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# CONSTRUCTIONS OF FACTORIZATION SYSTEMS IN CATEGORIES

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## 1. Introduction

In [2] we constructed homological localizations of spaces, groups, and $\pi$-modules; here we generalize those constructions to give "factorization systems" and "homotopy factorization systems" for maps in categories.

In Section 2 we recall the definition and basic properties of factorization systems, and in Section 3 we give our first existence theorem (3.1) for such systems. It can be viewed as a generalization of Deleanu's existence theorem [5] for localizations, and is best possible although it involves a hard-to-verify "solution set" condition. In Section 4 we give a second existence theorem (4.1) which is more specialized than the first, but is often easier to apply since it avoids the "solution set" condition. In Section 5 we use our existence theorems to construct various examples of factorization systems, and we consider the associated (co)localizations. As special cases, we obtain the Stone–Cech compactification for topological spaces, the homological localizations of groups and $\pi$-modules [2, Section 5], the Ext-completions for abelian groups [4, p. 171], and many new (co)localizations. In Section 6 we generalize the theory of factorization systems to the context of homotopical algebra [9]. Among the "homotopy factorization systems" in the category of simplicial sets are the Moore–Postnikov systems and the homological factorization systems of [2, Appendix]. In Section 7 we generalize 4.1 to give an existence theorem for homotopy factorization systems. This leads to "Anderson-like" localizations (7.3) and $p$-completions (7.4) in the pointed simplicial or CW homotopy category. It also leads to "colocalizations of spaces with respect to homotopy theories" (see 7.5).

We will use the language of Godel–Bernays set theory, distinguishing between "sets" and "classes". The objects of a category $\mathbf{C}$ will form a class, but $\mathbf{C}(X, Y)$ is required to be a set for each $X, Y \in \mathbf{C}$.

## 2. Factorization systems in a category

We now give a brief account of factorization systems in a category **C**, cf. [6], [8].

For $\varphi : A \to B$ and $\mu : X \to Y$ in **C**, we say $\varphi$ *has the unique left lifting property* (*the ULLP*) *for* $\mu$, or equivalently $\mu$ *has the unique right lifting property* (*the URLP*) *for* $\varphi$, if for each commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & X \arrow[d, "\mu"] \\ B \arrow[r, "\beta"'] \arrow[ur, dashed, "\gamma"] & Y \end{tikzcd}$$

in **C** there exists a unique map $\gamma$ such that $\gamma\varphi = \alpha$ and $\mu\gamma = \beta$. For a class $\boldsymbol{S}$ of maps in **C**, we let

$$\mathscr{E}(\boldsymbol{S}) = \{\varphi \mid \varphi \text{ has the ULLP for each } \mu \in \boldsymbol{S}\}$$

$$\mathscr{M}(\boldsymbol{S}) = \{\mu \mid \mu \text{ has the URLP for each } \varphi \in \boldsymbol{S}\}.$$

**2.1 Definition.** A *factorization system* $(\boldsymbol{E}, \boldsymbol{M})$ in **C** consists of classes of maps $\boldsymbol{E}$ and $\boldsymbol{M}$ such that:

(i) $\boldsymbol{E} = \mathscr{E}(\boldsymbol{M})$ and $\boldsymbol{M} = \mathscr{M}(\boldsymbol{E})$.

(ii) For every map $f$ in **C**, there exist $f_m \in \boldsymbol{M}$ and $f_e \in \boldsymbol{E}$ such that $f = f_m f_e$.

The factorization in 2.1 (ii) is clearly unique up to canonical isomorphism and is natural. To recognize factorization systems, one can use

**2.2 Lemma.** *Two classes* $(\boldsymbol{E}, \boldsymbol{M})$ *of maps in* **C** *form a factorization system if and only if the following hold*:

(i) *Every isomorphism is in both* **E** *and* **M**.

(ii) *Both* **E** *and* **M** *are closed under composition.*

(iii) *If* $\varphi \in \boldsymbol{E}$ *and* $\mu \in \boldsymbol{M}$, *then* $\varphi$ *has the* ULLP *for* $\mu$.

(iv) *For every map $f$ in* **C**, *there exist* $f_m \in \boldsymbol{M}$ and $f_e \in \boldsymbol{E}$ *such that* $f = f_m f_e$.

**Proof.** Assuming $(\boldsymbol{E}, \boldsymbol{M})$ satisfy (i)–(iv), we will show $\mathscr{E}(\boldsymbol{M}) \subset \boldsymbol{E}$. For $f \in \mathscr{E}(\boldsymbol{M})$, choose $f_e \in \boldsymbol{E}$ and $f_m \in \boldsymbol{M}$ such that $f = f_m f_e$. Then there exists a lifting $u$ such that $uf = f_e$ and $f_m u = 1$. Moreover, $uf_m = 1$ since $f_e$ has the ULLP for $f_m$. Thus $f_m$ is iso and $f$ is in $\boldsymbol{E}$. The rest of the proof is obvious.

Note that 2.2 remains valid if (i) and (ii) are replaced by the condition:
If $f$ is a retract of $g$ (in the category of maps) and $g$ is in $\boldsymbol{E}$ or $\boldsymbol{M}$, then so is $f$.

**2.3 Examples.** The following easy examples of factorization systems can be verified using 2.2.

(I) In the category of sets, groups, or modules over a ring: $\boldsymbol{E}$ = surjections and $\boldsymbol{M}$ = injections.

(II) In any category: $\boldsymbol{E}$ = isos and $\boldsymbol{M}$ = all maps; or vice versa.

Using the following lemma, it is easy to show that (I) and (II) are the *only* factorization systems in the category of sets or of vector spaces over a field.

**2.4 Lemma.** *If $(\boldsymbol{E}, \boldsymbol{M})$ is a factorization system in* $\mathbf{C}$ *(or more generally if* $\boldsymbol{E} = \mathscr{E}(\boldsymbol{S})$ *for some class* $\boldsymbol{S}$ *in* $\mathbf{C}$*), then:*

(A1) *Every isomorphism is in* $\boldsymbol{E}$.

(A2) $\boldsymbol{E}$ *is closed under composition.*

(A3) *If* $gf \in \boldsymbol{E}$ *and* $f \in \boldsymbol{E}$, *then* $g \in \boldsymbol{E}$.

(A4) *If*

$$\begin{tikzcd} V \arrow[r] \arrow[d, "i"] & X \arrow[d, "j"] \\ W \arrow[r] & Y \end{tikzcd}$$

*is a push-out diagram in* $\mathbf{C}$ *and* $i \in \boldsymbol{E}$, *then* $j \in \boldsymbol{E}$.

(A5) $\boldsymbol{E}$ *is closed under small colimits, i.e. if* $J$ *is a small index category and* $\{X(j) \to Y(j)\}_{j \in J}$ *is a diagram of maps in* $\boldsymbol{E}$, *then the induced map*

$$\operatorname*{Colim}_{i} X(j) \to \operatorname*{Colim}_{j} Y(j)$$

*is in* $\boldsymbol{E}$ *(provided those colimits exist).*

The proof is easy and there is an obvious dual result for a class $\boldsymbol{M}$. A factorization system $(\boldsymbol{E}, \boldsymbol{M})$ in a category $\mathbf{C}$ gives rise to

**2.5 Localizations and colocalizations.** If $\mathbf{C}$ has a terminal object $t$, there is a functor $T: \mathbf{C} \to \mathbf{C}$ and transformation $\eta: 1 \to T$ where $X \xrightarrow{\eta} TX \to t$ is "the" $(\boldsymbol{E}, \boldsymbol{M})$-factorization of $X \to t$. Call $(T, \eta)$ the *$(\boldsymbol{E}, \boldsymbol{M})$-localization* on $\mathbf{C}$, and note that it is idempotent. Moreover, it provides a left adjoint to the inclusion function Loc-$\mathbf{C} \xrightarrow{\subset} \mathbf{C}$ where Loc-$\mathbf{C}$ denotes the full subcategory given by all $X \in \mathbf{C}$ with $X \to t$ in $\boldsymbol{M}$. Note also, for $X \in \mathbf{C}$, that $\eta: X \to TX$ is the universal (terminal) example of a map in $\boldsymbol{E}$ with domain $X$. Dually, if $\mathbf{C}$ has initial object, one obtains an *$(\boldsymbol{E}, \boldsymbol{M})$-colocalization* on $\mathbf{C}$.

Finally, note that the factorization system $(\boldsymbol{E}, \boldsymbol{M})$ on $\mathbf{C}$ gives rise to obvious factorization systems on $\mathbf{C}/c$ and $c/\mathbf{C}$ for $c \in \mathbf{C}$. There are, of course, associated localizations and colocalizations on these categories.

## 3. An existence theorem for factorization systems

We now give our first existence theorem for factorization systems in *cocomplete* categories, i.e. those with colimits over arbitrary small index categories.

**3.1 Theorem.** *Let* $\mathbf{C}$ *be a cocomplete category and let* $\mathbf{E}$ *be a class of maps in* $\mathbf{C}$. *Then* $(\mathbf{E}, \mathcal{M}(\mathbf{E}))$ *is a factorization system in* $\mathbf{C}$ *if and only if* $\mathbf{E}$ *satisfies the conditions* (A1)–(A5) *of* 2.4 *together with the solution set condition*:

(SSC) *Each map* $f: X \to Y$ *in* $\mathbf{C}$ *has a set of factorizations* $\{X \xrightarrow{u_\alpha} B_\alpha \xrightarrow{v_\alpha} Y\}$ *with* $u_\alpha \in \mathbf{E}$ *for all* $\alpha$ *and such that any factorization,* $X \xrightarrow{u} B \xrightarrow{v} Y$ *with* $u \in \mathbf{E}$, *can be mapped (in the category* $\mathbf{F}_f$ *below) to some member of this set.*

**Proof.** For a map $f: X \to Y$ in $\mathbf{C}$, let $\mathbf{F}_f$ be the category whose objects are factorizations $X \xrightarrow{u} B \to Y$ of $f$ with $u \in \mathbf{E}$, and whose maps are commutative diagrams

$$\begin{tikzcd} X \arrow[r, "u_1"] \arrow[d, "1"] & B_1 \arrow[r, "v_1"] \arrow[d] & Y \arrow[d, "1"] \\ X \arrow[r, "u_2"] & B_2 \arrow[r, "v_2"] & Y. \end{tikzcd}$$

The "only if" part of 3.1 is obvious, and we now prove the "if" part. Using (A1)–(A5) it is straightforward to show that $\mathbf{F}_f$ is cocomplete. Since (SSC) holds, the existence theorem of [7, p. 116] now shows that $\mathbf{F}_f$ has a terminal object $X \xrightarrow{\hat{u}} B^\wedge \xrightarrow{\hat{v}} Y$. Clearly $\hat{u} \in \mathbf{E}$ and we claim that $\hat{v} \in \mathcal{M}(\mathbf{E})$. For this it suffices by a push-out argument to show that a unique lifting exists in each commutative diagram

$$\begin{tikzcd} B^\wedge \arrow[r, "1"] \arrow[d, "u"] & B^\wedge \arrow[d, "\hat{v}"] \\ C \arrow[r, "g"] \arrow[ur, dashed] & Y \end{tikzcd}$$

with $u \in \mathbf{E}$. This follows because the map

$$\begin{tikzcd} X \arrow[r, "\hat{u}"] \arrow[d, "1"] & B^\wedge \arrow[r, "\hat{v}"] \arrow[d, "u"] & Y \arrow[d, "1"] \\ X \arrow[r, "u\hat{u}"] & C \arrow[r, "g"] & Y \end{tikzcd}$$

in $\mathbf{F}_f$ has a unique left inverse since its domain is terminal.

In practice it is easy to obtain

**3.2 Classes $E$ satisfying (A1)–(A5).** Some general examples in a category $\mathbf{C}$ are:

(i) For any class $S$ of maps in $\mathbf{C}$, let $E = \mathscr{E}(S)$.

(ii) Given a class of factorization systems $\{(E_\alpha, M_\alpha)\}$ in $\mathbf{C}$, let $E = \bigcap_\alpha E_\alpha$.

(iii) Given a factorization system $(E', M')$ in a category $\mathbf{C}'$, and given a functor $T : \mathbf{C} \to \mathbf{C}'$ which preserves colimits, let $E$ be the class of maps $f$ in $\mathbf{C}$ with $Tf \in E'$.

Although the solution set condition (SSC) is automatic in a small category, it is often difficult to verify in practice. We conclude with a general example where 3.1 does apply. This involves a category $\mathbf{C}$ satisfying:

(3.3) For each object $M \in \mathbf{C}$, there exists a set of maps $\{i_\alpha : L_\alpha \to M\}$ such that each map $g : K \to M$ can be factored as $g = i_\alpha s$ for some $\alpha$ and some epimorphism $s : K \to L_\alpha$.

Note that (3.3) holds when $\mathbf{C}$ is the category of sets, groups, modules over a ring, topological spaces, etc.; in these categories, $\{i_\alpha : L_\alpha \to M\}$ can consist of inclusion maps from subobjects of $M$.

**3.4 Theorem.** *Let $\mathbf{C}$ be a cocomplete category satisfying* (3.3) *and having products over arbitrary index sets; let $\{B_\beta\}$ be a set of objects in $\mathbf{C}$; and let $E$ consist of all $u : V \to W$ in $\mathbf{C}$ such that*

$$u^* : \mathbf{C}(W, B_\beta) \approx \mathbf{C}(V, B_\beta)$$

*for all $\beta$. Then $(E, \mathcal{M}(E))$ is a factorization system in $\mathbf{C}$.*

**Proof.** Since $E$ clearly satisfies (A1)–(A5), it suffices by 3.1 to verify (SSC). $E$ consists of all $u : V \to W$ such that

$$u^* : \mathbf{C}(W, D) \approx \mathbf{C}(V, D)$$

for $D = \Pi_\beta B_\beta$. For $f : X \to Y$ in $\mathbf{C}$ let $\{D_\gamma\}$ be copies of $D$ indexed by the elements $\gamma \in \mathbf{C}(X, D)$, and let

$$\left\{ i_\alpha : L_\alpha \to Y \times \prod_\gamma D_\gamma \right\}_{\alpha \in J}$$

be a set of maps given by (3.3) for $M = Y \times \Pi_\gamma D_\gamma$. Let $R$ be the set of factorizations

$$X \xrightarrow{\quad} L_\alpha \xrightarrow{p i_\alpha} Y$$

of $f$ such that $\alpha \in J$, $r \in E$, and $p : Y \times \Pi_\gamma D_\gamma \to Y$ is the projection. Thus $R$ is a set of objects of $\mathbf{F}_f$, and it suffices to show that each object $X \xrightarrow{u} B \xrightarrow{v} Y$ of $\mathbf{F}_f$ maps to some member of $R$. First factor $v$ as

$$B \xrightarrow{w} Y \times \prod_\gamma D_\gamma \xrightarrow{p} Y$$

where $w$ is induced by $v : B \to Y$ and by the unique maps $\bar{\gamma} : B \to D_\gamma = D$ such that $\bar{\gamma}u = \gamma$ for $\gamma \in \mathbf{C}(X, D) \approx \mathbf{C}(B, D)$. Then factor $w$ as

$$B \xrightarrow{s} L_\alpha \xrightarrow{i_\alpha} Y \times \prod_\gamma D_\gamma$$

where $s$ is epi and $\alpha \in J$. Since

$$s^*(i_\alpha)^* = w^* : \mathbf{C}\Big(Y \times \prod_\gamma D_\gamma, D\Big) \to \mathbf{C}(B, D)$$

is onto and since $s$ is epi, it follows that

$$s^* : \mathbf{C}(L_\alpha, D) \approx \mathbf{C}(B, D).$$

Thus $s \in \boldsymbol{E}$ and we have found a map

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "1"] & B \arrow[r, "v"] \arrow[d, "s"] & Y \arrow[d, "1"] \\ X \arrow[r, "su"] & L_\alpha \arrow[r, "pi_\alpha"] & Y \end{tikzcd}$$

in $\mathbf{F}_f$ with target in $R$.

## 4. A second existence theorem for factorization systems

We will derive a second existence theorem (4.1) which avoids the solution set condition (SSC), and will then prove a technical lemma needed for applications. The theorem involves a cocomplete category $\mathbf{C}$ whose objects are "$s$-definite" (see 4.2); it applies, for instance, when $\mathbf{C}$ is the category of groups or modules over a ring.

**4.1 Theorem.** *Let $\mathbf{C}$ be a cocomplete category whose objects are $s$-definite, and let $\boldsymbol{S}$ be a set of maps in $\mathbf{C}$. Then $(\mathscr{S}(\boldsymbol{S}), \mathscr{M}(\boldsymbol{S}))$ is a factorization system in $\mathbf{C}$, where $\mathscr{S}(\boldsymbol{S})$ is the smallest class of maps in $\mathbf{C}$ containing $\boldsymbol{S}$ and satisfying* (A1)–(A5).

This will be proved in 4.5, and will be generalized in 7.1. We first explain our terminology.

**4.2 $s$-definite objects.** Roughly speaking, an object $X$ of a cocomplete category $\mathbf{C}$ is "$s$-definite" if the functor $\mathbf{C}(X, \ )$ preserves colimits of sufficiently long sequences. To be precise, for an infinite cardinal number $\beta$, let $\mathrm{Ord}[\beta]$ denote the smallest ordinal number of cardinality $\beta$, and let $\mathbf{Seq}[\beta]$ denote the well-ordered set of ordinals less than $\mathrm{Ord}[\beta]$. We regard $\mathbf{Seq}[\beta]$ as a category in the usual way, i.e. $\mathbf{Seq}[\beta](s, t)$ has one element if $s \leq t$ and is empty otherwise. Now an object $X \in \mathbf{C}$

is called *s-definite* if there exists an infinite cardinal $\alpha$ such that for each cardinal $\beta \geqslant \alpha$ and for each functor $F:\mathbf{Seq}[\beta]\to \mathbf{C}$ the canonical map

$$\operatorname{Colim} \mathbf{C}(X, F(s)) \to \mathbf{C}(X, \operatorname{Colim} F(s))$$

is a bijection.

To obtain examples we need

**4.3 Lemma.** *The s-definite objects of a cocomplete category* $\mathbf{C}$ *are closed under colimits* (*over small index categories*).

**Proof.** The $s$-definite objects of $\mathbf{C}$ are closed under finite colimits because, in the category of sets, finite limits commute with small filtered colimits [7, p. 211]. Thus, by [7, p. 109] it remains to show that $\coprod_{j\in J} X_j$ is $s$-definite whenever $\{X_j\}_{j\in J}$ is a set of $s$-definite objects of $\mathbf{C}$. This is easily proved using the following fact: If the cardinality of $J$ is less than $\beta$, then each set of objects of $\mathbf{Seq}[\beta]$ indexed by $J$ has an upper bound in $\mathbf{Seq}[\beta]$.

**4.4 Examples.** In the category **Grp** of groups, all objects are $s$-definite by 4.3, because every group can be built from infinite cyclic groups by using successive colimits. Similarly, in the category $R$-**Mod** of left modules over a ring $R$, all objects are $s$-definite. However, in the category **Top** of topological spaces, only the discrete spaces are $s$-definite.

**4.5 Proof of 4.1.** For maps $\varphi : A \to B$ and $\mu : X \to Y$, we say $\mu$ *has the right lifting property* (*the RLP*) *for* $\varphi$ if for each commutative diagram

$$\begin{tikzcd} A \arrow[r, "\theta"] \arrow[d, "\varphi"'] & X \arrow[d, "\mu"] \\ B \arrow[r, "\tau"'] \arrow[ur, dashed, "\lambda"] & Y \end{tikzcd}$$

there exists a map $\lambda$ such that $\lambda\varphi = \theta$ and $\mu\lambda = \tau$. For a map $\varphi : A \to B$ in $\mathbf{C}$, let $\varphi' : B \coprod_A B \to B$ denote the map induced by the commutative square

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & B \arrow[d, "1"] \\ B \arrow[r, "1"'] & B. \end{tikzcd}$$

Then a map $\mu : X \to Y$ in $\mathbf{C}$ has the URLP for $\varphi$ if and only if $\mu$ has the RLP for both $\varphi$ and $\varphi'$. Let $\sigma : M \to N$ denote the coproduct of all the maps in $\boldsymbol{S} \cup \boldsymbol{S}'$, where $\boldsymbol{S}' = \{u' \mid u \in \boldsymbol{S}\}$. Then $\sigma \in \mathcal{S}(\boldsymbol{S})$; moreover, a map $w$ is in $\mathcal{M}(\boldsymbol{S})$ if and only if $w$ has the RLP for $\sigma$. Since $M$ is $s$-definite, we can choose an infinite cardinal $\beta$ such that

$$\operatorname*{Colim}_s \mathbf{C}(M, F(s)) \xrightarrow{\approx} \mathbf{C}(M, \operatorname*{Colim}_s F(s))$$

for each functor $F : \mathbf{Seq}[\beta] \to \mathbf{C}$. We now proceed to construct an $(\mathcal{S}(\boldsymbol{S}), \mathcal{M}(\boldsymbol{S}))$-factorization of $f : X \to Y \in \mathbf{C}$. By transfinite induction, we form a ladder

$$\begin{tikzcd} X = X_0 \arrow[r, "i_0"] \arrow[d, "f = u_0"] & X_1 \arrow[r] \arrow[d, "u_1"] & \cdots \arrow[r] & X_t \arrow[r, "i_t"] \arrow[d, "u_t"] & X_{t+1} \arrow[r] \arrow[d, "u_{t+1}"] & \cdots \\ Y \arrow[r, "1"] & Y \arrow[r] & \cdots \arrow[r] & Y \arrow[r, "1"] & Y \arrow[r] & \cdots \end{tikzcd}$$

in $\mathbf{C}$ for $t \in \mathbf{Seq}[\beta]$ as follows. Given $u_t$ let $K$ be the set of maps from $\sigma$ to $u_t$. Using the obvious diagram

$$\begin{tikzcd} \coprod_{k \in K} M_k \arrow[r] \arrow[d] & X_t \arrow[d, "u_t"] \\ \coprod_{k \in K} N_k \arrow[r] & Y \end{tikzcd}$$

where $M_k \to N_k$ equals $\sigma : M \to N$, let $X_{t+1}$ be the push-out of the top and left maps, and define the maps $X_t \xrightarrow{i_t} X_{t+1} \xrightarrow{u_{t+1}} Y$ in the obvious way. If $\lambda \in \mathbf{Seq}[\beta]$ is a limit ordinal and the ladder is given for all $t < \lambda$, let $X_\lambda = \operatorname{Colim}_{t<\lambda} X_t$ and $u_\lambda = \operatorname{Colim}_{t<\lambda} u_t$. This completes the construction of the ladder, and we let $X \xrightarrow{\hat{\imath}} X^\wedge \xrightarrow{\hat{u}} Y$ be the associated factorization of $f$ with $X^\wedge = \operatorname{Colim}_{t \in \mathbf{Seq}[\beta]} X_t$. It is straightforward to show $\hat{\imath} \in \mathcal{S}(\boldsymbol{S})$, $\hat{u} \in \mathcal{M}(\boldsymbol{S})$, and thus $(\mathcal{S}(\boldsymbol{S}), \mathcal{M}(\boldsymbol{S}))$ is a factorization system in $\mathbf{C}$.

**4.6 Remark.** The "$s$-definite" hypothesis in 4.1 is actually needed only for the domains and codomains of maps in $\boldsymbol{S}$.

We conclude with a technical lemma needed for certain applications of 4.1. It will involve the notion of a *small filtered colimit*, i.e. a colimit whose index category is a small filtered category (see [7, p. 207]).

**4.7 Lemma.** *Let $\mathbf{D}$ be a cocomplete category, let $T : \mathbf{D} \to \mathbf{Set}_*$ ( = the category of sets with basepoint $*$) be a functor which preserves small filtered colimits, and let $\boldsymbol{S} \subset \operatorname{ob} \mathbf{D}$ denote the class of objects $X \in \mathbf{D}$ with $TX = *$. Suppose there exists a set $\boldsymbol{K} \subset \operatorname{ob} \mathbf{D}$ such that each object in $\mathbf{D}$ is a small filtered colimit of objects in $\boldsymbol{K}$. Then there exists a set $\boldsymbol{L} \subset \boldsymbol{S}$ such that each object in $\boldsymbol{S}$ is a small filtered colimit of objects in $\boldsymbol{L}$.*

**Proof.** Let $\alpha$ be an infinite cardinal which is an upper bound for the cardinalities of the sets $\{TX \mid X \in \boldsymbol{K}\}$. Let $\boldsymbol{L}' \subset \boldsymbol{S}$ be the class of all objects $W \in \boldsymbol{S}$ such that $W$ is a colimit of objects in $\boldsymbol{K}$ over a filtered index category $\mathbf{J}$ with $\#\mathbf{J} \leqslant \alpha$ (where $\#\mathbf{J}$ is the number of maps in $\mathbf{J}$). Let $\boldsymbol{L} \subset \boldsymbol{L}'$ be a set of representatives of the isomorphism classes in $\boldsymbol{L}'$. It remains to express an object $X \in \boldsymbol{S}$ as a small filtered colimit of objects in $\boldsymbol{L}$. We may suppose $X = \mathrm{Colim}_{i\in \mathbf{I}} F(i)$ where $\mathbf{I}$ is a small filtered category and $F(i) \in \boldsymbol{K}$ for all $i \in I$. We claim that for each set $\boldsymbol{G}$ of maps in $\mathbf{I}$ with $\#\boldsymbol{G} \leqslant \alpha$, there exists a filtered sub-category $\mathbf{H} \subset \mathbf{I}$ such that: (i) each $g \in \boldsymbol{G}$ is a map in $\mathbf{H}$, (ii) $\#\mathbf{H} \leqslant \alpha$, and (iii) $\mathrm{Colim}_{h\in\mathbf{H}} TF(h) = *$. This follows by taking $\mathbf{H} = \bigcup_n \mathbf{H}_n$ where $\mathbf{H}_1 \subset \cdots \subset \mathbf{H}_n \subset \mathbf{H}_{n+1} \subset \cdots$ is a sequence of filtered sub-categories of $\mathbf{I}$ constructed so that $\boldsymbol{G}$ is in $\mathbf{H}_1$, $\#\mathbf{H}_n \leqslant \alpha$, and the maps

$$\mathrm{Colim}_{h\in\mathbf{H}_n} TF(h) \to \mathrm{Colim}_{h\in\mathbf{H}_{n+1}} TF(h)$$

are trivial. Now let $\mathscr{H}$ be the partially ordered set of filtered sub-categories $\mathbf{H} \subset \mathbf{I}$ such that $\#\mathbf{H} \leqslant \alpha$ and $\mathrm{Colim}_{h\in\mathbf{H}} TF(h) = *$. By the above claim, $\mathscr{H}$ is filtered and

$$X \approx \mathrm{Colim}_{\mathbf{H}\in\mathscr{H}}(\mathrm{Colim}_{h\in\mathbf{H}} F(h)).$$

Thus $X$ can be expressed as a small filtered colimit of objects in $\boldsymbol{L}$.

## 5. Examples of factorization systems

We now give examples which illustrate the use of our existence theorems.

**5.1 Example.** In the category **Top** of topological spaces, let $\boldsymbol{E}_1$ be the class of all maps $X \to Y$ inducing a bijection $\mathbf{Top}(Y, I) \approx \mathbf{Top}(X, I)$ where $I$ is the closed unit interval. Then $(\boldsymbol{E}_1, \mathcal{M}(\boldsymbol{E}_1))$ is a factorization system in **Top** by 3.4. One can show that the $(\boldsymbol{E}_1, \mathcal{M}(\boldsymbol{E}_1))$-localization (2.5) on **Top** is just the Stone–Cech compactification (cf. [7, p. 127]).

**5.2 Example.** In the category **Top Grp** of topological groups, let $\boldsymbol{E}_2$ be the class of all maps $X \to Y$ inducing a bijection

$$\mathbf{Top\ Grp}(Y, G) \approx \mathbf{Top\ Grp}(X, G)$$

for each finite discrete group $G$. Then $(\boldsymbol{E}_2, \mathcal{M}(\boldsymbol{E}_2))$ is a factorization system in **Top Grp** by 3.4. One can show that the $(\boldsymbol{E}_2, \mathcal{M}(\boldsymbol{E}_2))$-localization on **Top Grp** is just the profinite completion.

**5.3 Example.** In the category **Grp** of groups, let $\boldsymbol{E}_3$ be the class of all maps $X \to Y$ inducing a bijection $\mathbf{Grp}(Y, G) \approx \mathbf{Grp}(X, G)$ for each finite group $G$. Then $(\boldsymbol{E}_3, \mathcal{M}(\boldsymbol{E}_3))$ is a factorization system in **Grp** by 3.4. The $(\boldsymbol{E}_3, \boldsymbol{M}_3)$-localization is a discrete analogue of the profinite completion.

**5.4 Example.** For an abelian group $G$, let $\boldsymbol{E}_4$ be the class of all maps $f : X \to Y$ in **Grp** with $H_1(X; G) \to H_1(Y; G)$ epi (using simple coefficients). Then $(\boldsymbol{E}_4, \mathcal{M}(\boldsymbol{E}_4))$ is a factorization system in **Grp** by 3.1, where (SSC) holds because a factorization $X \xrightarrow{u} B \xrightarrow{v} Y$ in $\mathbf{F}_f$ maps to the factorization $X \to v(B) \to Y$ in $\mathbf{F}_f$. If $G$ is a cyclic ring or subring of the rationals, one can show that $\mathcal{M}(\boldsymbol{E}_4)$ consists of all injections $i: X \to Y$ in **Grp** such that $i(X)$ is "$HM$-closed" in $Y$ in the sense of [3].

**5.5 Example.** For an abelian group $G$, let $\boldsymbol{E}_5$ be the class of all maps $f: X \to Y \in \mathbf{Grp}$ such that $f_*: H_i(X; G) \to H_i(Y; G)$ is iso for $i = 1$ and epi for $i = 2$. We will apply 4.1 to show that $(\boldsymbol{E}_5, \mathcal{M}(\boldsymbol{E}_5))$ is a factorization system in **Grp**. First let **D** be the category of maps in **Grp**, and let $T: \mathbf{D} \to \mathbf{Set}_*$ be the functor carrying the map $f: X \to Y \in \mathbf{Grp}$ to the underlying pointed set of $\ker_1 \oplus \operatorname{coker}_1 \oplus \operatorname{coker}_2$ where $\ker_n$ is the kernel of $f_*: H_n(X; G) \to H_n(Y; G)$ and $\operatorname{coker}_n$ is the cokernel. Then 4.7 shows the existence of a set $\boldsymbol{L} \subset \boldsymbol{E}_5$ such that each member of $\boldsymbol{E}_5$ is a small filtered colimit of members of $\boldsymbol{L}$. Thus $(\mathcal{S}(\boldsymbol{L}), \mathcal{M}(\boldsymbol{L}))$ is a factorization system by 4.1. It is straightforward to show $\mathcal{S}(\boldsymbol{L}) = \boldsymbol{E}_5$ and $\mathcal{M}(\boldsymbol{L}) = \mathcal{M}(\boldsymbol{E}_5)$, and consequently $(\boldsymbol{E}, \mathcal{M}(\boldsymbol{E}_5))$ is a factorization system in **Grp**. If $G$ is a cyclic ring or a subring of the rationals, then the $(\boldsymbol{E}_5, \mathcal{M}(\boldsymbol{E}_5))$-localization on **Grp** is the $HG$-localization studied in [2] and [3].

**5.6 Example.** In the category **R-Mod** of left modules over a ring $R$, let $\boldsymbol{E}_6$ be the class of all maps $f: X \to Y \in \mathbf{R}\text{-}\mathbf{Mod}$ such that $G \otimes_R X \to G \otimes_R Y$ is iso and $\operatorname{Tor}_1^R(G, X) \to \operatorname{Tor}_1^R(G, Y)$ is epi, where $G$ is a fixed right $R$-module. Then $(\boldsymbol{E}_6, \mathcal{M}(\boldsymbol{E}_6))$ is a factorization system in **R-Mod** by an argument like that in 5.5. If $R = Z$ and $G = Z/p$ for $p$ prime, then the $(\boldsymbol{E}_6, \mathcal{M}(\boldsymbol{E}_6))$-localization in **Z-Mod** is just the Ext completion $X \to \operatorname{Ext}(Z_{p^\infty}, X)$ as in [4, p. 171], and the $(\boldsymbol{E}_6, \mathcal{M}(\boldsymbol{E}_6))$-colocalization is given by $\operatorname{Hom}_Z(Z[1/p], X) \to X$. If $\pi$ is a group, $R = Z\pi$ (the group ring), and $G = Z$ with trivial $\pi$-action, then the $(\boldsymbol{E}_6, \mathcal{M}(\boldsymbol{E}_6))$-localization in **Z$\pi$-Mod** is just the $HZ$-localization introduced in [2, Section 8].

**5.7 Example.** For some $G \in \mathbf{R}\text{-}\mathbf{Mod}$, let $\boldsymbol{M}_7$ be the class of all maps $f: X \to Y \in \mathbf{R}\text{-}\mathbf{Mod}$ such that $f_*: \operatorname{Hom}_R(G, X) \to \operatorname{Hom}_R(G, Y)$ is iso and $f_*: \operatorname{Ext}_R^1(G, X) \to \operatorname{Ext}_R^1(G, Y)$ is mono. We will show that $(\mathscr{E}(\boldsymbol{M}_7), \boldsymbol{M}_7)$ is a factorization system in **R-Mod**. Choose a short exact sequence $0 \to W \xrightarrow{i} P \to G \to 0$ in **R-Mod** with $P$ projective. For a map $f: X \to Y \in \mathbf{R}\text{-}\mathbf{Mod}$, we have:

(i) $f \in \boldsymbol{M}_7$.

$\Longleftrightarrow$ (ii) $\operatorname{Hom}_{\mathbf{D}}(t, f) = 0 = \operatorname{Ext}_{\mathbf{D}}^1(t, f)$ where $t: G \to 0$ and **D** is the abelian category of maps in **R-Mod**.

$\Longleftrightarrow$ (iii) The map $\operatorname{Hom}_{\mathbf{D}}(1_P, f) \to \operatorname{Hom}_{\mathbf{D}}(i, f)$ is iso, i.e. $f \in \mathcal{M}(\{i\})$.

These equivalences follow using the $\operatorname{Ext}_{\mathbf{D}}(\ , f)$-sequences of $0 \to u \to 1_G \to t \to 0$ (with $u: 0 \to G$) and $0 \to i \to 1_P \to t \to 0$. We have shown $\boldsymbol{M}_7 = \mathcal{M}(\{i\})$, and thus $(\mathscr{E}(\boldsymbol{M}_7), \boldsymbol{M}_7)$ is a factorization system by 4.1. *When $R$ is a group ring and $G = Z$, the $(\mathscr{E}(\boldsymbol{M}_7), \boldsymbol{M}_7)$-colocalization is a cohomological analogue of the $HZ$-localization mentioned in* 5.6.

## 6. Homotopy factorization systems

We will introduce a homotopy theoretic notion of factorization system which generalizes the ordinary notion. It is convenient to use Quillen's framework of homotopical algebra [9], and we assume familiarity with

**6.1 Closed simplicial model categories.** These are defined in [9, II Section 2]. For a closed simplicial model category $\mathbf{C}$, let $\operatorname{ho}\mathbf{C}$ denote the associated homotopy category whose objects are the fibrant-cofibrant objects of $\mathbf{C}$ and whose maps are simplicial homotopy classes of maps in $\mathbf{C}$. Some basic examples are:

(i) The categories $\mathbf{S}$ of simplicial sets and $\mathbf{S}_*$ of pointed simplicial sets have standard closed simplicial model category structures [9]. Moreover, $\operatorname{ho}\mathbf{S}$ and $\operatorname{ho}\mathbf{S}_*$ are respectively equivalent to the homotopy categories of $CW$ complexes and pointed $CW$ complexes.

(ii) Any category $\mathbf{B}$ with finite limits and colimits has a "discrete" closed simplicial model category structure: fibrations = all maps; cofibrations = all maps; weak equivalences = isomorphisms; for $X, Y \in \mathbf{B}$, $\mathbf{Hom}(X, Y)$ = the discrete (i.e. constant) simplicial set on $\mathbf{B}(X, Y)$; for $X \in \mathbf{B}$ and finite $K \in \mathbf{S}$, $X \otimes K$ (resp. $X^K$) is a coproduct (resp. product) of copies of $X$ indexed by $\pi_0 K$. Clearly $\operatorname{ho}\mathbf{B} \approx \mathbf{B}$.

*In the rest of Section* 6, *let* $\mathbf{C}$ *be a closed simplicial model category.* For a cofibration $\varphi : A \to B$ and a fibration $\mu : X \to Y$ in $\mathbf{C}$, we say $\varphi$ *has the HLLP for* $\mu$, *or equivalently* $\mu$ *has the HRLP for* $\varphi$, if the Kan fibration

$$\mathbf{Hom}(B, X) \to \mathbf{Hom}(A, X) \times_{\mathbf{Hom}(A, Y)} \mathbf{Hom}(B, Y)$$

is a weak equivalence. For a class $\boldsymbol{T}$ of cofibrations and class $\boldsymbol{U}$ of fibrations let

$\mathscr{E}_H(\boldsymbol{T}) = \{\varphi \mid \varphi$ is a cofibration with the HLLP for each $\mu \in \boldsymbol{T}\}$

$\mathscr{M}_H(\boldsymbol{U}) = \{\mu \mid \mu$ is a fibration with the HRLP for each $\varphi \in \boldsymbol{U}\}$.

**6.2 Definition.** A *homotopy factorization system* $(\boldsymbol{E}, \boldsymbol{M})$ in $\mathbf{C}$ consists of classes $\boldsymbol{E}$ of cofibrations and $\boldsymbol{M}$ of fibrations such that:

(i) $\boldsymbol{E} = \mathscr{E}_H(\boldsymbol{M})$ and $\boldsymbol{M} = \mathscr{M}_H(\boldsymbol{E})$.

(ii) For every map $f$ in $\mathbf{C}$, there exist $f_m \in \boldsymbol{M}$ and $f_e \in \boldsymbol{E}$ such that $f = f_m f_e$.

The factorization in (ii) is unique up to a simplicial homotopy equivalence and is homotopically natural (i.e. in a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f_e"] \arrow[d, "u"] & V \arrow[r, "f_m"] \arrow[d, dashed, "h"] & X \arrow[d, "v"] \\ B \arrow[r, "g_e"] & W \arrow[r, "g_m"] & Y \end{tikzcd}$$

in $\mathbf{C}$ with $f_e, g_e \in \boldsymbol{E}$ and $f_m, g_m \in \boldsymbol{M}$, there exists a lifting $h$ unique up to simplicial homotopy).

**6.3 Examples.** (i) A homotopy factorization system in a "discrete" closed simplicial model category is just a factorization system in the underlying category.

(ii) In the category **S** (of simplicial sets) and for $n \geqslant 0$, let $\boldsymbol{E}$ be the class of $n$-connected cofibrations, and let $\boldsymbol{M}$ be the class of fibrations whose fibres have vanishing $i$th homotopy groups for all $i \geqslant n$. Then $(\boldsymbol{E}, \boldsymbol{M})$ is a (Moore–Postnikov) homotopy factorization system.

(iii) Let $h_*$ be an additive generalized homology theory on **S**, let $\boldsymbol{E}$ be the class of cofibrations which are $h_*$-equivalences, and let $\boldsymbol{M}$ be the class of $h_*$-fibrations ([2, 10.1]). Then $(\boldsymbol{E}, \boldsymbol{M})$ is a homotopy factorization system by [2].

Other examples are in Section 7, where we will need

**6.4 Lemma.** *If $(\boldsymbol{E}, \boldsymbol{M})$ is a homotopy factorization system in* **C** *(or more generally if $\boldsymbol{E} = \mathscr{E}_H(\boldsymbol{T})$ for a class $\boldsymbol{T}$ of fibrations), then:*

(B1) *Every trivial cofibration is in $\boldsymbol{E}$.*

(B2) *$\boldsymbol{E}$ is closed under composition.*

(B3) *If $gf \in \boldsymbol{E}$, $f \in \boldsymbol{E}$, and $g$ is a cofibration, then $g \in \boldsymbol{E}$.*

(B4) *If*

$$\begin{array}{ccc} V & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ W & \longrightarrow & Y \end{array}$$

*is a push-out diagram in* **C** *and $i \in \boldsymbol{E}$, then $j \in \boldsymbol{E}$.*

(B5) *$\boldsymbol{E}$ is closed under coproducts (when they exist).*

(B6) *For* $\mathbf{Seq}[\beta]$ *as in* 4.2, *let* $F : \mathbf{Seq}[\beta] \to \mathbf{C}$ *be a functor such that* $\operatorname{Colim}_{t<s} F(t) \approx F(s)$ *for each limit ordinal* $s \in \mathbf{Seq}[\beta]$. *If* $F(s) \to F(s+1)$ *is in* $\boldsymbol{E}$ *for each* $s \in \mathbf{Seq}[\beta]$, *then* $F(0) \to \operatorname{Colim} F$ *is in* $\boldsymbol{E}$ *(when* $\operatorname{Colim} F$ *exists).*

(B7) *$\boldsymbol{E}$ is closed under retracts.*

(B8) *If $f : X \to Y$ is in $\boldsymbol{E}$ and $K \subset L \in \mathbf{S}$ are finite, then the map*

$$f \otimes (L, K) : (X \otimes L) \amalg_{(X \otimes K)} (Y \otimes K) \to Y \otimes L$$

*is in $\boldsymbol{E}$.*

The proof is not hard. Of course, there is an obvious dual result for a class $\boldsymbol{M}$. A homotopy factorization system $(\boldsymbol{E}, \boldsymbol{M})$ in **C** gives rise to

**6.5 Localizations and colocalizations.** As in 2.5 there is an idempotent $(\boldsymbol{E}, \boldsymbol{M})$-*localization*, $T : \operatorname{ho}\mathbf{C} \to \operatorname{ho}\mathbf{C}$ and $\eta : 1 \to T$. Moreover, $T$ provides a left adjoint to the inclusion functor Loc-ho**C** $\xrightarrow{\subset}$ ho**C**, where Loc-ho**C** denotes the full subcategory given by all $X \in \operatorname{ho}\mathbf{C}$ such that the map from $X$ to the terminal object is in $\boldsymbol{M}$. Note also, for $X \in \operatorname{ho}\mathbf{C}$, that $\eta : X \to TX \in \operatorname{ho}\mathbf{C}$ is the universal (terminal) example of a homotopy class of a map in $\boldsymbol{E}$ with domain $X$. Dually, there is an $(\boldsymbol{E}, \boldsymbol{M})$-*colocalization* on ho**C**.

Finally, note that a homotopy factorization system $(\boldsymbol{E}, \boldsymbol{M})$ on $\mathbf{C}$ gives rise to obvious homotopy factorization systems on $\mathbf{C}/c$ and $c/\mathbf{C}$ for $c \in \mathbf{C}$ (see [9, II 2.8]). There are, of course, associated localizations and colocalizations on $\mathrm{ho}(\mathbf{C}/c)$ and $\mathrm{ho}(c/\mathbf{C})$.

## 7. Construction of homotopy factorization systems

We will generalize 4.1 to give an existence theorem for homotopy factorization systems. Let $\mathbf{C}$ be a closed simplicial model category such that:

(i) $\mathbf{C}$ is cocomplete.

(ii) The objects of $\mathbf{C}$ are $s$-definite (see 4.2).

(iii) There is a set $\boldsymbol{K}$ of trivial cofibrations in $\mathbf{C}$ such that a map is a fibration if it has the RLP for all members of $\boldsymbol{K}$.

Note that (i)–(iii) hold for the category of simplicial sets, and that (iii) holds for any "discrete" closed simplicial model category (see 6.1 (ii)).

**7.1 Theorem.** *If $\boldsymbol{T}$ is a set of cofibrations in $\mathbf{C}$, then $(\mathscr{S}_H(\boldsymbol{T}), \mathscr{M}_H(\boldsymbol{T}))$ is a homotopy factorization system in $\mathbf{C}$, where $\mathscr{S}_H(\boldsymbol{T})$ is the smallest class of cofibrations in $\mathbf{C}$ containing $\boldsymbol{T}$ and satisfying the conditions* (B1)–(B8) *of* 6.5.

**Proof.** Let $\boldsymbol{T}'$ be the set of cofibrations

$$\boldsymbol{T}' = \boldsymbol{K} \cup \{f \otimes (\Delta^n, \dot{\Delta}^n) \mid f \in \boldsymbol{T} \quad \text{and} \quad n \geq 0\}$$

where $\Delta^n$ is the standard $n$-simplex, $\dot{\Delta}^n$ is its "boundary", and $f \otimes (\Delta^n, \dot{\Delta}^n)$ is as in (B8). Then $\boldsymbol{T}' \subset \mathscr{S}_H(\boldsymbol{T})$, and a map is in $\mathscr{M}_H(\boldsymbol{T})$ iff it has the RLP for each member of $\boldsymbol{T}'$. The proof now proceeds as in 4.5 using $\boldsymbol{T}'$ in place of $\boldsymbol{S} \cup \boldsymbol{S}'$.

The hypothesis that the objects of $\mathbf{C}$ be $s$-definite is somewhat stronger than necessary (cf. 4.6).

**7.2 Corollary.** *For a set $\boldsymbol{T}$ of cofibrations in $\mathbf{C}$, the inclusion functor* $\text{Loc-ho}\mathbf{C} \xrightarrow{\subset} \mathrm{ho}\mathbf{C}$ *has a left adjoint, where* $\text{Loc-ho}\mathbf{C}$ *is the full subcategory given by all $X \in \mathrm{ho}\mathbf{C}$ such that*

$$\mathbf{Hom}(j, X) : \mathbf{Hom}(B, X) \to \mathrm{Hom}(A, X)$$

*is a weak equivalence for each $j : A \to B$ in $\boldsymbol{T}$.*

We illustrate 7.2 by two examples.

**7.3 Example.** In the category $\mathbf{S}_*$ and for a set $J$ of primes, let

$$\boldsymbol{T} = \{K(p, 1) : K(Z, 1) \to K(Z, 1)\}_{p \in J}$$

i.e. $T$ contains a cofibration of degree $p$ between "circles" for each $p \in J$. Then Loc-ho$\mathbf{S}_*$ is given by all $X \in \mathrm{ho}\,\mathbf{S}_*$ such that $\pi_n X$ is uniquely $p$-divisible for all $p \in J$ and $n \geq 1$. The left adjoint functor $\mathrm{ho}\,\mathbf{S}_* \to$ Loc-ho$\mathbf{S}_*$ is essentially Anderson's localization [1].

**7.4 Example.** In the category $\mathbf{S}_*$ and for $p$ prime, let $T$ consist of a cofibration corresponding to

$$K(f, 1) : K(F, 1) \to K(F, 1)$$

where $F$ is the free group on generators $x_0, x_1, x_2, \ldots$ and $f : F \to F$ is the homomorphism with $fx_i = x_i(x_{i+1})^{-p}$ for $i \geq 0$. Then Loc-ho$\mathbf{S}_*$ is given by all $X \in \mathrm{ho}\,\mathbf{S}_*$ such that $\pi_n X$, for $n \geq 1$, satisfies the Exp-$p$-completeness condition of [4, p. 175], i.e. the function

$$L : (\pi_n X \times \pi_n X \times \pi_n X \times \cdots) \to (\pi_n X \times \pi_n X \times \pi_n X \times \cdots)$$

is a bijection where

$$L(u_0, u_1, u_2, \ldots) = (u_0(u_1)^{-p}, u_1(u_2)^{-p}, u_2(u_3)^{-p}, \ldots).$$

The left adjoint functor $\mathrm{ho}\,\mathbf{S}_* \to$ Loc-ho$\mathbf{S}_*$ is an Anderson-like $p$-completion functor which reduces to the $p$-profinite completion on simply connected spaces with finitely generated homotopy groups.

We conclude with another easy corollary of 7.1.

**7.5 Corollary.** *For a set $\{A_\alpha\}$ of cofibrant objects in $\mathbf{C}$ and for $X \in \mathrm{ho}\,\mathbf{C}$, there is a terminal example, $UX \to X \in \mathrm{ho}\,\mathbf{C}$, among the maps $W \to X$ in $\mathrm{ho}\,\mathbf{C}$ which induce a weak equivalence $\mathbf{Hom}(A_\alpha, W) \to \mathbf{Hom}(A_\alpha, X)$ for each $A_\alpha$.*

One can regard $UX \to X \in \mathrm{ho}\,\mathbf{C}$ as a "colocalization of $X$ with respect to homotopy".

## References

[1] D.W. Anderson, Localizing CW-complexes, Illinois J. Math. 16 (1972), 519–525.
[2] A.K. Bousfield, The localization of spaces with respect to homology, Topology 14 (1975) 133–150.
[3] A.K. Bousfield, Homological localization towers for groups and $\pi$-modules, Memoirs Amer. Math. Soc., to appear.
[4] A.K. Bousfield and D.M. Kan, Homotopy Limits, Completions and Localizations, in: Lecture Notes in Mathematics, Vol. 304 (Springer-Verlag, New York, 1972).
[5] A. Deleanu, Existence of the Adams completion for objects of cocomplete categories, J. Pure Appl. Algebra 6 (1975) 31–39.
[6] M. Kelley, Monomorphisms, epimorphisms and pullbacks, J. Austr. Math. Soc. 9 (1969) 124–142.
[7] S. MacLane, Categories for the Working Mathematician (Springer-Verlag, New York, 1971).
[8] J. Meisen, On bicategories of relations and pullback spans, Comm. in Algebra 1, 5 (1974) 377–401.
[9] D.G. Quillen, Homotopical Algebra, in: Lecture Notes in Mathematics, Vol. 43 (Springer-Verlag, New York, 1967).