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Witt vectors and Tambara functors☆

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Abstract

We give a combinatorial description of the ring of G-Witt vectors on a polynomial algebra over the integers for every finite group G. Using this description we show that the functor, which takes a commutative ring with trivial action of G to its ring of Witt vectors, coincides with the left adjoint of the algebraic functor from the category of G-Tambara functors to the category of commutative rings with an action of G.

Keywords: Witt vectors; Mackey functors; Tambara functors; Colored theories

0. Introduction

In [16] Witt constructed an endofunctor on the category of commutative rings, which takes a commutative ring A to the ring $\mathbb{W}_p(A)$ of p-typical Witt vectors. This construction can be used to construct field extensions of the p-adic numbers [13], and it is essential in the construction of crystalline cohomology [2]. In [7] Dress and Siebeneicher constructed an endofunctor $\mathbb{W}_G$ on the category of commutative rings for every pro-finite group G. In the case where $G = \widehat{C}_p$ is the pro-p-completion of the infinite cyclic group the functors $\mathbb{W}_p$ and $\mathbb{W}_G$ agree. The functor $\mathbb{W}_G$ is constructed in such a way that $\mathbb{W}_G(\mathbb{Z})$ is an appropriately completed Burnside ring for the pro-finite group G. For an arbitrary commutative ring A, the ring $\mathbb{W}_G(A)$ is somewhat mysterious, even when $G = \widehat{C}_p$. The first aim of the present paper is to give a new description of

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the ring $\mathbb{W}_G(A)$ when A is a polynomial algebra over the integers. In the special case where G is finite our description is given in terms of Tambara's category U^G described in [15] and in Section 1. The following theorem is a special case of Theorem 23.

Theorem A. *Let G be a finite group and let X be a finite set with trivial action of G. The ring $U^G(X, G/e)$ is the polynomial ring $\mathbb{Z}[X]$ over $\mathbb{Z}$, with one indeterminate for each element in X, and the ring $\mathbb{W}_G(U^G(X, G/e)) = \mathbb{W}_G(\mathbb{Z}[X])$ is naturally isomorphic to a subring $\widetilde{U}^G(X, G/G)$ of the ring $U^G(X, G/G)$.*

The construction of the rings $U^G(X, G/G)$ and $\widetilde{U}^G(X, G/G)$ is similar to the construction of the Burnside ring for G. In particular, it involves group-completion. Since the underlying set of $\mathbb{W}_G(\mathbb{Z}[X])$ is a product of copies of $\mathbb{Z}[X]$ the above theorem can be considered as a computation of the underlying set of $\widetilde{U}^G(X, G/G)$. On the other hand, the ring-structure of $\widetilde{U}^G(X, G/G)$ is described by a simple combinatorial construction, and the theorem can be viewed as a combinatorial description of the ring-structure on the ring $\mathbb{W}_G(\mathbb{Z}[X])$. Our combinatorial description differs from the ones given by Metropolis and Rota [11], Graham [9] and Dress and Siebeneicher [8]. It incorporates the additional structure making the Witt ring construction into a Tambara functor.

Our second aim is to advertise the category of G-Tambara functors, that is, the category $[U^G, \mathcal{E}ns]_0$ of set-valued functors on U^G preserving finite products. (Tambara calls such a functor a TNR-functor, an acronym for "functor with trace, norm and restriction".) This category is intimately related to the Witt vectors of Dress and Siebeneicher. In order to explain this relation we consider the full subcategory U^{fG} of U^G with free G-sets as objects and the category of fG-Tambara functors, that is, the category $[U^{fG}, \mathcal{E}ns]_0$ of set-valued functors on U^{fG} preserving finite products. The functor $R \mapsto R(G/e)$ is an equivalence between the category of fG-Tambara functors and the category of commutative rings with an action of G through ring-automorphisms. We shall explain in Section 2 that the categories U^G and U^{fG} are colored theories in the sense of Boardman and Vogt [3]. As a consequence the forgetful functor $[U^G, \mathcal{E}ns]_0 \to [U^{fG}, \mathcal{E}ns]_0$ induced by the inclusion $j_G : U^{fG} \subseteq U^G$ has a left adjoint functor, which we shall denote by L_G.

Theorem B. *Let G be a finite group and let R be an fG-Tambara functor. If G acts trivially on $R(G/e)$ then, for every subgroup H of G, there is an isomorphism $\mathbb{W}_H(R(G/e)) \cong (L_G R)(G/H)$.*

We are also able to describe the ring $(L_G R)(G/H)$ in the case where G acts non-trivially on $R(G/e)$.

Theorem C. *Let G be a finite group. There is an epimorphism*

$$t : \mathbb{W}_H(R(G/e)) \to (L_G R)(G/H),$$

natural in the fG-Tambara functor R, whose kernel $\mathbb{I}_H(R(G/e))$ is explicitly described in Section 3.

There is a rich supply of G-Tambara functors coming from equivariant stable homotopy theory. In fact, every E_∞ ring G-spectrum gives rise to a G-Tambara functor by taking the zeroth homotopy group [5]. In the case where $G = e$ is the trivial group, the category of G-Tambara functors is equivalent to the category of commutative rings. It is well known that every commutative ring can be realized as the zeroth homotopy group of an E_∞ ring-spectrum. For an arbitrary finite group G one may speculate whether every G-Tambara functor can be realized as the zeroth homotopy group of an E_∞ ring-spectrum with an action of G.

The paper is organized as follows: In Section 1 we have collected some of the results from the papers [7,15] that we need in the rest of the paper. In Section 2 we note that the category of Tambara functors is the category of algebras for a colored theory. In Section 3 we construct a homomorphism relating Witt vectors and Tambara functors, which we have chosen to call the Teichmüller homomorphism because it is similar to the classical Teichmüller character. In Section 4 we prove the fundamental fact that the Teichmüller homomorphism is a ring-homomorphism. In Section 5 we prove that for free Tambara functors, the Teichmüller homomorphism is an isomorphism, and finally in Section 6 we prove Theorem C.

1. Prerequisites

In this section we fix some notation and recollect results from [7,15]. All rings are supposed to be both commutative and unital. Given a group G we only consider left actions of G. A G-ring is a ring with an action of G through ring-automorphisms.

Given a pro-finite group G we let $\mathcal{O}(G)$ denote the G-set of open subgroups of G with action given by conjugation and we let $\underline{\mathcal{O}}(G)$ denote the set of conjugacy classes of open subgroups of G. For a G-set X and a subgroup H of G we define $|X^H|$ to be the cardinality of the set X^H of H-invariant elements of X. The following is the main result of [7].

Theorem 1. *Let G be a pro-finite group. There exists a unique endofunctor $\mathbb{W}_G$ on the category of rings such that for a ring A the ring $\mathbb{W}_G(A)$ has the set $A^{\underline{\mathcal{O}}(G)}$ of maps from the set $\underline{\mathcal{O}}(G)$ to A as underlying set, in such a way that for every ring-homomorphism $h : A \to A'$ and every $x \in \mathbb{W}_G(A)$ one has $\mathbb{W}_G(h)(x) = h \circ x$, while for any subgroup U of G the family of G-maps*

$$\phi_U^A : \mathbb{W}_G(A) \to A$$

defined by

$$x = (x_V)'_{V \leqslant G} \mapsto \sum_{U \lesssim V \leqslant G}{}' |(G/V)^U| \cdot x_V^{(V:U)}$$

provides a natural transformation from the functor $\mathbb{W}_G$ into the identity functor. Here $U \lesssim V$ means that the subgroup U of G is sub-conjugate to V, i.e., there exists some $g \in G$ with $U \leqslant gVg^{-1}$, $(V : U)$ means the index of U in gVg^{-1} which coincides with $(G : U)/(G : V)$ and therefore is independent of g, and the symbol "$\sum'$" is meant to *indicate that for each conjugacy class of subgroups V with $U \lesssim V$ exactly one summand is taken. An element $a \in \mathbb{W}_G(A)$ is written on the form $a = (a_V)'_{V \leqslant G}$, where the prime means that $a_V = a_{gVg^{-1}}$ for $g \in G$.*

In Section 4 we give a slightly modified version of Dress and Siebeneicher's proof of Theorem 1 because it contains some of the main ingredients for our proof of Theorem C.

The rest of this section is a recollection of the work [15] of Tambara. We let $\mathcal{F}in$ denote the category of finite sets and we let $\mathcal{F}in^G$ denote the category of finite G-sets. Given a finite G-set X we denote by $\mathcal{F}in^G/X$ the category of objects over X in $\mathcal{F}in^G$. Given $f : X \to Y$ in $\mathcal{F}in^G$ the pull-back functor

$$\mathcal{F}in^G/Y \to \mathcal{F}in^G/X, \quad (B \to Y) \mapsto (X \times_Y B \to X)$$

has a right adjoint

$$\Pi_f : \mathcal{F}in^G/X \to \mathcal{F}in^G/Y, \quad (A \xrightarrow{p} X) \mapsto (\Pi_f A \xrightarrow{\Pi_f p} Y),$$

where $\Pi_f p$ is made from p as follows. For each $y \in Y$, the fiber $(\Pi_f p)^{-1}(y)$ is the set of maps $s : f^{-1}(y) \to A$ such that $p(s(x)) = x$ for all $x \in f^{-1}(y)$. If $g \in G$ and $s \in (\Pi_f p)^{-1}(y)$, the map ${}^g s : f^{-1}(gy) \to A$ taking x to $gs(g^{-1}x)$ belongs to $(\Pi_f p)^{-1}(gy)$. The operation $(g, s) \mapsto {}^g s$ makes $\Pi_f A$ a G-set and $\Pi_f p$ a G-map.

There is a commutative diagram of the form

$$\begin{tikzcd}
X \arrow[d, "f"'] & A \arrow[l, "p"'] & X \times_Y \Pi_f A \arrow[l, "e"'] \arrow[d, "f'"'] \\
Y & \Pi_f A \arrow[l, "\Pi_f p"'] \arrow[r, equal] & \Pi_f A,
\end{tikzcd}$$

where f' is the projection and e is the evaluation map $(x, s) \mapsto s(x)$. A diagram in $\mathcal{F}in^G$ which is isomorphic to a diagram of the above form is called an *exponential diagram*.

We say that two diagrams $X \leftarrow A \to B \to Y$ and $X \leftarrow A' \to B' \to Y$ in $\mathcal{F}in^G$ are equivalent if there exist G-isomorphisms $A \to A'$, $B \to B'$ making the diagram

$$\begin{tikzcd}
X \arrow[d, equal] & A \arrow[l] \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & Y \arrow[d, equal] \\
X & A' \arrow[l] \arrow[r] & B' \arrow[r] & Y
\end{tikzcd}$$

commutative, and we let $U_+^G(X, Y)$ be the set of the equivalence classes $[X \leftarrow A \to B \to Y]$ of diagrams $X \leftarrow A \to B \to Y$.

Tambara defines an operation $\circ : U_+^G(Y,Z) \times U_+^G(X,Y) \to U_+^G(X,Z)$ by

$$[Y \leftarrow C \to D \to Z]\circ[X \leftarrow A \to B \to Y] = [X \leftarrow A'' \to \widetilde{D} \to Z],$$

where the maps on the right are composites of the maps in the diagram

$$\begin{tikzcd}
X & A \arrow[l] \arrow[d] & A' \arrow[l] \arrow[d] & A'' \arrow[l] \arrow[d] \\
& B \arrow[d] & B' \arrow[l] \arrow[d] & \widetilde{C} \arrow[l] \arrow[d, equal] \\
& Y & C \arrow[l] \arrow[d] & \widetilde{C} \arrow[d] \\
& Z & D \arrow[l] & \widetilde{D}.\arrow[l]
\end{tikzcd}$$

Here the three squares are pull-back diagrams and the diagram

$$\begin{tikzcd}
C \arrow[d] & B' \arrow[l] & \widetilde{C} \arrow[l] \arrow[d] \\
D & \widetilde{D} \arrow[l] \arrow[r, equal] & \widetilde{D}
\end{tikzcd}$$

is an exponential diagram. He verifies that U_+^G is a category with $\circ$ as composition and given $f : X \to Y$ in $\mathcal{F}in^G$ he introduces the notation

$$R_f = [Y \xleftarrow{f} X \xrightarrow{=} X \xrightarrow{=} X],$$

$$T_f = [X \xleftarrow{=} X \xrightarrow{=} X \xrightarrow{f} Y]$$

and

$$N_f = [X \xleftarrow{=} X \xrightarrow{f} Y \xrightarrow{=} Y].$$

Every morphism in U_+^G is a composition of morphisms on the above form. He also shows:

Proposition 2. *Given objects X and Y in U_+^G, there is semi-ring-structure on $U_+^G(X,Y)$ given as follows*:

$$0 = [X \leftarrow \emptyset \to \emptyset \to Y],$$

$$1 = [X \leftarrow \emptyset \to Y \to Y],$$

$$[X \leftarrow A \rightarrow B \rightarrow Y] + [X \leftarrow A' \rightarrow B' \rightarrow Y]$$
$$= [X \leftarrow A \amalg A' \rightarrow B \amalg B' \rightarrow Y]$$

and

$$[X \leftarrow A \rightarrow B \rightarrow Y] \cdot [X \leftarrow A' \rightarrow B' \rightarrow Y]$$
$$= [X \leftarrow B \times_Y A' \amalg A \times_Y B' \rightarrow B \times_Y B' \rightarrow Y].$$

It is also shown in [15] that there is a unique category U^G satisfying the following conditions:

(i) $\mathrm{ob}\, U^G = \mathrm{ob}\, U^G_+$.
(ii) The morphism set $U^G(X, Y)$ is the group completion of the underlying additive monoid of $U^G_+(X, Y)$.
(iii) The group completion maps $k : U^G_+(X, Y) \rightarrow U^G(X, Y)$ and the identity on $\mathrm{ob}(U^G_+)$ form a functor $k : U^G_+ \rightarrow U^G$.
(iv) The functor k preserves finite products.

Proposition 3. (i) *If* $X_1 \xrightarrow{i_1} X \xleftarrow{i_2} X_2$ *is a sum diagram in* $\mathcal{F}in^G$, *then* $X_1 \xleftarrow{R_{i_1}} X \xrightarrow{R_{i_2}} X_2$ *is a product diagram in* U^G *and* $\emptyset$ *is final in* U^G.

(ii) *Let* X *be a* G*-set and* $\nabla : X \amalg X \rightarrow X$ *the folding map,* $i : \emptyset \rightarrow X$ *the unique map. Then* X *has the structure of a ring object of* U^G *with addition* T_∇, *additive unit* T_i, *multiplication* N_∇ *and multiplicative unit* N_i.

(iii) *If* $f : X \rightarrow Y$ *is a* G*-map, then the morphisms* R_f, T_f *and* N_f *of* U^G *preserve the above structures of ring, additive group and multiplicative monoid on* X *and* Y, *respectively*.

Given a category $\mathcal{C}$ with finite products, we shall denote the category of set-valued functors on $\mathcal{C}$ preserving finite products by $[\mathcal{C}, \mathcal{E}ns]_0$. The morphisms in $[\mathcal{C}, \mathcal{E}ns]_0$ are given by natural transformations.

Definition 4. The category of *G-Tambara functors* is the category $[U^G, \mathcal{E}ns]_0$.

Given a G-Tambara functor S and $[X \leftarrow A \rightarrow B \rightarrow Y] \in U^G(X, Y)$ we obtain a function $S[X \leftarrow A \rightarrow B \rightarrow Y] : S(X) \rightarrow S(Y)$. Since S is product-preserving, it follows from (ii) of Proposition 3 that $S(X)$ is a ring. Given a finite G-map $f : X \rightarrow Y$ we shall use the notation $S^*(f) = S(R_f)$, $S_+(f) = S(T_f)$ and $S_\bullet(f) = S(N_f)$. It follows from (iii) of Proposition 3 that $S^*(f)$ is a ring-homomorphism, that $S_+(f)$ is an additive homomorphism and that $S_\bullet(f)$ is multiplicative. A G-Tamara functor S is uniquely determined by the functions $S^*(f)$, $S_+(f)$ and $S_\bullet(f)$ for all $f : X \rightarrow Y$ in $\mathcal{F}in^G$.

Given subgroups $K \leqslant H \leqslant G$ we shall denote by $\pi^H_K : G/K \rightarrow G/H$ the projection induced by the inclusion $K \leqslant H$, and given $g \in G$ we shall let $c_g : G/H \rightarrow G/gHg^{-1}$ denote conjugation by g, $c_g(\sigma H) = \sigma g^{-1}(gHg^{-1})$.

2. Colored theories

In this section we shall explain that the category U^G is an $\underline{\mathcal{O}}(G)$-colored category in the sense of Boardman and Vogt [3].

Definition 5 (*Boardman and Vogt [3,2.3]*). (i) Let $\mathcal{O}$ be a finite set. An *$\mathcal{O}$-colored theory* is a category Θ together with a faithful functor $\sigma_\Theta : (\mathcal{F}in/\mathcal{O})^{\mathrm{op}} \to \Theta$ such that firstly σ_Θ preserves finite products and secondly every object of Θ is isomorphic to an object in the image of σ_Θ.

(ii) The *category of algebras* over a theory Θ is the category $[\Theta, \mathcal{E}ns]_0$ of product-preserving set-valued functors on Θ.

(iii) A morphism $\gamma : \Theta \to \Psi$ of colored theories is a functor preserving finite products together with a function $f : \mathcal{O} \to \mathcal{O}'$ such that $\gamma \circ \sigma_\Theta = \sigma_\Psi \circ f_*$.

Other authors, e.g. [1,2], use the name "sorted theory" for a colored theory.

Given a finite group G, choosing representatives G/H for the objects of $\underline{\mathcal{O}}(G)$, we can construct a functor

$$\sigma_{(\mathcal{F}in^G)^{\mathrm{op}}} : (\mathcal{F}in/\underline{\mathcal{O}}(G))^{\mathrm{op}} \to (\mathcal{F}in^G)^{\mathrm{op}},$$

$$(z : Z \to \underline{\mathcal{O}}(G)) \mapsto \coprod_{[G/H]\in\underline{\mathcal{O}}(G)} G/H \times z^{-1}([G/H]).$$

This way we give $(\mathcal{F}in^G)^{\mathrm{op}}$ the structure of an $\underline{\mathcal{O}}(G)$-colored theory. Composing $\sigma_{(\mathcal{F}in^G)^{\mathrm{op}}}$ with the functor $R : (\mathcal{F}in^G)^{\mathrm{op}} \to U^G$, $f \mapsto R_f$ we obtain a functor σ_{U^G} : $(\mathcal{F}in/\underline{\mathcal{O}}(G))^{\mathrm{op}} \to U^G$ preserving finite products by (i) of Proposition 3, making U^G an $\underline{\mathcal{O}}(G)$-colored theory and R a morphism of $\underline{\mathcal{O}}(G)$-colored theories.

Let $V^G \subseteq U^G$ denote the subcategory of U^G with the same class of objects as U^G and with $V^G(X,Y) \subseteq U^G(X,Y)$ the subgroup generated by morphisms of the form $[X \leftarrow A \xrightarrow{=} A \to Y]$. The inclusion $V^G \subseteq U^G$ preserves finite products, and we have morphisms $(\mathcal{F}in^G)^{\mathrm{op}} \to V^G \to U^G$ of $\underline{\mathcal{O}}(G)$-colored theories. The category V^G is strongly related to the category *spans* considered by Lindner in [10], and, in fact, the category of Mackey functors in the sense of Dress [6] is equal to the category of V^G-algebras.

Let $\mathcal{F}in^{fG}$ denote the full subcategory of $\mathcal{F}in^G$ with finite free G-sets as objects. The functor

$$\sigma_{(\mathcal{F}in^{fG})^{\mathrm{op}}} : \mathcal{F}in^{\mathrm{op}} \to (\mathcal{F}in^{fG})^{\mathrm{op}},$$

$$Z \mapsto G/e \times Z$$

gives $(\mathcal{F}in^{fG})^{\mathrm{op}}$ the structure of a theory. Similarly the full subcategories $U^{fG} \subseteq U^G$ and $V^{fG} \subseteq V^G$ with finite free G-sets as objects are colored theories. We have the following diagram of morphisms of colored theories:

$$\begin{array}{ccccc} (\mathcal{F}in^{fG})^{\mathrm{op}} & \longrightarrow & V^{fG} & \longrightarrow & U^{fG} \\ \downarrow & & \downarrow & & \downarrow \\ (\mathcal{F}in^{G})^{\mathrm{op}} & \longrightarrow & V^{G} & \longrightarrow & U^{G}, \end{array}$$

where the vertical functors are inclusions of full subcategories.

Lemma 6. (i) *The category* $[U^{fG}, \mathcal{E}ns]_0$ *of* fG*-Tambara functors is equivalent to the category of* G*-rings.*

(ii) *The category* $[V^{fG}, \mathcal{E}ns]_0$ *of* V^{fG}*-algebras is equivalent to the category of left* $\mathbb{Z}[G]$*-modules.*

(iii) *The category* $[(\mathcal{F}in^{fG})^{\mathrm{op}}, \mathcal{E}ns]_0$ *of* $(\mathcal{F}in^{fG})^{\mathrm{op}}$*-algebras is equivalent to the category of* G*-sets.*

Proof. Since the statements have similar proofs we only give the proof of (i). Given an fG-Tambara functor R, we construct a G-ring-structure on $A = R(G/e)$. Indeed by (ii) of Proposition 3 $R(G/e)$ is a ring, and given $g \in G$ the right multiplication $g : G/e \to G/e$, $x \mapsto xg$, induces a ring-automorphism $R^*(g^{-1})$ of $A = R(G/e)$. From the functoriality of R we obtain that A is a G-ring. Conversely, given a G-ring A' we shall construct an fG-Tambara functor R'. We define $R'(X)$ to be the set of G-maps from X to A'. Given $[X \xleftarrow{d} A \xrightarrow{f} B \xrightarrow{g} Y] \in U^G(X, Y)$, we define $R'[X \xleftarrow{d} A \xrightarrow{f} B \xrightarrow{g} Y] : R'(X) \to R'(Y)$ by the formula

$$R'[X \xleftarrow{d} A \xrightarrow{f} B \xrightarrow{g} Y](\phi)(y) = \sum_{b \in g^{-1}(y)} \left(\prod_{a \in f^{-1}(b)} \phi(d(a)) \right)$$

for $\phi \in R'(X)$ and $y \in Y$. We leave it to the reader to check that $R \mapsto A$ and $A' \mapsto R'$ are inverse functors up to isomorphism. $\square$

We refer to [12, Propositions 4.3 and 4.7] for a proof of the following two results. Alternatively, the reader may modify the proofs given in [4, 3.4.5 and 3.7.7] for their monochrome versions.

Proposition 7. *Let* Θ *be an* $\mathcal{O}$*-colored theory. The category of* Θ*-algebras is complete and cocomplete.*

Proposition 8. *Given a morphism* $\gamma : \Theta \to \Psi$ *of colored theories, the functor* $\gamma^* : [\Psi, \mathcal{E}ns]_0 \to [\Theta, \mathcal{E}ns]_0$, $A \mapsto A \circ \gamma$ *has a left adjoint* $\gamma_* : [\Theta, \mathcal{E}ns]_0 \to [\Psi, \mathcal{E}ns]_0$.

Definition 9. The category $[U^{fG}, \mathcal{E}ns]_0$ of U^{fG}-algebras is the category of *fG-Tambara functors.*

We let $L_G = j_{G*} : [U^{fG}, \mathcal{E}ns]_0 \to [U^G, \mathcal{E}ns]_0$ denote the left adjoint of the functor $j_G^* : [U^G, \mathcal{E}ns]_0 \to [U^{fG}, \mathcal{E}ns]_0$ induced by the inclusion $j_G : U^{fG} \subseteq U^G$. Note that L_G can be constructed as the left Kan extension along j_G, and that for $R \in [U^{fG}, \mathcal{E}ns]_0$, we have an isomorphism $(L_G R)(X) \cong R(X)$ for every finite free G-set X because U^{fG} is a full subcategory of U^G.

3. The Teichmüller homomorphism

We shall now give a connection between the category of G-Tambara functors and the category of rings with an action of a finite group G. Throughout this section we fix a G-Tambara functor S.

Definition 10. We call the ring-homomorphism

$$t : \mathbb{W}_G(S(G/e)) \to S(G/G), \quad (x_U)'_{U \leqslant G} \mapsto \sum_{U \leqslant G}' S_+(\pi_U^G) S_\bullet(\pi_e^U)(x_U)$$

the *unrestricted Teichmüller homomorphism*.

We shall prove the following proposition in the next section.

Proposition 11. *The unrestricted Teichmüller homomorphism*

$$t : \mathbb{W}_G(S(G/e)) \to S(G/G)$$

is a ring-homomorphism.

In general t will neither be injective nor surjective. However, in certain cases we can describe its kernel explicitly.

Definition 12. Let A be a commutative G-ring. We let $\mathbb{I}_G(A) \subseteq \mathbb{W}_G(A)$ denote the ideal generated by elements of the form $a - b$, where $a = (a_K)'_{K \leqslant G}$ and $b = (b_K)'_{K \leqslant G}$ satisfy the following condition: For every $K \leqslant G$ there exist $g_{1,K}, \dots g_{n,K}$ in the normalizer $N_K(G)$ of K in G, and $a_{1,K}, \dots, a_{n,K} \in A$, $n \geqslant 1$, such that

(1) $g_{1,K} K = \cdots = g_{n,K} K$,
(2) $a_K = a_{1,K} \cdots a_{n,K}$,
(3) $b_K = (g_{1,K} a_{1,K}) \cdots (g_{n,K} a_{n,K})$.

Let $\nabla : \coprod_1^n G/e \to G/e$ denote the fold map. For a, b and K as above we have

$$S_+(\pi_K^G) S_\bullet(\pi_e^K)(b_K) = S_+(\pi_K^G) S_\bullet(\pi_e^K)(g_{1,K} a_{1,K} \cdots g_{n,K} a_{n,K})$$

$$= S\left[\coprod_1^n G/e \xleftarrow{\coprod_{i=1}^n g_i} \coprod_1^n G/e \xrightarrow{\pi_e^K \circ \nabla} G/K \xrightarrow{\pi_K^G} G/G\right](a_{1,K},\dots,a_{n,K})$$

$$= S\left[\coprod_1^n G/e \xleftarrow{=} \coprod_1^n G/e \xrightarrow{\pi_e^K \circ \nabla} G/K \xrightarrow{\pi_K^G} G/G\right](a_{1,K},\dots,a_{n,K})$$

$$= S_+(\pi_K^G)S_\bullet(\pi_e^K)(a_{1,K}\cdots a_{n,K}) = S_+(\pi_K^G)S_\bullet(\pi_e^K)(a_K),$$

and therefore the unrestricted Teichmüller homomorphism $t : \mathbb{W}_G(S(G/e)) \to S(G/G)$ maps the ideal $\mathbb{I}_G(S(G/e))$ to zero.

Definition 13. The *Teichmüller homomorphism* is the ring-homomorphism τ: $\mathbb{W}_G(S(G/e))/\mathbb{I}_G(S(G/e)) \to S(G/G)$ induced by t.

The following theorem, proved in Section 5, implies the case $H = G$ of Theorem C.

Theorem 14. *For every fG-Tambara functor R the Teichmüller homomorphism τ : $\mathbb{W}_G((L_GR)(G/e))/\mathbb{I}_G((L_GR)(G/e)) \to (L_GR)(G/G)$ is an isomorphism. In particular, if G acts trivially on $R(G/e)$ then τ is an isomorphism of the form τ: $\mathbb{W}_G((L_GR)(G/e)) \to (L_GR)(G/G)$*

Recall that there is an isomorphism $(L_GR)(G/e) \cong R(G/e)$.

4. Witt polynomials

Theorem 15. *Let G be a finite group.*

(1) *There exist unique families $(s_U)'_{U \leqslant G}$, $(p_U)'_{U \leqslant G}$ of integral polynomials*

$$s_U = s_U^G, \quad p_U = p_U^G \in \mathbb{Z}[x_V, y_V \mid U \lesssim V \leqslant G]$$

in two times as many variables x_V, y_V $(U \lesssim V \leqslant G)$ as there are conjugacy classes of subgroups $V \leqslant G$ which contain a conjugate of U such that for every G-Tambara functor S:

$$\tau(x) + \tau(y) = \tau((s_U(x_V, y_V \mid U \lesssim V \leqslant G))'_{U \leqslant G}),$$

$$\tau(x)\cdot\tau(y) = \tau((p_U(x_V, y_V \mid U \lesssim V \leqslant G))'_{U \leqslant G})$$

for every $x = (x_U)'_{U \leqslant G}$ and $y = (y_U)'_{U \leqslant G}$ in $\mathbb{W}_G(S(G/e))$.

(2) *There exist polynomials $m_U = m_U^G \in \mathbb{Z}[a_V \mid U \lesssim V \leqslant G]$ such that for every G-Tambara functor S:*

$$-\tau(x) = \tau((m_U(x_V \mid U \lesssim V \leqslant G')_{U \leqslant G})$$

for every $x = (x_U)'_{U \leqslant G}$ in $\mathbb{W}_G(S(G/e))$.

(3) *For every subgroup H of G and every ring A we have*

$$\phi_H^A(x) + \phi_H^A(y) = \phi_H^A((s_U(x_V, y_V \mid U \lesssim V \leqslant G))'_{U \leqslant G}),$$
$$\phi_H^A(x) \cdot \phi_H^A(y) = \phi_H^A((p_U(x_V, y_V \mid U \lesssim V \leqslant G))'_{U \leqslant G})$$

for every $x = (x_U)'_{U \leqslant G}$ *and* $y = (y_U)'_{U \leqslant G}$ *in* $\mathbb{W}_G(A)$.
We shall call the polynomials s_U, p_U *and* m_U *the* Witt polynomials.

Theorem 15 is a version of [7, Theorem 3.2.1].

Proof of Theorem 1. We first consider the case where G is finite. Given a ring A, we define operations $+$ and $\cdot$ on $\mathbb{W}_G(A)$ by

$$(a_U)'_{U \leqslant G} + (b_U)'_{U \leqslant G} = (s_U(a_V, b_V \mid U \lesssim V \leqslant G))'_{U \leqslant G},$$
$$(a_U)'_{U \leqslant G} \cdot (b_U)'_{U \leqslant G} = (p_U(a_V, b_V \mid U \lesssim V \leqslant G))'_{U \leqslant G}.$$

In the case where A has no torsion, the map $\phi : \mathbb{W}_G(A) \to \prod'_{U \leqslant G} A$ with U'th component ϕ_U is injective, and hence $\mathbb{W}_G(A)$ is a sub-ring of $\prod'_{U \leqslant G} A$. In the case where A has torsion, we can choose a surjective ring-homomorphism $A' \to A$ from a torsion free ring A'. We obtain a surjection $\mathbb{W}_G(A') \to \mathbb{W}_G(A)$ respecting the operations $+$ and $\cdot$. Since $\mathbb{W}_G(A')$ is a ring we can conclude that $\mathbb{W}_G(A)$ is a ring, and by the above considerations it is uniquely determined. Given a surjective homomorphism $\gamma : G \to G'$ of finite groups we obtain a ring-homomorphism $\mathrm{restr}^G_{G'} : \mathbb{W}_G(A) \to \mathbb{W}_{G'}(A)$ with $\mathrm{restr}^G_{G'}((a_U)'_{U \leqslant G}) = ((b_V)')_{V \leqslant G'})$, where $b_V = a_{\gamma^{-1}(V)}$. (See [7, (3.3.11)].) The easiest way to see that $\mathrm{restr}^G_{G'}$ is a ring-homomorphism is to note that $(\gamma^{-1}H : \gamma^{-1}U) = (H : U)$ and that $\phi_{\gamma^{-1}H}(G/\gamma^{-1}U) = \phi_H(G'/H)$. For the case where G is a pro-finite group we note that for the ring $\mathbb{W}_G(A)$ has to be the limit $\lim_N \mathbb{W}_{G/N}(A)$ taken over all finite factor groups G/N, with respect maps on the form $\mathrm{restr}^G_{G'}$. □

Proof of Proposition 11. Proposition 11 follows from the first part of Theorem 15 because we use the Witt polynomials to define the ring-structure on the Witt vectors. □

We now turn to the proof of Theorem 15, and we fix a finite group G for the rest of this section. For the uniqueness of the Witt polynomials we consider the representable G-Tambara functor $\Omega := U^G(\emptyset, -)$ with $\Omega(G/e) = \mathbb{Z}$ and $\Omega(G/G)$ the Burnside ring for G. In [7, Theorem 2.12.7] it is shown that $\tau : \prod'_{U \leqslant G} \mathbb{Z} = \mathbb{W}_G(\Omega(G/e)) \to \Omega(G/G)$ is a bijection. Hence the Witt polynomials are unique. The following four lemmas establish the existence of Witt polynomials with the properties required in Theorem 15.

Lemma 16. *For a subset A of G, let* $U_A := \{g \in G \mid Ag = A\}$ *denote its stabilizer group and let* $i_A := |A/U_A|$ *denote the number of* U_A*-orbits in A. If the set* $\mathcal{U}(G)$ *of subsets of G is considered as a G-set via* $G \times \mathcal{U}(G) \to \mathcal{U}(G)$: $(g, A) \mapsto Ag^{-1}$*, then for any* $s, t \in S(G/e)$ *one has*

$$S_\bullet(\pi_e^G)(s+t) = \sum_{G\cdot A \in G\backslash \mathcal{U}(G)} S_+(\pi_{U_A}^G) S_\bullet(\pi_e^{U_A})(s^{i_A} \cdot t^{i_{G-A}}).$$

Proof. We let $i_1, i_2 : G/e \to G/e \amalg G/e$ denote the two natural inclusions. We have an exponential diagram

$$\begin{array}{ccccc} G/e & \xleftarrow{\nabla} & G/e \amalg G/e & \xleftarrow{d} & G/e \times \mathcal{U}(G) \\ \pi_e^G \downarrow & & & & \downarrow \mathrm{pr} \\ G/G & \longleftarrow & \mathcal{U}(G) & = & \mathcal{U}(G), \end{array}$$

where $d(g, A) = i_1(g)$ if $g^{-1} \in A$ and $d(g, A) = i_2(g)$ if $g^{-1} \notin A$. Let $Z = G/e \times A/U_A \amalg G/e \times (G - A)/U_A$. Since $\mathcal{U}(G) \cong \coprod_{G\cdot A \in G\backslash \mathcal{U}(G)} G\cdot A$, we have that

$$\begin{aligned} &S_\bullet(\pi_e^G)(s+t) \\ &\quad = S_\bullet(\pi_e^G) S_+(\nabla)(s,t) \\ &\quad = S[G/e \amalg G/e \xleftarrow{d} G/e \times \mathcal{U}(G) \longrightarrow \mathcal{U}(G) \to G/G](s,t) \\ &\quad = \sum_{G\cdot A \in G\backslash \mathcal{U}(G)} S[G/e \amalg G/e \xleftarrow{d} G/e \times GA \to GA \to G/G](s,t) \\ &\quad = \sum_{G\cdot A \in G\backslash \mathcal{U}(G)} S[G/e \amalg G/e \leftarrow Z \to G/U_A \to G/G](s,t) \\ &\quad = \sum_{G\cdot A \in G\backslash \mathcal{U}(G)} S[G/e \xleftarrow{=} G/e \xrightarrow{\pi_e^{U_A}} G/U_A \xrightarrow{\pi_{U_A}^G} G/G](s^{i_A} t^{i_{G-A}}) \\ &\quad = \sum_{G\cdot A \in G\backslash \mathcal{U}(G)} S_+(\pi_{U_A}^G) S_\bullet(\pi_e^{U_A})(s^{i_A} t^{i_{G-A}}), \end{aligned}$$

where the maps without labels are natural projections. □

Lemma 17. *With the notation of Lemma 16, we have for every subgroup U of G and for every* $s, t \in S(G/e)$:

$$(s+t)^{(G:U)} = \sum_{G\cdot A \in G\backslash \mathcal{U}(G)} |(G/U_A)^U| \cdot (s^{i_A} \cdot t^{i_{G-A}})^{(U_A:U)}.$$

Proof. We compute

$$(s+t)^{(G:U)} = \sum_{A\subseteq G/U} s^{|A|}t^{|G/U|-|A|} = \sum_{A\in\mathcal{U}(G),\, U\leqslant U_A} s^{|A/U|}t^{|(G-A)/U|}$$
$$= \sum_{G\cdot A\in G\backslash\mathcal{U}(G)} |(G/U_A)^U|\cdot(s^{i_A}\cdot t^{i_{G-A}})^{(U_A:U)}. \qquad \square$$

The following lemma is a variation on [7, Lemma 3.2.5], and the proof essentially identical to the one given in [7]. We include it for the reader's convenience.

Lemma 18. *Let $V_1,\dots,V_k\leqslant G$ be subgroups of G. For every subgroup $U\leqslant G$ and every $\varepsilon_1,\dots,\varepsilon_k\in\{\pm1\}$ there exists a polynomial $\xi_U=\xi^G_{(U;V_1,\dots,V_k;\varepsilon_1,\dots,\varepsilon_k)}\in\mathbb{Z}[x_1,\dots,x_k]$ satisfying*:

(1) *for every G-Tambara functor S and all $s_1,\dots,s_k\in S(G/e)$*:

$$\sum_{i=1}^{k}\varepsilon_i S_+(\pi^G_{V_i})S_\bullet(\pi^{V_i}_e)(s_i) = \sum_{U\leqslant G}{}' S_+(\pi^G_U)S_\bullet(\pi^U_e)(\xi_U(s_1,\dots,s_k)).$$

(2) *for every ring A, every $H\leqslant G$ and all $s_1,\dots,s_k\in A$*:

$$\sum_{i=1}^{k}\varepsilon_i|(G/V_i)^H|s_i^{(V_i:H)} = \sum_{U\leqslant G}{}' |(G/U)^H|(\xi_U(s_1,\dots,s_k))^{(U:H)}.$$

Proof. We first prove (1). If $\varepsilon_1=\varepsilon_2=\dots=\varepsilon_k=1$ and if V_i is not conjugate to V_j for $i\neq j$, then

$$\sum_{i=1}^{k}\varepsilon_i S_+(\pi^G_{V_i})S_\bullet(\pi^{V_i}_e)(s_i) = \sum\nolimits'_{U\leqslant G} S_+(\pi^G_U)S_\bullet(S^U_e)(s_U),$$

with $s_U=s_i$ if U is conjugate to V_i and $s_U=0$ if U is not conjugate to any of the $V_1,\dots,V_k$. So in this case we are done: $\xi_U(s_1,\dots,s_k)=s_U$. We prove the lemma by using triple induction. First with respect to $m_1=m_1(V_1,\dots,V_k;\varepsilon_1,\dots,\varepsilon_k)$ given by

$$m_1 := \max\{|V_i| \mid \varepsilon_i=-1 \text{ or there exists some } j\neq i \text{ with } V_j \text{ conjugate to } V_i\},$$

then with respect to

$$m_3 := |\{i \mid |V_i|=m_1 \text{ and there exists some } j\neq i \text{ with } V_j \text{ conjugate to } V_i\}|$$

and then with respect to $m_2 := |\{i \mid |V_i|=m_1 \text{ and } \varepsilon_i=-1\}|$.

We have just verified that the lemma holds in the case $m_1 = 0$. In case $m_1 > 0$ we have either $m_2 > 0$ or $m_3 > 0$. In case $m_2 > 0$, say $|V_1| = m_1$ and $\varepsilon_1 = -1$, we may use Lemma 16 with $G = V_1$, $s = -s_1$, $t = s_1$ to conclude that

$$0 = S_\bullet(\pi_e^{V_1})(0) = \sum_{V_1 \cdot A \in V_1 \backslash \mathcal{U}(V_1)} S_+(\pi_{U_A}^{V_1}) S_\bullet(\pi_e^{U_A})((-1)^{i_A} s_1^{(V_1:U_A)}).$$

Therefore, considering the two special summands $A = \emptyset$ and $A = V_1$ and putting $\mathcal{U}_0(V_1) := \{A \in \mathcal{U}(V_1) \mid A \neq \emptyset \text{ and } A \neq V_1\}$, one gets

$$-S_\bullet(\pi_e^{V_1})(s_1) = S_\bullet(\pi_e^{V_1})(-s_1) + \sum_{V_1 \cdot A \in V_1 \backslash \mathcal{U}_0(V_1)} S_+(\pi_{U_A}^{V_1}) S_\bullet(\pi_e^{U_A})((-1)^{i_A} s_1^{(V_1:U_A)}).$$

Hence, if $A_{k+1}, A_{k+2}, \ldots, A_{k'} \in \mathcal{U}_0(V_1)$ denote representatives of the V_1-orbits $V_1 A \subseteq \mathcal{U}_0(V_1)$ and we let $V_{k+1} := U_{A_{k+1}}, \ldots, V_{k'} := U_{A_{k'}}$ then $V_i \leqslant V_1$ and $V \neq V_1$ for $i \geqslant k+1$. If we put $\varepsilon_{k+1} = \cdots = \varepsilon_{k'} = 1$ and $s_{k+1} := (-1)^{i_{A_{k+1}}} s_1^{(V_1:V_{k+1})}, \ldots, s_{k'} = (-1)^{i_{A_{k'}}} s_1^{(V_1:V_{k'})}$, then the polynomial

$$\xi^G_{(U;V_1,\ldots,V_k;-1,\varepsilon_2,\ldots,\varepsilon_k)}(s_1, \ldots, s_k) := \xi^G_{(U;V_1,\ldots,V_{k'};1,\varepsilon_2,\ldots,\varepsilon_{k'})}(-s_1, s_2, \ldots, s_{k'})$$

makes the statement of the lemma hold. We can conclude that if the lemma holds for every (n_1, n_2, n_3) with either $n_1 < m_1$ or ($n_1 = m_1$, $n_2 < m_2$ and $n_3 \leqslant m_3$), then it also holds for (m_1, m_2, m_3).

Similarly, if $m_2 = 0$, but $m_3 > 0$, say V_1 is conjugate to V_2, then we may use Lemma 16 once more with $G = V_1$, $s = s_1$, and $t = s_2$ to conclude that

$$\begin{aligned} S_\bullet(\pi_e^{V_1})(s_1 + s_2) &= \sum_{V_1 \cdot A \in V_1 \backslash \mathcal{U}(V_1)} S_+(\pi_{U_A}^{V_1}) S_\bullet(\pi_e^{U_A})(s_1^{i_A} s_2^{i_{V_1 - A}}) \\ &= S_\bullet(\pi_e^{V_1})(s_1) + S_\bullet(\pi_e^{V_1})(s_2) \\ &\quad + \sum_{V_1 \cdot A \in V_1 \backslash \mathcal{U}_0(V_1)} S_+(\pi_{U_A}^{V_1}) S_\bullet(\pi_e^{U_A})(s_1^{i_A} s_2^{i_{V_1 - A}}), \end{aligned}$$

so with $V_{k+1}, \ldots, V_{k'}$ as above, but with $\varepsilon_{k+1} = \cdots = \varepsilon_{k'} = -1$ and with $s_{k+1} := s_1^{i_{A_{k+1}}} s_2^{i_{V_1 - A_{k+1}}}, \ldots, s_{k'} := s_1^{i_{A_{k'}}} s_2^{i_{V_1 - A_{k'}}}$, the polynomial

$$\xi^G_{(U;V_1,\ldots,V_k;\varepsilon_1,\ldots,\varepsilon_k)}(s_1, \ldots, s_k) := \xi^G_{(U;V_2,\ldots,V_{k'};1,\varepsilon_2,\ldots,\varepsilon_{k'})}(s_1 + s_2, s_3, \ldots, s_{k'})$$

makes the statement of the lemma hold. We can conclude that if the lemma holds for every (n_1, n_2, n_3) with either $n_1 < m_1$ or ($n_1 = m_1$, $n_2 = m_2 = 0$ and $n_3 < m_3$), then it also holds for $(m_1, 0, m_3)$. The statement of the lemma now follows by induction first on m_1, then on m_3 and finally on m_2.

The proof of (2) is similar to the proof of (1), the only difference being that we use Lemma 17 instead of Lemma 16. □

Lemma 19. *For subgroups $V, W \leqslant G$ one has the following modified Mackey formulas*:
(1) *For every G-Tambara functor S and all $s, t \in S(G/e)$*:

$$
\begin{aligned}
&S_+(\pi_V^G)S_\bullet(\pi_e^V)(s)\cdot S_+(\pi_W^G)S_\bullet(\pi_e^W)(t)\\
&\quad= \sum_{VgW \in V\backslash G/W} S_+(\pi^G_{V\cap gWg^{-1}})S_\bullet(\pi_e^{V\cap gWg^{-1}})(s^{(V:V\cap gWg^{-1})}\cdot t^{(W:g^{-1}Vg\cap W)}).
\end{aligned}
$$

(2) *For every ring A, every $s, t \in A$ and $H \leqslant G$*:

$$
\begin{aligned}
&|(G/V)^H| s^{(V:H)} |(G/W)^H| t^{(W:H)}\\
&\quad= \sum_{VgW \in V\backslash G/W} |(G/V\cap gWg^{-1})^H| (s^{(V:V\cap gWg^{-1})}\cdot t^{(W:g^{-1}Vg\cap W)})^{(V\cap gWg^{-1}:H)}.
\end{aligned}
$$

Proof. Statement (2) is [7,3.2.13]. To prove (1) consider the diagram

$$
\begin{tikzcd}
 & G/e\amalg G/e \arrow[d, "\pi_e^V\amalg\pi_e^W"'] & G/e\times G/W\amalg G/V\times G/e \arrow[l, "\mathrm{pr}_1\amalg\mathrm{pr}_2"'] \arrow[d, "\pi_e^V\times\mathrm{id}\amalg\mathrm{id}\times\pi_e^W"] \\
G/G\amalg G/G \arrow[d, "\nabla"'] & G/V\amalg G/W \arrow[l, "\pi_V^G\amalg\pi_W^G"'] & G/V\times G/W\amalg G/V\times G/W \arrow[l, "\mathrm{pr}_1\amalg\mathrm{pr}_2"'] \arrow[d, "\nabla"'] \\
G/G & G/V\times G/W \arrow[l] \arrow[r, equal] & G/V\times G/W,
\end{tikzcd}
$$

where ∇ is the fold map, the upper square is a pull-back and the lower rectangle is an exponential diagram. Concatenating with the diagram

$$
\begin{tikzcd}
G/e\times G/W\amalg G/V\times G/e \arrow[d, "\pi_e^V\times\mathrm{id}\amalg\mathrm{id}\times\pi_e^W"'] & \coprod_{gW\in G/W} G/e\amalg\coprod_{Vg\in V\backslash G} G/e \arrow[l, "\alpha_1\amalg\alpha_2"', "\cong"] \arrow[d, equal] \\
G/V\times G/W\amalg G/V\times G/W \arrow[d, "\nabla"'] & \coprod_{gW\in G/W} G/e\amalg\coprod_{Vg\in V\backslash G} G/e \arrow[d, "(\delta_1,\delta_2)"'] \\
G/V\times G/W & \coprod_{VgW\in V\backslash G/W} G/(V\cap gWg^{-1}) \arrow[l, "\gamma"', "\cong"]
\end{tikzcd}
$$

with maps defined by

$$
\begin{aligned}
\alpha_1(gW,\sigma) &= (\sigma, \sigma gW),\\
\alpha_2(Vg,\sigma) &= (\sigma g^{-1}V, \sigma),\\
\delta_1(gW,\sigma) &= (VgW, \sigma(V\cap gWg^{-1})),\\
\delta_2(Vg,\sigma) &= (VgW, \sigma g^{-1}(V\cap gWg^{-1}))
\end{aligned}
$$

and

$$\gamma(VgW, \sigma(V \cap gWg^{-1})) = (\sigma V, \sigma gW),$$

and using the notation $Z := \coprod_{VgW \in V\backslash G/W} G/(V \cap gWg^{-1})$ we get that

$$\begin{aligned}
&S_+(\pi_V^G)S_\bullet(\pi_e^V)(s) \cdot S_+(\pi_W^G)S_\bullet(\pi_e^W)(t) \\
&\quad = S[G/e \amalg G/e \leftarrow \coprod_{gW \in G/W} G/e \amalg \coprod_{Vg \in V\backslash G} G/e \to Z \to G/G](s,t) \\
&\quad = \sum_{VgW \in V\backslash G/W} S_+(\pi^G_{V\cap gWg^{-1}})S_\bullet(\pi_e^{V\cap gWg^{-1}})(s^{(V:V\cap gWg^{-1})} \cdot t^{(W:g^{-1}Vg\cap W)}). \quad \square
\end{aligned}$$

Proof of Theorem 15. Let $G = V_1, V_2, \ldots, V_k = U$ be a system of representatives of subgroups of G containing a conjugate of U. We define

$$s_U^G(a_{V_1}, b_{V_1}, \ldots, a_{V_k}, b_{V_k}) := \xi^G_{(U;V_1,V_1,\ldots,V_k,V_k;1,\ldots,1)}(a_{V_1}, b_{V_1}, \ldots, a_{V_k}, b_{V_k})$$

and

$$m_U^G(a_{V_1}, \ldots, a_{V_k}) := \xi^G_{(U;V_1,\ldots,V_k;-1,\ldots,-1)}(a_{V_1}, \ldots, a_{V_k}).$$

By Lemma 18 these are integral polynomials with the desired properties. For example we have:

$$\begin{aligned}
\phi_U^A(a) + \phi_U^A(b) &= \sum_{i=1}^k |(G/V_i)^U|(a_{V_i}^{(V_i:U)} + b_{V_i}^{(V_i:U)}) \\
&= \phi_U^A(\xi_{(U;V_1,V_1,\ldots,V_k,V_k;1,\ldots,1)}(a_{V_1}, b_{V_1}, \ldots, a_{V_k}, b_{V_k})) \\
&= \phi_U^A(s_U(a_{V_1}, b_{V_1}, \ldots, a_{V_k}, b_{V_k})).
\end{aligned}$$

To construct $p_U = p_U^G$ we first choose a system $x_1, x_2, \ldots, x_h$ of representatives of the G-orbits in

$$X := \coprod_{i,j=1}^k G/V_i \times G/V_j.$$

Next we put $W_r := G_{x_r}$ and

$$p_r = p_r(a_{V_1}, b_{V_1}, \ldots, a_{V_k}, b_{V_k}) := a_i^{(V_i:W_r)} \cdot b_j^{(V_j:W_r)}$$

in case $x_r = (g_r V_i, g_r' V_j) \in G/V_i \times G/V_j \subseteq X$. Using these conventions, we define

$$p_U^G(a_{V_1}, b_{V_1}, \ldots, a_{V_k}, b_{V_k}) := \xi^G_{(U;W_1,\ldots,W_h;1,\ldots,1)}(p_1, \ldots, p_r).$$

Using the Lemma 19 we see that p_U has the desired properties. $\square$

5. Free Tambara functors

In this section we prove Theorem 14. On the way we shall give a combinatorial description of the Witt vectors of a polynomial G-ring, that is, a G-ring of the form $U^G(X, G/e)$ for a finite G-set X. Recall from Lemma 6 that the functor $R \mapsto R(G/e)$ from the category $[U^{fG}, \mathcal{E}ns]_0$ of fG-Tambara functors to the category of G-rings is an equivalence of categories, and that there are morphisms $(\mathcal{F}in^{fG})^{\text{op}} \subseteq (\mathcal{F}in^G)^{\text{op}} \xrightarrow{R} U^G$ of colored theories. We let $F : \mathcal{E}ns^G \simeq [(\mathcal{F}in^{fG})^{\text{op}}, \mathcal{E}ns]_0 \to [U^G, \mathcal{E}ns]_0$ denote the left adjoint of the forgetful functor induced by the above composition of morphisms of colored theories.

Given finite G-sets X and Y we let $\widetilde{U}_+^G(X, Y) \subseteq U_+^G(X, Y)$ denote those elements of the form $[X \leftarrow A \to B \to Y]$, where G acts freely on A, and we let $\widetilde{U}^G(X, Y) \subseteq U^G(X, Y)$ denote the abelian subgroup generated by $\widetilde{U}_+^G(X, Y)$. The composition

$$\begin{aligned} &U^G(Y, Z) \times \widetilde{U}^G(X, Y) \cup \widetilde{U}^G(Y, Z) \times U^G(X, Y) \\ &\subseteq U^G(Y, Z) \times U^G(X, Y) \xrightarrow{\circ} U^G(X, Z) \end{aligned}$$

factors through the inclusion $\widetilde{U}^G(X, Z) \subseteq U^G(X, Z)$. We obtain a functor $\widetilde{U}^G : \mathcal{F}in^G \to [U^G, \mathcal{E}ns]_0$ with $\widetilde{U}^G(f : Y \to X) = \widetilde{U}^G(R_f, -)$.

Lemma 20. *Given a G-Tambara functor S and a finite free G-set A, there is an isomorphism $\mathcal{E}ns^G(A, S(G/e)) \xrightarrow{\cong} S^*(A)$, which is natural in A.*

Proof. Choosing an isomorphism $\phi : A \xrightarrow{\cong} G/e \times A_0$ we obtain an isomorphism

$$\begin{aligned} \mathcal{E}ns^G(A, S(G/e)) &\xrightarrow{\phi^{-1*}} \mathcal{E}ns^G(G/e \times A_0, S(G/e)) \\ &\cong \mathcal{E}ns(A_0, S(G/e)) \cong S(G/e \times A_0) \xrightarrow{S^*(\phi)} S(A). \end{aligned}$$

This isomorphism is independent of the choice of ϕ. $\square$

Lemma 21. *$FX \cong \widetilde{U}^G(X, -)$ for finite G-sets X.*

Proof. For every G-Tambara functor S we shall construct a bijection

$$\mathcal{E}ns^G(X, S(G/e)) \cong [U^G, \mathcal{E}ns]_0(\widetilde{U}^G(X, -), S).$$

Given $f : X \to S(G/e) \in \mathcal{E}ns^G(X, S(G/e))$ we let

$$\phi(f) \in [U^G, \mathcal{E}ns]_0(\widetilde{U}^G(X, -), S)$$

take $x = [X \xleftarrow{d} A \xrightarrow{b} B \xrightarrow{c} Y] \in \widetilde{U}^G(X, Y)$ to $\phi(f)(x) \in S(Y)$ constructed as follows: by Lemma 20 we obtain an element $a \in S(A)$, and we let $\phi(f)(x) =$

$S_+(c)S_\bullet(b)(a)$. Conversely, given $g \in [U^G, \mathcal{E}ns]_0(\widetilde{U}^G(X,-),S)$, we construct $\psi(g) \in \mathcal{E}ns^G(X, S(G/e))$ by letting $\psi(g)(x) = g([X \leftarrow G/e \to G/e \to G/e])$, where the map pointing left takes $e \in G$ to $x \in X$ and where the maps pointing right are identity maps. We leave it to the reader to check that ϕ and ψ are inverse bijections. $\square$

Corollary 22. *For every finite G-set X the functor $\widetilde{U}^G(X,-) : U^G \to \mathcal{E}ns$ is isomorphic to $L_G U^G(X, j_G(-))$, where $j_G : U^{fG} \subseteq U^G$ is the inclusion.*

Theorem 23. *Let X be a finite G-set and let $R = U^G(X, j_G(-))$. The Teichmüller homomorphism*

$$\tau : \mathbb{W}_G((L_G R)(G/e))/\mathbb{I}_G((L_G R)(G/e)) \to (L_G R)(G/G) = \widetilde{U}^G(X, G/G)$$

is an isomorphism.

Proof of Theorem 14. Let $A = R(G/e)$. Given $\alpha = [W \leftarrow C \to D \to X] \in U^{fG}(W,X)$ we have an *fG*-Tambara map $\alpha^* : U^G(X,-) \to U^G(W,-)$ and we have the map $R(\alpha) : R(W) \to R(X)$. Hence we obtain maps

$$U^G(X,G/G) \times R(X) \leftarrow U^G(X,G/G) \times R(W) \to U^G(W,G/G) \times R(W).$$

The value $(L_G R)(G/G)$ at G/G of the left Kan extension $L_G R$ of R along j_G is isomorphic to the coequalizer of the diagram

$$\coprod_{X,Y \in \mathrm{ob} U^{fG}} U^G(X,G/G) \times R(Y) \rightrightarrows \coprod_{X \in \mathrm{ob} U^{fG}} U^G(X,G/G) \times R(X),$$

induced by the above maps. We shall construct a map

$$\rho : (L_G R)(G/G) \to \mathbb{W}_G(A)/\mathbb{I}_G(A)$$

by specifying explicit maps $\rho_X : U^G(X,G/G) \times R(X) \to \mathbb{W}_G(A)/\mathbb{I}_G(A)$. Given $\underline{r} \in R(X)$, we have an *fG*-Tambara morphism $\mathrm{ev}_{\underline{r}} : U^{fG}(X,-) \to R$. Since G acts freely on X we have $U^G(X,G/G) = \widetilde{U}^G(X,G/G)$ and by Theorem 23 we get an induced ring-homomorphism

$$U^G(X,G/G) \cong \mathbb{W}_G(U^{fG}(X,-))/\mathbb{I}_G(U^{fG}(X,-)) \to \mathbb{W}_G(A)/\mathbb{I}_G(A).$$

By adjunction we obtain a map

$$\rho_X : U^G(X,G/G) \times R(X) \to \mathbb{W}_G(A)/\mathbb{I}_G(A).$$

We need to check that these ρ_X induce a map on the coequalizer $(L_G R)(G/G)$ of the above coequalizer diagram, that is, for α as above we need to show that the diagram

$$\begin{array}{ccc} U^G(X,G/G) \times R(W) & \xrightarrow{\alpha^* \times \mathrm{id}} & U^G(W,G/G) \times R(W) \\ {\scriptstyle \mathrm{id} \times R(\alpha)} \downarrow & & \downarrow {\scriptstyle \rho_W} \\ U^G(X,G/G) \times R(X) & \xrightarrow{\rho_X} & \mathbb{W}_G(A)/\mathbb{I}_G(A) \end{array}$$

commutes. For this we note that the diagram

$$\begin{array}{ccc} \mathbb{W}_G(U^G(X,G/e)) & \xrightarrow{t} & U^G(X,G/G) \\ {\scriptstyle\alpha^*}\downarrow & & {\scriptstyle\alpha^*}\downarrow \\ \mathbb{W}_G(U^G(W,G/e)) & \xrightarrow{t} & U^G(W,G/G) \end{array}$$

commutes, and therefore it will suffice to show that the diagram

$$\begin{array}{ccc} \mathbb{W}_G(U^G(X,G/e)) \times R(W) & \xrightarrow{\alpha^* \times \mathrm{id}} & \mathbb{W}_G(U^G(W,G/e)) \times R(W) \\ {\scriptstyle \mathrm{id}\times R(\alpha)}\downarrow & & \downarrow \\ \mathbb{W}_G(U^G(X,G/e)) \times R(X) & \longrightarrow & \mathbb{W}_G(A) \end{array}$$

commutes, where the arrows without labels are constructed using the homomorphisms $\mathbb{W}_G(\mathrm{ev}_{\underline{r}}(G/e))$ for $\underline{r}$ an element of either $R(X)$ or $R(W)$. Using diagonal inclusions of the form

$$\mathbb{W}_G(T) \times Z \to \mathbb{W}_G(T) \times \prod_{U \leqslant G}{}' Z \approx \prod_{U \leqslant G}{}' (T \times Z)$$

we see that it suffices to note that the diagram

$$\begin{array}{ccc} \prod'_{U \leqslant G}(U^G(X,G/e) \times R(W)) & \xrightarrow{\prod'_{U \leqslant G}(\alpha^* \times \mathrm{id})} & \prod'_{U \leqslant G}(U^G(W,G/e) \times R(W)) \\ {\scriptstyle \prod'_{U \leqslant G}(\mathrm{id}\times R(\alpha))}\downarrow & & \downarrow \\ \prod'_{U \leqslant G}(U^G(X,G/e) \times R(X)) & \longrightarrow & \prod'_{U \leqslant G} R(G/e) \approx \mathbb{W}_G(A) \end{array}$$

commutes. This ends the construction of $\rho : (L_G R)(G/G) \to \mathbb{W}_G(A)/\mathbb{I}_G(A)$.

We leave it to the reader to check that ρ and τ are inverse bijections. For this it might be helpful to note that

$$\sum_{U \leqslant G}{}' ([G/e \overset{=}{\leftarrow} G/e \xrightarrow{\pi_e^U} G/U \to G/G] \circ [Y \leftarrow A_U \to B_U \to G/e])$$

$$= \left[\coprod_{U \leqslant G}{}' G/e \overset{=}{\leftarrow} \coprod_{U \leqslant G}{}' G/e \to \coprod_{U \leqslant G}{}' G/U \to G/G \right]$$

$$\circ \left[Y \leftarrow \coprod_{U \leqslant G}{}' A_U \to \coprod_{U \leqslant G}{}' B_U \to \coprod_{U \leqslant G}{}' G/e \right]. \qquad \square$$

For the proof of Theorem 23 we need to introduce filtrations of both sides.

Definition 24. Let A be a ring and let $U \leqslant G$ be a subgroup of G. We let $I_U(A) \subseteq \mathbb{W}_G(A)$ denote the ideal generated by those $a = (a_K)'_{K \leqslant G} \in \mathbb{W}_G(A)$ for which

$a_K \neq 0$ implies that $K \lesssim U$. We let $\widetilde{I}_U(A) \subseteq I_U(A)$ denote the sub-ideal $\widetilde{I}_U(A) = \sum_{V \subsetneq U} I_V(A) \subseteq I_U(A)$.

Definition 25. Given a G-set X and $U \leqslant G$, we let $J_U^+ \subseteq \widetilde{U}^G(X, G/G)$ denote the subset of elements of the form

$$[X \leftarrow A \rightarrow B \rightarrow G/G] \in \widetilde{U}_+^G(X, G/G) \subseteq \widetilde{U}^G(X, G/G),$$

for which $B^K = \emptyset$ when U is a conjugate to a proper subgroup of K. We let $J_U \subseteq U^G(X, G/G)$ denote the ideal generated by J_U^+ and we let $\widetilde{J}_U \subseteq J_U$ denote the sub-ideal $\widetilde{J}_U = \sum_{V \subsetneq U} J_V \subseteq J_U$.

Lemma 26. (i) *Any element in J_U is of the form $x - y$ for $x, y \in J_U^+$.*

(ii) *Every element x in the image of the map $J_U^+ \rightarrow J_U/\widetilde{J}_U$ is of the form*

$$x = [X \xleftarrow{d} G/e \times A \xrightarrow{\pi_e^U \times f} G/U \times B \xrightarrow{q} G/G] + \widetilde{J}_U,$$

with $f : A \rightarrow B$ a map of (non-equivariant) sets and d a G-map, where q is the composition $G/U \times B \xrightarrow{\mathrm{pr}} G/U \xrightarrow{\pi_U^G} G/G$.

(iii) *If*

$$x = [X \xleftarrow{d} G/e \times A \xrightarrow{\pi_e^U \times f} G/U \times B \xrightarrow{q} G/G] + \widetilde{J}_U,$$

and

$$x' = [X \xleftarrow{d'} G/e \times A' \xrightarrow{\pi_e^U \times f'} G/U \times B' \xrightarrow{q'} G/G] + \widetilde{J}_U,$$

then $x = x'$ if and only if there exist bijections $\alpha : A \rightarrow A'$ and $\beta : B \rightarrow B'$ and for every $a \in A$ there exists $g_a \in N_G(U)$ such that

(a) $f'\alpha = \beta f$,

(b) $d'(e, \alpha a) = d(g_a, a)$ *and*

(c) $g_{a_1}U = g_{a_2}U$ *if* $f(a_1) = f(a_2)$.

Proof. A straightforward verification yields that the multiplication in $\widetilde{U}_+^G(X, G/G)$ induces a map $J_U^+ \times \widetilde{U}_+^G(X, G/G) \rightarrow J_U^+$ and that J_U^+ is closed under sum. It follows that J_U is the abelian subgroup of $U^G(X, G/G)$ generated by J_U^+. Statement (i) is a direct consequence of this. For (ii) we note that for every element

$$r = [X \xleftarrow{d} D \xrightarrow{c} E \xrightarrow{t} G/G]$$

in $\widetilde{U}_+^G(X, G/G)$, we have a decomposition $E \cong \coprod'_{K \leqslant G} E_K$, where $E_K \cong G/K \times B_K$ for some B_K. This decomposition induces an isomorphism

$$\widetilde{U}^G(X, G/G) \cong \bigoplus_{K \leqslant G}{}' J_K/\widetilde{J}_K$$

of abelian groups. Given an element x of the form

$$x = [X \xleftarrow{d} D \xrightarrow{c} G/U \times B \xrightarrow{q} G/G] + \widetilde{J}_U,$$

we can choose a G-bijection of the form $c^{-1}(G/U \times \{b\}) \cong G/e \times A_b$ for every $b \in B$. It follows that x is represented by an element of the form

$$x = [X \xleftarrow{d} G/e \times A \xrightarrow{p \times f} G/U \times B \xrightarrow{q} G/G] + \widetilde{J}_U.$$

We leave the straightforward verification of part (iii) to the reader. $\square$

Lemma 27. *Let U be a subgroup of G and let $a = (a_V)'_{V \leqslant G} \in I_U(\widetilde{U}^G(X, G/e))$ with*

$$a_U = [X \xleftarrow{d} A \times G/e \xrightarrow{f \times 1} B \times G/e \xrightarrow{\mathrm{pr}} G/e].$$

Then

$$\tau(a) \equiv [X \xleftarrow{d} A \times G/e \xrightarrow{f \times \pi_e^U} B \times G/U \to G/G] \mod \widetilde{J}_U.$$

Proof. The lemma follows from the diagram:

$$\begin{tikzcd}
X & A \times G/e \arrow[l, "d"'] \arrow[d, "f \times \mathrm{id}"'] & W \arrow[l] \arrow[d] & G/e \times A \arrow[l] \arrow[d] \\
G/e \arrow[d, "\pi_e^U"'] & G/e \times B \arrow[l, "\mathrm{pr}"'] & Z \arrow[l] \arrow[d] & G/e \times B \arrow[l] \arrow[d] \\
G/U & \coprod_{gU \in G/U} \mathrm{map}(gU, B) \arrow[l] \arrow[r, equal] & Y & G/U \times B, \arrow[l]
\end{tikzcd}$$

where the lower rectangle is an exponential diagram and the squares are pull-backs. We use that the map $G/U \times B \to Y$ which takes (gU, b) to the constant map $gU \to B$ with value b is an isomorphism on G/U-parts and that $Y^H = \emptyset$ for $U \lesssim H$ and $U \neq H$. $\square$

Corollary 28. *Let X be a G-set and let $R = \widetilde{U}^G(X, j_G(-))$. The map*

$$\tau : \mathbb{W}_G(U^G(X, G/e)) \to (L_G R)(G/G) = \widetilde{U}^G(X, G/G)$$

satisfies that $\tau(I_U(R(G/e))) \subseteq J_U$ and that $\tau(\widetilde{I}_U(R(G/e))) \subseteq \widetilde{J}_U$.

Proposition 29. *Let X be a G-set, let $R = \widetilde{U}^G(X, j_G(-))$ and let $A = R(G/e)$. For every $U \leqslant G$ the map $\tau : \mathbb{W}_G(A)/\mathbb{I}_G(A) \to (L_G R)(G/G) = \widetilde{U}^G(X, G/G)$ induces an isomorphism $\tau_U : (\mathbb{I}_G(A) + I_U(A))/(\mathbb{I}_G(A) + \widetilde{I}_U(A)) \to J_U/\widetilde{J}_U$.*

Proof. Let $x \in I_U$ with $x_U = [X \xleftarrow{d} A \times G/e \xrightarrow{f \times 1} B \times G/e \xrightarrow{\mathrm{pr}} G/e]$. Then by Lemma 27 $\tau(x) \equiv [X \xleftarrow{d} A \times G/e \xrightarrow{f \times \pi_e^U} B \times G/U \xrightarrow{q} G/G] \mod \widetilde{J}_U$, with the notation introduced there, and it follows from Lemma 26 that τ_U is onto. On the other hand, to prove injectivity, we pick $x_1, x_2 \in I_U$ with $\tau(x_1) \equiv \tau(x_2) \bmod \widetilde{J}_U$. Suppose that $x_{i,U}$ has the form

$$x_{i,U} = [Z \xleftarrow{d_i} A_i \times G/e \xrightarrow{f_i \times 1} B_i \times G/e \xrightarrow{=} G/e]$$

for $i = 1, 2$. Let

$$y_i = [Z \xleftarrow{d_i} A_i \times G/e \xrightarrow{f_i \times \pi_e^U} B_i \times G/U \xrightarrow{q} G/G]$$

for $i = 1, 2$. Then by Lemma 27 $y_i \equiv \tau(x_i) \bmod \widetilde{J}_U$ for $i = 1, 2$. It follows from Lemma 26 that there exist bijections $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$ with $f_2\alpha = \beta f_1$ and for every $a \in A_1$ there exists $g_a \in N_G(U)$ such that firstly $d_1(\alpha a) = g_a d(a)$ and secondly, if $a_1, a_2 \in A$ satisfy that $f(a_1) = f(a_2)$, then $g_{a_1}U = g_{a_2}U$. Given $a \in A_i$, let $z_{i,a} \in \widetilde{U}^G(X, G/e)$ denote the element $[X \xleftarrow{d_{i,a}} G/e \xrightarrow{p} G/e \xrightarrow{=} G/e]$, where $d_{i,a}(e) = d_i(a, e)$. Then $z_{2,\alpha(a)} = g_a z_{1,a}$ and $x_{i,U} = \sum_{b \in B_i} (\prod_{a \in f_i^{-1}(b)} z_{i,a})$ for $i = 1, 2$, where an empty product is 1 and an empty sum is 0. We can conclude that $x_{1,U} - x_{2,U} \in \mathbb{I}_G(R)$, and hence $x_1 - x_2 \in \mathbb{I}_G(R) + \widetilde{I}_U$. In the general case $\tau(x_1 - x_1') \equiv \tau(x_2 - x_2') \bmod \widetilde{J}_U$ we easily obtain that $x_1 - x_1' \equiv x_2 - x_2' \bmod \mathbb{I}_G(R) + \widetilde{I}_U$ by collecting the positive terms. $\square$

Proof of Theorem 23. We start by noting that $\widetilde{I}_V = \sum_{U \subsetneq V} I_U \cong \operatorname{colim}_{U \subsetneq V} I_U \subseteq I_V$ and that $\widetilde{J}_V = \sum_{U \subsetneq V} J_U \cong \operatorname{colim}_{U \subsetneq V} J_U \subseteq J_V$. The result now follows by induction on the cardinality of V using the above proposition and the five lemma on the following map of short exact sequences:

$$\begin{array}{ccccc}
\widetilde{\mathbb{I}}_G(R(G/e)) + I_V & \longrightarrow & \mathbb{I}_G(R(G/e)) + I_V & \longrightarrow & \frac{(\mathbb{I}_G(R(G/e)) + I_V)}{(\mathbb{I}_G(R(G/e)) + \widetilde{I}_V)} \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{J}_V & \longrightarrow & J_V & \longrightarrow & J_V/\widetilde{J}_V.
\end{array}$$

$\square$

6. The Witt Tambara-functor

In this section we finally prove Theorem C. Given a subgroup $H \leqslant G$ and an H-set X, we can construct a $G \times H$-set $G/e \times X$, where G acts by multiplication on the left on G/e, and where $h \cdot (g, x) := (gh^{-1}, hx)$. We let $\operatorname{ind}_H^G X$ denote the G-set $G \times_H X = H \backslash (G/e \times X)$.

Lemma 30. *Let H be a subgroup of G. The functor $\operatorname{ind}_H^G : \mathcal{F}in^H \to \mathcal{F}in^G$ induces functors $\operatorname{ind}_H^G : U^H \to U^G$, and $\operatorname{ind}_{fH}^{fG} : U^{fH} \to U^{fG}$.*

Proof. Since the functor $\operatorname{ind}_H^G : \mathcal{F}in^H \to \mathcal{F}in^G$ preserves pull-back diagrams and exponential diagrams it induces a functor $\operatorname{ind}_H^G : U^H \to U^G$ that takes $X \leftarrow A \to B \to Y$ to $\operatorname{ind}_H^G X \leftarrow \operatorname{ind}_H^G A \to \operatorname{ind}_H^G B \to \operatorname{ind}_H^G Y$. $\square$

Given a G-Tambara functor S we construct an H-Tambara functor $\mathrm{res}_H^G S = S \circ \mathrm{ind}_H^G$. Similarly, given an fG-Tambara functor R we can construct an fH-Tambara functor $\mathrm{res}_{fH}^{fG} S = S \circ \mathrm{ind}_{fH}^{fG}$.

Theorem 31. *Given an fG-Tambara functor R, the Teichmüller homomorphism*

$$\tau : \mathbb{W}_H(\mathrm{res}_{fH}^{fG} R(H/e))/\mathbb{I}_H(\mathrm{res}_{fH}^{fG} R(H/e)) \to \mathrm{res}_H^G L_G R(H/H)$$

is an isomorphism.

Proof of Theorem C. If we consider $A = R(G/e)$ as an H-ring, then

$$\mathbb{W}_H(\mathrm{res}_{fH}^{fG} R(H/e))/\mathbb{I}_H(\mathrm{res}_{fH}^{fG} R(H/e)) \cong \mathbb{W}_H(A)/\mathbb{I}_H(A),$$

and $(\mathrm{res}_H^G L_G R)(H/H) = (L_G R)(G/H)$. Combining these observations with Theorem 31 we obtain the statement of Theorem C. $\square$

Lemma 32. *Let H be a subgroup of G. The forgetful functor $i^* : \mathcal{F}in^G \to \mathcal{F}in^H$ which takes a G-set Y to the same set considered as an H-set induces functors $i^* : U^G \to U^H$ and $i_f^* : U^{fG} \to U^{fH}$.*

Lemma 33. *Let H be a subgroup of G. The functor $i^* : U^G \to U^H$ is left adjoint to the functor $\mathrm{ind}_H^G : U^H \to U^G$ and the functor $i_f^* : U^{fG} \to U^{fH}$ is left adjoint to the functor $\mathrm{ind}_{fH}^{fG} : U^{fH} \to U^{fG}$.*

Proof. We prove only the first part of the lemma. Given $X \leftarrow A \to B \to G \times_H Y$ in $U^G(X, G \times_H Y)$ we construct an element in $U^H(i^*X, Y)$ by the following diagram where the two squares furthest to the right are pull-back squares:

$$\begin{array}{ccccccc} i^*X & \longleftarrow & A_H & \longrightarrow & B_H & \longrightarrow & H \times_H Y \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & A & \longrightarrow & B & \longrightarrow & G \times_H Y. \end{array}$$

Conversely, given $i^*X \leftarrow E \to F \to Y$ in $U^H(i^*X, Y)$ we construct the element $X \leftarrow G \times_H E \to G \times_H F \to G \times_H Y$ in $U^G(X, G \times_H Y)$. Here the arrow pointing to the left is the composite $G \times_H E \to G \times_H i^*X \to X$. We leave it to the reader to check that the maps are inverse bijections in an adjunction. $\square$

We have the following commutative diagram of categories:

$$\begin{array}{ccc} U^{fG} & \xrightarrow{i_f^*} & U^{fH} \\ \downarrow & & \downarrow \\ U^G & \xrightarrow{i^*} & U^H, \end{array}$$

where the vertical functors are the natural inclusions. Since ind_H^G is right adjoint to i^*, res_H^G is left adjoint to $[i^*, \mathcal{E}ns]_0$ (see for example [14,Proposition 16.6.3]). Similarly res_{fH}^{fG} is left adjoint to $[i_f^*, \mathcal{E}ns]_0$. From the commutative diagram of functor categories

$$\begin{tikzcd} {[U^{fG}, \mathcal{E}ns]_0} & {[U^{fH}, \mathcal{E}ns]_0} \arrow[l, "{[i_f^*, \mathcal{E}ns]_0}"'] \\ {[U^{G}, \mathcal{E}ns]_0} \arrow[u] & {[U^{H}, \mathcal{E}ns]_0,} \arrow[l, "{[i^*, \mathcal{E}ns]_0}"'] \arrow[u] \end{tikzcd}$$

where the vertical maps are the forgetful functors induced by the inclusions $j_G : U^{fG} \subseteq U^G$ and $j_H : U^{fH} \subseteq U^H$ we can conclude that there is a natural isomorphism $\mathrm{res}_H^G L_G \cong L_H \mathrm{res}_{fH}^{fG}$.

Proof of Theorem 31. By Theorem 14 we have an isomorphism

$$\begin{aligned} &\mathbb{W}_H(\mathrm{res}_{fH}^{fG} R(H/e))/\mathbb{I}_H(\mathrm{res}_{fH}^{fG} R(H/e)) \\ &\xrightarrow{\tau} L_H \mathrm{res}_{fH}^{fG} R(H/H) \cong \mathrm{res}_H^G L_G R(H/H). \qquad \square \end{aligned}$$

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