{S}_{T(n)}[\omega_8^{(n)}]$; see Theorem 5.33.

1.1.4 Faithfulness and descent Taking the colimit over all $(p^r)^{\mathrm{th}}$ cyclotomic extensions, we obtain the infinite cyclotomic extension

$$R_n := \mathbb{S}_{K(n)}[\omega_{p^\infty}^{(n)}] = \varinjlim \mathbb{S}_{K(n)}[\omega_{p^r}^{(n)}].$$

This continuous $\mathbb{Z}_p^\times$–Galois extension of $\mathbb{S}_{K(n)}$, which is classified by $\det\colon \mathbb{G}_n \to \mathbb{Z}_p^\times$, enables several key constructions in $\mathrm{Sp}_{K(n)}$. Among them, are the class $\zeta_n \in \pi_{-1}(\mathbb{S}_{K(n)})$ (see [18, Section 8]) and the determinant sphere $\mathbb{S}_{K(n)}\langle\det\rangle \in \mathrm{Pic}(\mathrm{Sp}_{K(n)})$; see Barthel, Beaudry, Goerss and Stojanoska [7], Westerland [51] and Goerss, Henn, Mahowald and Rezk [21]. Using our results, we can similarly construct the $T(n)$–local infinite cyclotomic extension $R_n^f := \mathbb{S}_{T(n)}[\omega_{p^\infty}^{(n)}]$. Assuming R_n^f is *faithful*, one could lift ζ_n and $\mathbb{S}_{K(n)}\langle\det\rangle$ to the $T(n)$–local world. However, while all *finite* Galois extensions of $\mathbb{S}_{T(n)}$ are faithful, we do not know whether the infinite Galois extension R_n^f is faithful.[6] As far as we know, R_n^f might be even $K(n)$–local, in which case the faithfulness of R_n^f would be equivalent to the telescope conjecture. As an example, one can argue directly to show that R_1^f is both faithful and isomorphic to R_1, which leads to a new proof of the telescope conjecture at height $n=1$. A more detailed account of this circle of ideas will appear elsewhere.

1.2 Conventions

Throughout the paper, we work in the framework of ∞–categories (a.k.a. quasicategories), and in general follow the notation of Lurie [34; 35]. The terminology and notation for all concepts related to higher semiadditivity and (semiadditive) height are as in Carmeli, Schlank and Yanovski [13]. In addition,

(1) We use the notation $\hom(X,Y)$ for the enriched/internal hom-objects, as opposed to $\mathrm{Map}(X,Y)$ which always denotes the mapping space.

[6] In an earlier stage of this project, we believed that we have a proof for the faithfulness of R_n^f, which led to [9, Remark 8.5.3]. However, while writing this paper we have discovered a crucial gap in the argument. For a more thorough discussion see [8].

(2) For an object X in a monoidal ∞–category $\mathscr{C}$, we write $\Omega^\infty X$ for $\operatorname{Map}(\mathbb{1}, X)$ and $\pi_0 X$ for $\pi_0 \operatorname{Map}(\mathbb{1}, X)$.

(3) We denote by Pr the ∞–category of presentable ∞–categories and colimit-preserving functors, and by

$$\mathrm{Pr}_{\mathfrak{T}_n} \subseteq \mathrm{Pr}_{\mathrm{st}}^{\oplus-n} \subseteq \mathrm{Pr}_{\mathrm{st}} \subseteq \mathrm{Pr}_{\mathrm{add}} \subseteq \mathrm{Pr}$$

the full subcategories spanned by ∞–categories which are additive, stable, stable n–semiadditive and stable n–semiadditive of semiadditive height n (with respect to an implicit prime p).[7]

(4) For an abelian group A and a natural number m we denote by $A[m]$ the subgroup of m–torsion elements in A.

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2 Galois theory

We begin by discussing some special features of Galois extensions following Rognes [49], under the assumption that the classifying space of the acting group is ambidextrous with respect to the ∞–category. We shall work mainly in the setting of *additive* presentable ∞–categories. These include the stable presentable ∞–categories as well as ordinary additive presentable categories, such as that of abelian groups.

The main result of Section 2.1 is that the Galois property can be detected by nil-conservative functors (Proposition 2.9), and of Section 2.2, that it can be characterized by the fact that a certain associated lax symmetric monoidal functor is strong monoidal (Proposition 2.13).

2.1 Definition and detection

We begin by recalling some terminology and notation regarding local systems. Given an ∞–category $\mathscr{C}$ and a space A we denote by $\mathscr{C}^A$ the ∞–category of functors $A \to \mathscr{C}$ and refer to its objects as $\mathscr{C}$–valued local systems on A. If $\mathscr{C}$ is (symmetric) monoidal then $\mathscr{C}^A$ is (symmetric) monoidal with respect to the

[7] In the language of [13, Section 5.2], these properties are classified by the *modes* $\mathrm{Sp}^{\mathrm{cn}}$, Sp, $\mathfrak{T}^{[n]}$ and $\mathfrak{T}_n$, respectively.

pointwise structure. For a map $f: A \to B$, restriction along it provides a functor $f^*: \mathscr{C}^B \to \mathscr{C}^A$, which is (symmetric) monoidal when $\mathscr{C}$ is. When $\mathscr{C}$ admits A–shaped limits/colimits, f^* admits a left/right adjoint, which we denote by $f_!$ and f_*, respectively.

When $f: A \to B$ is weakly $\mathscr{C}$–ambidextrous in the sense of [27, Definition 4.1.11], there is a canonical norm map $\mathrm{Nm}_f: f_! \to f_*$ and f is called $\mathscr{C}$–ambidextrous if Nm_f is an isomorphism. In this case, we obtain for every pair of objects $X, Y \in \mathscr{C}^B$ an integration operation [14, Definition 2.1.11]

$$\int_f: \mathrm{Map}(f^*X, f^*Y) \to \mathrm{Map}(X, Y).$$

In the special case of $f: A \to \mathrm{pt}$ we write $\int_f \mathrm{Id}_X: X \to X$ by $|A|$ and think of it as "multiplication by the cardinality of A". Recall also that $\mathscr{C}$ is said to be m–semiadditive if every m–finite map is $\mathscr{C}$–ambidextrous.

We next recall the definition of a Galois extension from [49]:

Definition 2.1 (Rognes) Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$, let G be a finite group and let $R \in \mathrm{CAlg}(\mathscr{C}^{BG})$. We say that R is a G–*Galois* extension (or just Galois) if it satisfies the following two conditions:

(1) The canonical map $\mathbb{1} \to R^{hG}$ is an isomorphism.

(2) The canonical map $R \otimes R \to \prod_G R$, given informally by $x \otimes y \mapsto (x \cdot \sigma y)_{\sigma \in G}$, is an isomorphism.

A Galois extension R is called *faithful* if in addition the functor

$$(-) \otimes R: \mathscr{C} \longrightarrow \mathscr{C}$$

is conservative. We denote by $\mathrm{CAlg}^{G\text{–gal}}(\mathscr{C}) \subseteq \mathrm{CAlg}(\mathscr{C}^{BG})$ the full subcategory spanned by G–Galois extensions.

Remark 2.2 For $S \in \mathrm{CAlg}(\mathscr{C})$, by a G–Galois extension of S, we shall mean a G–Galois extension in the symmetric monoidal ∞–category $\mathrm{Mod}_S(\mathscr{C})$ in the sense of Definition 2.1.

Remark 2.3 It is proved in [49, Proposition 6.3.3] that faithfulness of a Galois extension R is equivalent to the condition that the norm map

$$\mathrm{Nm}: R_{hG} \to R^{hG}$$

is an isomorphism. We shall be particularly interested in situations where BG is $\mathscr{C}$–ambidextrous (eg when $\mathscr{C}$ is 1–semiadditive or $|G|$ is invertible in $\mathscr{C}$), in which case this condition is satisfied automatically.

Unlike in the classical Galois theory for fields, Galois extensions are not required to be *connected*. In particular, for every group G, there is always the "trivial" G–extension:

Example 2.4 (split-Galois extension) Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$ and let G be a finite group with $e\colon \mathrm{pt} \to BG$ the inclusion of the basepoint. The functor $e_*\colon \mathscr{C} \to \mathscr{C}^{BG}$ is lax symmetric monoidal and the induced object

$$e_*\mathbb{1} \simeq \prod_G \mathbb{1} \in \mathrm{CAlg}(\mathscr{C}^{BG})$$

is a G–Galois extension, where G acts by permuting the factors according to the regular action of G on itself. We say that a G–Galois extension R is *split* if it is isomorphic to $e_*\mathbb{1}$ as a G–equivariant commutative algebra.

The underlying object of a Galois extension is always dualizable; see [49, Proposition 6.2.1] and [38, Proposition 6.14]. To detect Galois extensions, it will be useful to establish certain closure properties for dualizable objects.

Proposition 2.5 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr})$ and let $I \in \mathrm{Cat}_\infty$. If the tensor product of $\mathscr{C}$ preserves I^{op}–shaped limits in each variable, then the dualizable objects in $\mathscr{C}$ are closed under I–shaped colimits.*

Proof We denote by $\hom(X,Y) \in \mathscr{C}$ the internal hom-object of $X, Y \in \mathscr{C}$. An object $X \in \mathscr{C}$ is dualizable if and only if for every $Y \in \mathscr{C}$, the canonical map

$$\hom(X,\mathbb{1}) \otimes Y \to \hom(X, \mathbb{1} \otimes Y) \simeq \hom(X,Y)$$

is an isomorphism; see eg [45, Theorem 2.2]. Let $X = \varinjlim_{a \in I} X_a$ be such that $X_a \in \mathscr{C}$ is dualizable for all $a \in I$. For every $Y \in \mathscr{C}$ the canonical map above fits into a commutative diagram

$$\begin{tikzcd}
\hom(\varinjlim_{a\in I} X_a, \mathbb{1}) \otimes Y \arrow[rr] \arrow[d, "\wr"] & & \hom(\varinjlim_{a\in I} X_a, Y) \arrow[d, "\wr"] \\
(\varprojlim_{a\in I^{\mathrm{op}}} \hom(X_a,\mathbb{1})) \otimes Y \arrow[r, "\sim"] & \varprojlim_{a\in I^{\mathrm{op}}} (\hom(X_a,\mathbb{1}) \otimes Y) \arrow[r, "\sim"] & \varprojlim_{a\in I^{\mathrm{op}}} \hom(X_a, Y).
\end{tikzcd}$$

The vertical arrows are isomorphisms since $\hom(-,-)$ takes colimits in the first variable into limits. The bottom left arrow is an isomorphism because the tensor product preserves I^{op}–limits in each variable and the bottom right arrow is an isomorphism because each X_a is dualizable. It follows that the top map is an isomorphism and hence that X is dualizable. □

Remark 2.6 For $\mathscr{C}$ stable, the tensor product preserves finite limits, and we recover the classical fact that dualizable objects in $\mathscr{C}$ are closed under finite colimits; see eg [40].

Corollary 2.7 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr})$ and let A be a $\mathscr{C}$–ambidextrous space. The dualizable objects in $\mathscr{C}$ are closed under A–shaped limits and colimits.*

Proof Since A is $\mathscr{C}$–ambidextrous, A–shaped limits coincide with A–shaped colimits. It therefore suffices to show that the dualizable objects are closed under A–shaped colimits. Moreover, since the tensor product preserves A–shaped colimits in each variable, it also preserves A–shaped limits in each variable; see [13, Proposition 2.1.8]. Therefore, the claim follows from Proposition 2.5. □

Remark 2.8 The special case of a constant A–shaped colimit was treated in [14, Proposition 3.3.6], where we have further shown that $\dim(A \otimes \mathbb{1}) = |LA| \in \pi_0\mathbb{1}$; see [14, Corollary 3.3.10].

The following proposition shows that in the stable setting, the Galois property can be detected by *nil-conservative* functors; see [14, Definition 4.4.1].

Proposition 2.9 *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a nil-conservative functor in $\mathrm{CAlg}(\mathrm{Pr}_{st})$, let G be a finite group such that BG is $\mathscr{C}$–ambidextrous, and let $R \in \mathrm{CAlg}(\mathscr{C}^{BG})$. If $F(R) \in \mathrm{CAlg}(\mathscr{D}^{BG})$ is Galois and R is dualizable in $\mathscr{C}$, then R is Galois.*

Proof First, by [14, Corollary 3.3.2], the space BG is also $\mathscr{D}$–ambidextrous. Now, since BG is $\mathscr{C}$– and $\mathscr{D}$–ambidextrous and F preserves colimits, F also preserves BG–shaped limits by [13, Proposition 2.1.8]. Thus, applying F to the maps

$$\mathbb{1} \to R^{hG}, \quad R \otimes R \to \prod_G R,$$

in conditions (1) and (2) of Definition 2.1, we get the corresponding maps

$$\mathbb{1} \to F(R)^{hG}, \quad F(R) \otimes F(R) \to \prod_G F(R),$$

for $F(R) \in \mathrm{CAlg}(\mathscr{D}^{BG})$. Since $F(R)$ is Galois, these maps are isomorphisms. Since the underlying object of R is dualizable in $\mathscr{C}$, all the objects $\mathbb{1}$, R^{hG}, $R \otimes R$ and $\prod_G R$ are dualizable as well. Indeed, R^{hG} is dualizable by Corollary 2.7, and the other three by standard arguments. Thus, as nil-conservative functors are conservative on dualizable objects [14, Proposition 4.4.4], we get that R is Galois. □

2.2 Twisting functors

It will be useful for the sequel to observe that the Galois property can be characterized using the following notion:

Definition 2.10 For every $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$ and $R \in \mathrm{CAlg}(\mathscr{C}^{BG})$, we define the *twisting functor* of R to be the composition

$$T_R\colon \mathscr{C}^{BG} \xrightarrow{R\otimes(-)} \mathscr{C}^{BG} \xrightarrow{(-)^{hG}} \mathscr{C}.$$

The functor T_R is lax symmetric monoidal as a composition of the lax symmetric monoidal functor $(-)^{hG}$ and the functor $R \otimes (-)$, which is itself lax symmetric monoidal as a composition of the functors in the free-forgetful symmetric monoidal adjunction

$$\mathscr{C}^{BG} \xrightarrow{F} \mathrm{Mod}_R(\mathscr{C}^{BG}) \xrightarrow{U} \mathscr{C}^{BG}.$$

We note the following immediate consequence of assuming that BG is $\mathscr{C}$–ambidextrous:

Lemma 2.11 *If BG is $\mathscr{C}$–ambidextrous, then T_R preserves colimits and is $\mathscr{C}$–linear in the sense that for all $X \in \mathscr{C}^{BG}$ and $Z \in \mathscr{C}$, the canonical map*

$$T_R(X) \otimes Z = (R \otimes X)^{hG} \otimes Z \xrightarrow{\beta_*} (R \otimes X \otimes Z)^{hG} = T_R(X \otimes Z)$$

is an isomorphism.

Proof The norm map

$$\mathrm{Nm}\colon (R \otimes X)_{hG} \to (R \otimes X)^{hG} = T_R(X)$$

is an isomorphism; hence the functor T_R is isomorphic to the colimit-preserving functor $X \mapsto (R \otimes X)_{hG}$. Furthermore, for every $Z \in \mathscr{C}$ consider the colimit-preserving functor $(-) \otimes Z : \mathscr{C} \to \mathscr{C}$ and the associated commutative norm diagram [14, Theorem 3.2.3], which for every $X \in \mathscr{C}^{BG}$ is of the form

$$\begin{tikzcd}
(R \otimes X \otimes Z)_{hG} \arrow[r, "\mathrm{Nm}", "\sim"'] & (R \otimes X \otimes Z)^{hG} \arrow[d, "\beta_*"] \\
(R \otimes X)_{hG} \otimes Z \arrow[u, "\beta_!", "\wr"'] \arrow[r, "\mathrm{Nm}", "\sim"'] & (R \otimes X)^{hG} \otimes Z
\end{tikzcd}$$

It follows that β_* is an isomorphism as well. □

Remark 2.12 Using the free-forgetful adjunction $\mathrm{Pr} \leftrightarrows \mathrm{Mod}_{\mathscr{C}}(\mathrm{Pr})$, the $\mathscr{C}$–linearity of T_R can be rephrased as follows. Let $\overline{T}_R$ be the restriction of T_R along $\mathscr{S}^{BG} \to \mathscr{C}^{BG}$. The functor $\overline{T}_R : \mathscr{S}^{BG} \to \mathscr{C}$ is colimit-preserving and

$$T_R : \mathscr{C}^{BG} \simeq \mathscr{C} \otimes \mathscr{S}^{BG} \to \mathscr{C}$$

is the corresponding $\mathscr{C}$–linear functor.

The lax symmetric monoidal structure of T_R can be used to characterize the Galois property for R.

Proposition 2.13 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$ and let G be a finite group such that BG is $\mathscr{C}$–ambidextrous. A G–equivariant commutative ring $R \in \mathrm{CAlg}(\mathscr{C}^{BG})$ is Galois if and only if T_R is (strong) symmetric monoidal.*

Proof The Galois property of R can be related to the properties of the functor T_R as follows. First, for $BG \xrightarrow{q} \mathrm{pt}$, the unitality of T_R amounts to the unit map

$$\mathbb{1} \to T_R(q^*\mathbb{1}) \simeq R^{hG}$$

being an isomorphism, ie it is equivalent to the first Galois condition for R. Second, let $\mathrm{pt} \xrightarrow{e} BG$ denote the basepoint. We have, on the one hand,

$$T_R(e_*\mathbb{1}) \otimes T_R(e_*\mathbb{1}) \simeq \Big(\prod_G R\Big)^{hG} \otimes \Big(\prod_G R\Big)^{hG} \simeq R \otimes R$$

and on the other,

$$T_R(e_*\mathbb{1} \otimes e_*\mathbb{1}) \simeq \Big(\prod_{G \times G} R\Big)^{hG} \simeq \prod_G R.$$

The second isomorphism follows from the fact that the diagonal action of G on $G \times G$ is free with quotient G. Moreover, the canonical map, induced by T_R being lax symmetric monoidal,

$$R \otimes R \simeq T_R(e_*\mathbb{1}) \otimes T_R(e_*\mathbb{1}) \to T_R(e_*\mathbb{1} \otimes e_*\mathbb{1}) \simeq \prod_G R$$

is exactly the map appearing in the second Galois condition for R. Hence, the second condition is equivalent to the structure map $T_R(X) \otimes T_R(Y) \to T_R(X \otimes Y)$ to be an isomorphism in the special case $X = Y = e_*\mathbb{1} \simeq e_!\mathbb{1}$. In particular, if T_R is strong symmetric monoidal, then R is Galois.

Conversely, assume that R is Galois. By the $\mathscr{C}$–linearity of T_R (Lemma 2.11), to show that T_R is strong symmetric monoidal, it suffices to show that the restriction $\overline{T}_R \colon \mathscr{S}^{BG} \to \mathscr{C}$ is strong symmetric monoidal (Remark 2.12). By the above, the structure map for the symmetric monoidality of $\overline{T}_R$ is an isomorphism in the case $X = Y = e_!(\mathrm{pt})$. The local system $e_!(\mathrm{pt})$ generates $\mathscr{S}^{BG}$ under colimits, and $\overline{T}_R$ is colimit-preserving. It follows that $\overline{T}_R$ is strong symmetric monoidal and hence so is T_R. □

The strong symmetric monoidality of the twisting functor implies that it induces a "descent" map $\mathrm{Pic}(\mathscr{C}^{BG}) \to \mathrm{Pic}(\mathscr{C})$. This allows one to construct Picard objects in $\mathscr{C}$ by twisting Picard objects in $\mathscr{C}^{BG}$; see eg [7]). Though we shall adopt a somewhat different perspective, our construction of Picard objects from Galois extensions in the next section fits into this paradigm.

3 Kummer theory

In this section, we study the relationship between abelian Galois extensions and the Picard spectrum. As in Section 2, we shall work mainly in the setting of additive ∞–categories.

In Section 3.1, we review the notion of (primitive) roots of unity (Definition 3.3) and prove a general form of the "orthogonality of characters" (Proposition 3.11). In Section 3.2, we give a general form of the discrete Fourier transform (Proposition 3.13) and use the results of Section 2.2 to characterize the Galois property of a commutative algebra in terms of its Fourier transform (Corollary 3.17). In Section 3.3, we use this characterization to establish the general form of Kummer theory (Theorem 3.18), and analyze the special case of a cyclic group (Proposition 3.23). We conclude with a certain variant for constructing Picard objects out of $\mathbb{Z}/2$–Galois extensions (Proposition 3.27), which will play a role in the chromatic world when $p = 2$.

3.1 Character theory

3.1.1 Roots of unity Following [1, Section 1.3], for every $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr})$, there is a unique symmetric monoidal colimit-preserving functor $\mathscr{S} \to \mathscr{C}$ which induces an adjunction

$$\mathrm{CAlg}(\mathscr{S}) \leftrightarrows \mathrm{CAlg}(\mathscr{C}).$$

Furthermore, $\mathrm{CAlg}(\mathcal{S}) \simeq \mathrm{CMon}(\mathcal{S})$ contains a coreflective full subcategory of grouplike commutative monoids $\mathrm{CMon}^{\mathrm{gp}}(\mathcal{S}) \subseteq \mathrm{CMon}(\mathcal{S})$, which is equivalent to the full subcategory of connective spectra $\mathrm{Sp}^{\mathrm{cn}} \subseteq \mathrm{Sp}$. Composing these adjunctions, we get an adjunction of the form

$$\mathbb{1}[-]\colon \mathrm{Sp}^{\mathrm{cn}} \leftrightarrows \mathrm{CAlg}(\mathcal{C}) : (-)^{\times}.$$

We think of the left adjoint $\mathbb{1}[-]$ as the *group algebra* functor, and of the right adjoint $(-)^{\times}$ as the commutative *group of units*.

Specializing to the case $\mathcal{C} = \mathrm{Cat}_\infty$ with its Cartesian symmetric monoidal structure, $\mathrm{CAlg}(\mathrm{Cat}_\infty)$ is the ∞–category of symmetric monoidal ∞–categories. In this case, the functor

$$\mathbb{1}[-]\colon \mathrm{Sp}^{\mathrm{cn}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

is fully faithful with essential image those symmetric monoidal ∞–categories in which all morphisms are invertible and all objects are $\otimes$–invertible. We shall thus abuse notation and regard a connective spectrum also as a symmetric monoidal ∞–category via this fully faithful embedding.

Definition 3.1 For $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Cat}_\infty)$, the *Picard spectrum* of $\mathcal{C}$ is given by

$$\mathrm{pic}(\mathcal{C}) := \mathcal{C}^{\times} \in \mathrm{Sp}^{\mathrm{cn}},$$

and the *Picard group* is

$$\mathrm{Pic}(\mathcal{C}) := \pi_0(\mathrm{pic}(\mathcal{C})) \in \mathrm{Ab}.$$

Less formally, the Picard spectrum of $\mathcal{C}$ consists of tensor invertible objects of $\mathcal{C}$ with the tensor product as a coherently commutative group operation. This is a (usually nontrivial) delooping of the connective spectrum $\mathbb{1}_{\mathcal{C}}^{\times}$ in the following sense:

$$(\Omega\mathrm{pic}(\mathcal{C}))_{\geq 0} \simeq \mathbb{1}_{\mathcal{C}}^{\times} \in \mathrm{Sp}^{\mathrm{cn}}.$$

The counit of the adjunction $\mathrm{Sp}^{\mathrm{cn}} \leftrightarrows \mathrm{CAlg}(\mathrm{Cat}_\infty)$ provides a symmetric monoidal functor $\mathrm{pic}(\mathcal{C}) \to \mathcal{C}$, which is the nonfull embedding of the $\otimes$–invertible objects and the isomorphisms between them into $\mathcal{C}$.

Remark 3.2 For a large symmetric monoidal ∞–category $\mathcal{C}$, the spectrum $\mathrm{pic}(\mathcal{C})$ might a priori be large as well. However, if $\mathcal{C}$ is presentable, the spectrum $\mathrm{pic}(\mathcal{C})$ is (essentially) small; see for instance [39, Remark 2.1.4].

Having introduced the space of units of a commutative algebra, we can now further consider *roots of unity*.

Definition 3.3 (roots of unity) Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$ and let $R \in \mathrm{CAlg}(\mathcal{C})$. For every $m \in \mathbb{N}$:

(1) We define the space of m^{th} *roots of unity* in R by

$$\mu_m(R) := \mathrm{Map}_{\mathrm{Sp}^{\mathrm{cn}}}(C_m, R^{\times}),$$

where C_m is the cyclic group of order m.

(2) We say that an m^{th} root of unity $C_m \xrightarrow{\omega} R^\times$ is *primitive* if R is m–divisible (ie m is invertible in $\pi_0 R$), and for every d which strictly divides m, the only commutative R–algebra S for which there exists a dotted arrow rendering the diagram of connective spectra

$$\begin{tikzcd} C_m \arrow[r] \arrow[d, two heads] & R^\times \arrow[d] \\ C_d \arrow[r, dashed] & S^\times \end{tikzcd}$$

commutative, is $S=0$. We denote by $\mu_m^{\text{prim}}(R) \subseteq \mu_m(R)$ the union of connected components of primitive m^{th} roots of unity.

By convention, a (primitive) m^{th} root of unity of $\mathscr{C}$ is a (primitive) m^{th} root of unity of $\mathbb{1}_{\mathscr{C}}$.

Employing the adjunction $\mathbb{1}[-] \dashv (-)^\times$, the functor $\mu_m\colon \mathrm{CAlg}(\mathscr{C}) \to \mathscr{S}$ is corepresented by the group algebra $\mathbb{1}[C_m]$. If we further assume that $\mathbb{1}$ is m–divisible, then for every divisor $d \mid m$, the map $\mathbb{1}[C_m] \to \mathbb{1}[C_d]$ can be identified with $\mathbb{1}[C_m] \to \mathbb{1}[C_m][\varepsilon_d^{-1}]$, for the idempotent

$$\varepsilon_d = \frac{d}{m} \sum_{a \in d \cdot C_m} a \ \in \ \pi_0(\mathbb{1}[C_m]).$$

Definition 3.4 (cyclotomic extensions) Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$, such that m is invertible in $\mathscr{C}$. We define the m^{th} *cyclotomic extension* to be

$$\mathbb{1}[\omega_m] := \mathbb{1}[C_m][\varepsilon^{-1}], \quad \text{where } \varepsilon = \prod_{1 \le d < m,\, d \mid m} (1-\varepsilon_d) \ \in \ \pi_0(\mathbb{1}[C_m]).$$

The commutative algebra $\mathbb{1}[\omega_m]$ carries a tautological (primitive) m^{th} root of unity denoted by ω_m.

By the above discussion, the cyclotomic extension $\mathbb{1}[\omega_m]$ corepresents the functor of primitive m^{th} roots of unity $\mu_m^{\text{prim}}\colon \mathrm{CAlg}(\mathscr{C}) \to \mathscr{S}$. Namely, for all $R \in \mathrm{CAlg}(\mathscr{C})$ we have a natural isomorphism

$$\mu_m^{\text{prim}}(R) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathscr{C})}(\mathbb{1}[\omega_m], R).$$

Example 3.5 For the ∞–category $\mathscr{C} = \mathrm{Sp}^{\mathrm{cn}}$ of connective spectra, the m^{th} cyclotomic extension

$$\mathbb{S}\Big[\frac{1}{m}\Big] \to \mathbb{S}\Big[\frac{1}{m}, \omega_m\Big]$$

is the unique étale extension which on π_0 induces the ordinary m–cyclotomic extension

$$\mathbb{Z}\Big[\frac{1}{m}\Big] \to \mathbb{Z}\Big[\frac{1}{m}, \omega_m\Big] := \mathbb{Z}\Big[\frac{1}{m}, t\Big] \big/ \Phi_m(t).$$

See [35, Theorem 7.5.0.6]. Here, $\Phi_m(t)$ is the m^{th} cyclotomic polynomial.

For $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$, we have a unique symmetric monoidal colimit-preserving functor $\mathrm{Sp}^{\mathrm{cn}} \to \mathscr{C}$, whose right adjoint (the "underlying connective spectrum") we denote by $X \mapsto \underline{X}$.

Lemma 3.6 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$ and let $R \in \mathrm{CAlg}(\mathscr{C})$. For every m, there is a canonical isomorphism $\mu_m(R) \simeq \mu_m(\underline{R})$, which restricts to an isomorphism $\mu_m^{\mathrm{prim}}(R) \simeq \mu_m^{\mathrm{prim}}(\underline{R})$, if R is m–divisible.*

Proof The first claim follows from the adjunction $\mathrm{CAlg}(\mathrm{Sp}^{\mathrm{cn}}) \leftrightarrows \mathrm{CAlg}(\mathscr{C})$ as follows:

$$\mu_m(R) = \mathrm{Map}_{\mathrm{CAlg}(\mathscr{C})}(\mathbb{1}[C_m], R) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp}^{\mathrm{cn}})}(\mathbb{S}[C_m], \underline{R}) = \mu_m(\underline{R}).$$

Assuming R is m–divisible, we can without loss of generality assume that $\mathbb{1}$ is also m–divisible, by replacing $\mathscr{C}$ with $\mathrm{Mod}(R)$. Thus, the second claim follows similarly:

$$\mu_m^{\mathrm{prim}}(R) = \mathrm{Map}_{\mathrm{CAlg}(\mathscr{C})}(\mathbb{1}[\omega_m], R) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp}^{\mathrm{cn}})}\left(\mathbb{S}\left[\frac{1}{m}, \omega_m\right], \underline{R}\right) = \mu_m^{\mathrm{prim}}(\underline{R}). \qquad \square$$

We deduce that Example 3.5 is universal in the sense that a primitive m^{th} root of unity in $\mathscr{C}$ is the same as a symmetric monoidal colimit-preserving functor from $\mathrm{Mod}_{\mathbb{S}[1/m,\omega_m]}(\mathrm{Sp}^{\mathrm{cn}})$ to $\mathscr{C}$.

Proposition 3.7 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$. For every m, we have*

$$\mu_m^{\mathrm{prim}}(\mathscr{C}) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr})}(\mathrm{Mod}_{\mathbb{S}[1/m,\omega_m]}(\mathrm{Sp}^{\mathrm{cn}}), \mathscr{C}) \in \mathcal{S}.$$

Proof By [19, Corollary 4.8] we have an equivalence $\mathrm{Pr}_{\mathrm{add}} \simeq \mathrm{Mod}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathrm{Pr})$. Thus, by [35, Theorems 4.8.5.11, 4.8.5.16 and Corollary 4.8.5.21], we have an adjunction

$$\mathrm{Mod}_{(-)}(\mathrm{Sp}^{\mathrm{cn}}) \colon \mathrm{CAlg}(\mathrm{Sp}^{\mathrm{cn}}) \leftrightarrows \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}}) \colon \mathbb{1}_{(-)}.$$

Applying this to $\mathbb{S}[1/m, \omega_m] \in \mathrm{CAlg}(\mathrm{Sp}^{\mathrm{cn}})$ and $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$, we get by Lemma 3.6,

$$\mu_m^{\mathrm{prim}}(\mathbb{1}_{\mathscr{C}}) \simeq \mu_m^{\mathrm{prim}}(\underline{\mathbb{1}_{\mathscr{C}}}) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp}^{\mathrm{cn}})}\left(\mathbb{S}\left[\frac{1}{m}, \omega_m\right], \underline{\mathbb{1}_{\mathscr{C}}}\right) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})}(\mathrm{Mod}_{\mathbb{S}[1/m,\omega_m]}(\mathrm{Sp}^{\mathrm{cn}}), \mathscr{C}). \ \square$$

We also deduce that for an m–divisible commutative algebra R, the space of (primitive) m^{th} roots of unity is discrete and depends only on $\pi_0(R)$.

Proposition 3.8 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$ and let $R \in \mathrm{CAlg}(\mathscr{C})$ which is m–divisible. We have a canonical bijection $\mu_m(R) \simeq \mu_m(\pi_0 R)$, which restricts to a bijection $\mu_m^{\mathrm{prim}}(R) \simeq \mu_m^{\mathrm{prim}}(\pi_0 R)$.*

Proof By Lemma 3.6, it suffices to consider the universal case $\mathscr{C} = \mathrm{Sp}^{\mathrm{cn}}$. In this case, we have that

$$\Omega^\infty(R^\times) \subseteq \Omega^\infty(R) \in \mathcal{S}$$

is an inclusion of connected components. Thus, $\pi_n R \simeq \pi_n R^\times$ for all $n \geq 1$. Namely, the fiber $R^\times_{\geq 1}$ (in Sp) of the truncation map $R^\times \to \pi_0 R^\times$ has the same homotopy groups as the spectrum $R_{\geq 1}$. We therefore deduce that $R^\times_{\geq 1}$ is m–divisible and hence $\mathrm{Map}_{\mathrm{Sp}^{\mathrm{cn}}}(C_m, R_{\geq 1}) = 0$. It follows that

$$\mu_m(R) = \mathrm{Map}_{\mathrm{Sp}^{\mathrm{cn}}}(C_m, R^\times) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{Sp}^{\mathrm{cn}}}(C_m, \pi_0 R^\times) = \mu_m(\pi_0 R).$$

Since the invertibility of an idempotent is a condition on π_0, under this bijection primitive roots correspond to primitive roots. $\square$

Of specific importance for us, is the following special case:

Corollary 3.9 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$ and let $R \in \mathrm{CAlg}(\mathscr{C})$ be p–complete*[8] *for some prime p. For every $m \mid (p-1)$, the commutative algebra R admits a primitive m^{th} root of unity.*

Proof First of all, since R is p–complete, it is m–divisible. Now, by Proposition 3.8, it suffices to show that $\pi_0 R$ admits a primitive m^{th} root of unity. This follows from the fact that $\pi_0 R$ is a $\mathbb{Z}_p$–algebra, and $\mathbb{Z}_p$ admits primitive m^{th} roots of unity given by Teichmüller lifts. □

3.1.2 Characters As in ordinary commutative algebra, primitive roots of unity in $\mathscr{C}$ allow us to set up a *character theory* for $\mathscr{C}$. Let A be a finite m–torsion abelian group with Pontryagin dual denoted by

$$A^* := \hom(A, C_m) = \hom(A, \mathbb{Q}/\mathbb{Z}).$$

Given $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$ with a choice of a primitive m^{th} root of unity $\omega\colon C_m \to \mathbb{1}^\times$ (so that in particular m is invertible in $\mathscr{C}$), the canonical pairing of A with A^* induces a map of spectra

$$A^* \otimes A \to C_m \xrightarrow{\omega} \mathbb{1}^\times \to \Omega\mathrm{pic}(\mathscr{C}).$$

This map corresponds to a map of connective spectra

$$A^* \to \hom(A, \Omega\mathrm{pic}(\mathscr{C})) \simeq \hom(\Sigma A, \mathrm{pic}(\mathscr{C})).$$

Definition 3.10 Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$ with a primitive m^{th} root of unity ω, and let A be a finite m–torsion abelian group. We define a map of connective spectra

$$\mathbb{1}(-)\colon A^* \to \mathrm{pic}(\mathscr{C}^{BA})$$

to be the composition

$$A^* \to \hom(\Sigma A, \mathrm{pic}(\mathscr{C})) \to \mathrm{pic}(\mathscr{C})^{BA} \simeq \mathrm{pic}(\mathscr{C}^{BA}),$$

where the first map is the one given above and the second is induced from the counit $\mathbb{S}[BA] \to \Sigma A$ by precomposition. Even though the construction of $\mathbb{1}(-)$ depends on ω, we shall keep this dependence implicit.

Intuitively, for every character $A \xrightarrow{\varphi} C_m$, the object $\mathbb{1}(\varphi) \in \mathscr{C}^{BA}$ is the unit $\mathbb{1} \in \mathscr{C}$, on which the group A acts through the composition of the character φ with $C_m \xrightarrow{\omega} \mathbb{1}^\times$. The fact that $\mathbb{1}(-)$ is a map of connective spectra encodes in a coherent way the A–equivariant identities

$$\mathbb{1}(0) \simeq \mathbb{1} \quad \text{and} \quad \mathbb{1}(\varphi + \psi) \simeq \mathbb{1}(\varphi) \otimes \mathbb{1}(\psi).$$

For $X \in \mathscr{C}^{BA}$, we define its twist by a character $\varphi \in A^*$ to be

$$X(\varphi) := X \otimes \mathbb{1}(\varphi) \ \in \ \mathscr{C}^{BA}.$$

[8] That is, $\hom_{\mathscr{C}}(X, R) \in \mathrm{Sp}^{\mathrm{cn}}$ is p–complete for all $X \in \mathscr{C}$.

We shall implicitly treat an object $X \in \mathscr{C}$ as an object of $\mathscr{C}^{BA}$ with a trivial action. The main fact we shall need about this construction is the following analogue of the "orthogonality of characters" from classical algebra:

Proposition 3.11 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$ with a primitive m^{th} root of unity and let A be a finite m–torsion abelian group.*

(1) *For every $X \in \mathscr{C}$ and $\varphi \in A^*$, we have*

$$X(\varphi)^{hA} \simeq \begin{cases} X & \text{if } \varphi = 0, \\ 0 & \text{else.} \end{cases}$$

(2) *For every $X \in \mathscr{C}$, we have*

$$\prod_{a \in A} X \simeq \bigoplus_{\varphi \in A^*} X(\varphi) \ \in \ \mathscr{C}^{BA},$$

where on the left side we have the induced representation, ie A acts by permuting the factors.

Proof (1) Since BA is $\mathscr{C}$–ambidextrous, we have (by [14, Proposition 3.3.1])

$$X(\varphi)^{hA} = (X \otimes \mathbb{1}(\varphi))^{hA} \simeq X \otimes (\mathbb{1}(\varphi)^{hA}).$$

Thus, it suffices to show the claim for $X = \mathbb{1}$. By Proposition 3.7, we have a colimit-preserving symmetric monoidal functor $F\colon \mathrm{Mod}_{\mathbb{S}[1/m,\omega_m]}(\mathrm{Sp}^{\mathrm{cn}}) \to \mathscr{C}$, which in particular takes the unit $\mathbb{S}[1/m, \omega_m]$ to the unit $\mathbb{1}$. Since A is m–torsion, by [13, Proposition 2.1.8], F also preserves A–homotopy fixed points, thus it suffices to prove the claim for $\mathbb{S}[1/m, \omega_m]$. Since π_* preserves A–homotopy fixed points for m–divisible spectra, the result for $\mathbb{S}[1/m, \omega_m]$ follows from the analogous fact for $\pi_*(\mathbb{S}[1/m, \omega_m]) = \pi_*(\mathbb{S})[1/m, \omega_m]$. Finally, in the case $\varphi = 0$ the action of A on $\pi_*(\mathbb{S})[1/m, \omega_m]$ is trivial so the statement is clear. For $\varphi \neq 0$, there is $a \in A$ such that

$$\varphi(a) \neq 0 \ \in \ \mathbb{Z}/m\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z},$$

and since ω_m is a primitive m^{th} root of unity, $\omega_m^{\varphi(a)} - 1$ is invertible in $\pi_0(\mathbb{S})[1/m, \omega_m] \simeq \mathbb{Z}[1/m, \omega_m]$. Thus, $\omega_m^{\varphi(a)} - 1$ acts invertibly on $\pi_*(\mathbb{S})[1/m, \omega_m]$, and therefore $(\pi_*(\mathbb{S})[1/m, \omega_m](\varphi))^{hA} \simeq 0$.

(2) It again suffices to consider the case $X = \mathbb{1}$. Under the free-forgetful adjunction $\mathscr{C} \leftrightarrows \mathscr{C}^{BA}$, the nonequivariant map $\mathbb{1}(\varphi) \xrightarrow{\sim} \mathbb{1}$ corresponds to the map

$$\iota_\varphi\colon \mathbb{1}(\varphi) \to \prod_{a \in A} \mathbb{1},$$

which on the a–factor is given by multiplication with $\omega_m^{\varphi(a)}$. Hence, the induced map

$$\iota\colon \bigoplus_{\varphi \in A^*} \mathbb{1}(\varphi) \to \prod_{a \in A} \mathbb{1}$$

is represented by the discrete Fourier transform $(A^* \times A)$–matrix $c_{\varphi,a} = \omega_m^{\varphi(a)}$. The square of the determinant of this Fourier transform matrix is $|A|^{|A|}$, which is invertible in the ring $\mathbb{Z}[1/m, \omega_m]$ since A is an m–torsion group. Hence, this matrix is invertible also in $\pi_0(\mathbb{1})$. □

3.2 Fourier transform

3.2.1 Construction Consider the composition

$$A^* \xrightarrow{\mathbb{1}(-)} \operatorname{pic}(\mathscr{C}^{BA}) \to \mathscr{C}^{BA},$$

of symmetric monoidal functors, in which the second functor is the counit map described below Definition 3.1. We shall denote it again by $\mathbb{1}(-)$. Since the dual of an invertible object coincides with its inverse we have

$$\mathbb{1}(\varphi)^\vee \simeq \mathbb{1}(-\varphi).$$

Consider the following composition of functors

$$\hat{\mathfrak{F}}\colon A^* \times \mathscr{C}^{BA} \xrightarrow{\mathbb{1}(-)^\vee \times \mathrm{Id}} \mathscr{C}^{BA} \times \mathscr{C}^{BA} \xrightarrow{\otimes} \mathscr{C}^{BA} \xrightarrow{(-)^{hA}} \mathscr{C}.$$

On the level of objects, for every $X \in \mathscr{C}^{BA}$ and $\varphi \in A^*$ we have

$$\hat{\mathfrak{F}}(\phi, X) \simeq X(-\varphi)^{hA}.$$

This should be thought of as extracting from X the eigenspace corresponding to the character φ. Taking the mate of the above functor under the exponential law, we get:

Definition 3.12 (Fourier transform) Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$ with a choice of a primitive m^{th} root of unity and let A be a finite m–torsion abelian group. We define the *$\mathscr{C}$-Fourier transform* to be the functor

$$\mathfrak{F}\colon \mathscr{C}^{BA} \to \mathrm{Fun}(A^*, \mathscr{C})$$

given by $\mathfrak{F}(X)_\varphi := \hat{\mathfrak{F}}(\phi, X)$.

The category of functors from A^* to $\mathscr{C}$ can be endowed with the *Day convolution* symmetric monoidal structure, which we denote by $\mathrm{Fun}(A^*, \mathscr{C})_{\mathrm{Day}}$; see [35, Section 2.2.6] and [20]. By [35, Example 2.2.6.9], the construction of $\mathrm{Fun}(A^*, \mathscr{C})_{\mathrm{Day}}$ is a special case of the norm construction for ∞–operads, in the sense of [35, Definition 2.2.6.1]. Thus, by its universal property, we have an equivalence of ∞–categories

$$\mathrm{Fun}^{\mathrm{lax}}(A^* \times \mathscr{C}^{BA}, \mathscr{C}) \simeq \mathrm{Fun}^{\mathrm{lax}}(\mathscr{C}^{BA}, \mathrm{Fun}(A^*, \mathscr{C})_{\mathrm{Day}}).$$

Since $\hat{\mathfrak{F}}$ is lax symmetric monoidal, as a composition of functors that are canonically such, the functor $\mathfrak{F}$ acquires a lax symmetric monoidal structure as well. In fact,

Proposition 3.13 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$ with a choice of a primitive m^{th} root of unity and let A be a finite m–torsion abelian group. The $\mathscr{C}$–Fourier transform*

$$\mathfrak{F}\colon \mathscr{C}^{BA} \to \mathrm{Fun}(A^*, \mathscr{C})_{\mathrm{Day}}$$

is a (strong) symmetric monoidal equivalence.

Proof We first show that $\mathfrak{F}$ is an equivalence of ∞–categories (ignoring the symmetric monoidal structure) by showing that it admits a fully faithful and essentially surjective left adjoint. The functor $\mathfrak{F}$ admits a left adjoint

$$\mathfrak{F}^{-1}\colon \mathrm{Fun}(A^*, \mathscr{C}) \to \mathscr{C}^{BA},$$

given by tensoring pointwise with the functor $\mathbb{1}(-)\colon A^* \to \mathscr{C}^{BA}$ followed by taking the direct sum over A^*. Thus, its value on objects is given by

$$\mathfrak{F}^{-1}(\{X_\varphi\}) = \bigoplus_{\varphi \in A^*} X_\varphi(\varphi).$$

To show that $\mathfrak{F}^{-1}$ is fully faithful, it suffices to show that the unit of the adjunction $\mathfrak{F}^{-1} \dashv \mathfrak{F}$ is an isomorphism. Unwinding the definitions and using Proposition 3.11(1), we get

$$\mathfrak{F}(\mathfrak{F}^{-1}(\{X_\varphi\}))_\psi \simeq \Big(\bigoplus_{\varphi \in A^*} X_\varphi(\varphi - \psi)\Big)^{hA} \simeq X_\psi,$$

and that the unit map under this identification is the identity. Now, we observe that for all $X \in \mathscr{C}$, the induced representations (see Proposition 3.11(2))

$$\prod_A X \simeq \bigoplus_{\varphi \in A^*} X(\varphi)$$

are in the essential image of $\mathfrak{F}^{-1}$. Since these generate $\mathscr{C}^{BA}$ under colimits (by [27, Proposition 4.3.8]), and $\mathfrak{F}^{-1}$ is fully faithful, we deduce that $\mathfrak{F}^{-1}$ is essentially surjective and hence is an equivalence.

We now turn to the preservation of the symmetric monoidal structure. To show that $\mathfrak{F}$ is strong symmetric monoidal, it suffices to consider objects of the form $X(\varphi)$ for $X \in \mathscr{C}$ and $\varphi \in A^*$, as they generate $\mathscr{C}^{BA}$ under colimits and $\mathfrak{F}$ is colimit-preserving (being an equivalence). For such objects, we have by Proposition 3.11(1)

$$\mathfrak{F}(X(\varphi))_\psi \simeq X(\varphi - \psi)^{hA} \simeq \begin{cases} X & \text{if } \psi = \varphi, \\ 0 & \text{else,} \end{cases}$$

and the structure map is the obvious isomorphism

$$\mathfrak{F}(X(\varphi)) \otimes \mathfrak{F}(Y(\psi)) \to \mathfrak{F}((X \otimes Y)(\varphi + \psi)).$$

One can similarly show that $\mathfrak{F}$ is unital and hence strong symmetric monoidal. □

3.2.2 Fourier of rings In the situation of Proposition 3.13, the symmetric monoidal equivalence

$$\mathfrak{F}\colon \mathscr{C}^{BA} \xrightarrow{\sim} \mathrm{Fun}(A^*, \mathscr{C})_{\mathrm{Day}}$$

induces an equivalence of the ∞–categories of commutative algebra objects. By [35, Example 2.2.6.9], we have

$$\mathrm{CAlg}(\mathscr{C})^{BA} \simeq \mathrm{CAlg}(\mathscr{C}^{BA}) \xrightarrow{\sim} \mathrm{CAlg}(\mathrm{Fun}(A^*, \mathscr{C})_{\mathrm{Day}}) \simeq \mathrm{Fun}^{\mathrm{lax}}(A^*, \mathscr{C}).$$

Remark 3.14 Informally, this equivalence expresses the fact that for $R \in \mathrm{CAlg}(\mathscr{C})^{BA}$, the A–equivariant decomposition into eigenspaces

$$R \simeq \bigoplus_{\varphi \in A^*} R_\varphi(\varphi) \in \mathscr{C}^{BA}$$

is also compatible with the *multiplicative* structure. Namely, the unit and multiplication maps of R respectively decompose, in a coherent way, through maps

$$\mathbb{1} \to R_0 \quad \text{and} \quad R_\varphi \otimes R_\psi \to R_{\varphi+\psi}.$$

Given $R \in \mathrm{CAlg}(\mathscr{C}^{BA})$, we shall now express the lax symmetric monoidal functor $\mathfrak{F}(R)\colon A^* \to \mathscr{C}$ in terms of the twisting functor T_R of Definition 2.10. For this, we first discuss the following general setting. Let $\mathscr{D} \in \mathrm{CAlg}(\mathrm{Pr})$ and let $R \in \mathrm{CAlg}(\mathscr{D})$. The functor $R \otimes (-)\colon \mathscr{D} \to \mathscr{D}$ can be made lax symmetric monoidal in two ways. First, as a composition of the functors in the free-forgetful symmetric monoidal adjunction

$$\mathscr{D} \xrightarrow{F_R} \mathrm{Mod}_R(\mathscr{D}) \xrightarrow{U_R} \mathscr{D}.$$

Second, the tensor product functor $\mathscr{D} \times \mathscr{D} \xrightarrow{\otimes} \mathscr{D}$ corresponds to a lax symmetric monoidal functor

$$S_{(-)}\colon \mathscr{D} \to \mathrm{Fun}(\mathscr{D}, \mathscr{D})_{\mathrm{Day}},$$

which on objects is given by $S_X(Y) = X \otimes Y$. This induces a functor on the ∞–categories of commutative algebras

$$S_{(-)}\colon \mathrm{CAlg}(\mathscr{D}) \to \mathrm{Fun}^{\mathrm{lax}}(\mathscr{D}, \mathscr{D}),$$

so that $S_R(-) = R \otimes (-)$, becomes lax symmetric monoidal. We shall need the fact that these two lax symmetric monoidal structures on the functor $R \otimes (-)$ are in fact equivalent.

Proposition 3.15 *Let $\mathscr{D} \in \mathrm{CAlg}(\mathrm{Pr})$ and let $R \in \mathrm{CAlg}(\mathscr{D})$. We have an isomorphism*

$$U_R \circ F_R \simeq S_R \;\in\; \mathrm{Fun}^{\mathrm{lax}}(\mathscr{D}, \mathscr{D}).$$

Proof For convenience we write $U := U_R$ and $F := F_R$. We observe that the composition

$$\mathscr{D} \times \mathrm{Mod}_R(\mathscr{D}) \xrightarrow{F \times \mathrm{Id}} \mathrm{Mod}_R(\mathscr{D}) \times \mathrm{Mod}_R(\mathscr{D}) \xrightarrow{\otimes} \mathrm{Mod}_R(\mathscr{D})$$

induces the lax symmetric monoidal functor

$$S_{(-)} \circ F\colon \mathrm{Mod}_R(\mathscr{D}) \to \mathrm{Fun}(\mathscr{D}, \mathrm{Mod}_R(\mathscr{D}))_{\mathrm{Day}}.$$

Consider the following diagram of symmetric monoidal ∞–categories and lax symmetric monoidal functors:

$$\begin{tikzcd}
\mathscr{D} \arrow[r, "S_{(-)}"] \arrow[d, "F"] & \mathrm{Fun}(\mathscr{D}, \mathscr{D})_{\mathrm{Day}} \arrow[d, "F\circ(-)"] \\
\mathrm{Mod}_R(\mathscr{D}) \arrow[r, "S_{(-)}\circ F"] \arrow[d, "U"] & \mathrm{Fun}(\mathscr{D}, \mathrm{Mod}_R(\mathscr{D}))_{\mathrm{Day}} \arrow[d, "U\circ(-)"] \\
\mathscr{D} \arrow[r, "S_{(-)}"] & \mathrm{Fun}(\mathscr{D}, \mathscr{D})_{\mathrm{Day}}
\end{tikzcd}$$

The top square commutes by construction. The bottom square is obtained from the top square by taking right adjoints of the vertical functors. Thus, it canonically "lax commutes" in the sense that we have the Beck–Chevalley natural transformation of lax symmetric monoidal functors

$$\beta\colon S_{U(-)} \to U \circ F \circ S_{U(-)} \xrightarrow{\sim} U \circ S_{FU(-)} \circ F \to U \circ S_{(-)} \circ F,$$

where the first and last maps are the unit and counit of the respective adjunctions. For every $M \in \mathrm{Mod}_R(\mathscr{D})$ and $X \in \mathscr{D}$ this is the composition

$$M \otimes X \to R \otimes (M \otimes X) \xrightarrow{\sim} (R \otimes M) \otimes_R (R \otimes X) \to M \otimes_R (R \otimes X),$$

where the first map is induced by the unit $\mathbb{1} \to R$ and the last by the action $R \otimes M \to M$, and hence is an isomorphism for all M and X. Therefore the diagram commutes up to homotopy. Applying $\mathrm{CAlg}(-)$ to it, we get that the composition

$$\mathrm{CAlg}(\mathscr{D}) \xrightarrow{S_{(-)}} \mathrm{Fun}^{\mathrm{lax}}(\mathscr{D}, \mathscr{D}) \xrightarrow{F \circ (-)} \mathrm{Fun}^{\mathrm{lax}}(\mathscr{D}, \mathrm{Mod}_R(\mathscr{D})) \xrightarrow{U \circ (-)} \mathrm{Fun}^{\mathrm{lax}}(\mathscr{D}, \mathscr{D})$$

can be identified with the composition

$$\mathrm{CAlg}(\mathscr{D}) \xrightarrow{F} \mathrm{CAlg}_R(\mathscr{D}) \xrightarrow{U} \mathrm{CAlg}(\mathscr{D}) \xrightarrow{S_{(-)}} \mathrm{Fun}^{\mathrm{lax}}(\mathscr{D}, \mathscr{D}).$$

Applying this to $\mathbb{1} \in \mathrm{CAlg}(\mathscr{D})$, we get $U \circ F \simeq S_R \in \mathrm{Fun}^{\mathrm{lax}}(\mathscr{D}, \mathscr{D})$. □

Proposition 3.16 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$ with a choice of a primitive m^{th} root of unity, and let A be a finite m–torsion abelian group. For $R \in \mathrm{CAlg}(\mathscr{C}^{BA})$, the functor $\mathfrak{F}(R) \colon A^* \to \mathscr{C}$ is homotopic, as a lax symmetric monoidal functor, to the composition*

$$A^* \xrightarrow{\mathbb{1}(-)^\vee} \mathscr{C}^{BA} \xrightarrow{T_R} \mathscr{C}.$$

Proof Unwinding the definitions, for all $\varphi \in A^*$ we have

$$\mathfrak{F}(R)_\varphi \simeq R(-\varphi)^{hA} \simeq T_R(\mathbb{1}(-\varphi)).$$

More precisely, we have

$$T_R \simeq (-)^{hA} \circ (U_R \circ F_R) \quad \text{and} \quad \mathfrak{F}(R) \simeq (-)^{hA} \circ S_R \circ \mathbb{1}(-)^\vee$$

as lax symmetric monoidal functors. Thus, the claim follows from Proposition 3.15. □

As a consequence, we obtain a characterization of the Galois property of $R \in \mathrm{CAlg}(\mathscr{C}^{BA})$, in terms of its Fourier transform $\mathfrak{F}(R) \colon A^* \to \mathscr{C}$.

Corollary 3.17 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$ with a choice of a primitive m^{th} root of unity and let A be a finite m–torsion abelian group. A commutative algebra $R \in \mathrm{CAlg}(\mathscr{C}^{BA})$ is Galois if and only if the lax symmetric monoidal functor $\mathfrak{F}(R) \colon A^* \to \mathscr{C}$ is strong symmetric monoidal.*

Proof Since BA is $\mathscr{C}$–ambidextrous, by Proposition 2.13, R is Galois if and only if T_R is strong symmetric monoidal. By Proposition 3.16 we have an equivalence of lax symmetric monoidal functors $\mathfrak{F}(R) \simeq T_R \circ \mathbb{1}(-)^\vee$. Thus, we wish to show that T_R is strong symmetric monoidal if and only if its precomposition with $\mathbb{1}(-)^\vee$ is strong symmetric monoidal. Since T_R is $\mathscr{C}$–linear and colimit-preserving (by Lemma 2.11), it remains to show that the image of $\mathbb{1}(-)^\vee$, or equivalently, of $\mathbb{1}(-)$, generates $\mathscr{C}^{BA}$

under colimits and tensoring with objects of $\mathscr{C}$. The ∞–category $\mathscr{C}^{BA}$ is generated under colimits by the induced objects

$$e_*X = \prod_A X \simeq X \otimes \prod_A \mathbb{1} \quad \text{for } X \in \mathscr{C},$$

where $e\colon \mathrm{pt} \to BA$ is a basepoint. Consequently, it is generated under colimits and tensoring with objects of $\mathscr{C}$ by the single object $\prod_A \mathbb{1}$. Finally, by Proposition 3.11(2) applied to $X = \mathbb{1}$, we have an isomorphism

$$\prod_A \mathbb{1} \simeq \bigoplus_{\varphi \in A^*} \mathbb{1}(\varphi),$$

and hence the generator $\prod_A \mathbb{1}$ is a direct sum of objects in the image of $\mathbb{1}(-)$. □

3.3 Galois and Picard

Using the results of the previous subsection, we obtain the following ∞–categorical version of Kummer theory:

Theorem 3.18 (Kummer theory) *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$ with a choice of a primitive m^{th} root of unity $\omega \in \mu_m^{\mathrm{prim}}(\mathscr{C})$ and let A be a finite m–torsion abelian group. The $\mathscr{C}$–Fourier transform induces an isomorphism*

$$\mathrm{CAlg}^{A\text{–gal}}(\mathscr{C}) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{Sp}^{\mathrm{cn}}}(A^*, \mathrm{pic}(\mathscr{C})),$$

natural in the pair $(\mathscr{C}, \omega)$. Moreover, one can replace $\mathrm{pic}(\mathscr{C})$ with its 1–truncation $\mathrm{pic}(\mathscr{C})_{\le 1}$ in the above isomorphism.

Proof In view of Corollary 3.17, the natural equivalence

$$\mathfrak{F}\colon \mathrm{CAlg}(\mathscr{C}^{BA}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lax}}(A^*, \mathscr{C})$$

restricts to a natural equivalence

$$\mathrm{CAlg}^{A\text{–gal}}(\mathscr{C}) \xrightarrow{\sim} \mathrm{Fun}^{\otimes}(A^*, \mathscr{C}).$$

Since A^* is an abelian group, we have

$$\mathrm{Fun}^{\otimes}(A^*, \mathscr{C}) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Cat}_\infty)}(A^*, \mathscr{C}) \simeq \mathrm{Map}_{\mathrm{Sp}^{\mathrm{cn}}}(A^*, \mathrm{pic}(\mathscr{C})).$$

Finally, for $n \ge 2$, we have

$$\pi_n \mathrm{pic}(\mathscr{C}) \simeq \pi_{n-1}(\mathbb{1}^\times) \simeq \pi_{n-1}(\mathbb{1}),$$

which is m–divisible (since $\mathscr{C}$ admits a primitive m^{th} root of unity). Thus, we get

$$\mathrm{CAlg}^{A\text{–gal}}(\mathscr{C}) \simeq \mathrm{Map}_{\mathrm{Sp}^{\mathrm{cn}}}(A^*, \mathrm{pic}(\mathscr{C})) \simeq \mathrm{Map}_{\mathrm{Sp}^{\mathrm{cn}}}(A^*, \mathrm{pic}(\mathscr{C})_{\le 1}).$$ □

To summarize, given $R \in \mathrm{CAlg}(\mathscr{C}^{BA})$, we have a decomposition into eigenspaces $R \simeq \bigoplus_{\varphi \in A^*} R_\varphi$ as objects of $\mathscr{C}$, and the unit and multiplication of R are induced from maps

$$\mathbb{1} \to R_0 \quad \text{and} \quad R_\varphi \otimes R_\psi \to R_{\varphi+\psi}.$$

Now, R is *Galois* if and only if those maps are *isomorphisms*, in which case the R_φ are invertible and assemble into a map $R_{(-)}\colon A^* \to \mathrm{pic}(\mathscr{C})$.

Remark 3.19 The equivalence provided by Theorem 3.18 induces an abelian group structure on the set $\pi_0(\mathrm{CAlg}^{A\text{–gal}}(\mathscr{C}))$. In fact, this set always admits a canonical group structure, even without assuming the existence of primitive roots of unity. The objects of $\mathrm{CAlg}^{A\text{–gal}}(\mathscr{C})$ can be viewed as local systems of commutative algebras on BA. The external product $R \boxtimes S$ of two such, as a local system on $BA \times BA$, can be pushed forward along the addition map $BA \times BA \xrightarrow{\alpha} BA$ to produce a new local system $R +_A S := \alpha_*(R \boxtimes S)$ of commutative algebras on BA. It can be shown that if R and S are A–Galois extensions, then $R +_A S$ is an A–Galois extension and that this operation endows $\pi_0(\mathrm{CAlg}^{A\text{–gal}}(\mathscr{C}))$ with an abelian group structure. In the situation of Theorem 3.18, this group structure coincides with the one induced from $\mathrm{pic}(\mathscr{C})$.

3.3.1 Cyclic group We shall now analyze the case $A = \mathbb{Z}/m$ in greater detail. For a symmetric monoidal ∞–category $\mathscr{C}$ and a dualizable object $X \in \mathscr{C}$, we can form the symmetric monoidal dimension (a.k.a. Euler characteristic) $\dim(X) \in \pi_0(\mathbb{1})$; see for instance [44, Defintion 2.2]. The symmetric monoidal dimension satisfies

$$\dim(\mathbb{1}) = 1 \quad \text{and} \quad \dim(X \otimes Y) = \dim(X)\cdot\dim(Y).$$

Hence, it restricts to a group homomorphism $\dim\colon \mathrm{Pic}(\mathscr{C}) \to (\pi_0\mathbb{1})^\times$. We shall now describe this homomorphism in terms of the spectrum $\mathrm{pic}(\mathscr{C})$.

Proposition 3.20 *Let $\mathscr{C}$ be a symmetric monoidal ∞–category. The homomorphism*

$$\pi_0\mathrm{pic}(\mathscr{C}) \simeq \mathrm{Pic}(\mathscr{C}) \xrightarrow{\dim} (\pi_0\mathbb{1})^\times \simeq \pi_1\mathrm{pic}(\mathscr{C})$$

is given by precomposition with the Hopf map $\eta \in \pi_1(\mathbb{S})$.

Proof The space $\Omega^\infty\mathbb{S}$ admits a structure of a commutative monoid in $\mathscr{S}$ that we can regard as a symmetric monoidal ∞–category. An element $Z \in \mathrm{Pic}(\mathscr{C})$ is classified by a map of connective spectra $\mathbb{S} \to \mathrm{pic}(\mathscr{C})$, which corresponds to a symmetric monoidal functor $\Omega^\infty\mathbb{S} \to \mathscr{C}$ sending $1 \in \mathbb{Z} = \pi_0\mathbb{S}$ to Z. Since both the dimension and precomposition with η are natural in $\mathscr{C}$, it suffices to prove the claim for $\mathscr{C} = \mathbb{S}$ and $Z = 1$.

In this case, we have

$$\dim(1) \in \pi_1\mathbb{S} \simeq \mathbb{Z}/2\cdot\eta,$$

so we only need to show that $\dim(1) \neq 0$. For this, it suffices to produce some example of an invertible object with a nontrivial dimension. For example, in $\mathscr{C} = \mathrm{Sp}$ we have

$$\dim(\Sigma\mathbb{S}) = -1 \ \in \ \mathbb{Z}^\times = \pi_0\mathbb{S}^\times. \qquad \square$$

Corollary 3.21 *Let $\mathscr{C}$ be a symmetric monoidal ∞–category. For every $X \in \mathrm{Pic}(\mathscr{C})$, we have $\dim(X)^2 = 1$. In particular, if $\pi_0\mathbb{1}$ is a connected ring and 2 is invertible in $\pi_0\mathbb{1}$, then $\dim(X) = \pm 1$.*

Proof The first part follows from Proposition 3.20 and the fact that $\eta \in \pi_1\mathbb{S}$ is 2–torsion. Now, if 2 is invertible and $\pi_0\mathbb{1}$ admits no nontrivial idempotents, then the only solutions to the equation $t^2 - 1 = 0$ are $t = \pm 1$. □

Given the above, we shall be interested in the following variant of the Picard group:

Definition 3.22 The *even Picard group* of a symmetric monoidal ∞–category $\mathscr{C}$, is the subgroup $\mathrm{Pic}^{\mathrm{ev}}(\mathscr{C}) \leq \mathrm{Pic}(\mathscr{C})$ given by the kernel of the map $\mathrm{Pic}(\mathscr{C}) \xrightarrow{\dim} (\pi_0\mathbb{1}_{\mathscr{C}})^\times$.

We shall now describe the collection of isomorphism classes of $\mathbb{Z}/m$–Galois extensions in $\mathscr{C}$ in terms of the homotopy groups of the Picard spectrum of $\mathscr{C}$.

Proposition 3.23 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{add}})$ with a choice of a primitive m^{th} root of unity $\omega \in \mu_m^{\mathrm{prim}}(\mathscr{C})$. We have a short exact sequence of abelian groups*

$$0 \to (\pi_0\mathbb{1}^\times)/(\pi_0\mathbb{1}^\times)^m \to \pi_0\,\mathrm{CAlg}^{\mathbb{Z}/m\text{–gal}}(\mathscr{C}) \to \mathrm{Pic}^{\mathrm{ev}}(\mathscr{C})[m] \to 0$$

which is natural in the pair $(\mathscr{C}, \omega)$. Moreover, this sequence splits (though not naturally).

Proof Throughout the proof, we work in the ∞–category $\mathrm{Sp}^{\mathrm{cn}}$. In particular, for $X, Y \in \mathrm{Sp}^{\mathrm{cn}}$ we denote by $\hom(X, Y)$ the internal mapping object in *connective* spectra. By Theorem 3.18, we have a natural isomorphism

$$\pi_0\,\mathrm{CAlg}^{\mathbb{Z}/m\text{–gal}}(\mathscr{C}) \simeq \pi_0 \hom(\mathbb{Z}/m, \mathrm{pic}(\mathscr{C})_{\leq 1}).$$

Let $\mathbb{S}/\eta$ be the cofiber of the map $\Sigma\mathbb{S} \xrightarrow{\eta} \mathbb{S}$. Since $(\mathbb{S}/\eta)_{\leq 1} \simeq \mathbb{Z}$, we get

$$\hom(\mathbb{Z}, \mathrm{pic}(\mathscr{C})_{\leq 1}) \simeq \hom(\mathbb{S}/\eta, \mathrm{pic}(\mathscr{C})_{\leq 1}).$$

Hence, $\hom(\mathbb{Z}, \mathrm{pic}(\mathscr{C})_{\leq 1})$ is the fiber of the map

$$\mathrm{pic}(\mathscr{C})_{\leq 1} \xrightarrow{\eta} \Omega\mathrm{pic}(\mathscr{C})_{\leq 1} \;\in\; \mathrm{Sp}^{\mathrm{cn}}.$$

By Proposition 3.20, we have

$$\pi_0 \hom(\mathbb{Z}, \mathrm{pic}(\mathscr{C})_{\leq 1}) \simeq \ker(\mathrm{Pic}(\mathscr{C}) \xrightarrow{\dim} (\pi_0\mathbb{1})^\times) \simeq \mathrm{Pic}^{\mathrm{ev}}(\mathscr{C})$$

and we also have

$$\pi_1 \hom(\mathbb{Z}, \mathrm{pic}(\mathscr{C})_{\leq 1}) \simeq \pi_1\mathrm{pic}(\mathscr{C})_{\leq 1} \simeq (\pi_0\mathbb{1})^\times,$$
$$\pi_n \hom(\mathbb{Z}, \mathrm{pic}(\mathscr{C})_{\leq 1}) = 0 \quad \text{for all } n \geq 2.$$

Thus, inspecting the long exact sequence in homotopy groups associated with the natural fiber sequence

$$\hom(\mathbb{Z}/m, \mathrm{pic}(\mathscr{C})_{\leq 1}) \to \hom(\mathbb{Z}, \mathrm{pic}(\mathscr{C})_{\leq 1}) \xrightarrow{m} \hom(\mathbb{Z}, \mathrm{pic}(\mathscr{C})_{\leq 1}),$$

we get a natural short exact sequence of abelian groups

$$0 \to (\pi_0 \mathbb{1}^\times)/(\pi_0 \mathbb{1}^\times)^m \to \pi_0 \operatorname{hom}(\mathbb{Z}/m, \operatorname{pic}(\mathscr{C})_{\le 1}) \to \operatorname{Pic}^{\mathrm{ev}}(\mathscr{C})[m] \to 0.$$

Further, since $\operatorname{hom}(\mathbb{Z}, \operatorname{pic}(\mathscr{C})_{\le 1})$ is a $\mathbb{Z}$–module, it splits (noncanonically) as a direct sum

$$\operatorname{hom}(\mathbb{Z}, \operatorname{pic}(\mathscr{C})_{\le 1}) \simeq \operatorname{Pic}^{\mathrm{ev}}(\mathscr{C}) \oplus \Sigma(\pi_0 \mathbb{1})^\times$$

and thus we get a splitting for the above exact sequence. □

The following example shows that Theorem 3.18 indeed generalizes classical Kummer theory for field extensions.

Example 3.24 For a field k and $\mathscr{C} = \mathrm{Vect}_k$, we have $\operatorname{Pic}(\mathscr{C}) = 0$. Hence, if k contains a primitive m^{th} root of unity, Proposition 3.23 reduces to the classical fact that the isomorphism classes of $\mathbb{Z}/m$–Galois extensions of k are in bijection with the set $(k^\times)/(k^\times)^m$.

At the other extreme, we have the following:

Example 3.25 Let C be a smooth projective algebraic curve over an algebraically closed field k whose characteristic is prime to m (and hence, admits primitive m^{th} roots of unity), and let $\mathscr{C}$ be the category of quasicoherent sheaves on C. We have $(k^\times)/(k^\times)^m = 0$, while $\operatorname{Pic}^{\mathrm{ev}}(\mathscr{C})[m]$ is the m–torsion of the Jacobian of C. In this case, Proposition 3.23 recovers the classification of cyclic m–covers of C by the m–torsion points on the Jacobian.

3.3.2 A $\mathbb{Z}/2$–variant In the case $A = \mathbb{Z}/2$, one can carry out the construction of Picard objects out of $\mathbb{Z}/2$–Galois extensions with fewer assumptions on the ambient category than in Proposition 3.23. For convenience, we shall use here the multiplicative notation $\mu_2 = \{\pm 1\}$, instead of the additive $\mathbb{Z}/2$, for the group of order 2. For simplicity, we shall assume that all the ∞–categories under consideration are stable.

Definition 3.26 Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}})$ and let $R \in \mathrm{CAlg}^{\mu_2\text{–gal}}(\mathscr{C})$. We denote by $\bar{R}$ the cofiber of the unit map $\mathbb{1} \to R$.

When 2 is invertible in $\mathscr{C}$, and hence $-1 \in \pi_0 \mathbb{1}$ is a primitive second root of unity, we have by Kummer theory a splitting $R \simeq \mathbb{1} \oplus \bar{R}$, and furthermore, $\bar{R} \in \operatorname{Pic}(\mathscr{C})$; see the discussion after Theorem 3.18. It turns out that the invertibility of $\bar{R}$ holds regardless of whether 2 is invertible.

Proposition 3.27 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{st})$. For every $R \in \mathrm{CAlg}^{\mu_2\text{–gal}}(\mathscr{C})$, we have $\bar{R} \in \operatorname{Pic}(\mathscr{C})$.*

Proof If R is split-Galois then $\bar{R} \simeq \mathbb{1} \in \operatorname{Pic}(\mathscr{C})$. We now reduce the general case to the split case. The object $\bar{R} \in \mathscr{C}$ is the cofiber of a map between dualizable objects and hence dualizable; see Remark 2.6. Hence, it suffices to show that the evaluation map $\bar{R} \otimes \bar{R}^\vee \to \mathbb{1}$ is an isomorphism. This can be checked after applying the conservative symmetric monoidal functor

$$R \otimes (-) \colon \mathscr{C} \to \mathrm{Mod}_R(\mathscr{C}).$$

The image of R under this functor is split-Galois, so the general case follows from the split case. □

We stress, however, that unlike the case where 2 is invertible in $\mathscr{C}$, the element $\overline{R} \in \mathrm{Pic}(\mathscr{C})$ need not be 2–torsion.

Example 3.28 [49, Proposition 5.3.1] We have $KU \in \mathrm{CAlg}^{\mu_2\text{-gal}}(\mathrm{Mod}_{KO}(\mathrm{Sp}))$. The unit map $KO \to KU$ fits into the (nonsplit) Bott periodicity cofiber sequence

$$\Sigma KO \xrightarrow{\eta} KO \to KU \ \in \ \mathrm{Mod}_{KO}(\mathrm{Sp}).$$

It follows that $\overline{KU} \simeq \Sigma^2 KO$. Hence, by real Bott periodicity, $\overline{KU} \in \mathrm{Pic}(\mathrm{Mod}_{KO}(\mathrm{Sp}))$ is of order 4.

Warning 3.29 More generally, when 2 is not invertible in $\mathscr{C}$, the function

$$\overline{(-)}\colon \pi_0(\mathrm{CAlg}^{\mu_2\text{-gal}}(\mathscr{C})) \to \mathrm{Pic}(\mathscr{C})$$

need *not* be a group homomorphism with respect to the group structure on the source given by Remark 3.19.

4 Higher cyclotomic theory

In this section, we define and study "higher" cyclotomic extensions in the setting of higher semiadditive stable ∞–categories. These are the higher (semiadditive) height analogues of the cyclotomic extensions of Definition 3.4. We shall work primarily in $\mathrm{Pr}_{\mathrm{st}}^{\oplus-n} \subseteq \mathrm{Pr}$ for some $n \geq 0$, which is the full subcategory of Pr, spanned by stable n–semiadditive ∞–categories. We also fix an implicit prime p, with respect to which one can consider semiadditive height. We recall from [13, Theorem C], that every ∞–category in $\mathrm{Pr}_{\mathrm{st}}^{\oplus-n}$ splits into a product of ∞–categories according to height. Moreover, the finite-height factors are ∞–semiadditive[9] [13, Theorem A]. We shall mainly concentrate on the full subcategory $\mathrm{Pr}_{\mathfrak{z}_n} \subseteq \mathrm{Pr}_{\mathrm{st}}^{\oplus-n}$ of those ∞–categories which are of height n.

We begin in Section 4.1, by discussing primitive higher roots of unity (Definition 4.2), and continue in Section 4.2, with the higher cyclotomic extensions which corepresent them (Definition 4.7 and Proposition 4.8).

4.1 Higher roots of unity

In Definition 3.3, we have recalled the space $\mu_m(R)$ of m^{th} roots of unity of a commutative algebra object R in a symmetric monoidal ∞–category $\mathscr{C}$. By decomposing m into a product of distinct prime powers $m = p_1^{r_1} \cdots p_s^{r_s}$, we obtain a decomposition of the functor $\mu_m\colon \mathrm{CAlg}(\mathscr{C}) \to \mathscr{S}$ into a product

$$\mu_m \simeq \mu_{p_1^{r_1}} \times \cdots \times \mu_{p_s^{r_s}}.$$

We may thus restrict attention to the case $m = p^r$. While the definition of $(p^r)^{\text{th}}$ roots of unity is rather general, the notion of *primitive* roots behaves well only when R is *p–divisible*, in which case $\mu_{p^r}(R)$ is *discrete*; see Proposition 3.8. In the terminology of [13, Definition 3.1.6], the condition that R is

[9] To be precise, the height $n = 0$ factor is only *p–typically* ∞–semiadditive.

p–divisible amounts to R having (semiadditive) height 0. More generally, when $\mathscr{C}$ is higher semiadditive, the properties of the construction $\mu_{p^r}(R)$ turn out to be closely related to the height of R. To begin with:

Proposition 4.1 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{st}^{\oplus-n})$ and let $R \in \mathrm{CAlg}(\mathscr{C})$. If R is of height $\leq n$, then for all $r \in \mathbb{N}$ the space $\mu_{p^r}(R)$ is n–truncated.*

Proof By [13, Proposition 2.4.7], we have $R[B^{n+1}C_{p^r}] \simeq R$. We thus get a sequence of isomorphisms

$$\begin{aligned}\Omega^{n+1}\mu_{p^r}(R) &\simeq \Omega^{n+1}\mathrm{Map}_{\mathrm{CAlg}(\mathscr{S})}(C_{p^r}, R^\times) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathscr{S})}(B^{n+1}C_{p^r}, R^\times)\\ &\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathscr{C})}(\mathbb{1}[B^{n+1}C_{p^r}], R) \simeq \mathrm{Map}_{\mathrm{CAlg}_R(\mathscr{C})}(R[B^{n+1}C_{p^r}], R)\\ &\simeq \mathrm{Map}_{\mathrm{CAlg}_R(\mathscr{C})}(R, R) \simeq \mathrm{pt}.\end{aligned}$$

Since all connected components of the space $\mu_{p^r}(R)$ are isomorphic, it follows that it is n–truncated. □

As we shall demonstrate, when R is of height exactly n, the set $\pi_n(\mu_{p^r}(R))$ serves as a good substitute for the set $\pi_0(\mu_{p^r}(R))$ of ordinary $(p^r)^{\mathrm{th}}$ roots of unity of R. With that in mind, we introduce the following generalization of Definition 3.3:

Definition 4.2 (higher roots of unity) Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^{\oplus-n})$ and let $R \in \mathrm{CAlg}(\mathscr{C})$. For every prime p and $r \in \mathbb{N}$:

(1) We define the space of $(p^r)^{th}$ *roots of unity of height n* in R to be

$$\mu_{p^r}^{(n)}(R) := \Omega^n\mu_{p^r}(R) \simeq \mathrm{Map}_{\mathrm{Sp}^{\mathrm{cn}}}(C_{p^r}, \Omega^n R^\times).$$

(2) We say that a higher root of unity $C_{p^r} \xrightarrow{\omega} \Omega^n R^\times$ is *primitive* if R is of height n and the only commutative R–algebra S for which there exists a dotted arrow rendering the diagram of spectra

$$\begin{tikzcd} C_{p^r} \arrow[r] \arrow[d, two heads] & \Omega^n R^\times \arrow[d] \\ C_{p^{r-1}} \arrow[r, dashed] & \Omega^n S^\times \end{tikzcd}$$

commutative, is $S=0$. We denote by $\mu_{p^r}^{(n),\mathrm{prim}}(R) \subseteq \mu_{p^r}^{(n)}(R)$ the union of connected components of height n primitive $(p^r)^{\mathrm{th}}$ roots of unity.

By convention, a height n (primitive) $(p^r)^{\mathrm{th}}$ root of unity of $\mathscr{C}$ is a height n (primitive) $(p^r)^{\mathrm{th}}$ root of unity of $\mathbb{1}_{\mathscr{C}}$.

The (higher) $(p^r)^{\mathrm{th}}$ roots of unity for various r are interrelated in two ways. First, for all $k \leq r$, the *surjective* group homomorphisms $C_{p^r} \twoheadrightarrow C_{p^k}$ induce, by precomposition, natural transformations

$$\mu_{p^k}^{(n)}(R) \simeq \mathrm{Map}(C_{p^k}, \Omega^n R^\times) \to \mathrm{Map}(C_{p^r}, \Omega^n R^\times) \simeq \mu_{p^r}^{(n)}(R).$$

We can think of this as the inclusion of the (higher) $(p^k)^{\text{th}}$ roots of unity into the (higher) $(p^r)^{\text{th}}$ roots of unity. Second, the *injective* group homomorphisms $C_{p^{r-k}} \hookrightarrow C_{p^r}$ induce, by precomposition, natural transformations

$$(-)^{p^k} : \mu^{(n)}_{p^r}(R) \simeq \operatorname{Map}(C_{p^r}, \Omega^n R^\times) \to \operatorname{Map}(C_{p^{r-k}}, \Omega^n R^\times) \simeq \mu^{(n)}_{p^{r-k}}(R).$$

We can think of this as raising a (higher) $(p^r)^{\text{th}}$ root of unity to the $(p^k)^{\text{th}}$ power to get a (higher) $(p^{r-k})^{\text{th}}$ root of unity.

Proposition 4.3 *Let $\mathscr{C} \in \operatorname{CAlg}(\operatorname{Pr}_{\Sigma_n})$ and let $R \in \operatorname{CAlg}(\mathscr{C})$. For $0 \le k < r$, a higher root of unity $\omega \in \mu^{(n)}_{p^r}(R)$ is primitive, if and only if $\omega^{p^k} \in \mu^{(n)}_{p^{r-k}}(R)$ is primitive.*

Proof This follows from the definition of primitivity (Definition 4.2) and the fact that we have a pushout diagram in $\mathrm{Sp}^{\mathrm{cn}}$ of the form

$$\begin{tikzcd} C_{p^{r-k}} \arrow[r, hook] \arrow[d, two heads] & C_{p^r} \arrow[d, two heads] \\ C_{p^{r-k-1}} \arrow[r, hook] & C_{p^{r-1}} \end{tikzcd}$$

□

4.2 Higher cyclotomic extensions

4.2.1 Definition and properties We shall now mimic the construction of cyclotomic extensions, which corepresent primitive roots of unity, to produce *higher* cyclotomic extensions, which corepresent primitive *higher* roots of unity. For $\mathscr{C} \in \operatorname{CAlg}(\operatorname{Pr}_{\mathrm{st}})$ and a fixed $r \in \mathbb{N}$, the functor

$$\mu^{(n)}_{p^r} : \operatorname{CAlg}(\mathscr{C}) \to \mathscr{S}$$

is corepresented by the group algebra $\mathbb{1}[B^n C_{p^r}]$. The group homomorphism $q : C_{p^r} \twoheadrightarrow C_{p^{r-1}}$ induces a map of commutative groups in spaces $q_n : B^n C_{p^r} \to B^n C_{p^{r-1}}$ and hence a map of group algebras

$$\bar{q}_n : \mathbb{1}[B^n C_{p^r}] \to \mathbb{1}[B^n C_{p^{r-1}}] \in \operatorname{CAlg}(\mathscr{C}).$$

The map $\bar{q}_n$ corepresents the inclusion $\mu^{(n)}_{p^{r-1}}(R) \hookrightarrow \mu^{(n)}_{p^r}(R)$ discussed above. The key point is that if $\mathscr{C}$ is higher semiadditive of *height* n, then we can realize $\bar{q}_n$ as a splitting of an idempotent in $\pi_0(\mathbb{1}[B^n C_{p^r}])$. To translate between local systems and modules we need the following general fact, which seems to be well known, but for which we could not find a reference in the literature.

Proposition 4.4 *Let $\mathscr{C} \in \operatorname{CAlg}(\operatorname{Pr})$ and let B be a pointed connected space. There is a natural equivalence of $\mathscr{C}$–linear ∞–categories*

$$\mathscr{C}^B \simeq \operatorname{LMod}_{\mathbb{1}[\Omega B]}(\mathscr{C}) \in \operatorname{Mod}_{\mathscr{C}}(\operatorname{Pr}).$$

Proof We first consider the case $\mathscr{C} = \mathscr{S}$. Let $\mathrm{pt} \xrightarrow{e} B$ be the base point of B and let $M = e_!(\mathrm{pt})$ in $\mathscr{S}^B$. The functor $F_M : \mathscr{S} \to \mathscr{S}^B$, which is given by pointwise product with M, is left adjoint to the pullback functor $e^*\colon \mathcal{S}^B \to \mathcal{S}$. Since e^* is itself a symmetric monoidal, conservative left adjoint, it follows from [35, Proposition 4.8.5.8], that $\mathcal{S}^B$ is equivalent to $\mathrm{LMod}_{\mathrm{End}(M)}(\mathcal{S})$. Finally, under the Grothendieck construction equivalence $\mathcal{S}^B \simeq \mathcal{S}_{/B}$, the object $M = e_!(\mathrm{pt})$ corresponds to $\mathrm{pt} \xrightarrow{e} B$ and its endomorphisms are given by $\Omega B \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{S})$. For a general $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr})$, we shall deduce the claim by tensoring the equivalence

$$\mathcal{S}^B \simeq \mathrm{LMod}_{\Omega B}(\mathcal{S}) \;\in\; \mathrm{Pr}$$

with $\mathcal{C}$ in Pr. Indeed, it follows from [35, Proposition 4.8.1.17] that $\mathcal{C} \otimes \mathcal{S}^B \simeq \mathcal{C}^B$, and from [35, Theorems 4.8.4.6 and 4.8.5.16] that

$$\mathcal{C} \otimes \mathrm{LMod}_{\Omega B}(\mathcal{S}) \simeq \mathrm{LMod}_{\Omega B}(\mathcal{C}) \simeq \mathrm{LMod}_{\mathbb{1}[\Omega B]}(\mathcal{C}). \qquad \square$$

This allows us to use the results of [13, Section 4.3], to deduce the following:

Proposition 4.5 *Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathfrak{S}_n})$. There exists an idempotent $\varepsilon \in \pi_0(\mathbb{1}[B^n C_{p^r}])$, such that*

$$\mathbb{1}[B^n C_{p^r}][\varepsilon^{-1}] \simeq \mathbb{1}[B^n C_{p^{r-1}}],$$

and under this isomorphism, the canonical map $\mathbb{1}[B^n C_{p^r}] \to \mathbb{1}[B^n C_{p^r}][\varepsilon^{-1}]$ is identified with $\overline{q}_n$.

Proof By the naturality of the equivalence of ∞–categories in Proposition 4.4

$$\mathcal{C}^{B^{n+1}C_{p^r}} \simeq \mathrm{Mod}_{\mathbb{1}[B^n C_{p^r}]}(\mathcal{C}),$$

restriction of scalars along $\overline{q}_n$ is identified with the functor $q_{n+1}^*\colon \mathcal{C}^{B^{n+1}C_{p^{r-1}}} \to \mathcal{C}^{B^{n+1}C_{p^r}}$. By [13, Theorem 4.3.2], this induces an equivalence of ∞–categories

$$\mathcal{C}^{B^{n+1}C_{p^r}} \xrightarrow{\sim} \mathcal{C}^{B^{n+1}C_{p^{r-1}}} \times (\mathcal{C}^{B^{n+1}C_{p^{r-1}}})^{\perp},$$

where $(\mathcal{C}^{B^{n+1}C_{p^{r-1}}})^{\perp} \subseteq \mathcal{C}^{B^{n+1}C_{p^r}}$ is the full subcategory spanned by the objects X, for which $(q_{n+1})_* X = 0$. Let

$$\varepsilon\colon \mathrm{Id}_{\mathcal{C}^{B^{n+1}C_{p^r}}} \to \mathrm{Id}_{\mathcal{C}^{B^{n+1}C_{p^r}}}$$

be the idempotent natural endomorphism which projects onto the essential image of $\mathcal{C}^{B^{n+1}C_{p^{r-1}}}$ under the functor q_{n+1}^*. This corresponds to a natural endomorphism

$$\overline{\varepsilon}\colon \mathrm{Id}_{\mathrm{Mod}_{\mathbb{1}[B^n C_{p^r}]}(\mathcal{C})} \to \mathrm{Id}_{\mathrm{Mod}_{\mathbb{1}[B^n C_{p^r}]}(\mathcal{C})},$$

which evaluates at $\mathbb{1}[B^n C_{p^r}]$ to an idempotent element in the commutative ring $\pi_0(\mathbb{1}[B^n C_{p^r}])$. By construction, the decomposition

$$\mathbb{1}[B^n C_{p^r}] \xrightarrow{\sim} \mathbb{1}[B^n C_{p^r}][\overline{\varepsilon}^{-1}] \times \mathbb{1}[B^n C_{p^r}][(1-\overline{\varepsilon})^{-1}]$$

identifies the projection onto the first factor with $\overline{q}_n$. $\square$

In [13, Proposition 4.3.4], we also provided an explicit description of the idempotent ε of Proposition 4.5 in the language of local systems. Translating into the language of rings, we get the following description of $\varepsilon \in \pi_0(\mathbb{1}[B^n C_{p^r}])$, in terms of the higher semiadditive structure of $\mathscr{C}$. The fiber of $q_n\colon B^n C_{p^r} \to B^n C_{p^{r-1}}$ is isomorphic to $B^n C_p$, and we thus get a map of spaces $\iota\colon B^n C_p \to \Omega^\infty \mathbb{1}[B^n C_{p^r}]$. The idempotent ε can be identified with the "average of ι", in the sense that

$$\varepsilon = \frac{1}{|B^n C_p|} \int_{B^n C_p} \iota \ \in \ \pi_0(\mathbb{1}[B^n C_{p^r}]).$$

When $n = 0$, we recover the classical formula of Definition 3.4 for the case $m = p^r$; see also [13, Example 4.3.3].

Remark 4.6 Consider the ∞–group $G = B^n C_p$ and the canonical maps

$$f\colon \mathbb{1}[G]^{hG} \to \mathbb{1}[G] \quad \text{and} \quad g\colon \mathbb{1}[G] \to \mathbb{1}[G]_{hG} \simeq \mathbb{1}.$$

It can be shown that if $\mathscr{C}$ is ∞–semiadditive of height n (and hence in particular G is *$\mathscr{C}$–stably dualizable* in the sense of [48, Definition 2.3.1]), then $h = g \circ f$ is invertible and $f \circ h^{-1} = \varepsilon$. In the case $\mathscr{C} = \mathrm{Sp}_{K(n)}$, the fact that h is invertible, which suffices for the construction of ε, was first observed in [48, Example 5.4.6]. We thank John Rognes for explaining to us this alternative description of ε.

We are now ready to give the main definition of the paper.

Definition 4.7 (higher cyclotomic extensions) Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{ᛒ_n})$. For every integer $r \geq 1$, we define

$$\mathbb{1}[\omega_{p^r}^{(n)}] := \mathbb{1}[B^n C_{p^r}][(1-\varepsilon)^{-1}] \ \in \ \mathrm{CAlg}(\mathscr{C}),$$

where $\varepsilon \in \pi_0(\mathbb{1}[B^n C_{p^r}])$ is the idempotent provided by Proposition 4.5. For every $R \in \mathrm{CAlg}(\mathscr{C})$, we define

$$R[\omega_{p^r}^{(n)}] := R \otimes \mathbb{1}[\omega_{p^r}^{(n)}] \ \in \ \mathrm{CAlg}_R(\mathscr{C}).$$

We refer to it as the (height n) $(p^r)^{\mathrm{th}}$ *cyclotomic extension* of R.

As promised, the higher cyclotomic extensions indeed corepresent the higher primitive roots:

Proposition 4.8 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{ᛒ_n})$. The object $\mathbb{1}[\omega_{p^r}^{(n)}] \in \mathrm{CAlg}(\mathscr{C})$ corepresents the functor*

$$\mu_{p^r}^{(n),\mathrm{prim}}\colon \mathrm{CAlg}(\mathscr{C}) \to \mathrm{Set} \subseteq \mathscr{S}.$$

Proof By Proposition 4.1, the essential image of $\mu_{p^r}^{(n),\mathrm{prim}}$ is contained in the full subcategory $\mathrm{Set} \subseteq \mathscr{S}$. Using the adjunction

$$\mathbb{1}[-]\colon \mathrm{CMon}(\mathscr{S}) \leftrightarrows \mathrm{CAlg}(\mathscr{C})\colon (-)^\times,$$

we see that for $R \in \mathrm{CAlg}(\mathscr{C})$, a higher root of unity $\mathbb{1}[B^n C_{p^r}] \xrightarrow{\omega} R$ is primitive if and only if

$$\mathbb{1}[B^n C_{p^{r-1}}] \otimes_{\mathbb{1}[B^n C_{p^r}]} R \simeq 0.$$

By the decomposition

$$\mathbb{1}[B^n C_{p^r}] \xrightarrow{\sim} \mathbb{1}[B^n C_{p^{r-1}}] \times \mathbb{1}[\omega_{p^r}^{(n)}],$$

the property above holds if and only if the map $\omega\colon \mathbb{1}[B^n C_{p^r}] \to R$ factors through the projection map $\mathbb{1}[B^n C_{p^r}] \to \mathbb{1}[\omega_{p^r}^{(n)}]$. □

The higher cyclotomic extensions enjoy some additional pleasant properties:

Proposition 4.9 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathfrak{Z}_n})$.*

(1) *$\mathbb{1}[\omega_{p^r}^{(n)}]$ is dualizable as an object of $\mathscr{C}$ for all $r \in \mathbb{N}$.*

(2) *$\mathbb{1}[\omega_p^{(n)}]$ is faithful.*

Proof (1) We have a fiber sequence

$$\mathbb{1}[\omega_{p^r}^{(n)}] \to \mathbb{1}[B^n C_{p^r}] \to \mathbb{1}[B^n C_{p^{r-1}}] \ \in \ \mathscr{C}.$$

Since $\mathscr{C}$ is ∞–semiadditive, $B \otimes \mathbb{1}$ is a dualizable object of $\mathscr{C}$ for every π–finite space B (Corollary 2.7), and dualizable objects in a stable category are closed under (co)fibers (Remark 2.6).

(2) For $n = 0$ the object $\mathbb{1}[\omega_p^{(n)}]$ is the fiber of the fold map

$$\mathbb{1}^{\oplus p} \simeq \mathbb{1}[C_p] \to \mathbb{1}$$

and hence isomorphic to $\mathbb{1}^{\oplus(p-1)}$. Tensoring with this object is conservative since it contains the unit as a direct summand. Assume now that $n \geq 1$. For every object $X \in \mathscr{C}$ we have a fiber sequence

$$X \otimes \mathbb{1}[\omega_p^{(n)}] \to X \otimes \mathbb{1}[B^n C_p] \to X \ \in \ \mathscr{C}.$$

Therefore, $X \otimes \mathbb{1}[\omega_p^{(n)}] = 0$ if and only if the fold map $X \otimes B^n C_p \to X$ is an isomorphism. We wish to deduce that $X = 0$. Indeed, for $n \geq 1$ by [13, Proposition 2.4.7], we get that X is of height $< n$. Since $\mathscr{C}$ is of height n, the only object $X \in \mathscr{C}$ which is of height $< n$ is $X = 0$. □

4.2.2 Infinite cyclotomic extensions From Proposition 4.8, it follows in particular that we have canonical maps $\mathbb{1}[\omega_{p^{r-1}}^{(n)}] \to \mathbb{1}[\omega_{p^r}^{(n)}]$ corepresenting the natural transformation $\omega \mapsto \omega^p$ on primitive roots of unity; see Proposition 4.3. Gathering the cyclotomic extensions for all $r \geq 0$ along these maps we get:

Definition 4.10 Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathfrak{Z}_n})$. We define

$$\mathbb{1}[\omega_{p^\infty}^{(n)}] := \varinjlim_{r \in \mathbb{N}} \mathbb{1}[\omega_{p^r}^{(n)}].$$

Loosely speaking, the commutative algebra $\mathbb{1}[\omega^{(n)}_{p^\infty}]$ corepresents choices of compatible systems of height n primitive roots of unity $\omega_p, \omega_{p^2}, \omega_{p^3}, \dots$ such that $\omega_{p^r}^p = \omega_{p^{r-1}}$ for all $r \in \mathbb{N}$. The infinite cyclotomic extension $\mathbb{1}[\omega^{(n)}_{p^\infty}]$ can also be constructed directly by splitting off an idempotent from a group algebra. The group homomorphism $\mathbb{Z}_p \xrightarrow{\times p} \mathbb{Z}_p$ induces a map of commutative algebras

$$\overline{q}\colon \mathbb{1}[B^{n+1}\mathbb{Z}_p] \to \mathbb{1}[B^{n+1}\mathbb{Z}_p].$$

The following is a direct analogue, and a consequence, of Proposition 4.5 for the case $r = \infty$:

Proposition 4.11 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathfrak{Z}_n})$ for some $n \geq 1$. Then there exists an idempotent element $\varepsilon \in \pi_0(\mathbb{1}[B^{n+1}\mathbb{Z}_p])$ such that*

$$\mathbb{1}[B^{n+1}\mathbb{Z}_p][(1-\varepsilon)^{-1}] \simeq \mathbb{1}[\omega^{(n)}_{p^\infty}] \quad \textit{and} \quad \mathbb{1}[B^{n+1}\mathbb{Z}_p][\varepsilon^{-1}] \simeq \mathbb{1}[B^{n+1}\mathbb{Z}_p],$$

and under the second isomorphism, the canonical map $\mathbb{1}[B^{n+1}\mathbb{Z}_p] \to \mathbb{1}[B^{n+1}\mathbb{Z}_p][\varepsilon^{-1}]$ is identified with $\overline{q}$.

Proof Using Proposition 4.5 for every $r \geq 0$, and taking the colimit, we get that $C_{p^\infty} \xrightarrow{\times p} C_{p^\infty}$ induces an idempotent $\varepsilon \in \pi_0(\mathbb{1}[B^n C_{p^\infty}])$, such that

$$\mathbb{1}[B^n C_{p^\infty}][\varepsilon^{-1}] \simeq \mathbb{1}[B^n C_{p^\infty}] \quad \text{and} \quad \mathbb{1}[B^n C_{p^\infty}][(1-\varepsilon)^{-1}] \simeq \mathbb{1}[\omega^{(n)}_{p^\infty}].$$

Now, the short exact sequence of abelian groups

$$0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to C_{p^\infty} \to 0$$

induces a Bockstein homomorphism

$$B^n C_{p^\infty} \to B^{n+1}\mathbb{Z}_p,$$

which becomes an isomorphism upon p–completion. Since $\mathscr{C}$ is assumed to be of height ≥ 1, it is p–complete and the result follows. $\square$

4.2.3 Equivariance and Galois For every $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}})$ and $R \in \mathrm{CAlg}(\mathscr{C})$, the space

$$\mu^{(n)}_{p^r}(R) = \mathrm{Map}_{\mathrm{Sp}^{\mathrm{cn}}}(C_{p^r}, \Omega^n R^\times)$$

admits a canonical action of the group $(\mathbb{Z}/p^r)^\times$ by precomposition. If $\mathscr{C}$ is higher semiadditive and of height n, then $\mu^{(n)}_{p^r}(R)$ is discrete (Proposition 4.1). Furthermore, since for every commutative R–algebra S the subset $\mu^{(n)}_{p^{r-1}}(S) \subseteq \mu^{(n)}_{p^r}(S)$ is closed under the action of $(\mathbb{Z}/p^r)^\times$, so is the subset of primitive roots $\mu^{(n),\mathrm{prim}}_{p^r}(R) \subseteq \mu^{(n)}_{p^r}(R)$. We therefore obtain an action of $(\mathbb{Z}/p^r)^\times$ on the corepresenting object $\mathbb{1}[\omega^{(n)}_{p^r}] \in \mathrm{CAlg}(\mathscr{C})$ making the map $\mathbb{1}[B^n C_{p^r}] \to \mathbb{1}[\omega^{(n)}_{p^r}]$ equivariant with respect to $(\mathbb{Z}/p^r)^\times$. Given $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathfrak{Z}_n})$, it is natural to ask whether the objects

$$\mathbb{1}[\omega^{(n)}_{p^r}] \in \mathrm{CAlg}(\mathscr{C}^{B(\mathbb{Z}/p^r)^\times})$$

are *Galois*. For $n = 0$, this is always the case. However, for $n = 1$ a counterexample was constructed by Yuan in [52]. In the next section, we shall address this question for higher semiadditive ∞–categories arising in chromatic homotopy theory.

5 Chromatic applications

In this final section, we apply the general theory of higher cyclotomic extensions to the chromatic world and deduce the main results of the paper. We begin in Section 5.1 by showing that the higher cyclotomic extensions in $\mathrm{Sp}_{K(n)}$ are Galois, and deduce using the results of Section 2.1, that the same holds for $\mathrm{Sp}_{T(n)}$ (Proposition 5.2). Then, in Section 5.2, we review the Galois theory of $\mathrm{Sp}_{K(n)}$, and identify the quotients of the Morava stabilizer group corresponding to the (higher) cyclotomic extensions of $\mathbb{S}_{K(n)}$ (Theorem 5.8 and Corollary 5.15). In particular, we deduce that all the abelian Galois extensions of $\mathbb{S}_{K(n)}$ can be obtained as a combination of ordinary and higher cyclotomic extensions. In Section 5.3, we apply the results of Section 3 to relate the higher cyclotomic extensions of $\mathbb{S}_{K(n)}$ to the $K(n)$–local Picard group (Proposition 5.23 and Proposition 5.30). Finally, in Section 5.4, we establish the consequences of the above for the Galois extensions of $\mathrm{Sp}_{T(n)}$ (Theorem 5.31) and its Picard group (Theorem 5.32 and Theorem 5.33).

5.1 Cyclotomic Galois extensions

We fix a natural number $n \geq 1$, a prime number p, and a formal group law Γ of height n over $\mathbb{F}_p$ (which will be kept implicit throughout). We denote by $K(n)$ and E_n the Morava K–theory and Lubin–Tate ring spectra associated to Γ. In particular, the homotopy groups of $K(n)$ and E_n are given by[10]

$$\begin{aligned} \pi_* K(n) &= \mathbb{F}_p[v_n^{\pm}], & &\text{where } |v_n| = 2(p^n - 1), \\ \pi_* E_n &= W(\overline{\mathbb{F}}_p)[\![u_1, \dots, u_{n-1}]\!][u^{\pm}], & &\text{where } |u_i| = 0,\ |u| = 2. \end{aligned}$$

We view E_n as an object of $\mathrm{CAlg}(\mathrm{Sp}_{K(n)})$ (see [23; 29]) and denote the symmetric monoidal ∞–category of $K(n)$–local E_n–modules as

$$\Theta_n := \mathrm{Mod}_{E_n}(\mathrm{Sp}_{K(n)}).$$

For $M \in \Theta_n$, we consider $\pi_*(M)$ as a graded module over the twisted continuous group algebra of $\mathbb{G}_n$ over $\pi_* E_n$. Namely, as an object in the category of *Morava modules*; see [6, Definition 3.37]. If $\pi_{\mathrm{odd}}(M) = 0$, we say that the Morava module of M is *even*. In this case, one can consider the equivalent data of $\pi_0(M)$ as a module over the twisted continuous group algebra of $\mathbb{G}_n$ over $\pi_0(E_n)$, which we call the *even Morava module* of M. For $X \in \mathrm{Sp}$, we refer to the (even) Morava module of $L_{K(n)}(E_n \otimes X)$, simply as the (even) Morava module of X.

[10] In the literature, E_n often denotes a closely related ring spectrum whose homotopy groups are $W(\mathbb{F}_{p^n})[\![u_1, \dots, u_{n-1}]\!][u^{\pm}]$.

In addition, we let $F(n)$ be some finite spectrum of type n, with a v_n–self map $v\colon \Sigma^d F(n) \to F(n)$, and an associated "telescope"

$$T(n) := F(n)[v^{-1}] = \varinjlim\bigl(F(n) \xrightarrow{v} \Sigma^{-d} F(n) \xrightarrow{v} \Sigma^{-2d} F(n) \xrightarrow{v} \cdots\bigr).$$

The ∞–categories $\mathrm{Sp}_{K(n)}$, $\mathrm{Sp}_{T(n)}$, and Θ_n are all ∞–semiadditive and of height n [13, Propostion 4.4.4 and Theorem 4.4.5]; see also [27, Theorem 5.2.1] and [14, Theorem A]. That is, we have

$$\mathrm{Sp}_{K(n)},\ \mathrm{Sp}_{T(n)},\ \Theta_n\ \in\ \mathrm{CAlg}(\mathrm{Pr}_{\mathfrak{z}_n}).$$

Thus, we can consider height n cyclotomic extensions in each one of them. Our first goal is to show that all these extensions are Galois. We begin by showing that in Θ_n, the higher cyclotomic extensions are, in fact, *split-Galois* (Example 2.4).

Proposition 5.1 *For every $r \in \mathbb{N}$, there is a $(\mathbb{Z}/p^r)^\times$–equivariant commutative ring isomorphism*

$$E_n[\omega^{(n)}_{p^r}] \simeq \prod_{(\mathbb{Z}/p^r)^\times} E_n.$$

Proof For every finite abelian p–group A, we denote by

$$A^* \simeq \hom(A, \mathbb{Q}_p/\mathbb{Z}_p)$$

the Pontryagin dual of A. By [27, Corollary 5.3.26], we have an isomorphism

$$E_n[B^n A] \simeq E_n^{A^*}\ \in\ \mathrm{CAlg}(\mathrm{Sp}_{K(n)}),$$

which is furthermore natural in A. In particular, when A is of exponent p^r, this isomorphism is equivariant with respect to the $(\mathbb{Z}/p^r)^\times$–action on A given by scalar multiplications. Consider the $(\mathbb{Z}/p^r)^\times$–equivariant decomposition

$$E_n[B^n C_{p^r}] \simeq E_n[B^n C_{p^{r-1}}] \times E_n[\omega^{(n)}_{p^r}].$$

The group homomorphism $C_{p^r} \twoheadrightarrow C_{p^{r-1}}$ induces an injection on Pontryagin duals, which we can identify with the embedding $C_{p^{r-1}} \hookrightarrow C_{p^r}$, whose image is pC_{p^r}. Noting that

$$C_{p^r} \setminus pC_{p^r} \simeq (\mathbb{Z}/p^r)^\times,$$

it follows that we have a $(\mathbb{Z}/p^r)^\times$–equivariant isomorphism

$$E_n[\omega^{(n)}_{p^r}] \simeq \prod_{(\mathbb{Z}/p^r)^\times} E_n\ \in\ \mathrm{CAlg}(\mathrm{Sp}_{K(n)}). \qquad \square$$

Using nil-conservativity, we can now deduce that the higher cyclotomic extensions of $\mathrm{Sp}_{T(n)}$ (and hence $\mathrm{Sp}_{K(n)}$) are Galois as well.

Proposition 5.2 *For all $r \in \mathbb{N}$, the $(p^r)^{th}$ cyclotomic extensions in $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$ are faithful Galois extensions.*

Proof The nilpotence theorem [30] implies that the functors

$$\mathrm{Sp}_{T(n)} \xrightarrow{L_{K(n)}} \mathrm{Sp}_{K(n)} \xrightarrow{E_n \otimes (-)} \Theta_n$$

are nil-conservative; see [14, Corollary 5.1.17]. Moreover, $\mathbb{S}_{T(n)}[\omega_{p^r}^{(n)}]$ and $\mathbb{S}_{K(n)}[\omega_{p^r}^{(n)}]$ are dualizable by Proposition 4.9(1). Thus, the Galois property of these extensions follows from Propositions 5.1 and 2.9. Since $\mathrm{Sp}_{T(n)}$ and $\mathrm{Sp}_{K(n)}$ are ∞–semiadditive, these extensions are faithful; see Remark 2.3. □

5.2 The $K(n)$–local cyclotomic character

In classical algebra, Galois theory allows one to classify the Galois extensions of a commutative ring in terms of its Galois group. For example, the sequence of $(p^r)^{\text{th}}$ cyclotomic extensions $\mathbb{Q}_p(\omega_{p^r})$ is classified by the (p–adic) cyclotomic character

$$\chi\colon \mathrm{Gal}(\mathbb{Q}_p) \to \mathbb{Z}_p^\times.$$

By the work of Devinatz and Hopkins [18], Rognes [49], Baker and Richter [5] and Mathew [38], the Galois extensions of $\mathbb{S}_{K(n)}$ can be similarly classified in terms of the (extended) Morava stabilizer group. In this subsection, we define the higher analogue of the p–adic cyclotomic character for $\mathrm{Sp}_{K(n)}$, which classifies the higher cyclotomic extensions $\mathbb{S}_{K(n)}[\omega_{p^r}^{(n)}]$, and prove that it identifies with the determinant map of the Morava Stabilizer group (see Theorem 5.8). We also show that the canonical map $\mathbb{G}_n \to \widehat{\mathbb{Z}}$ classifies the (ordinary) prime to p cyclotomic extensions (Corollary 5.15), by analogy with the map $\mathrm{Gal}(\mathbb{Q}_p) \to \widehat{\mathbb{Z}}$, which classifies the maximal unramified extension of $\mathbb{Q}_p$.

5.2.1 Morava stabilizer group We begin with a recollection of the Galois theory of $\mathrm{Sp}_{K(n)}$. The commutative ring $\pi_0(E_n)$ carries the universal deformation of the formal group $\overline{\Gamma} = \Gamma \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$. As such, it is acted on by the following group:

Definition 5.3 For every integer $n \geq 1$, the height n (extended) *Morava stabilizer group* $\mathbb{G}_n$ is defined to be the group of automorphisms of $\overline{\Gamma}$ over $\mathbb{F}_p$. That is, the group of pairs (σ, φ), where $\sigma \in \mathrm{Gal}(\mathbb{F}_p)$ and $\varphi\colon \sigma^*\overline{\Gamma} \xrightarrow{\sim} \overline{\Gamma}$. We denote by

$$\pi\colon \mathbb{G}_n \twoheadrightarrow \mathrm{Gal}(\mathbb{F}_p) \simeq \widehat{\mathbb{Z}}$$

the projection $(\sigma, \varphi) \mapsto \sigma$, whose kernel is $\mathrm{Aut}(\overline{\Gamma}/\overline{\mathbb{F}}_p)$.

In [18], Devinatz and Hopkins have lifted the canonical continuous action of the pro-finite group $\mathbb{G}_n$ on the $(p, u_1, \dots, u_{n-1})$–adic ring $\pi_0(E_n)$, to a continuous action on E_n itself as an object of $\mathrm{CAlg}(\mathrm{Sp}_{K(n)})$, which allows taking continuous fixed points with respect to closed subgroups. This action was shown to exhibit E_n as a pro-finite Galois extension of $\mathbb{S}_{K(n)}$ in [49, Theorem 5.4.4]. Furthermore, E_n itself is algebraically closed (by [5, Theorem 1.1] for p odd, and [38, Section 10.2 and Theorem 6.29] for all p). Consequently, $\mathbb{G}_n$ classifies Galois extensions of $\mathbb{S}_{K(n)}$ via a version of the "Galois correspondence" that we recall (with some paraphrasing) from [38]. For every finite group G and a continuous group homomorphism $\rho\colon \mathbb{G}_n \to G$, we equip

$$C(G, E_n) := \prod_G E_n \in \mathrm{CAlg}(\mathrm{Sp}_{K(n)})$$

with the (continuous) *ρ–twisted action* of $\mathbb{G}_n$, which, in addition to the standard action on each factor, permutes the factors through ρ and the left regular action of G on itself. In particular, on homotopy groups, $g \in G$ acts by the formula

$$g \cdot (x_h)_{h\in G} = (g x_{\rho(g)^{-1}h})_{h\in G} \quad \text{for } (x_h)_{h\in G} \in \prod_G \pi_*(E_n).$$

In addition, the group G acts on $C(G, E_n)$ by permuting the factors through the *right* regular action of G on itself, and the two actions clearly commute. Thus, $C(G, E_n)^{h\mathbb{G}_n}$ acquires a G–action.

Proposition 5.4 ("$K(n)$–local Galois correspondence" [38, Theorem 10.9, Proposition 5.32]) *Let G be a finite group. Taking $\mathbb{G}_n \to G$ to $C(G, E_n)^{h\mathbb{G}_n}$ establishes a bijection*

$$\{\text{continuous homomorphisms } \mathbb{G}_n \to G\}/\text{conjugation} \simeq \{G\text{–Galois extensions of } \mathbb{S}_{K(n)}\}/\text{ isomorphism}.$$

In particular, for a surjective homomorphism with kernel $U \le \mathbb{G}_n$, the corresponding Galois extension is given by E_n^{hU} (with the residual G–action).

We deduce that the Morava module of a Galois extension R can be described in terms of the ρ–twisted action.

Proposition 5.5 *For $R \in \mathrm{CAlg}^{G\text{–gal}}(\mathrm{Sp}_{K(n)})$ classified by $\rho\colon \mathbb{G}_n \to G$, the Morava module of R is even, and there is an isomorphism of even Morava modules*

$$\pi_0(E_n \otimes R) \simeq C(G, \pi_0(E_n)),$$

where on the left-hand side the action of $\mathbb{G}_n$ is induced from the action on E_n and the trivial action on R, and on the right-hand side, it is the ρ–twisted action. In particular, the $\mathbb{G}_n$ action on $\pi_0(E_n \otimes R)$ determines ρ up to conjugation.

Proof By Proposition 5.4, we have $R \simeq C(G, E_n)^{h\mathbb{G}_n}$. By [18, Theorem 1(iii)], we have a canonical G–equivariant isomorphism

$$\pi_*(E_n \otimes C(G, E_n)^{h\mathbb{G}_n}) \xrightarrow{\sim} C(G, C(\mathbb{G}_n, \pi_* E_n))^{hG} \simeq \pi_* C(G, E_n),$$

so in particular this holds on the level of π_0. □

5.2.2 The p–adic cyclotomic character By Proposition 5.4, the cyclotomic extensions $\mathbb{S}_{K(n)}[\omega_{p^r}^{(n)}]$ are classified by a sequence of homomorphisms $\chi_r : \mathbb{G}_n \to (\mathbb{Z}/p^r)^\times$. Since χ_{r+1} identifies with χ_r upon reduction modulo p^r for every $r \ge 0$, these assemble into a single continuous group homomorphism

$$\chi\colon \mathbb{G}_n \to \mathbb{Z}_p^\times \simeq \varprojlim_{r\in\mathbb{N}} (\mathbb{Z}/p^r)^\times.$$

The map χ thus classifies, via the $K(n)$–local Galois correspondence, the infinite cyclotomic extension

$$\mathbb{S}_{K(n)}[\omega^{(n)}_{p^\infty}] = \varinjlim_{r\in\mathbb{N}} \mathbb{S}_{K(n)}[\omega^{(n)}_{p^r}].$$

Definition 5.6 We refer to $\chi\colon \mathbb{G}_n \to \mathbb{Z}_p^\times$ as the *p–adic cyclotomic character* of $\mathrm{Sp}_{K(n)}$.

We would like to describe the p–adic cyclotomic character in terms of the description of $\mathbb{G}_n$ as the group of automorphisms of the formal group $\overline{\Gamma}$. The case of an odd prime p for the Honda formal group law was carried out in [51]. The general case follows similarly using the results of [27] expressing the $K(n)$–homology of Eilenberg–Mac Lane spaces in terms of alternating powers of the associated formal group following [47]. For completeness, we shall provide the details of the argument.

From now on, we shall consider $\overline{\Gamma}$ as a connected p–divisible group $\overline{\Gamma} = \varinjlim \overline{\Gamma}[p^r]$ with $\overline{\Gamma}[p^r]$ the corresponding finite flat group schemes of p^r–torsion. As in [27, Construction 3.2.1], for a finite flat group scheme G and an integer $d \geq 1$, one associates a finite flat group scheme $\mathrm{Alt}^{(d)}_G$ called the d^{th} alternating power of G. Again as in [27, Corollary 3.5.4], given a p–divisible group Υ of dimension 1, we can now assemble the finite flat group schemes of p–power torsion $\mathrm{Alt}^{(d)}_{\Upsilon[p^r]}$ to a p–divisible group

$$\mathrm{Alt}^{(d)}_{\Upsilon} := \varinjlim \mathrm{Alt}^{(d)}_{\Upsilon[p^r]}.$$

Moreover, when Υ is of height m the p–divisible group $\mathrm{Alt}^{(d)}_{\Upsilon}$ is of height $\binom{m}{d}$ and dimension $\binom{m-1}{d}$. In particular, when $d = m$ the p–divisible group $\mathrm{Alt}^{(m)}_{\Upsilon}$ is étale of height 1.

Returning to our p–divisible group $\overline{\Gamma}$, the top alternating power $\mathrm{Alt}^{(n)}_{\overline{\Gamma}}$ is an étale p–divisible group of height 1 over the algebraically closed field $\overline{\mathbb{F}}_p$, and hence its $\overline{\mathbb{F}}_p$–points identify noncanonically with the group $\mathbb{Q}_p/\mathbb{Z}_p$. As a result there is a *canonical* isomorphism

$$\mathrm{Aut}(\mathrm{Alt}^{(n)}_{\overline{\Gamma}}(\overline{\mathbb{F}}_p)) \simeq \mathbb{Z}_p^\times.$$

The group $\mathbb{G}_n$ acts on both $\mathrm{Alt}^{(n)}_{\overline{\Gamma}}$ (by functoriality) and on $\overline{\mathbb{F}}_p$ via $\pi\colon \mathbb{G}_n \twoheadrightarrow \mathrm{Gal}(\mathbb{F}_p)$, hence it acts on the group of $\overline{\mathbb{F}}_p$–points $\mathrm{Alt}^{(n)}_{\overline{\Gamma}}(\overline{\mathbb{F}}_p)$.

Proposition 5.7 *The cyclotomic character $\chi\colon \mathbb{G}_n \to \mathbb{Z}_p^\times$ identifies with the map*

$$\mathbb{G}_n \to \mathrm{Aut}(\mathrm{Alt}^{(n)}_{\overline{\Gamma}}(\overline{\mathbb{F}}_p)) \simeq \mathbb{Z}_p^\times,$$

classifying the action discussed above.

Proof It suffices to show that for each $r \geq 0$ the map in the statement agrees with χ after reduction modulo p^r, which we denote by $\chi_r\colon \mathbb{G}_n \to (\mathbb{Z}/p^r)^\times$. Via the Galois correspondence χ_r corresponds to the finite cyclotomic extension $\mathbb{S}_{K(n)}[\omega^{(n)}_{p^r}]$. Hence, by Proposition 5.5, we have a $\mathbb{G}_n$–equivariant isomorphism

$$\pi_0(E_n \otimes \mathbb{S}_{K(n)}[\omega^{(n)}_{p^r}]) \simeq C((\mathbb{Z}/p^r)^\times, \pi_0 E_n),$$

where on the right-hand side we have the so-called χ_r–twisted action. By collecting together the terms in the decomposition

$$\mathbb{S}_{K(n)}[B^n C_{p^r}] \simeq \mathbb{S}_{K(n)}[\omega_{p^0}^{(n)}] \oplus \mathbb{S}_{K(n)}[\omega_{p^1}^{(n)}] \oplus \cdots \oplus \mathbb{S}_{K(n)}[\omega_{p^r}^{(n)}],$$

and reducing modulo the maximal ideal of $\pi_0(E_n)$, we similarly get a $\mathbb{G}_n$–equivariant isomorphism

$$\pi_0(K(n) \otimes \mathbb{S}_{K(n)}[B^n C_{p^r}]) \simeq C(\mathbb{Z}/p^r, \overline{\mathbb{F}}_p),$$

where again on the right-hand side we have the χ_r–twisted action.

On the other hand, by [27, Theorem 2.0.1] there is a $\mathbb{G}_n$–equivariant isomorphism

$$\pi_0(K(n) \otimes \mathbb{S}_{K(n)}[B^n C_{p^r}]) \simeq \mathcal{O}(\mathrm{Alt}^{(n)}_{\overline{\Gamma}[p^r]}),$$

where $\mathcal{O}(-)$ stands for the algebra of regular functions on a scheme. Since $\mathrm{Alt}^{(n)}_{\overline{\Gamma}[p^r]}$ is an étale finite flat group scheme over $\overline{\mathbb{F}}_p$, its algebra of regular functions can be described as

$$\mathcal{O}(\mathrm{Alt}^{(n)}_{\overline{\Gamma}}[p^r]) \simeq C(\mathrm{Alt}^{(n)}_{\overline{\Gamma}}[p^r](\overline{\mathbb{F}}_p), \overline{\mathbb{F}}_p) \simeq C(\mathbb{Z}/p^r, \overline{\mathbb{F}}_p),$$

and this isomorphism is $\mathbb{G}_n$–equivariant, where the action on the right is twisted by the reduction modulo p^r of the map in the statement. □

The above result provides a completely algebraic description of the cyclotomic character χ. To compute it more explicitly, we need to recall some facts about the structure of the group $\mathbb{G}_n$. Let $\mathcal{D}_n$ be a division algebra over $\mathbb{Q}_p$ of invariant $1/n \in \mathbb{Q}/\mathbb{Z}$, and $\mathcal{O}_n \subseteq \mathcal{D}_n$ the maximal order. The group of units $\mathcal{O}_n^\times \subseteq \mathcal{O}_n$ is isomorphic to $\mathrm{Aut}(\overline{\Gamma}/\overline{\mathbb{F}}_p)$, which is the kernel of $\mathbb{G}_n \overset{\pi}{\twoheadrightarrow} \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. As for any finite dimensional division algebra, there is a determinant (a.k.a reduced norm) multiplicative map $\det\colon \mathcal{D}_n \to \mathbb{Q}_p$, which restricts to a group homomorphism $\det\colon \mathcal{O}_n^\times \to \mathbb{Z}_p^\times$.

Theorem 5.8 *The restriction of the p–adic cyclotomic character $\chi\colon \mathbb{G}_n \to \mathbb{Z}_p^\times$ to the subgroup $\mathcal{O}_n^\times \lhd \mathbb{G}_n$ is the determinant map.*

Proof By Proposition 5.7, we have to show that the action of $\mathcal{O}_n^\times \subseteq \mathbb{G}_n$ on $\mathrm{Alt}^{(n)}_{\overline{\Gamma}}(\overline{\mathbb{F}}_p) \simeq \mathbb{Z}_p^\times$ is via the determinant map. Recall that to the p–divisible group $\overline{\Gamma}$ over $\overline{\mathbb{F}}_p$ we can associate its *Dieudonné module* $\mathrm{DM}(\overline{\Gamma})$, which is in particular a free $\mathbb{W}(\overline{\mathbb{F}}_p)$–module of rank n (see [27, Section 1.3]). The action of $\mathcal{O}_n \simeq \mathrm{End}_{\overline{\mathbb{F}}_p}(\overline{\Gamma})$ on the Dieudonné module $\mathrm{DM}(\overline{\Gamma})$ gives rise to a $\mathbb{Z}_p$–algebra map

$$i\colon \mathcal{O}_n \to \mathrm{End}_{\mathbb{W}(\overline{\mathbb{F}}_p)}(\mathrm{DM}(\overline{\Gamma})) \simeq \mathrm{M}_{n\times n}(\mathbb{W}(\overline{\mathbb{F}}_p)).$$

Extending scalars along $\mathbb{Z}_p \to \mathbb{Q}_p$ we get a $\mathbb{Q}_p$–algebra map

$$\mathcal{D}_n \to \mathrm{End}_{\mathbb{W}(\overline{\mathbb{F}}_p)}(\mathrm{DM}(\overline{\Gamma})) \otimes \mathbb{Q}_p \simeq \mathrm{M}_{n\times n}(\widehat{\mathbb{Q}}_p^{\mathrm{ur}}),$$

where $\widehat{\mathbb{Q}}_p^{\mathrm{ur}} = \mathbb{W}(\overline{\mathbb{F}}_p)[1/p]$ is the completion of the maximal unramified extension of $\mathbb{Q}_p$. Since $\mathcal{D}_n$ is a central division algebra of dimension n^2 over $\mathbb{Q}_p$, after extending scalars on the source along $\mathbb{Q}_p \to \widehat{\mathbb{Q}}_p^{\mathrm{ur}}$, we get an isomorphism of $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$–algebras

$$\mathcal{D}_n \otimes_{\mathbb{Q}_p} \widehat{\mathbb{Q}}_p^{\mathrm{ur}} \xrightarrow{\sim} \mathrm{M}_{n\times n}(\widehat{\mathbb{Q}}_p^{\mathrm{ur}}).$$

By the definition of the reduced norm, the map $\det\colon \mathbb{D}_n \to \mathbb{Q}_p^\times$ is therefore the restriction of the ordinary determinant $\det\colon \mathrm{M}_{n\times n}(\widehat{\mathbb{Q}}_p^{\mathrm{ur}}) \to \widehat{\mathbb{Q}}_p^{\mathrm{ur}}$ along the inclusion $\mathbb{D}_n \hookrightarrow \mathrm{M}_{n\times n}(\widehat{\mathbb{Q}}_p^{\mathrm{ur}})$ above. Since the determinant of a matrix is given by its action on the top alternating power of a vector space, we deduce that the map

$$\det\colon \mathbb{O}_n^\times \to \mathbb{Z}_p^\times \hookrightarrow \mathbb{W}(\overline{\mathbb{F}}_p)^\times$$

can be written as the composition

$$\mathbb{O}_n^\times = \mathrm{Aut}_{\overline{\mathbb{F}}_p}(\overline{\Gamma}) \xrightarrow{\mathrm{DM}} \mathrm{Aut}_{\mathbb{W}(\overline{\mathbb{F}}_p)}(\mathrm{DM}(\overline{\Gamma})) \xrightarrow{\wedge^n(-)^\vee} \mathrm{Aut}_{\mathbb{W}(\overline{\mathbb{F}}_p)}(\wedge^n \mathrm{DM}(\overline{\Gamma})^\vee) \simeq \mathbb{W}(\overline{\mathbb{F}}_p)^\times.$$

Finally, by [27, Theorem 3.3.1] we have a natural identification

$$\wedge^n_{\mathbb{W}(\overline{\mathbb{F}}_p)} \mathrm{DM}(\overline{\Gamma})^\vee \simeq \mathrm{DM}(\mathrm{Alt}^{(n)}_{\overline{\Gamma}}).$$

We deduce that the action of $\mathbb{O}_n$ on the Dieudonné module $\mathrm{DM}(\mathrm{Alt}^{(n)}_{\overline{\Gamma}})$ is via the determinant map. Since the map

$$\mathbb{Z}_p^\times \simeq \mathrm{Aut}(\mathrm{Alt}^{(n)}_{\overline{\Gamma}}) \xrightarrow{\mathrm{DM}} \mathrm{Aut}(\mathrm{DM}(\mathrm{Alt}^{(n)}_{\overline{\Gamma}})) \simeq \mathbb{W}(\overline{\mathbb{F}}_p)^\times$$

is the canonical inclusion, this implies that the action of $\mathbb{O}_n^\times$ on $\mathrm{Alt}^{(n)}_{\overline{\Gamma}}$ is via the determinant map as well. □

Theorem 5.8 identifies the p–adic cyclotomic character $\chi\colon \mathbb{G}_n \to \mathbb{Z}_p^\times$ only on the kernel of the map $\pi\colon \mathbb{G}_n \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \simeq \widehat{\mathbb{Z}}$. However, it is possible to identify χ on the entire group $\mathbb{G}_n$ as well. The choice of Γ (namely, the choice of an $\mathbb{F}_p$–form of $\overline{\Gamma}$) yields a section of π, and hence, a semidirect product decomposition $\mathbb{G}_n \simeq \widehat{\mathbb{Z}} \ltimes \mathbb{O}_n^\times$. It therefore remains to identify the restriction of χ to the subgroup $\widehat{\mathbb{Z}} \le \mathbb{G}_n$ under this decomposition, which we denote by

$$\chi_{\mathrm{gal}}\colon \widehat{\mathbb{Z}} \longrightarrow \mathbb{Z}_p^\times.$$

While the p–divisible group $\mathrm{Alt}^{(n)}_{\overline{\Gamma}}$ is isomorphic to the constant p–divisible group $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$, this no longer necessarily holds for $\mathrm{Alt}^{(n)}_{\Gamma}$. In fact, the isomorphism class of $\mathrm{Alt}^{(n)}_{\Gamma}$ depends on Γ, and might or might not be split. In general, $\mathrm{Alt}^{(n)}_{\Gamma}$ is an $\mathbb{F}_p$–form of $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$, and therefore corresponds to a continuous cohomology class in

$$H^1_c(\mathrm{Gal}(\mathbb{F}_p), \mathrm{Aut}(\underline{\mathbb{Q}_p/\mathbb{Z}_p})) \simeq \hom_c(\mathrm{Gal}(\mathbb{F}_p), \mathbb{Z}_p^\times).$$

By the classical theory of Galois forms, we have the following:

Proposition 5.9 *The cohomology class classifying* $\mathrm{Alt}^{(n)}_{\Gamma}$ *is* $\chi_{\mathrm{gal}}\colon \widehat{\mathbb{Z}} \to \mathbb{Z}_p^\times$.

Proof By Proposition 5.7, the group $\mathrm{Gal}(\mathbb{F}_p) \simeq \widehat{\mathbb{Z}}$ acts on the $\overline{\mathbb{F}}_p$–points of $\mathrm{Alt}^{(n)}_{\Gamma}$ via χ_{gal}. By inspecting the construction of the cohomology class corresponding to an $\mathbb{F}_p$–form, this action is given by the mentioned cohomology class. □

Combining Theorem 5.8 with Proposition 5.9 we get a complete algebraic description of the p–adic cyclotomic character

$$\chi\colon \mathbb{G}_n \simeq \widehat{\mathbb{Z}} \ltimes \mathbb{O}_n^\times \to \mathbb{Z}_p^\times.$$

Namely,

$$\chi(u,a) = \det(a)\chi_{\mathrm{gal}}(a) \quad \text{for all} \quad a \in \mathbb{O}_n^\times,\ u \in \hat{\mathbb{Z}},$$

where χ_{gal} is as in Proposition 5.9.

Example 5.10 Assume that Γ is *normalizable* in the sense of [27, Definition 5.3.1]. That is, we have an isomorphism $\mathrm{Alt}_\Gamma^{(n)} \simeq \underline{\mathbb{Q}_p/\mathbb{Z}_p}$ defined over $\mathbb{F}_p$. In this case, with respect to the splitting $\mathbb{G}_n \simeq \hat{\mathbb{Z}} \ltimes \mathbb{O}_n^\times$ defined by Γ, the map χ_{gal} is trivial and

$$\chi(u,a) = \det(a) \quad \text{for all} \quad a \in \mathbb{O}_n^\times,\ u \in \hat{\mathbb{Z}}.$$

The next example is a reformulation of a computation carried out in [51, Proposition 3.20].

Example 5.11 (Westerland) Let p be an odd prime and let Γ be the Honda formal group law of height n over $\mathbb{F}_p$. The form $\mathrm{Alt}_\Gamma^{(n)}$ is classified in this case by the cocycle $\chi_{\mathrm{gal}}(u) = (-1)^{u(n-1)}$. Consequently, with respect to the splitting

$$\mathbb{G}_n \simeq \hat{\mathbb{Z}} \ltimes \mathbb{O}_n^\times$$

defined by Γ, the cyclotomic character χ is given by

$$\chi(u,a) = (-1)^{u(n-1)} \det(a) \quad \text{for all} \quad a \in \mathbb{O}_n^\times,\ u \in \hat{\mathbb{Z}}.$$

This is the map denoted by $\det_\pm$ in [51, Section 1.1]. Namely, for n even the Honda formal group is not normalizable, which introduces the sign factor in $\det_\pm$.

For future use, we record here a mild variation on [11, Lemma 1.33] regarding the fixed points of the action of $\mathbb{G}_n$ on the ring $\pi_0 E_n$.

Proposition 5.12 *Let $N \lhd \mathbb{G}_n$ be the kernel of the cyclotomic character $\chi\colon \mathbb{G}_n \to \mathbb{Z}_p^\times$. We have*

$$(\pi_0 E_n)^N = \mathbb{Z}_p \subseteq \pi_0 E_n.$$

Proof Recall that $\pi_0(E_n) = W(\overline{\mathbb{F}}_p)[\![u_1, \dots, u_{n-1}]\!]$, and that $W(\overline{\mathbb{F}}_p)[1/p]$ is isomorphic to $\hat{\mathbb{Q}}_p^{\mathrm{ur}}$, the completion of the maximal unramified extension of $\mathbb{Q}_p$. Since $\pi_0 E_n$ is torsion-free, it embeds in $\pi_0 E_n[1/p]$. Therefore, it suffice to show that $(\pi_0 E_n[1/p])^N = \mathbb{Q}_p$. Consider the subgroup $\mathbb{O}_n^\times \le \mathbb{G}_n$, and recall that the algebra $\mathbb{O}_n$ has the following presentation:

$$\mathbb{O}_n \simeq W(\mathbb{F}_{p^n})\{S\}/(S^n = p,\ Sx = \varphi(x)S \quad \text{for all } x \in W(\mathbb{F}_{p^n})),$$

where S is a noncommutative indeterminate and $\varphi\colon W(\mathbb{F}_{p^n}) \to W(\mathbb{F}_{p^n})$ is the (unique) lift of the Frobenius endomorphism of $\mathbb{F}_{p^n}$. By [17, Proposition 3.3], we have an $\mathbb{O}_n^\times$–equivariant embedding[11]

$$\pi_0 E_n[1/p] \hookrightarrow \hat{\mathbb{Q}}_p^{\mathrm{ur}}[\![w_1, \dots, w_{n-1}]\!],$$

[11]This embedding exhibits the target as the completion of the source with respect to its unique maximal ideal. We also remark that the w_i do not belong to the image of $\pi_0 E_n$.

such that the action of $\mathbb{O}_n^\times$ on the right-hand side is $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$–linear, and each $x \in W(\mathbb{F}_{p^n})^\times \le \mathbb{O}_n^\times$ acts on a power series $f = f(w_1, \dots, w_{n-1})$ by

$$(x \cdot f)(w_1, \dots, w_{n-1}) = f\left(\frac{\varphi(x)}{x} w_1, \dots, \frac{\varphi^{n-1}(x)}{x} w_{n-1}\right).$$

It will suffice to show that

$$\widehat{\mathbb{Q}}_p^{\mathrm{ur}}[\![w_1, \dots, w_{n-1}]\!]^N = \mathbb{Q}_p.$$

Consider now the subgroup

$$W^{(1)}(\mathbb{F}_{p^n})^\times := W(\mathbb{F}_{p^n})^\times \cap N \le \mathbb{O}_n^\times.$$

If f is fixed by N, and hence by $W^{(1)}(\mathbb{F}_{p^n})^\times$, the only monomials $w_1^{d_1} w_2^{d_2} \cdots w_{n-1}^{d_{n-1}}$, that can appear in f with nonzero coefficients, are those for which

$$x^{d_1+d_2+\cdots+d_{n-1}} = \varphi(x)^{d_1} \varphi^2(x)^{d_2} \cdots \varphi^{n-1}(x)^{d_{n-1}} \quad \text{for all } x \in W^{(1)}(\mathbb{F}_{p^n})^\times.$$

For a general element $x \in W(\mathbb{F}_{p^n})^\times \le \mathbb{O}_n^\times$, the determinant $\det(x)$ coincides with the norm $\mathrm{Nm}(x) := \prod_{i=0}^{n-1} \varphi^i(x)$. Taking p–adic logarithm on the above displayed formula, this implies that the equation

$$(*) \qquad (d_1 + d_2 + \cdots + d_{n-1}) y = d_1 \varphi(y) + d_2 \varphi^2(y) + \cdots + d_{n-1} \varphi^{n-1}(y)$$

holds for every $y \in W(\mathbb{F}_{p^n})$ with $\mathrm{Tr}(y) = \sum_{i=0}^{n-1} \varphi^i(y) = 0$ and a sufficiently high p–adic valuation. Since $(*)$ is a linear equation, it in fact holds for all $y \in \mathbb{Q}_p(\omega_{p^n-1}) = W(\mathbb{F}_{p^n})[1/p]$ such that $\mathrm{Tr}(y) = 0$. We deduce, by the linear independence of the φ^i, that $d_1 = \cdots = d_{n-1} = 0$. This means that f has to be constant, ie an element of $\widehat{\mathbb{Q}}_p^{\mathrm{ur}} \subseteq \pi_0 E_n[1/p]$.

Finally, we have a semidirect product decomposition $\mathbb{G}_n \simeq \widehat{\mathbb{Z}} \ltimes \mathbb{O}_n^\times$, by which we identify the topological generator $1 \in \widehat{\mathbb{Z}}$ with an element $\sigma \in \mathbb{G}_n$. Since $\det = \mathrm{Nm} \colon W(\mathbb{F}_{p^n})^\times \to \mathbb{Z}_p^\times$ is surjective ([42, Proposition III.1.2]), there exists an element $a \in W(\mathbb{F}_{p^n})^\times$, with $\det(a) = \det(\sigma)$. Thus, we get an element $a^{-1}\sigma \in N$, which acts on $\widehat{\mathbb{Q}}_p^{\mathrm{ur}} \subseteq \pi_0 E_n[1/p]$ as the Frobenius (see [6, Section 3.2.2]). By the Ax–Sen–Tate theorem [3], the fixed points of $a^{-1}\sigma$ on $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$ are $\mathbb{Q}_p \subseteq \pi_0 E_n[1/p]$. □

5.2.3 The total cyclotomic character We conclude this subsection by discussing the Galois extensions classified by the map $\pi \colon \mathbb{G}_n \twoheadrightarrow \widehat{\mathbb{Z}}$ from Definition 5.3. Roughly speaking, π classifies the *ordinary*, ie height 0, cyclotomic extensions of $\mathbb{S}_{K(n)}$ of order prime to p (see Corollary 5.15 for the precise statement). This perspective is originally due to Rognes (see [49, Section 5.4.6]) and we review it for completeness.

We begin by considering the Galois extensions of the p–complete sphere $\mathbb{S}_p \in \mathrm{Sp}$. Since $\mathbb{S}_p$ is connective, by [38, Theorem 6.17], all Galois extensions of $\mathbb{S}_p$ (ie of $\mathrm{Mod}_{\mathbb{S}_p}$) are *algebraic*. Namely, they are étale and, by applying π_0, correspond bijectively to the (ordinary) Galois extensions of the ring $\pi_0(\mathbb{S}_p) = \mathbb{Z}_p$. The Galois extensions of $\mathbb{Z}_p$ are in turn classified by the Galois group

$$\mathrm{Gal}(\mathbb{Z}_p) \simeq \mathrm{Gal}(\mathbb{F}_p) \simeq \widehat{\mathbb{Z}}.$$

More concretely, the finite quotients $\widehat{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}/m$ correspond to the rings of Witt vectors $W(\mathbb{F}_{p^m})$ with the action given by the lift of Frobenius. Hence, the corresponding Galois extensions of $\mathbb{S}_p$ are the rings of *spherical* Witt vectors $\mathbb{S}W(\mathbb{F}_{p^m})$, which are characterized by being étale over $\mathbb{S}_p$ and having $\pi_0(\mathbb{S}W(\mathbb{F}_{p^m})) \simeq W(\mathbb{F}_{p^m})$; see [36, Example 5.2.7].

Proposition 5.13 *For every $m \in \mathbb{N}$, the composition*

$$\mathbb{G}_n \xrightarrow{\pi} \widehat{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}/m$$

classifies the $\mathbb{Z}/m$–Galois extension $L_{K(n)}\mathbb{S}W(\mathbb{F}_{p^m})$ of $\mathbb{S}_{K(n)}$.

Proof By Proposition 5.5, it suffices to show that $L_{K(n)}\mathbb{S}W(\mathbb{F}_{p^m})$ is Galois and the even Morava module

$$\pi_0(E_n \otimes L_{K(n)}\mathbb{S}W(\mathbb{F}_{p^m})) \simeq \pi_0(E_n) \otimes W(\mathbb{F}_{p^m})$$

is equivariantly isomorphic to $C(\mathbb{Z}/m, \pi_0(E_n))$ with the π–twisted $\mathbb{G}_n$–action. The second claim follows from the fact that the action of $\mathbb{G}_n$ on the coefficient ring $W(\overline{\mathbb{F}}_p) \subseteq \pi_0(E_n)$ factors through π and is given again by the lift of Frobenius; see [6, Section 3.2.2]. We now observe that the first claim follows from the second. Indeed, by [51, Theorem 3.24], if a $K(n)$–local commutative ring spectrum R has a Morava module isomorphic to $C(G, \pi_0(E_n))$ for some $\rho\colon \mathbb{G}_n \twoheadrightarrow G$, then R is isomorphic to the Galois extension $E_n^{h\ker(\rho)}$ and hence in particular Galois. □

Remark 5.14 In the language of [38, Definition 6.8], the map $\pi\colon \mathbb{G}_n \twoheadrightarrow \widehat{\mathbb{Z}}$ is the map induced on (weak) Galois groups by the functor $L_{K(n)}\colon \mathrm{Mod}_{\mathbb{S}_p} \to \mathrm{Sp}_{K(n)}$.

The relation to cyclotomic extensions of order prime to p (ie of height zero) is as follows:

Corollary 5.15 (Rognes) *For every $m \in \mathbb{N}$, the composition*

$$\mathbb{G}_n \overset{\pi}{\twoheadrightarrow} \widehat{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}/m \xhookrightarrow{p^{(-)}} \mathbb{Z}/(p^m-1)^\times$$

classifies the (nonconnected) cyclotomic Galois extension $\mathbb{S}_{K(n)}[\omega_{p^m-1}]$.

Proof By Proposition 5.13, it suffices to show that the composition

$$f\colon \widehat{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}/m \xhookrightarrow{p^{(-)}} \mathbb{Z}/(p^m-1)^\times$$

classifies $\mathbb{S}_p[\omega_{p^m-1}]$. Since all Galois extensions of $\mathbb{S}_p$ are algebraic [38, Theorem 6.17], it suffices to show that the Galois extension of $\mathbb{Z}_p = \pi_0\mathbb{S}_p$ classified by f is $\mathbb{Z}_p[\omega_{p^m-1}]$. The splitting of the cyclotomic polynomial $\Phi_{p^m-1}(t)$ into irreducible factors over $\mathbb{Z}_p$ induces an isomorphism of the ring

$$\mathbb{Z}_p[\omega_{p^m-1}] \simeq \mathbb{Z}_p[t]/\Phi_{p^m-1}(t)$$

with a product of $\phi(p^m-1)/m$ copies of $W(\mathbb{F}_{p^m})$. Moreover, as a $\mathbb{Z}/(p^m-1)^\times$–equivariant ring, $\mathbb{Z}_p[\omega_{p^m-1}]$ is isomorphic to the induction of $W(\mathbb{F}_{p^m})$ along the group homomorphism

$$p^{(-)}\colon \mathbb{Z}/m \hookrightarrow \mathbb{Z}/(p^m-1)^\times,$$

and hence the claim follows. □

Remark 5.16 For every $N \in \mathbb{N}$ with $(N, p) = 1$, we have $N \mid (p^m - 1)$ for some $m \in \mathbb{N}$. Thus, $\pi : \mathbb{G}_n \twoheadrightarrow \hat{\mathbb{Z}}$ accounts for all *prime to* p cyclotomic extensions of $\mathbb{S}_{K(n)}$.

Taken together, π and χ assemble into a single map

$$\chi_{\mathrm{tot}} : \mathbb{G}_n \twoheadrightarrow \hat{\mathbb{Z}} \times \mathbb{Z}_p^\times,$$

which we call the (*total*) *cyclotomic character*. We recall the following standard fact:

Proposition 5.17 *The map χ_{tot} exhibits $\hat{\mathbb{Z}} \times \mathbb{Z}_p^\times$ as the* (*profinite*) *abelianization of $\mathbb{G}_n$.*

Proof Let $\mathcal{D}_n$ be a division algebra over $\mathbb{Q}_p$ of invariant $1/n$, so that $\mathbb{O}_n$ is the maximal order in $\mathcal{D}_n$. We may present $\mathbb{O}_n$ as

$$\mathbb{O}_n \simeq W(\mathbb{F}_{p^n})\{S\}/\big(S^n = p,\ Sx = \varphi(x)S \text{ for all } x \in W(\mathbb{F}_{p^n})\big).$$

Then, S is a uniformizer of $\mathcal{D}_n$ and hence there is a split short exact sequence

$$1 \to \mathbb{O}_n^\times \hookrightarrow \mathcal{D}_n^\times \to \mathbb{Z} \to 1$$

in which the second map is the S–adic valuation map. Since conjugation by S acts by φ on $W(\mathbb{F}_{p^n})$, after profinite completion, the above short exact sequence identifies with

$$1 \to \mathbb{O}_n^\times \hookrightarrow \mathbb{G}_n \xrightarrow{\pi} \hat{\mathbb{Z}} \to 1.$$

By [43] the map

$$\mathcal{D}_n^\times \xrightarrow{\det} \mathbb{Q}_p^\times$$

exhibits $\mathbb{Q}_p^\times$ as the abelianization of $\mathcal{D}_n^\times$. Taking profinite completions, we obtain, as claimed, that

$$\mathbb{G}_n^{\mathrm{ab}} \simeq (\widehat{\mathcal{D}_n^\times})^{\mathrm{ab}} \xrightarrow[\sim]{\det} \widehat{\mathbb{Q}_p^\times} \simeq \mathbb{Z}_p^\times \times \hat{\mathbb{Z}}. \qquad \square$$

Consequently, every abelian Galois extension of $\mathbb{S}_{K(n)}$ is a subextension of a cyclotomic extension, obtained by adding an ordinary root of unity of some order prime to p and a higher root of unity of some p–power order.

Remark 5.18 For $\mathbb{Q}_p \in \mathrm{CAlg}(\mathrm{Sp}_{\mathbb{Q}})$, considered as the extrapolation to height $n = 0$ of the sequence $\mathbb{S}_{K(n)} \in \mathrm{CAlg}(\mathrm{Sp}_{K(n)})$, we have a completely analogous picture. By the (p–local) Kronecker–Weber theorem, every abelian extension of $\mathbb{Q}_p$ is contained in a cyclotomic extension. Moreover, we have $\mathrm{Gal}(\mathbb{Q}_p)^{\mathrm{ab}} \simeq \hat{\mathbb{Z}} \times \mathbb{Z}_p^\times$, where the $\hat{\mathbb{Z}}$ component corresponds to the maximal *unramified* cyclotomic extension $\mathbb{Q}_p^{\mathrm{un}} = \bigcup_m \mathbb{Q}_p(\omega_{p^m-1})$, and the $\mathbb{Z}_p^\times$ component corresponds to the maximal *ramified* cyclotomic extension $\mathbb{Q}_p(\omega_{p^\infty})$.

5.3 Picard groups

In this subsection we relate the higher cyclotomic extensions of $\mathbb{S}_{K(n)}$ to the Picard group of $\mathrm{Sp}_{K(n)}$.

Definition 5.19 Let $\mathrm{Pic}_n := \mathrm{Pic}(\mathrm{Sp}_{K(n)})$, and let $\mathrm{Pic}_n^0 \le \mathrm{Pic}_n$ be the (index 2) subgroup of objects $X \in \mathrm{Pic}_n$, such that $E_n \otimes X \simeq E_n$ as E_n–modules.

We denote by $\mathrm{Pic}_n^{\mathrm{alg},0}$ the Picard group of the category of even Morava modules. The functor $\pi_0(E_n\otimes -)$ induces a map $\mathrm{Pic}_n^0 \to \mathrm{Pic}_n^{\mathrm{alg},0}$ (whose kernel is known as the *exotic* Picard group). Furthermore, there is a canonical isomorphism [22, Proposition 2.5]

$$\mathrm{Pic}_n^{\mathrm{alg},0} \simeq H_c^1(\mathbb{G}_n;(\pi_0 E_n)^\times).$$

Remark 5.20 Since it will play a role in the sequel, we recall briefly how this identification goes. Given $M\in \mathrm{Pic}_n^{\mathrm{alg},0}$, we have $M\simeq \pi_0 E_n$ as $\pi_0 E_n$–modules. By choosing a generator $x\in M$, we associate with M the function $\alpha_M\colon \mathbb{G}_n\to \pi_0 E_n^\times$ given by $\alpha_M(\sigma):=\sigma(x)/x$. This function is a 1–cocycle, whose cohomology class $[\alpha_M]\in H_c^1(\mathbb{G}_n;(\pi_0E_n)^\times)$ is independent of the generator $x\in M$.

5.3.1 Odd prime

We begin by considering the case where the prime p is odd. First:

Lemma 5.21 *If p is odd, then $\mathrm{Pic}_n^0 = \mathrm{Pic}^{\mathrm{ev}}(\mathrm{Sp}_{K(n)})$.*

Proof Since $(\pi_0\mathbb{S}_{K(n)})^{\mathrm{red}}\simeq \mathbb{Z}_p$ (see for instance [13, Proposition 2.2.6]), the commutative ring $\pi_0\mathbb{S}_{K(n)}$ is connected with 2 invertible. Hence, by Corollary 3.21, every $X\in\mathrm{Pic}_n$ satisfies $\dim(X)=\pm 1$. Applying the symmetric monoidal functor

$$E_n\otimes(-)\colon \mathrm{Sp}_{K(n)}\to \mathrm{Mod}_{E_n}(\mathrm{Sp}_{K(n)}),$$

we can test whether $\dim(X)$ is 1 or -1, by looking at $\dim(E_n\otimes X)$. Finally, by [4, Theorem 8.7], we have $\mathrm{Pic}(E_n)\simeq \mathbb{Z}/2$, with representatives given by E_n and ΣE_n, which have dimensions 1 and -1, respectively. $\square$

We can now apply the Kummer theory developed in Section 3 to relate the p^{th} cyclotomic extension to the $(p-1)$–torsion in the Picard group of $\mathrm{Sp}_{K(n)}$. Namely, since the p^{th} cyclotomic extension is Galois it provides us with a distinguished Picard object.

Definition 5.22 For p odd, let $Z_n\in \mathrm{Pic}_n^0[p-1]$ be the Picard object corresponding to the $\mathbb{Z}/(p-1)$–Galois extension $\mathbb{S}_{K(n)}[\omega_p^{(n)}]$ in $\mathrm{Sp}_{K(n)}$, under the map of Proposition 3.23.

That is, Z_n is a $(p-1)$–torsion Picard object of dimension 1 in $\mathrm{Sp}_{K(n)}$ such that

$$\mathbb{S}_{K(n)}[\omega_p^{(n)}]\simeq \bigoplus_{k=0}^{p-2} Z_n^{\otimes k}\ \in\ \mathrm{Sp}_{K(n)}.$$

The Picard object Z_n can be characterized in an intrinsic way to Pic_n as follows:

Proposition 5.23 *For p odd, the group $\mathrm{Pic}_n^0[p-1]$ is isomorphic to $\mathbb{Z}/(p-1)$ and is generated by Z_n.*

Proof Using Lemma 5.21 and Proposition 3.23 together with its naturality with respect to the symmetric monoidal the functor

$$L_{K(n)}\colon \mathrm{Mod}_{\mathbb{S}_p}(\mathrm{Sp})\to \mathrm{Sp}_{K(n)},$$

we obtain the following commutative diagram of abelian groups:

$$\begin{tikzcd}
0 \arrow[r] & (\pi_0\mathbb{S}_p^\times)/(\pi_0\mathbb{S}_p^\times)^{p-1} \arrow[r] \arrow[d, "f"] & \pi_0\,\mathrm{CAlg}^{\mathbb{Z}/(p-1)\text{-gal}}(\mathbb{S}_p) \arrow[r] \arrow[d, "g"] & \mathrm{Pic}^{\mathrm{ev}}(\mathbb{S}_p)[p-1] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\pi_0\mathbb{S}_{K(n)}^\times)/(\pi_0\mathbb{S}_{K(n)}^\times)^{p-1} \arrow[r] & \pi_0\,\mathrm{CAlg}^{\mathbb{Z}/(p-1)\text{-gal}}(\mathrm{Sp}_{K(n)}) \arrow[r] & \mathrm{Pic}_n^0[p-1] \arrow[r] & 0
\end{tikzcd}$$

First, it is well known that $\mathrm{Pic}(\mathbb{S}_p) \simeq \mathbb{Z}$ (see for instance [12, Proposition 4.13]), so the upper-right corner vanishes. In the top-left corner, we have

$$(\pi_0\mathbb{S}_p^\times)/(\pi_0\mathbb{S}_p^\times)^{p-1} \simeq (\mathbb{Z}_p^\times)/(\mathbb{Z}_p^\times)^{p-1} \simeq \mathbb{Z}/(p-1).$$

Furthermore, the left vertical map f is an isomorphism. Indeed, the map

$$\mathbb{Z}_p \simeq \pi_0\widehat{\mathbb{S}}_p \to \pi_0\mathbb{S}_{K(n)}$$

admits a retract $r\colon \pi_0\mathbb{S}_{K(n)} \to \mathbb{Z}_p$, whose kernel consists of nilpotent elements [13, Proposition 2.2.6]. In particular, every element in the kernel of $r^\times\colon \pi_0\mathbb{S}_{K(n)}^\times \to \mathbb{Z}_p^\times$, is of the form $x = (1+\varepsilon)$ for some nilpotent $\varepsilon \in \pi_0\mathbb{S}_{K(n)}$. Since $p-1$ is invertible in $\pi_0\mathbb{S}_{K(n)}$ and the power series expansion of $(1+t)^{1/(p-1)}$ belongs to $\mathbb{Z}[1/(p-1)][\![t]\!]$, every such element x has a $(p-1)^{\mathrm{st}}$ root. Hence, $r^\times$ induces an isomorphism after modding out the $(p-1)^{\mathrm{st}}$ powers. Since this induced isomorphism is a left-inverse of f, it follows that f is an isomorphism as well.

Next, by Proposition 5.17, the map

$$(\pi,\chi)\colon \mathbb{G}_n \to \hat{\mathbb{Z}} \times \mathbb{Z}_p^\times$$

exhibits the target as the abelianization of the source. Hence, g can be identified with the inclusion (see Remark 5.14)

$$\hom(\hat{\mathbb{Z}}, \mathbb{Z}/(p-1)) \hookrightarrow \hom(\hat{\mathbb{Z}}, \mathbb{Z}/(p-1)) \oplus \hom(\mathbb{Z}_p^\times, \mathbb{Z}/(p-1)).$$

Since $\hom(\hat{\mathbb{Z}}, \mathbb{Z}/(p-1)) \simeq \mathbb{Z}/(p-1)$, the entire diagram can be identified with

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/(p-1) \arrow[r, "="] \arrow[d, "="] & \mathbb{Z}/(p-1) \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}/(p-1) \arrow[r, hook] & \mathbb{Z}/(p-1) \oplus \hom(\mathbb{Z}_p^\times, \mathbb{Z}/(p-1)) \arrow[r] & \mathrm{Pic}_n^0[p-1] \arrow[r] & 0
\end{tikzcd}$$

where both inclusions of $\mathbb{Z}/(p-1)$ are as the first summand of the target. Thus, the bottom right map restricts to an isomorphism

$$\mathbb{Z}/(p-1) \simeq \hom(\mathbb{Z}_p^\times, \mathbb{Z}/(p-1)) \xrightarrow{\sim} \mathrm{Pic}_n^0[p-1].$$

Chasing through the identifications, the generator $1 \in \mathbb{Z}/(p-1)$ corresponds to the $\mathbb{Z}/(p-1)$–Galois extension $\mathbb{S}_{K(n)}[\omega_p^{(n)}]$, and thus its image, Z_n, generates $\mathrm{Pic}_n^0[p-1]$. $\square$

Remark 5.24 By [51, Section 3.3], the image of Z_n in $\mathrm{Pic}_n^{\mathrm{alg},0}$ is classified by the composition

$$\mathbb{G}_n \xrightarrow{\chi} \mathbb{Z}_p^\times \twoheadrightarrow \mathbb{F}_p^\times \xrightarrow{\tau} \mathbb{Z}_p^\times \subseteq (\pi_0 E_n)^\times,$$

where τ is the Teichmüller lift.

5.3.2 Even prime In the case $p=2$, we cannot rely on Kummer theory to produce Picard objects in $\mathrm{Sp}_{K(n)}$. However, we can use instead the variant afforded by Definition 3.26. Recall that given $R \in \mathrm{CAlg}^{\mu_2\text{–gal}}(\mathrm{Sp}_{K(n)})$, where $\mu_2=\{\pm 1\}$, the cofiber of the unit map $\mathbb{1}\to R$, denoted by $\overline{R}$, belongs to Pic_n (Proposition 3.27). In fact, we have a somewhat stronger statement:

Lemma 5.25 *For every $R \in \mathrm{CAlg}^{\mu_2\text{–gal}}(\mathrm{Sp}_{K(n)})$, we have $\overline{R}\in \mathrm{Pic}_n^0$.*

Proof Let $R \in \mathrm{CAlg}^{\mu_2\text{–gal}}(\mathrm{Sp}_{K(n)})$. We need to show that $E_n\otimes \overline{R}\simeq E_n$ as an E_n–module. For this, we first observe that $R\otimes\overline{R}$ is isomorphic to the cofiber of the unit map $\mathbb{1}\to R$ tensored with R. Since this map can be identified with the diagonal $R\to R\times R$, whose cofiber is R, we get that $R\otimes \overline{R}\simeq R$ as R–modules. Since R is a Galois extension of $\mathbb{S}_{K(n)}$, there exists a map of commutative algebras $R\to E_n$. Base-changing from R to E_n along this map, we get that $E_n\otimes\overline{R}\simeq E_n$. □

Thus, we get a function

$$\Xi\colon \hom_c(\mathbb{G}_n,\mu_2)\simeq \pi_0\,\mathrm{CAlg}^{\mu_2\text{–gal}}(\mathrm{Sp}_{K(n)}) \xrightarrow{\overline{(-)}} \mathrm{Pic}_n^0.$$

To analyze the image of Ξ, we shall consider its further image in $\mathrm{Pic}_n^{\mathrm{alg},0}$. For this, it will be convenient to identify $\hom_c(\mathbb{G}_n,\mu_2)$ with $H_c^1(\mathbb{G}_n;\mu_2)$ for the trivial $\mathbb{G}_n$–action on μ_2.

Proposition 5.26 *The composition*

$$H_c^1(\mathbb{G}_n;\mu_2)\simeq \hom_c(\mathbb{G}_n,\mu_2)\xrightarrow{\Xi}\mathrm{Pic}_n^0\to \mathrm{Pic}_n^{\mathrm{alg},0}\simeq H_c^1(\mathbb{G}_n;\pi_0E_n^\times)$$

is induced by the inclusion $\mu_2\subseteq \pi_0E_n^\times$.

Proof Let $\mathbb{G}_n\xrightarrow{\rho}\mu_2$ be a homomorphism, and let $R\in \mathrm{CAlg}^{\mu_2\text{–gal}}(\mathrm{Sp}_{K(n)})$ be the Galois extension classified by ρ by the Galois correspondence (Proposition 5.4). We have an isomorphism of $\mathbb{G}_n$–equivariant E_n–modules $E_n\otimes R\simeq \prod_{\mu_2}E_n$, where $\mathbb{G}_n$ acts on the right-hand side via the ρ–twisted action (Proposition 5.5). Hence, we can identify $\pi_0(E_n\otimes\overline{R})$ with the cokernel of the diagonal map $\pi_0E_n\to\prod_{\mu_2}\pi_0E_n$. This cokernel can be further identified with π_0E_n, via the difference map $\prod_{\mu_2}\pi_0E_n\to\pi_0E_n$. Choosing the generator $x_0\in\pi_0(E_n\otimes\overline{R})$, that corresponds via this identification to $1\in\pi_0E_n$, we get that the action of $\sigma\in\mathbb{G}_n$ on x_0 is given by $\sigma(x_0)=\rho(\sigma)x_0$. This implies that the image of $\overline{R}$ in $H_c^1(\mathbb{G}_n,\pi_0E_n^\times)$ is the 1–cocycle $\mathbb{G}_n\xrightarrow{\rho}\mu_2\subseteq\pi_0E_n^\times$; see Remark 5.20. □

Remark 5.27 The above shows that the composition

$$\pi_0\,\mathrm{CAlg}^{\mu_2\text{–gal}}(\mathscr{C})\xrightarrow{\overline{(-)}}\mathrm{Pic}_n^0\to\mathrm{Pic}_n^{0,\mathrm{alg}}$$

is a group homomorphism. This is in contrast to the fact that $\overline{(-)}$ itself is not; see Example 3.28.

From Proposition 5.26 we deduce the following:

Proposition 5.28 *The composition*

$$\mathrm{hom}_c(\mathbb{Z}_2^\times, \mu_2) \xrightarrow{\chi^*} \mathrm{hom}_c(\mathbb{G}_n, \mu_2) \xrightarrow{\Xi} \mathrm{Pic}_n^0$$

is injective.

Proof It suffices to show that composing further with $\mathrm{Pic}_n^0 \to \mathrm{Pic}_n^{\mathrm{alg},0}$ yields an injective map. By Proposition 5.26, this reduces to showing that the composition

$$H_c^1(\mathbb{Z}_2^\times; \mu_2) \xrightarrow{\chi^*} H_c^1(\mathbb{G}_n; \mu_2) \to H_c^1(\mathbb{G}_n; \pi_0 E_n^\times)$$

is injective. Let $N \lhd \mathbb{G}_n$ denote the kernel of the cyclotomic character $\chi\colon \mathbb{G}_n \twoheadrightarrow \mathbb{Z}_2^\times$. By Proposition 5.12, we have $\mu_2 \subseteq \mathbb{Z}_2^\times = (\pi_0 E_n^\times)^N$, so this composition fits into the commutative diagram

$$\begin{tikzcd}
H_c^1(\mathbb{Z}_2^\times; \mu_2) \arrow[r] \arrow[d, "\chi^*"'] & H_c^1(\mathbb{Z}_2^\times; \mathbb{Z}_2^\times) \arrow[r, equal] & H_c^1(\mathbb{Z}_2^\times; (\pi_0 E_n^\times)^N) \arrow[d, "\chi^*"] \\
H_c^1(\mathbb{G}_n; \mu_2) \arrow[rr] \arrow[dr] & & H_c^1(\mathbb{G}_n; (\pi_0 E_n^\times)^N) \arrow[dl] \\
& H_c^1(\mathbb{G}_n; \pi_0 E_n^\times) &
\end{tikzcd}$$

The top left horizontal map is injective because the residual action of $\mathbb{G}_n/N = \mathbb{Z}_2^\times$ on $(\pi_0 E_n^\times)^N = \mathbb{Z}_2^\times$ is trivial. The composition of the right vertical map χ^* with the right diagonal map is the inflation map

$$H_c^1(\mathbb{Z}_2^\times; (\pi_0 E_n^\times)^N) \to H_c^1(\mathbb{G}_n; \pi_0 E_n^\times).$$

The injectivity of this map is part of the inflation-restriction exact sequence in (continuous) group cohomology. It follows that the composition of the left vertical map χ^* and the left diagonal map is injective as well. □

In concrete terms, we have

$$\mathrm{hom}_c(\mathbb{Z}_2^\times, \mu_2) \simeq \mathrm{hom}_c((\mathbb{Z}/8)^\times, \mu_2) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2.$$

The three nonzero elements correspond to the $\mathbb{Z}/2$–Galois subextensions of the $(\mathbb{Z}/8)^\times$–Galois cyclotomic extension $\mathbb{S}_{K(n)}[\omega_8^{(n)}]$, which we denote by R_1, R_2 and R_3. The zero element corresponds of course to the split $\mathbb{Z}/2$–Galois extension $R_0 := \prod_{\mu_2} \mathbb{S}_{K(n)}$.

Definition 5.29 For $i = 0, \dots, 3$, we define the Picard objects $W_i := \overline{R}_i \in \mathrm{Pic}_n^0$.

Proposition 5.28 implies that $W_0 (= \mathbb{S}_{K(n)})$, W_1, W_2 and W_3 are all different. We shall now show further that all of their (de)suspensions are different as well.

Proposition 5.30 *The various (de)suspensions of $W_0 (= \mathbb{S}_{K(n)})$, W_1, W_2 and W_3 are all different elements of* Pic_n.

Proof We need to show that if $\Sigma^{k_i} W_i = \Sigma^{k_j} W_j$, then $i = j$ and $k_i = k_j$. By (de)suspending, we may assume that $k_j = 0$, and by Proposition 5.28, it suffices to show that we must have $k_i = 0$ as well. Let $k = k_i$ and let $R = R_i$ and $R' = R_j$. By Lemma 5.25, we have

$$E_n \simeq E_n \otimes \overline{R}' \simeq E_n \otimes \Sigma^k \overline{R} \simeq \Sigma^k E_n$$

as E_n–modules. Thus, we get that $k = 2m$ for some $m \in \mathbb{Z}$. To show that m must be zero, we shall consider the image of $\Sigma^{2m} \overline{R}$ in $\mathrm{Pic}_n^{\mathrm{alg},0}$. More specifically, since the center $\mathbb{Z}_2^\times \leq \mathbb{G}_n$ acts trivially on $\pi_0 E_n^\times$ (see [6, Section 3.2.2]), restriction along its inclusion into $\mathbb{G}_n$ is a map of the form

$$\theta_{(-)} \colon \mathrm{Pic}_n^{\mathrm{alg},0} \simeq H_c^1(\mathbb{G}_n; \pi_0 E_n^\times) \to H_c^1(\mathbb{Z}_2^\times; \pi_0 E_n^\times) \simeq \hom_c(\mathbb{Z}_2^\times, \pi_0 E_n^\times).$$

Every element of the center $a \in \mathbb{Z}_2^\times \lhd \mathbb{G}_n$ acts on the polynomial generator $u \in \pi_2(E_n)$ by multiplication $u \mapsto au$; see [6, Section 3.2.2]. Thus, the object $\pi_{2m} E_n \in \mathrm{Pic}_n^{\mathrm{alg},0}$ is mapped to

$$\theta_{\pi_{2m}(E_n)} = (-)^{-m} \colon \mathbb{Z}_2^\times \to \mathbb{Z}_2^\times \subseteq \pi_0 E_n^\times.$$

Since we have

$$\pi_0(E_n \otimes \Sigma^{2m} \overline{R}) \simeq (\pi_{2m} E_n) \otimes_{\pi_0 E_n} \pi_0(E_n \otimes \overline{R}),$$

we get

$$\theta_{\Sigma^{2m} \overline{R}}(a) = a^{-m} \theta_{\overline{R}}(a) \quad \text{for all } a \in \mathbb{Z}_2^\times.$$

If $\theta_{\Sigma^{2m} \overline{R}}$ were to be equal to $\theta_{\overline{R}'}$, it would in particular have to factor through the finite group $\mu_2 \subseteq \mathbb{Z}_2^\times$. However, this cannot happen unless $m = 0$. □

5.4 Telescopic lifts

We can now combine the results of the previous subsections to deduce the main results of the paper regarding the Galois extensions and Picard groups of the telescopic categories $\mathrm{Sp}_{T(n)}$. Recall that by Remark 2.3 in higher semiadditive ∞–categories such as $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$ all finite Galois extensions are automatically faithful. First, we have:

Theorem 5.31 *Let G be a finite abelian group. For every G–Galois extension R in $\mathrm{Sp}_{K(n)}$, there exists a G–Galois extension R^f in $\mathrm{Sp}_{T(n)}$ such that $L_{K(n)} R^f \simeq R$.*

Proof By Proposition 5.17, the abelian Galois extensions of $\mathrm{Sp}_{K(n)}$ are classified by the group $\mathbb{G}_n^{\mathrm{ab}} \simeq \widehat{\mathbb{Z}} \times \mathbb{Z}_p^\times$, through the homomorphism

$$\chi_{\mathrm{tot}} \colon \mathbb{G}_n \twoheadrightarrow \widehat{\mathbb{Z}} \times \mathbb{Z}_p^\times.$$

Thus, it suffices to show that the Galois extensions corresponding to the finite quotients

$$\mathbb{G}_n \twoheadrightarrow \widehat{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}/m \quad \text{and} \quad \mathbb{G}_n \twoheadrightarrow \mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/p^r)^\times$$

can be lifted to $\mathrm{Sp}_{T(n)}$. For the first kind, we can take $L_{T(n)}\mathbb{S}W(\mathbb{F}_{p^m})$, which is Galois by the nil-conservativity of $L_{K(n)}\colon \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{K(n)}$ (see [14, Proposition 5.1.15]) and Propositions 5.13 and 2.9. For the second kind, it follows from Proposition 5.2 that we can take $\mathbb{S}_{T(n)}[\omega^{(n)}_{p^r}]$. □

The proof of Theorem 5.31 shows in fact a bit more. Namely, that the telescopic lifts of the abelian Galois extensions in $\mathrm{Sp}_{K(n)}$ can be chosen in a "compatible way". In the language of [38], the situation can be described as follows. The functor $L_{K(n)}\colon \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{K(n)}$ induces a continuous homomorphism on weak Galois groups [38, Definition 6.8]

$$\pi_1^{\mathrm{weak}}(\mathrm{Sp}_{K(n)}) \to \pi_1^{\mathrm{weak}}(\mathrm{Sp}_{T(n)})$$

and after passing to abelianizations, this homomorphism admits a left-inverse. Hence, $\pi_1^{\mathrm{weak}}(\mathrm{Sp}_{T(n)})^{\mathrm{ab}}$ contains

$$\pi_1^{\mathrm{weak}}(\mathrm{Sp}_{K(n)})^{\mathrm{ab}} \simeq \widehat{\mathbb{Z}} \times \mathbb{Z}_p^\times$$

as a direct summand.

Consider now the telescopic Picard group $\mathrm{Pic}_n^f := \mathrm{Pic}(\mathrm{Sp}_{T(n)})$ and its subgroup $\mathrm{Pic}_n^{f,0} \le \mathrm{Pic}_n^f$ of objects that map to Pic_n^0 under $K(n)$–localization. When p is odd, the cyclotomic extension $\mathbb{S}_{T(n)}[\omega_p^{(n)}]$ provides us with the following:

Theorem 5.32 *For every $n \ge 1$ and an odd prime p, there exists a $Z_n^f \in \mathrm{Pic}_n^{f,0}[p-1]$ such that $L_{K(n)}Z_n^f \simeq Z_n$; see Definition 5.22. In particular, $\mathrm{Pic}_n^{f,0}[p-1]$ contains $\mathrm{Pic}_n^0[p-1] \simeq \mathbb{Z}/(p-1)$ as a direct summand.*

Proof We define Z_n^f to be the image of the $\mathbb{Z}/(p-1)$–Galois extension $\mathbb{S}_{T(n)}[\omega_p^{(n)}]$ under the map of Proposition 3.23. By the naturality with respect to the functor $L_{K(n)}\colon \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{K(n)}$, we have $L_{K(n)}Z_n^f \simeq Z_n$. In view of Proposition 5.23, this provides a section to the map

$$\mathrm{Pic}_n^{f,0}[p-1] \to \mathrm{Pic}_n^0[p-1] \simeq \mathbb{Z}/(p-1),$$

which proves the last claim. □

In the case $p=2$, the cyclotomic extension $\mathbb{S}_{T(n)}[\omega_8^{(n)}]$ provides the following:

Theorem 5.33 *For every $n \ge 1$ and $p=2$, there exist objects $W_1^f, W_2^f, W_3^f \in \mathrm{Pic}_n^{f,0}$ such that $L_{K(n)}W_i^f = W_i$; see Definition 5.29. In particular, all the (de)suspensions of the W_i^f are different and nontrivial.*

Proof Let $R_1^f, R_2^f, R_3^f \in \mathrm{Pic}_n^f$ be the nontrivial $\mathbb{Z}/2$–Galois subextensions of the $(\mathbb{Z}/8)^\times$–Galois cyclotomic extension $\mathbb{S}_{T(n)}[\omega_8^{(n)}]$, corresponding to the three order 2 subgroups of

$$(\mathbb{Z}/8)^\times \simeq \mathbb{Z}/2 \times \mathbb{Z}/2.$$

We define $W_i^f = \overline{R_i^f} \in \mathrm{Pic}_n^f$ for $i=1,2,3$. Since $L_{K(n)}W_i^f \simeq W_i$, the last claim follows from Proposition 5.30. □

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