CHAPTER VIII

POWER OPERATIONS IN H_∞^d RING THEORIES

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It was shown in Chapter I that an H_∞^d ring structure on a spectrum E induces certain operations $\mathcal{P}_j$ in E-cohomology. In this chapter we investigate these operations in some important special cases, namely ordinary cohomology, K-theory, and cobordism.

In section 1 we collect the properties of the $\mathcal{P}_j$ and their internal variants P_j; most of these have already been shown in Chapter I. We also show that the results of Chapter VII allow one to construct an H_∞^d structure on E by giving space-level operations with certain properties. The section concludes with a brief account of a multiplicative transfer in E-cohomology which generalizes the norm map of Evens [35].

In section 2 we show that the general facts given in section 1 are strong enough to prove the usual properties of the Steenrod operations without any use of chain-level arguments. In section 3 we show that the same arguments applied to the spectrum $HZ_p \wedge X$ give the Dyer-Lashof operations in $H_*(X;Z_p)$ with all of their usual properties; in particular, we give new proofs of the Adem and Nishida relations which involve less calculation than the standard proofs.

In section 4 we show that the power operations in K-theory induced by the H_∞^d structures on KU and KO are precisely those defined by Atiyah [17]; this gives a rather concrete description of these H_∞^d structures. In section 5 we show that cobordism operations defined by tom Dieck in [31] lead to H_∞^d structures on the classical cobordism spectra which agree with their E_∞ structures; again, this fact gives a rather concrete homotopical description of the E_∞ structure. In section 6 we show that the Atiyah-Bott-Shapiro orientations are H_∞^d ring maps; it is still an open question whether they are E_∞ maps.

In section 7 we show that questions about H_∞^d ring maps simplify considerably when the spectra involved are p-local. We use this to show that the Adams operations are H_∞ ring maps (a fact which will be important in Chapter IX) and that the Adams summand of p-local K-theory is an H_∞^2 ring spectrum. We also give a sufficient condition for BP to be an H_∞^2 ring spectrum; however the question of whether it actually is an H_∞^2 ring spectrum remains open.

Notation. In chapters VIII and IX we shall write ΣX for $S^1 \wedge X$, instead of $X \wedge S^1$ as in chapters I-VII. We shall also use Σ to denote the suspension isomorphism $\tilde{E}^n X \to \tilde{E}^{n+1} X$. In particular, if E is a ring spectrum the fundamental class in $\tilde{E}^n S^n$ will be denoted by $\Sigma^n 1$.

§1. General properties of power operations

Let E and F be spectra, let π be a subgroup of Σ_k, and let d be a fixed positive integer. By a power operation on $\bar{h}\mathcal{S}$ in the most general sense we mean simply a sequence $\mathcal{P}_\pi$ of natural transformations

$$E^{di}X \to F^{dik}D_\pi X,$$

one for each $i \in Z$, which are defined for all $X \in \bar{h}\mathcal{S}$. We shall also call $\mathcal{P}_\pi$ an (E,π,F) power operation when it is necessary to be more specific. In this section we consider the relation between power operations, extended pairings, and H_∞^d ring structures. In particular, we collect the properties of the canonical power operations associated to an H_∞^d ring structure and of the related internal operations.

The most important class of power operations for us will be the operations

$$\mathcal{P}_\pi : E^{di}X \to E^{dij}D_\pi X$$

determined by an H_∞^d ring structure on E. As usual, we abbreviate $\mathcal{P}_{\Sigma_j}$ by $\mathcal{P}_j$. Recall the definition from I§4: if $x \in E^{di}X$ is represented by $f: X \to \Sigma^{di}E$ then $\mathcal{P}_\pi x$ is represented by the composite

$$D_\pi X \xrightarrow{\iota} D_k X \xrightarrow{D_k f} D_k \Sigma^{di} E \xrightarrow{\xi_{k,i}} \Sigma^{dik} E.$$

Our first result collect the properties of these operations.

Proposition 1.1. Let E be an H_∞^d ring spectrum and let $x \in E^{di}X$, $y \in E^{dj}Y$, $\pi \subset \Sigma_k$,

(i) $\alpha^* \mathcal{P}_{j+k} x = (\mathcal{P}_j x)(\mathcal{P}_k x) \in E^{d(j+k)i}(D_j X \wedge D_k X)$.

(ii) $\beta^* \mathcal{P}_{jk} x = \mathcal{P}_j \mathcal{P}_k x \in E^{djki}(D_j D_k X)$

(iii) $\delta^*[(\mathcal{P}_\pi x)(\mathcal{P}_\pi y)] = \mathcal{P}_\pi(xy) \in E^{d(i+j)k} D_\pi(X \wedge Y)$.

(iv) $i^* \mathcal{P}_\pi x = x^k \in E^{dij}(X^{(k)})$

(v) If $1 \in E^0 S$ is the unit then $\mathcal{P}_\pi 1$ is the unit in $E^0(D_\pi S) = E^0(B\pi^+)$.

(vi) If X = Y and i = j then

$$\mathcal{P}_k(x + y) = \mathcal{P}_k x + \mathcal{P}_k y + \sum_{0 < \ell < k} \tau^*_{\ell,k-\ell}[(\mathcal{P}_\ell x)(\mathcal{P}_{k-\ell} y)]$$

in $E^{dik}D_k X$, where

$$\tau_{\ell,k-\ell}: D_k X \longrightarrow D_\ell X \wedge D_{k-\ell} X$$

is the transfer defined in II.1.4.

(vii) If E is p-local then $\mathcal{P}_\pi x = \frac{1}{|\pi|} \tau_\pi^* x^k$ whenever $|\pi|$ is prime to p, where τ_π is the transfer $D_\pi X \to X^{(k)}$ of II.1.4.

(viii) If E is p-local then

$$\mathcal{P}_p(x+y) = \mathcal{P}_p x + \mathcal{P}_p y + \tau_p^*[\frac{1}{p!}((x+y)^p - x^p - y^p)].$$

Proof. (i), (ii), and (iii) are immediate from Definition I.4.3. Part (iv) follows from Remark I.4.4. Part (v) follows from I.3.4(i). Parts (vi) and (viii) were shown in II.2.1 and II.2.2, and part (vii) follows from the proof of the latter.

We shall also want to go in the other direction, that is, to start from a set of operations having certain properties and deduce the existence of an H_∞^d ring structure. Let E be a ring spectrum. We say that a set $\{\mathcal{P}_j\}_{j\geq 0}$ of (E, Σ_j, E) power operations is consistent if it satisfies 1.1(i), (ii), and (iii). Given a consistent set of operations $\mathcal{P}_j$ on E we can define maps

$$\xi_{j,i}: D_j \Sigma^{di} E \to \Sigma^{dij} E$$

by applying $\mathcal{P}_j$ to the classes represented by the identity maps $\Sigma^{di}E \to \Sigma^{di}E$. It is easy to see that the $\xi_{j,i}$ form an H_∞^d ring structure on E whose induced power operations are the given $\mathcal{P}_j$. On the other hand, two H_∞^d ring structures on E which determine the same power operations are clearly equal. Thus there is a one-to-one correspondence between H_∞^d ring structures on E and consistent sets of (E,Σ_j,E) power operations.

Next we consider a more general situation. Let π be a subgroup of Σ_k and let F be a π-oriented ring spectrum with orientation $\mu: D_\pi S^d \to \Sigma^{dk} F$ (see VII§3). The class in $F^{dk}(D_\pi S^d)$ represented by the orientation will also be denoted by μ. An (E,π,F) power operation $\mathcal{P}_\pi$ is stable if the equation

$$\mathcal{P}_\pi(\Sigma^d x) = \delta^*(\mu \cdot \mathcal{P}_\pi x) \tag{1}$$

holds in $F^{d(i+1)k}(D_\pi \Sigma^d X)$ for all $x \in E^{di}X$. 1.1(iii) implies that the (E,π,E) power operations determined by an H_∞^d ring structure on E are stable. More generally, let $\xi: D_\pi E \to F$ be any map (in the terminology of VII§3, ξ is called an extended pairing). If $x \in E^{di}X$ is represented by $f: X \to \Sigma^{di}E$ define $\mathcal{P}_\pi x \in F^{dik} D_\pi X$ to be the element represented by the composite

$$D_\pi X \xrightarrow{D_\pi f} D_\pi \Sigma^{di} E \xrightarrow{\delta} (D_\pi S^d)^{(i)} \wedge D_\pi E \xrightarrow{\mu^{(i)} \wedge \xi} (\Sigma^{dk} F)^{(i)} \wedge F \xrightarrow{\phi} \Sigma^{dik} F,$$

where ϕ is the product map for F. Then $\mathcal{P}_\pi$ is a stable power operation. Conversely, given a stable operation $\mathcal{P}_\pi$ we obtain a map $\xi:D_\pi E \to F$ by applying $\mathcal{P}_\pi$ to the identity map $E \to E$. Clearly, this gives a one-to-one correspondence between maps $\xi:D_\pi E \to F$ and stable power operations. To sum up, we have shown

Proposition 1.2. (i) There is a one-to-one correspondence between consistent sets of (E,Σ_j,E) power operations and H_∞^d ring structures on E.

(ii) If F is a π-oriented ring spectrum and E is any spectrum, there is a one-to-one correspondence between stable (E,π,F) power operations and maps $\xi:D_\pi E \to F$.

For applications of 1.2 it is usually easiest to work with space-level instead of spectrum-level power operations. Our next result will allow us to reduce to this case. Let $\mathcal{C}$ be the homotopy category of finite CW complexes. Let $\{(E\pi)_\alpha\}_{\alpha\in A}$ be the set of finite π-subcomplexes of $E\pi$. By an (E,π,F) power operation on $\mathcal{C}$ we mean a sequence $\mathcal{P}_\pi$ of natural transformations

$$\tilde{E}^{di}X \to \lim_{\alpha} \tilde{F}^{dik}((E\pi)_\alpha^+ \wedge_\pi X^{(k)}),$$

one for each $i \in Z$, which are defined for all $X \in \mathcal{C}$. $\mathcal{P}_\pi$ is stable if it satisfies equation (1). A set $\{\mathcal{P}_j\}_{j\geq 0}$ of (E,Σ_j,E) power operations on $\mathcal{C}$ is consistent if it satisfies 1.1(i),(ii) and (iii). Recall the cylinder construction Z from VII§1.

Proposition 1.3. (i) Let T be a prespectrum and suppose that each T_{di} has the homotopy type of a countable CW-complex. Let F be a ring spectrum. If the pair (T,F) is $\lim^1$-free in the sense of VII.4.1 then every stable (ZT,π,F) operation on $\mathcal{C}$ extends uniquely to a stable operation on $\overline{h}\mathcal{S}$.

(ii) Let E be a ring spectrum and suppose that each E_{di} has the homotopy type of a countable CW-complex and that zE is $\lim^1$-free. Then every consistent set $\{\mathcal{P}_j\}$ of (E,Σ_j,E) operations on $\mathcal{C}$ extends uniquely to a consistent set of operations on $\overline{h}\mathcal{S}$.

Proof. For part (i), let $\{X_{i,\beta}\}$ be the set of finite subcomplexes of T_{di} and let $x_{i,\beta} \in \tilde{E}^{di}X_{i,\beta}$ be the class of the inclusion map $X_{i,\beta} \to T_{di}$. The elements $\mathcal{P}_\pi(x_{i,\beta})$ determine an element of $\lim_{\alpha,\beta} \tilde{F}^{dik}((E\pi)_\alpha^+ \wedge_\pi X_{i,\beta})$ and hence of $\tilde{F}^{dik}D_\pi T_{di}$ by VII.4.10 and VII.4.12. It is easy to see that the maps $\zeta_i:D_\pi T_{di} \to F_{dik}$ representing these elements form an extended pairing of prespectra as defined in VII.3.2. Part (i) now follows from VII.3.4. For part (ii), a similar argument shows that the set $\{\mathcal{P}_j\}$ determines an H_∞^d ring structure on the prespectrum zE and the result follows from VII.6.3.

The definitions we have given are closely related to tom Dieck's axioms for "generalized Steenrod operations" [31]. Let E be a ring spectrum. In tom Dieck's terminology, a generalized Steenrod operation is what we have called an (E,π,E) power operation. His axioms P1 and P2 are 1.1(iv) and 1.1(ii) respectively. In particular, if $\mathcal{P}_\pi$ satisfies P1 then $\mathcal{P}_\pi\Sigma^d 1$ is a π-orientation for E. Axiom P3 is equation (1) above with $\mu = \mathcal{P}_\pi\Sigma^d 1$. Thus an operation satisfying P1 and P3 is stable in our sense (but not conversely). tom Dieck's final axiom P4 will also be of interest in what follows. If q is a vector bundle over X then $E\pi \times_\pi q^k$ is a vector bundle over $E\pi \times_\pi X^k$ whose Thom complex is homeomorphic to $D_\pi T(q)$. If ν is an E-orientation for q and $\mathcal{P}_\pi$ is an operation satisfying P1 then $\mathcal{P}_\pi(\nu)$ is clearly an E-orientation for $E\pi \times_\pi q^k$. Axiom P4 is the statement that E has canonical orientations for some class of vector bundles and that $\mathcal{P}_\pi$ takes the canonical orientation for q to that for $E\pi \times_\pi q^k$. This axiom will be satisfied in all of the particular cases considered in this chapter.

From now on we fix an H_∞^d ring spectrum E and let $\mathcal{P}_\pi$ denote the associated power operations. Let X be a space. Let Δ be the diagonal map

$$X \wedge B\pi^+ = X \wedge D_\pi S^0 \to D_\pi(X \wedge S^0) = D_\pi X$$

defined in II.3.1. We define the <u>internal power operation</u>

$$P_\pi : \tilde{E}^{di}X \to \tilde{E}^{dik}(X \wedge B\pi^+)$$

to be the composite

$$\tilde{E}^{di}X \xrightarrow{\mathcal{P}_\pi} \tilde{E}^{dik}D_\pi X \xrightarrow{\Delta^*} \tilde{E}^{dik}(X \wedge B\pi^+).$$

Since $X^+ \wedge B\pi^+ = (X \times B\pi)^+$ we obtain an unreduced operation

$$P_\pi : E^{di}X \to E^{dik}(X \times B\pi).$$

Our next result summarizes the properties of the unreduced operations; similar statements hold for the reduced ones.

<u>Proposition 1.4</u>. Let $x \in E^{di}X$, $y \in E^{dj}X$, $\pi \subset \Sigma_k$.

(i) $\iota^* P_\pi x = x^k \in E^{dik}X$

(ii) $P_\pi 1 = 1 \in E^0(X \times B\pi)$

(iii) $P_\pi(xy) = (P_\pi x)(P_\pi y) \in E^{d(i+j)k}(X \times B\pi)$

(iv) If i = j then

$$P_k(x+y) = P_k x + P_k y + \sum_{0<\ell<k} (\tau_{\ell,k-\ell})^* [(P_\ell x)(P_{k-\ell} y)]$$

(v) If E is p-local and $|\pi|$ is prime to p then $P_\pi x = \frac{1}{|\pi|} x^k \tau_\pi^* 1$.

(vi) If E is p-local then

$$P_p(x+y) = P_p x + P_p y + \frac{1}{p!}\,[(x+y)^p - x^p - y^p)](\tau_p^* 1).$$

(vii) If $\pi \subset \Sigma_k$ is generated by a k-cycle and $\pi' \subset \Sigma_\ell$ is generated by an ℓ-cycle then

$$(1 \times \gamma)^* P_\pi P_{\pi'} x = P_{\pi'} P_\pi x \in E^{dik\ell}(X \times B\pi \times B\pi'),$$

where $\gamma: B\pi \times B\pi' \to B\pi' \times B\pi$ switches the factors .

Proof. All parts except (vii) are immediate from 1.1. For (vii) we use the argument of [100, VIII.1.3]. If we give the set $\pi \times \pi'$ its lexicographic order we obtain a faithful action of $\Sigma_{k\ell}$ on it. Let $g \in \Sigma_{k\ell}$ be the element which switches the factors π and π'. The following diagram is readily seen to commute.

$$\begin{array}{ccccccc}
\pi \times \pi' & \xrightarrow{d} & \pi \int \pi' & \overset{\iota}{\hookrightarrow} & \Sigma_k \int \Sigma_\ell & \xrightarrow{\beta_{k,\ell}} & \Sigma_{k\ell} \\
\downarrow{\scriptstyle\gamma} & & & & & & \downarrow{\scriptstyle c_g} \\
\pi' \times \pi & \xrightarrow{d} & \pi' \int \pi & \overset{\iota}{\hookrightarrow} & \Sigma_\ell \int \Sigma_k & \xrightarrow{\beta_{\ell,k}} & \Sigma_{k\ell}
\end{array}$$

Here d is the evident diagonal and c_g is conjugation by g. By 1.1(ii) we have

$$P_{\pi'} P_\pi x = \Delta^* d^* \iota^* \beta^*_{k,\ell} \mathcal{P}_{k\ell} x = (1 \times \beta_{k,\ell} \circ \iota \circ d)^* P_{k\ell} x$$

and similarly

$$P_\pi P_{\pi'} x = (1 \times \beta_{\ell,k} \circ \iota \circ d)^* P_{k\ell} x.$$

But $(1 \times c_g)^* P_{k\ell} x = P_{k\ell} x$ since $c_g: B\Sigma_{k\ell} \to B\Sigma_{k\ell}$ is homotopic to the identity.

We conclude this section with a brief description of another kind of operation induced by H_∞^d structures, namely a multiplicative version of the transfer for finite coverings. The definition is due to May. First recall the definition of the ordinary (additive) transfer. If $p: X \to B$ is a j-fold covering then one can construct a map

$$\tilde{p}: B \to E\Sigma_j \times_{\Sigma_j} X^j$$

as in [8, p.112]. If $x \in F^i X$ is represented by $f: X \to F_i$ then $p_! x \in F^i B$ is represented by

$$B \xrightarrow{\tilde{p}} E\Sigma_j \times_{\Sigma_j} X^j \xrightarrow{1 \times f^j} E\Sigma_j \times_{\Sigma_j} (F_i)^j \longrightarrow F_i\ ,$$

where the last map is the Dyer-Lashof map determined by the infinite loop space structure on F_i. B Now if F is an H_∞^d ring spectrum and if $x \in F^{di} X$ is represented by $f: \Sigma(X^+) \to \Sigma^{di} F$ we define $p_\otimes x \in F^{dij} B$ to be the element represented by

$$\Sigma^\infty(B^+) \xrightarrow{\Sigma^\infty(\tilde{p}^+)} \Sigma^\infty(E\Sigma_j \times_{\Sigma_j} X^j)^+ \simeq D_j \Sigma^\infty X^+ \xrightarrow{D_j f} D_j \Sigma^{di} F \xrightarrow{\xi_{j,i}} \Sigma^{dij} F.$$

If F is merely H_∞ one can give the same definition in degree zero. Our next result records some properties of $p_\otimes$.

<u>Proposition 1.5</u> (i) $p_\otimes 1 = 1$, $p_\otimes 0 = 0$.

(ii) $p_\otimes(xy) = (p_\otimes x)(p_\times y)$

(iii) If $q:Y \to X$ is a k-fold covering then $(pq)_\otimes = p_\otimes q_\otimes$

(iv) $f^* p'_\otimes = p_\otimes g^*$ for a pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & X' \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p'} \\ B & \xrightarrow{\quad f \quad} & B' \end{array}$$

(v) If Y is any space and $x \in F^{di}X$, $y \in F^{dk}Y$ then

$$(1 \times p)_\otimes(y \times x) = [(1 \times h)^* P_j y](p_\otimes x) \in F^{dj(i+k)}(Y \times B)$$

where $h:B \to B\Sigma_j$ is the classifying map of p.

<u>Proof</u>. Part (i) is trivial and parts (iii) and (iv) have the same proofs as in the additive case. For part (ii) let $f:\Sigma^\infty(X^+) \to \Sigma^{di}F$ and $g:\Sigma^\infty(X^+) \to \Sigma^{dk}F$ represent x and y. It suffices to show commutativity of the following diagram, in which Σ^∞ has been suppressed to simplify the notation.

$$\begin{array}{ccccccccc} B^+ & \xrightarrow{\tilde{p}^+} & D_j X^+ & \xrightarrow{D_j\Delta} & D_j(X^+ \wedge X^+) & \xrightarrow{D_j(f\ g)} & D_j(\Sigma^{di}F \wedge \Sigma^{dk}F) & \longrightarrow & D_j\Sigma^{d(i+k)} \\ \Big\downarrow{\scriptstyle \Delta} & & & & \Big\downarrow{\scriptstyle \delta} & & \Big\downarrow{\scriptstyle \delta} & & \searrow \\ & & & & & & & & \Sigma^{dj(i+k)}F \\ & & & & & & & & \nearrow \\ B^+ \wedge B^+ & \xrightarrow{\tilde{p}^+ \wedge \tilde{p}^+} & & & D_j X^+ \wedge D_j X^+ & \xrightarrow{D_j f \wedge D_j g} & D_j\Sigma^{di}F \wedge D_j\Sigma^{dk}F & \longrightarrow & \Sigma^{dji}F \wedge \Sigma^{djk}F \end{array}$$

The pentagon commutes by I.4.3 and the remaining pieces by naturality. For part (v) it suffices by (ii) to show

$$(1 \times p)_\otimes(\pi^* y) = (1 \times h)^* P_j y$$

where $\pi:Y \times X \to Y$ is the projection. An inspection of [8, p.112] shows that the diagram

$$\begin{array}{ccc} Y^+ \wedge B^+ & \xrightarrow{\quad \widetilde{(1 \times p)}^+ \quad} & D_j(Y^+ \wedge X^+) \\ \Big\downarrow{\scriptstyle 1 \wedge h^+} & & \Big\downarrow{\scriptstyle D_j \pi^+} \\ Y^+ \wedge B\Sigma_j^+ & \xrightarrow{\quad \Delta^+ \quad} & D_j Y^+ \end{array}$$

commutes and the results follows.

Remarks 1.6.(i) Formula (v) is due to Brian Sanderson (also cf. [35, remark 6.2]). If we let $p:X \to B\Sigma_j$ be the j-fold cover associated to $E\Sigma_j \to B\Sigma_j$ and let $x = 1$ then the formula gives

$$(1 \times p)_{\otimes}(y \times 1) = P_j y,$$

so that the internal operation P_j is completely determined by the multiplicative transfer, an observation also due to Sanderson.

(ii) If $p:X \to B$ and $q:Y \to C$ are any two coverings then $p \times q$ is a covering which factors as $(p \times 1)(1 \times q)$. We can therefore compute $(p \times q)_{\otimes}(x \times y)$ in principle by using formulas (ii), (iii) and (v), but there is no simple external analog of formula (ii).

(iii) If F is H_∞^d then $\bigvee_{i \in Z} \Sigma^{di} F$ is H_∞ by II.1.3. Thus we can define a map

$$p_{\otimes} : \prod_{i \in Z} F^{di} X \longrightarrow \prod_{i \in Z} F^{di} B$$

which agrees on homogeneous elements with that already given. We leave it as an exercise for the reader to show that if x has nonzero degree then $p_{\otimes}(1 + x)$ has components $p_! x$ in degree $|x|$ and $p_{\times} x$ in degree $j|x|$ (cf. [35, Theorem 7.1]).

(iv) In the case $F = HZ_p$ a multiplicative version of the transfer was first defined by Evens, who called it the norm [35]. It seems likely that this agrees with $p_{\otimes}$, but we shall not give a proof. Note that in this case one always has $p_! p^* x = jx$, but it is not true that $p_{\otimes} p^* x = x^j$. For example, formula (v) gives

$$(1 \times p)_{\otimes}(1 \times p)^*(y \times 1) = (1 \times h)^* P_j y.$$

which is certainly not equal to $y^j \times 1$ in general.

2. Steenrod Operations in Ordinary Cohomology.

In this section we use the framework of §1 to construct the Steenrod operations in mod p cohomology and prove their usual properties. The construction will be similar to one given by Milgram [37, Chapter 27], except that we use stable extended powers instead of space-level ones. On the other hand, the proofs will be quite close to those of Steenrod and Epstein [100] except that we make no use of chain-level arguments.

Throughout this section and the next we write H for HZ_p, H^* for mod-p cohomology, and π for the subgroup of Σ_p generated by a p-cycle. If p is an odd prime we write m for $\frac{p-1}{2}$ as usual. For odd primes the spectrum HZ_p is H_∞^2 but not H_∞^1 (see VII.6.1), hence the power operation $\mathcal{P}_p$ can be defined in even degrees but not in odd degrees (unless one uses some form of local coefficients). The operation $\mathcal{P}_\pi$ does extend to odd degrees, as we shall now show.

Proposition 2.1. For each $i \in Z$ there is a unique map $\xi : D_\pi \Sigma^i H \to \Sigma^{pi} H$ for which the diagram

$$\begin{array}{ccc} (\Sigma^i H)^{(p)} & \xrightarrow{\iota} & D_\pi \Sigma^i H \\ & \phi \searrow \quad \swarrow \xi & \\ & \Sigma^{pi} H & \end{array}$$

commutes, where ϕ is the iterated product map. For each $i,j \in Z$ the diagram

$$\begin{array}{ccc} D_\pi(\Sigma^i H \wedge \Sigma^j H) & \xrightarrow{\delta} & D_\pi \Sigma^i H \wedge D_\pi \Sigma^j H \\ \downarrow D_\pi \phi & & \downarrow \xi \wedge \xi \\ D_\pi(\Sigma^{i+j} H) & & \Sigma^{pi} H \wedge \Sigma^{pj} H \\ & \xi \searrow \quad \swarrow \phi & \\ & \Sigma^{p(i+j)} H & \end{array}$$

commutes up to the sign $(-1)^{mij}$.

The proof is the same as for I.4.5. One can in fact replace π in this result by any subgroup of the alternating group A_j, but we shall have no occasion to do so.

Using the map ξ we obtain an external operation

$$\mathcal{P}_\pi : \tilde{H}^i X \to \tilde{H}^{pi} D_\pi X$$

and an internal operation

$$P_\pi : \tilde{H}^i X \to \tilde{H}^{pi}(X \wedge B\pi^+)$$

as in §1. The uniqueness property in 2.1 implies that these operations agree with those already defined when i is even.

Since $\iota^* \mathcal{P}_\pi \Sigma 1 \in \tilde{H}^p S^p$ is the canonical generator $\Sigma^p 1$, we see that $\mathcal{P}_\pi \Sigma 1$ is an orientation for the real regular representation bundle

$$E\pi \times_\pi (\mathbf{R}^1)^p \to B\pi .$$

It follows that the element $\chi \in H^{p-1} B\pi$ defined by

$$\Sigma \chi = P_\pi \Sigma 1$$

is the Euler class of the real reduced regular representation (i.e., the sum of the nontrivial real irreducibles). In particular, χ is nonzero since each nontrivial real irreducible has nonzero Euler class.

Our next result gives the basic properties of the operation P_π. Note that $\tilde{H}^*(X \wedge B\pi^+)$ is an $H^*(B\pi)$-module.

Proposition 2.2. (i) $\iota^* P_\pi x = x^p$

(ii) $P_\pi(xy) = (-1)^{m|x||y|}(P_\pi x)(P_\pi y)$

(iii) $P_\pi \Sigma x = (-1)^{m|x|}\chi(\Sigma P_\pi x)$

(iv) $P_\pi(x + y) = P_\pi x + P_\pi y$

(v) $\beta P_\pi x = 0$ if p is odd or $|x|$ is even.

Proof. Parts (i) and (ii) are immediate from 2.1 and part (iii) follows from part (ii). For part (iv) we assume first that $|x|$ is even. Then we may apply 1.4(vi) to get

$$P_p(x + y) = P_p x + P_p y + \frac{1}{p!}\,[(x + y)^p - x^p - y^p](\tau_p^* 1).$$

But $\tau_p^* 1 = \tau^* \iota^* 1 = p!1 = 0$ and the result follows in this case. If $|x|$ is odd this gives

$$P_\pi(\Sigma x + \Sigma y) = P_\pi \Sigma x + P_\pi \Sigma y.$$

Applying part (iii) gives the equation

$$(-1)^{m|x|}\chi(\Sigma P_\pi(x + y)) = (-1)^{m|x|}\chi(\Sigma(P_\pi x + P_\pi y))$$

and the result follows since χ is not a zero divisor in $H^*B\pi$. For part (v) we need a lemma. Let $\beta: H \to \Sigma H$ represent the Bockstein operation.

Lemma 2.3. The composite

$$D_\pi \Sigma^{2i} H \xrightarrow{\ \xi\ } \Sigma^{2pi} H \xrightarrow{\ \Sigma^{2pi}\beta\ } \Sigma^{2pi+1} H$$

factors through the transfer

$$\tau_\pi : D_\pi \Sigma^{2i} H \longrightarrow (\Sigma^{2i} H)^{(p)}.$$

The proof of 2.3 is rather technical and will be given at the end of this section. For the moment we use it to prove part (v). Let $x \in \tilde{H}^{2i} X$ be represented by $f: \Sigma^\infty X \to \Sigma^{2i} H$ and consider the following diagram, where we have suppressed Σ^∞ to simplify the notation.

$$\begin{array}{ccccccccc}
X \wedge B\pi^+ & \xrightarrow{\ \Delta\ } & D_\pi X & \xrightarrow{\ D_\pi f\ } & D_\pi \Sigma^{2i} H & \xrightarrow{\ \xi\ } & \Sigma^{2pi} H & \xrightarrow{\ \Sigma^{2pi}\beta\ } & \Sigma^{2pi+1} H \\
\Big\downarrow 1\ \tau_\pi & & \Big\downarrow \tau_\pi & & \Big\downarrow \tau_\pi & & & \nearrow & \\
X & \xrightarrow{\ \Delta\ } & X^{(p)} & \xrightarrow{\ f^{(p)}\ } & (\Sigma^{2i} H)^{(p)} & \dashrightarrow & & &
\end{array}$$

The dotted arrows exist by 2.3 and the diagram commutes. The top row represents $\beta P_\pi x$. Thus $\beta P_\pi x$ is in the image of the transfer

$$(1 \wedge \tau_\pi)^* : \tilde{H}^* X \to \tilde{H}^*(X \wedge B\pi^+).$$

But the composite of $(1 \wedge \tau_\pi)^*$ with the restriction

$$(1 \wedge \iota)^* : \tilde{H}^*(X \wedge B\pi^+) \to \tilde{H}^* X$$

is multiplication by p and hence vanishes. Since $(1 \wedge \iota)^*$ is clearly onto we see that $(1 \wedge \tau_\pi)^* = 0$ so that $\beta P_\pi x = 0$ as required. Finally, if p is odd and $x \in H^{2i-1}X$ we have

$$0 = \beta P_\pi(\Sigma x) = \beta(\chi \cdot \Sigma P_\pi x) = -\chi \cdot \Sigma(\beta P_\pi x)$$

since $\beta\chi = 0$. The result follows in this case since χ is not a zero divisor. This completes the proof of 2.2.

Now let $x \in H^q X$. If $p = 2$ we define $P^i x \in H^{q+i}X$ to be the coefficient of χ^{q-i} in $P_\pi x$. If p is odd we define $P^i x \in H^{q+2i(p-1)}X$ to be $(-1)^{mi+mq(q-1)/2}$ times the coefficient of χ^{q-2i} in $P_\pi x$. We also define an element $\omega \in H^{p-2}B\pi$ for p odd by the equation $\beta\omega = \chi$.

<u>Proposition 2.4</u>. (i) $P^i(x + y) = P^i x + P^i y$

(ii) $P^i(\Sigma x) = \Sigma P^i x$

(iii) $P^i x = x^p$ if $q = 2i$ and p is odd or if $q = i$ and $p = 2$. $P^i x = 0$ if $q < 2i$ and p is odd or if $q < i$ and $p = 2$.

(iv) $P^0 x = x$.

(v) If $p = 2$ then $\beta P^{2i} x = P^{2i+1} x$; in particular, $\beta x = P^1 x$.

(vi) If $p = 2$ then $P_\pi x = \Sigma(P^i x)\chi^{q-i}$. If p is odd then

$$P_\pi x = \Sigma(-1)^{mi+mq(q-1)/2}[(P^i x)\chi^{q-2i} + (-1)^q(\beta P^i x)\omega\chi^{q-2i-1}].$$

(vii) $P^i xy = \Sigma(P^j x)(P^{i-j} y)$.

<u>Proof</u>. (i), (ii) and (iii) follow from 2.2(iv), 2.2(iii) and 2.2(i) respectively. For part (iv), we observe that P^0 is a stable operation of degree 0 and hence represents an element of $H^0 H \cong Z_p$. Thus P^0 is a constant multiple of the identity and the result follows since $P^0 1 = 1^p = 1$ by part (iii). In part (v) we can use part (ii) to reduce to the case where q is even. The result follows in that case from 2.2(v) and the relation $\beta\chi = \chi^2$. In part (vi) the p = 2 case is true by definition. If p is odd we can use part (ii) and 2.2(iii) to reduce to the case where q is even. We then have $P_\pi x = \iota^* P_p x$. We recall from [68, Lemma 1.4] that the image of

$$\iota^* : H^* B\Sigma_p \to H^* B\pi$$

is nonzero only in dimensions of the form $2i(p-1)$ and $2i(p-1)-1$. Thus this image is generated as a ring by χ and ω and we have

$$P_\pi x = \sum (-1)^{mi+mq(q-1)/2}[(P^i x)\chi^{q-2i} + y_i\omega\chi^{q-2i-1}]$$

for some elements $y_i \in H^{q+2i(p-1)+1}X$. Now 2.2(v) imples that $y_i = (-1)^q \beta P^i x$ as required. Finally, part (vii) follows from 2.2(iv) and part (vi). This completes the proof of 2.4.

Next we shall prove the Adem relations for p odd. We use the method of proof of Bullett and MacDonald [26, §4], where the case p = 2 may be found. However, in our context the relations arise more naturally in the form given by Steiner [102]. Let U and V denote indeterminates of degree 2p-2 and define S and T by

$$S = U(1 - V^{-1}U)^{p-1}$$

$$T = V(1 - U^{-1}V)^{p-1}.$$

We shall prove that the equations

(1) $$\sum_{i,j} (P^j P^i x)U^{-j}T^{-i} = \sum_{i,j} (P^j P^i x)V^{-j}S^{-i}$$

(2) $$\sum_{i,j} (P^j \beta P^i x)U^{-j}T^{-i} = (1 - U^{-1}V)\sum_{i,j} (\beta P^j P^i x)V^{-j}S^{-i} + U^{-1}V\sum_{i,j} (P^j \beta P^i x)V^{-j}S^{-i}$$

hold for all x. The usual Adem relations can easily be obtained from these as in [102, p. 163]; the basic idea is simply to expand the right sides of (1) and (2) as power series in U and T and compare coefficients. The proof of (1) and (2), like any proof of the Adem relations, is based on the relation

(3) $$\gamma^* P_\pi P_\pi x = P_\pi P_\pi x$$

given by 1.4(vii). In order to compute $P_\pi P_\pi x$ in terms of the P^i we need to know more about the element $\chi \in H^{p-1}B\pi$. We have mentioned that χ is the Euler class of the real reduced regular representation of π, and that this representation is the sum of the nontrivial real irreducibles of π. Choose one such irreducible, and let $u \in H^2 B\pi$ denote its Euler class. Then the Euler classes of the remaining irreducibles (suitably oriented) are 2u, 3u,...,mu, and thus $\chi = \pm m!u^m$. The ambiguity in the sign arises from the question of whether the various orientations have been chosen consistently, but it turns out that we shall not need to eliminate this ambiguity. Thus we shall assume $\chi = m!u^m$ (it is in fact possible to choose the orientations so that this holds) and leave it to the reader to check that the other possibility leads to the same relations (1) and (2). We define $b \in H^1 B\pi$ by the equation $\beta b = u$, so that $\omega = m!bu^{m-1}$. Then the equation 2.4(v) may be written as follows.

(4) $$P_\pi x = \sum (-1)^{i+mq(q-1)/2}(m!)^q[P^i x + (-1)^q(\beta P^i x)bu^{-1}]u^{m(q-2i)}.$$

Since both sides of (1) and (2) are stable we may assume that q has the form 2r with r even. We define $U = -u^{2m}$, so that (4) becomes

$$(5) \qquad P_\pi x = \sum (-1)^r [P^i x + (\beta P^i x) b u^{-1}] U^{r-i}.$$

Now 2.2(ii) and 2.2(iv) give

$$(6) \qquad P_\pi P_\pi x = \sum (-1)^r [P_\pi P^i x + (-1)^m (P_\pi \beta P^i x)(P_\pi b)(P_\pi u)^{-1} (P_\pi U)^{r-i}]$$

in $H^*X \otimes H^*B\pi \otimes H^*B\pi$. We denote the copies of b and u in the second copy of $B\pi$ by c and v, and we let $V = -v^{p-1}$. Equation (4) gives the following formulas.

$$(7) \qquad P_\pi b = m! [b - ucv^{-1}] v^m$$

$$(8) \qquad P_\pi u = u^p - uv^{p-1} = u(V - U)$$

$$(9) \qquad P_\pi U = -(P_\pi u)^{p-1} = U(V - U)^{p-1} = V^{p-1} S$$

$$(10) \qquad P_\pi P^i x = \sum (-1)^r [P^j P^i x + (\beta P^j P^i x) c v^{-1}] V^{r+2im-j}$$

$$(11) \qquad P_\pi \beta P^i x = \sum (-1)^r m! [P^j \beta P^i x - ((\beta P^j \beta P^i x) c v^{-1}] V^{r+2im-j} v^m.$$

We therefore have

$$(12) \qquad P_\pi P_\pi x = (V^p S)^r \sum [P^j P^i x + (\beta P^j P^i x) c v^{-1} + (P^j \beta P_i x)(bu^{-1} - cv^{-1}) V(V - U)^{-1}$$
$$+ (\beta P^j \beta P^i x) bcu^{-1} v^{-1} V(V - U)^{-1}] V^{-j} S^{-i}.$$

Now we apply equation (3). We have $\gamma^* u = v$, $\gamma^* U = V$, and $\gamma^* S = T$. Since $V^p S = U^p T = \gamma^*(V^p S)$ we have

$$(13) \qquad P_\pi P_\pi x = \gamma^* P_\pi P_\pi x = (V^p S)^r \sum [P^j P^i x - (\beta P^j P^i x) b u^{-1}$$
$$+ (P^j \beta P^i x)(cv^{-1} - bu^{-1}) U(U - V)^{-1} - (\beta P^j \beta P^i) bcu^{-1} v^{-1} U(U-V)^{-1}] U^{-j} T^{-i}.$$

Collecting the terms in (12) and (13) which do not involve b or c gives equation (1), and the terms which involve c but not b give (2). This completes the proof.

Finally, we give the proof of Lemma 2.3. Let M be the Moore spectrum $S \cup_p e^1$ and let $i: S \to M$ be the inclusion of the bottom cell.

<u>Lemma 2.5</u>. $H^1(D_\pi M)$ has a basis $\{x, y\}$ such that $(D_\pi i)^* x = 0$, $(D_\pi i)^* y \neq 0$, and x is in the image of the transfer

$$\tau_\pi^*: H^1 M^{(p)} \to H^1 D_\pi M.$$

Proof of 2.5. We use the spectral sequence

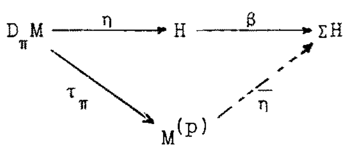
$$H^i(\pi; H^j(M^{(p)})) \Rightarrow H^{i+j} D_\pi M$$

of I.2.4. Each of the groups $E_2^{0,1}$ and $E_2^{1,0}$ is generated by a single element. The generator of the latter group clearly survives to E_∞ and represents an element $y \in H^1 D_\pi M$. Since $(i^{(p)})^*: H^0 M^{(p)} \to H^0 S$ is an isomorphism, so is the map induced by $D_\pi i$ on $E_2^{1,0}$. Hence $(D_\pi i)^* y \neq 0$. Now let $z \in H^1 M^{(p)}$ be a generator of $H^1 M \otimes H^0 M \otimes \cdots \otimes H^0 M$ and let $x = \tau_\pi^* z$. Clearly, x is represeneted by a generator of $E_2^{0,1}$, and $(D_\pi i)^* x = (D_\pi i)^* \tau_\pi^* z = \tau_\pi^* (i^{(p)})^* z$ which is zero since $H^1 S = 0$.

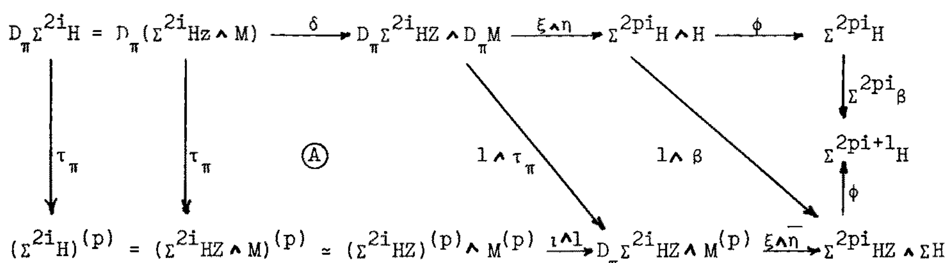
Proof of 2.3. Let HZ be the spectrum representing integral cohomology. Then $H = HZ \wedge M$. Let $e: S \to HZ$ be the unit and let η be the composite

$$D_\pi M = D_\pi(S \wedge M) \xrightarrow{D_\pi(e \wedge 1)} D_\pi(HZ \wedge M) = D_\pi H \xrightarrow{\xi} H.$$

Let w be the element of $H^0 D_\pi M$ represented by η. Then $(D_\pi i)^* \beta w = 0$ since β vanishes on $H^0 D_\pi S = H^0 B\pi$. Hence by Lemma 2.5, βw is a multiple of x and in particular it is in the image of the transfer. Thus we have a factorization



Now consider the diagram



The uniqueness clause in 2.1 imples that the composite of the top row is $\xi: D_\pi \Sigma^{2i} H \to \Sigma^{2pi} H$, so it suffices to show that the diagram commutes. Part (A) commutes by VI.3.10 of the sequel, and the other parts clearly commute.

§3. Dyer-Lashof operations and the Nishida relations

An interesting feature of the treatment of Steenrod operations in §2 is that is generalizes to give the properties of Dyer-Lashof operations; thus homology operations are a special case of cohomology operations (cf. [68]). The use of stable instead of space-level extended powers is crucial for this since homology does not have a simple space-level description. We give the details in this section; IX§1 will give another approach to homology operations which generalizes to extraordinary theories. We continue to use the notations of §2, so that H denotes HZ_p.

First let M be any module spectrum over H and let Y be an arbitrary spectrum. There is a natural transformation

$$A: M^*Y \to \mathrm{Hom}(H_*Y, \pi_*M)$$

defined as follows: if $y \in M^*Y$ is represented by $f: Y \to \Sigma^i M$ then A(y) is the composite

$$H_*Y = \pi_*(H \wedge Y) \xrightarrow{(1\wedge f)_*} \pi_*(H \wedge M) \longrightarrow \pi_*M,$$

which is a homomorphism raising degrees by i. Clearly A is a morphism of cohomology theories. Since it is an isomorphism for Y = S we have

Lemma 3.1. A is an isomorphism.

Now let X be a fixed H_∞ ring spectrum with structural maps θ_j (for example, X might have the form $\Sigma^\infty Z^+$ for an infinite loop space Z) and let $M = H \wedge X$. Then M is an H_∞^2 ring spectrum with structural maps

$$D_j(\Sigma^{2i} H \wedge X) \xrightarrow{\delta} D_j \Sigma^{2i} H \wedge D_j X \xrightarrow{\xi_{j,i} \wedge \theta_j} \Sigma^{2ij} H \wedge X$$

and we obtain power operations

$$\mathcal{R}_j : M^{2i}Y \to M^{2ij} D_j Y$$

and $$R_j : M^{2i} Y \to M^{2ij}(Y \times B\Sigma_j).$$

The operation $\mathcal{R}_\pi$ can be extended to odd degrees by means of the maps

$$D_\pi \Sigma^i M = D_\pi(\Sigma^i H \wedge X) \xrightarrow{\delta} D_\pi \Sigma^i H \wedge D_\pi X \xrightarrow{\xi \wedge \theta} \Sigma^{pi} H \wedge X$$

where ξ is the map given by 2.1. The unit of X gives an H_∞^2 ring map $h: H \to H \wedge X = M$ and h_* also preserves $\mathcal{P}_\pi$ in odd degrees.

Define $\bar{b}$, $\bar{u}$, $\bar{\chi}$ and $\bar{\omega}$ in $M^*B\pi$ to be the images under h_* of the elements b, u, χ and ω in $H^*B\pi$ defined in §2. Thus $\Sigma\bar{\chi} = R_\pi \Sigma 1$. Lemma 3.1 gives the following isomorphisms for any space Y.

$$M^*(Y \times B\pi) \cong (M^*Y)[[\overline{\chi}]] \quad \text{if } p = 2.$$

$$M^*(Y \times B\pi) \cong (M^*Y)[[\overline{b},\overline{u}]] \quad \text{if } p \text{ is odd.}$$

Thus we can define operations $R^i y$ for $y \in \tilde{M}^q Y$ as follows: if $p = 2$ let $R^i y$ be the coefficient of $\overline{\chi}^{q-i}$ in $R_\pi y$, and if p is odd let $R^i y$ be $(-1)^{mi+mq(q-1)/2}$ times the coefficient of $\overline{\chi}^{q-2i}$ in $R_\pi y$. Now if $Y = S^0$ there is an isomorphism $H_q X \cong \tilde{M}^{-q}S^0$ which we shall always denote by $x \longmapsto \underline{x}$. We define the Dyer-Lashof operations

$$Q^i : H_q X \to H_{q+i} X \quad \text{when } p = 2$$

$$Q^i : H_q X \to H_{q+2i(p-1)} X \quad \text{when } p \text{ is odd}$$

by the equation $\underline{Q^i x} = R^{-i}\underline{x}$. The properties of Q^i will follow from those of R_π and R^i. Our next result gives the basic facts about R_π.

<u>Proposition 3.2</u>. (i) $i^* R_\pi y = y^p$

(ii) $R_\pi(yz) = (-1)^{m|y||z|}(R_\pi y)(R_\pi z)$

(iii) $R_\pi(\Sigma y) = (-1)^{m|y|}\,\overline{\chi}\cdot\Sigma R_\pi y$

(iv) $R_\pi(y + z) = R_\pi y + R_\pi z$.

(v) $\beta R_\pi y = 0$ if p is odd or $|y|$ is even.

<u>Proof</u>. (i) and (ii) are immediate from the definitions and (iii) follows from (ii). In the proof of 2.2(v) it was observed that the transfer

$$\tau_\pi^* : H^*Y \to H^*(Y \times B\pi)$$

vanishes for all spaces Y. By 3.1 it follows that

$$\tau_\pi^* : M^*Y \to M^*(Y \times B\pi)$$

also vanishes. In particular, the map

$$\tau_p^* : M^*(\text{pt.}) \to M^*(B\Sigma_p)$$

vanishes. Part (iv) now follows by the proof of 2.2(iv). To complete the proof of part (v) it suffices to give a suitable substitute for Lemma 2.3. That lemma gives a map

$$F : (\Sigma^{2i}H)^{(p)} \to \Sigma^{2pi+1}H$$

such that $F \circ \tau_\pi$ is the composite

$$D_\pi \Sigma^{2i}H \xrightarrow{\xi} \Sigma^{2pi}H \xrightarrow{\Sigma^{2pi}\beta} \Sigma^{2pi+1}H.$$

Consider the following diagram

$$\begin{array}{ccccccc}
D_\pi(\Sigma^{2i}H \wedge X) & \xrightarrow{\delta} & D_\pi\Sigma^{2i}H \wedge D_\pi X & \xrightarrow{\xi \wedge \theta} & \Sigma^{2pi}H \wedge X & \xrightarrow{\Sigma^{2pi}\beta \wedge 1} & \Sigma^{2pi+1}H \wedge X \\
\downarrow{\scriptstyle \tau_\pi} & & & \searrow{\scriptstyle \tau_\pi \wedge 1} & & \nearrow{\scriptstyle F \wedge \theta} & \\
(\Sigma^{2i}H \wedge X)^{(p)} & \simeq & (\Sigma^{2i}H)^{(p)} \wedge X^{(p)} & \xrightarrow{1 \wedge \iota} & (\Sigma^{2i}H)^{(p)} \wedge D_\pi X & &
\end{array}$$

The left part commutes by VI.3.10 of the sequel and the right part commutes by definition of F. Thus the top row of the diagram factors through τ_π. Using this fact in place of Lemma 2.3, the proof of 2.2(v) now goes through to prove part (v).

If we now replace P^i, χ and ω in Proposition 2.4 by R^i, $\overline{\chi}$ and $\overline{\omega}$ then every part except (iv) remains true with the same proof. If we replace U,V,S and T in the Adem relations (equations (1) and (2) of Section 2) by $\overline{U} = h_*U$, $\overline{V} = h_*V$, $\overline{S} = h_*S$ and $\overline{T} = h_*T$ then these relations remain true and have the same proof.

Proposition 3.3. (i) $Q^i(x + y) = Q^ix + Q^iy$

(ii) If p is odd then $Q^ix = 0$ for $2i < |x|$ and $Q^ix = x^p$ for $2i = |x|$.

If $p = 2$ then $Q^ix = 0$ for $i < |x|$ and $Q^ix = x^2$ for $i = |x|$.

(iii) $\beta Q^{2s} = Q^{2s-1}$ if $p = 2$

(iv) $Q^i(xy) = \sum_i (Q^jx)(Q^{i-j}y)$

(v) The Adem relations hold: if U and V are indeterminates of dimension $2-2p$, $S = U(1 - V^{-1}U)^{p-1}$ and $T = V(1 - U^{-1}V)^{p-1}$ then the equations

$$\sum_{i,j} (Q^iQ^jx)U^iT^j = \sum_{i,j} (Q^iQ^jx)V^iS^j$$

and if p is odd

$$\sum_{i,j} (Q^i\beta Q^jx)U^iT^j = (1 - U^{-1}V)\sum_{i,j} (\beta Q^iQ^jx)V^iS^j$$
$$+ U^{-1}V \sum_{i,j} (Q^i\beta Q^jx)V^iS^j$$

are valid for all x.

(vi) If X has the form $\Sigma^\infty Z^+$ for an E_∞ space Z and

$$\sigma:\tilde{H}_q\Omega Z \to H_{q+1}Z$$

is the homology suspension then $Q^i\sigma = \sigma Q^i$.

<u>Proof</u>. We shall prove part (vi); the remaining parts are immediate from the properties of R^i. For any space Z the retraction of Z to a point splits the cofibre sequence

$$\Sigma^\infty S^0 \longrightarrow \Sigma^\infty Z^+ \xrightarrow{\lambda} \Sigma^\infty Z$$

and gives a map

$$\nu : \Sigma^\infty Z \to \Sigma^\infty Z^+$$

Now let Z be an E_∞ space and let $X = \Sigma^\infty Z^+$, $\tilde{X} = \Sigma^\infty Z$, $W = \Sigma^\infty(\Omega Z)^+$, $\tilde{W} = \Sigma^\infty \Omega Z$. Then X and W are H_∞ ring spectra but $\tilde{X}$ and $\tilde{W}$ are not. Let ζ denote either of the composites

$$D_\pi \tilde{X} \xrightarrow{D_\pi \nu} D_\pi X \longrightarrow X \xrightarrow{\lambda} \tilde{X}$$

and

$$D_\pi \tilde{W} \xrightarrow{D_\pi \nu} D_\pi W \longrightarrow W \xrightarrow{\lambda} \tilde{W},$$

where the unmarked arrows come from the H_∞ structures on X and W. We can use the maps ζ to define operations $\tilde{R}_\pi$ in the theories represented by $H \wedge \tilde{X}$ and $H \wedge \tilde{W}$ and it is easy to see that

(1) $$(1 \wedge \nu)_* \tilde{R}_\pi y = R_\pi (1 \wedge \nu)_* y$$

for all y. Now if $x \in \tilde{H}_q \Omega Z$ then $\underline{x} \in (H \wedge \tilde{W})^{-q} S \subset (H \wedge W)^{-q} S$, and (1) and the definition of Q^i give

(2) $$\tilde{R}_\pi \underline{x} = \sum_i (-1)^{mi + mq(q+1)/2} \underline{Q^i x}\, \chi^{2i-q}$$

since $(1 \wedge \nu)_*$ is monic. The natural map $\varepsilon : \Sigma \Omega Z \to Z$ induces a map $\Sigma \tilde{W} \to \tilde{X}$ which will also be called ε. A fairly tedious diagram chase (given at the end of IX§7) shows that the following diagram commutes.

$$\begin{array}{ccc}
\Sigma D_\pi \tilde{W} & \xrightarrow{\Delta} & D_\pi \Sigma \tilde{W} \\
 & & \downarrow{\scriptstyle D_\pi \varepsilon} \\
\downarrow{\scriptstyle \Sigma\zeta} & & D_\pi \tilde{X} \\
 & & \downarrow{\scriptstyle \zeta} \\
\Sigma \tilde{W} & \xrightarrow{\varepsilon} & \tilde{X}
\end{array}$$

Hence the following diagram commutes, where $f : S \to \Sigma^{-q} H \wedge \tilde{W}$ represents $\underline{x}$.

$$\begin{array}{ccccccccc}
\Sigma B\pi^+ & \xrightarrow{\Sigma D_\pi f} & \Sigma D_\pi(\Sigma^{-q} H \wedge \tilde{W}) & \xrightarrow{\delta} & D_\pi \Sigma^{-q} H \wedge \Sigma D_\pi \tilde{W} & \longrightarrow & \Sigma^{-pq} H \wedge \Sigma \tilde{W} & \xrightarrow{1 \wedge \varepsilon} & \Sigma^{-pq} H \wedge \tilde{X} \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle 1 \wedge \Delta} & & & \nearrow{\scriptstyle 1 \wedge \zeta} & \\
D_\pi S^1 & \xrightarrow{D_\pi \Sigma f} & D_\pi(\Sigma^{-q+1} H \wedge \tilde{W}) & \longrightarrow & D_\pi \Sigma^{-q} H \wedge D_\pi \Sigma \tilde{W} & \longrightarrow & \Sigma^{-pq} H \wedge D_\pi \tilde{X} & &
\end{array}$$

The top row of this diagram represents $(1\wedge\varepsilon)_{\pi}\Sigma\widetilde{R}_{\pi}\underline{x}$ and the other composite represents $\widetilde{R}_{\pi}(1\wedge\varepsilon)_{*}\Sigma\underline{x}$. Thus we have

(3) $$(1\wedge\varepsilon)_{*}\Sigma\widetilde{R}_{\pi}\underline{x} = \widetilde{R}_{\pi}(1\wedge\varepsilon)_{*}\Sigma\underline{x}.$$

Combining this with (1) gives

(4) $$(1\wedge\nu\varepsilon)_{*}\Sigma\widetilde{R}_{\pi}\underline{x} = R_{\pi}(1\wedge\nu\varepsilon)_{*}\Sigma\underline{x} .$$

Now the definition of σ gives

(5) $$\Sigma\underline{\sigma x} = (1\wedge\nu\varepsilon)_{*}\Sigma\underline{x} .$$

Combining (5) and (2) gives

(6) $$(1\wedge\nu\varepsilon)_{*}\Sigma\widetilde{R}_{\pi}\underline{x} = \sum_i (-1)^{mi+mq(q+1)/2}\underline{\Sigma\sigma Q^i x}\,\chi^{2i-q}$$

Finally, by 3.2(iii) we have

(7) $$R_{\pi}(1\wedge\nu\varepsilon)_{*}\Sigma\underline{x} = R_{\pi}\Sigma\underline{\sigma x} = (-1)^{m(q+1)}\chi(\Sigma R_{\pi}\underline{\sigma x})$$

$$= \sum_i (-1)^{mi+mq(q+1)/2}\underline{\Sigma Q^i\sigma x}\,\chi^{2i-q}.$$

The result follows from (4), (6), and (7). This completes the proof of 3.3.

We conclude this section with a proof of the Nishida relations in the form given by Steiner:

(8) $$\sum_{i,j} (\overline{P}^i_{*}Q^j x)V^{-i}S^j = \sum_{i,j} (Q^i\overline{P}^j_{*}x)U^iT^{-j}$$

and if p is odd

(9) $$\sum_{i,j} (\overline{P}^i_{*}\beta Q^j x)V^{-i}S^j = (1 - UV^{-1}) \sum_{i,j} (\beta Q^i\overline{P}^j_{*}x)U^iT^{-j}$$

$$+ UV^{-1} \sum_{i,j} (Q^i\beta\overline{P}^j_{*}x)U^iT^{-j} ,$$

where $\overline{P}^i_{*}$ is the dual of the conjugate Steenrod operation $\overline{P}^i$ and U, V, S and T are as in 3.3(v). The usual Nishida relations can easily be obtained from these by first translating from $\overline{P}^i_{*}$ to P^i_{*} and then writing both sides as power series in U and V; see [102, p. 164]. We shall prove (8) and (9) for p odd; there is a similar proof for p = 2. The basic idea will be to show that the total Steenrod operation

$$H \longrightarrow \bigvee_{i\in Z} \Sigma^i H$$

is an H_∞ ring map, and this in turn will follow easily from 1.4(vii). To make this work, however, we need a particular H_∞ structure on $\bigvee_{i\in Z} \Sigma^i H$ which we now construct.

Let E^*X be the functor $H^*(X \times B\pi)$ on the category of spaces. We denote the generators of $H^1 B\pi$ and $H^2 B\pi$ by c and v, so that E^*X is the polynomial ring $(H^*X)[c,v]$. E^* is a multiplicative cohomology theory and hence is represented by a ring spectrum E. The projection $X \times B\pi \to X$ gives a natural transformation $H^*X \to E^*X$ which is represented by a map $g:H \to E$. Of course, E is equivalent to $\bigvee_{i\leq 0} \Sigma^i H$ with its usual ring structure and g is the inclusion of H in this wedge.

Next we define power operations in E^*. Let $\mathcal{P}_j^E$ be the composite

$$\tilde{E}^{2i}X = \tilde{H}^{2i}(X \wedge B\pi^+) \xrightarrow{j} \tilde{H}^{2ij}(D_j(X \wedge B\pi^+)) \xrightarrow{\Delta^*} \tilde{H}^{2ij}((D_jX) \wedge B\pi^+) = \tilde{E}^{2ij}D_jX.$$

It is easy to see that the $\mathcal{P}_j^E$ are consistent in the sense of Definition 1.2 and thus they determine an H_∞^2 ring structure on E by 1.3 (compare II.1.3). The operation $\mathcal{P}_\pi^E$ extends to odd degrees since $\mathcal{P}_\pi$ does, and g is an H_∞^2 ring map which also preserves $\mathcal{P}_\pi$ in odd degrees. An inspection of the definitions gives the following description of the internal operation $\mathcal{P}_\pi$.

(10) $$P_\pi^E = (1\wedge\gamma)^* P_\pi : \tilde{H}^i(X \wedge B\pi^+) \to \tilde{H}^{pi}(X \wedge B\pi^+ \wedge B\pi^+)$$

Note that, with the conventions we have adopted, c and v are the generators in the second copy of $B\pi$ in this situation. As in Section 2 write b and u for the generators in the first copy of $B\pi$; thus $g_*:H^*B\pi \to E^*B\pi$ takes v to u and c to b.

Now let F^*X be the Laurent series ring $(H^*X)[[c,v,v^{-1}]] = E^*X[[v^{-1}]]$. F^* is a multiplicative cohomology theory and hence is represented by a ring spectrum F, and the inclusion $H^*X \to F^*X$ is represented by a ring map $H \to F$ which we again call g; of course F is equivalent as a ring spectrum to $\bigvee_{i\in Z} \Sigma^i H$ and g is the inclusion of H in this wedge. Now observe that the element $\mathcal{P}_j^E v$ $(H^*B\Sigma_j)[[c,v,v^{-1}]]$ is a Laurent series which is bounded above, and that by 1.1(iv) it has leading coefficient $1 \in H^0 B\Sigma_j$. Hence $\mathcal{P}_j^E v$ is invertible, and it follows that we can extend the operations $\mathcal{P}_j^E$ to operations $\mathcal{P}_j^F$ in the F-cohomology of finite complexes. The $\mathcal{P}_j^F$ are consistent in the sense of 1.2 and hence give an H_∞^2 structure for F by 1.3.

Next we define the total Steenrod operation $t:H \to F$ by letting t_* be the composite

$$H^qX \xrightarrow{P_\pi} H^{pq}(X \times B\pi) = E^{pq}X \longrightarrow F^qX,$$

where the last map is multiplication by $(-1)^{mq(q-1)/2}(m!)^{-q}\,\nu^{-mq}$. By 2.4(vi) we have the formula

(11) $$t_*x = \sum_i [g_*P^ix + (-1)^q g_*(\beta P^i x)cv^{-1}]V^{-i},$$

where $V = -v^{p-1}$ as in Section 2. In particular, the projection of $t: \to \bigvee_{i\in Z}\Sigma^i H$ on

$\Sigma^{2k(p-1)}H$ is P^k. Either from the definition or from formula (11) we get the following equations.

(12) $$t_*c = b - cuv^{-1}$$

(13) $$t_*v = u + u^pV^{-1} = u(1 - UV^{-1})$$

(14) $$t_*V = -u^{p-1}(1 - UV^{-1})^{p-1} = U(1 - UV^{-1})^{p-1} = S.$$

t is clearly a ring map, but it turns out not to be an H^2_∞ map. However, we have

<u>Proposition 3.4</u>. Let Y be any spectrum and let $y \in H^qY$. Let $w = (1 - UV^{-1})^m$. Then

$$t_* \mathcal{P}_\pi y = w^q \mathcal{P}^F_\pi t_* y.$$

This fact will suffice for our purposes but we remark that by combining it with 7.2 below one can show that t is actually an H_∞ map. It is certainly not H^2_∞ since it does not preserve $\mathcal{P}_\pi$.

For the proof of 3.4 we need a standard lemma.

<u>Lemma 3.5</u>. For any space Y the map

$$\iota^* \oplus \Delta^*: \tilde{H}^*D_\pi Y \to \tilde{H}^*Y^{(p)} \oplus \tilde{H}^*(Y \wedge B\pi^+)$$

is monic.

For completeness we shall give a proof of 3.5 at the end of this section.

<u>Proof of 3.4</u>. Since both sides of the equation are stable (H,π,F) operations in the sense of 1.2 and 1.3 it suffices to show that they agree on finite complexes. By 3.5 it suffices to show

$$\iota^* t_* \mathcal{P}_\pi y = \iota^* w^q \mathcal{P}^F_\pi t_* y$$

and

(15) $$t_* P_\pi y = w^q P^F_\pi t_* y$$

for all y. Since $\iota^* w = 1$ and t is a ring map the first equation follows from 1.1 (iv). For the second, we first let $y = \Sigma 1$. Then

$$t_* P_\pi \Sigma 1 = t_*(\chi \cdot \Sigma 1) = (t_*\chi)\cdot(t_*\Sigma 1) = \chi w \cdot \Sigma 1$$

while

$$w P^F_\pi t_* \Sigma 1 = w P^F_\pi g_* \Sigma 1 = w g_* P_\pi \Sigma 1 = w\chi \cdot \Sigma 1.$$

Since χw is not a zero divisor, it suffices to show (15) when q is even, say

$q = 2r$. Then as elements of $(H^*Y)[[b,c,u,v,v^{-1}]]$ we have

$$t_* P_\pi y = (-1)^{mr}(m!)^{-2r}v^{-2mpr}P_\pi P_\pi y = V^{-pr}P_\pi P_\pi y$$

and

$$\begin{aligned} w^{2r}P_\pi^F t_* y &= w^{2r}(1\wedge\gamma)^* P_\pi(U^{-r}P_\pi y) \quad \text{by (10)} \\ &= w^{2r}V^{-pr}(1 - UV^{-1})^{-2mr}(1 \quad \gamma)^* P_\pi P_\pi y \\ &= V^{-pr}P_\pi P_\pi y \quad \text{by 1.4(vii),} \end{aligned}$$

and the result follows.

If we let y be the class of the identity map $H \to H$ we obtain

Corollary 3.6. The diagram

$$\begin{array}{ccc} D_\pi H & \longrightarrow & H \\ \Big\downarrow D_\pi t & & \Big\downarrow t \\ D_\pi F & \longrightarrow & F \end{array}$$

commutes, where the unmarked arrows come from the H_∞ structures of H and F.

Now let X be an H_∞ ring spectrum. Then, as we have seen, $H\wedge X$ is an H_∞^2 ring spectrum and there is an operation

$$\mathcal{R}_\pi : (H\wedge X)^q Y \to (H\wedge X)^{pq}D_\pi Y$$

for $Y \in \overline{h}\mathcal{S}$. Similarly, $F\wedge X$ is an H_∞^2 ring spectrum and we obtain an operation

$$\mathcal{R}_\pi^R : (F\wedge X)^q Y \to (F\wedge X)^{pq}D_\pi Y,$$

The unit of X induces H_∞ ring maps $h: H \to H\wedge X$ and $h': F \to F\wedge X$.

Corollary 3.7. If Y is any spectrum and $y \in (H\wedge X)^q Y$ then the equation

$$(t\wedge 1)_* \mathcal{R}_\pi y = w^q \mathcal{R}_\pi^R (t\wedge 1)_* y$$

holds in $(F\wedge X)^{pq}Y$.

Proof. For $q = 0$ this is immediate from 3.6. If $y = \Sigma 1$ we have

$$(t\wedge 1)_* \mathcal{R}_\pi \Sigma 1 = (t\wedge 1)_* \mathcal{R}_\pi h_* \Sigma 1 = h_*' t_*\ {}_\pi \Sigma 1 = w\,\mathcal{R}_\pi^F (t\wedge 1)_* \Sigma 1$$

by 3.4. For general y let $z = \Sigma^{-q}y \in (H\wedge X)^0(\Sigma^{-q}Y)$. Then $y = (\Sigma 1)^q z$ and we have

$$\begin{aligned}(t\wedge 1)_* \mathcal{R}_\pi y &= (t\wedge 1)_*\delta^*[(\mathcal{R}_\pi\Sigma 1)^q\mathcal{R}_\pi z] \\ &= \delta^*[w^q(\mathcal{R}^F_\pi\Sigma 1)^q\mathcal{R}^F_\pi(t\wedge 1)_* z] \\ &= w^q\mathcal{R}^F_\pi(t\wedge 1)_* y\end{aligned}$$

as required.

Corollary 3.7 gives the following relation between the internal operations.

$$(t\wedge 1)_* R_\pi y = w^q R^F_\pi (t\wedge 1)_* y \tag{16}$$

To prove the relations (8) and (9) one simply evaluates both sides in the special case when Y is a point. First we recall that the operation in homology induced by $P^i : H \to \Sigma^{2i(p-1)}H$ is not P^i_* but its conjugate $\overline{P}^i_*$. Since $\overline{\beta} = -\beta$ we have in particular $\beta\underline{z} = -\underline{\beta z}$. Thus (11) gives

$$(t\wedge 1)_*\underline{z} = \sum_i [g_*\underline{\overline{P}^i_* z} - (-1)^q g_*\underline{\beta\overline{P}^i_* z}\ cv^{-1}]V^{-i} \tag{17}$$

for any $z \in H_qX$. Now let $x \in H_qX$, $y = \underline{x}$. Then we have

$$\begin{aligned}(t\wedge 1)_* R_\pi y &= (t\wedge 1)_* \sum_j (-1)^{mq(q+1)/2}(m!)^{-q}[\underline{Q^j x}-(-1)^q(\underline{\beta Q^j x})bv^{-1}]v^{-mq}V^j \\ &= (-1)^{mq(q+1)/2}(m!)^{-q}\sum_j [(t\wedge 1)_*\underline{Q^j x}-(-1)^q(t\wedge 1)_*\beta\underline{Q^j x}(t_*b)(t_*v)^{-1}](t_*v)^{-mq}(t_*V)^j \\ &= (-1)^{mq(q+1)/2}(m!)^{-q}u^{-mq}w^{-q}\sum_{i,j} [\underline{\overline{P}^i_* Q^j x} - (-1)^q\underline{\beta\overline{P}^i_* Q^j x}cv^{-1} \\ &\quad - (-1)^q\underline{\overline{P}^i_*}\beta Q^j x(bu^{-1} - cv^{-1})(1 - UV^{-1})^{-1} + \underline{\beta\overline{P}^i_*\beta Q^j}xbcu^{-1}v^{-1}(1 - UV^{-1})^{-1}]S^jV^{-i}\end{aligned} \tag{18}$$

On the other hand, we have

$$\begin{aligned}w^{-q}R^F_\pi(t\wedge 1)_* y &= w^{-q}R^F_\pi \sum_j [g_*\underline{\overline{P}^j_* x} - (-1)^q g_*\underline{\beta\overline{P}^j_*}xcv^{-1}]V^{-j} \\ &= w^{-q}\sum_j [R_\pi\overline{P}^j_* x - (-1)^q(-1)^{m(q+1)}R_\pi\underline{\beta\overline{P}^j_* x}\ (P^F_\pi c)(P^F_\pi v)^{-1}](P^F_\pi V)^{-j} \\ &= (-1)^{mq(q+1)/2}(m!)^{-q}u^{-mq}w^{-q}\sum_{i,j} [\underline{Q^i\overline{P}^j_* x} - (-1)^q\ \underline{\beta Q^i\overline{P}^j_*}xbu^{-1} \\ &\quad + (-1)^q Q^i\beta\overline{P}^j_* x(bu^{-1}-cv^{-1})(1-U^{-1}V)^{-1} - \underline{\beta Q^i\beta\overline{P}^j_*}xbcu^{-1}v^{-1}(1-U^{-1}V)^{-1}]U^iT^{-j}\end{aligned} \tag{19}$$

If we collect the terms in (18) and (19) not involving b or c we get (8). Collecting the terms involving b but not c gives (9).

It remains to show 3.5.

Proof of 3.5. Let p be odd; the p = 2 case is similar. We use the spectral sequence I.2.4
$$H^i(\pi;\tilde{H}^j X^{(p)}) \Longrightarrow \tilde{H}^{i+j}(D_\pi X).$$

Let $\{x_\alpha\}_{\alpha\in A}$ be an ordered basis for $\tilde{H}^*X$. Let $|\alpha|$ denote the degree of x_α. The graded group $\tilde{H}^*(X^{(p)}) \cong \tilde{H}^*(X)^{\otimes p}$ has the basis $\{x_{\alpha_1}\otimes\cdots\otimes x_{\alpha_p} \mid \alpha_1,\ldots,\alpha_p \in A\}$ and the E_2-term has a basis consisting of representatives for the elements $\{b^\varepsilon u^i \mathcal{P}_\pi x_\alpha \mid \alpha\in A,\ \varepsilon = 0 \text{ or } 1,\ i \geq 0\}$ and $\{\tau_\pi(x_{\alpha_1}\otimes\cdots\otimes x_{\alpha_p}) \mid \alpha_1 = \min \alpha_i \neq \max \alpha_i\}$ (in particular, the spectral sequence collapses, as we also know from I.2.3). Hence these elements form a basis for $\tilde{H}^*(D_\pi X)$. Let $z\in\tilde{H}^* D_\pi X$ be a nonzero element with $\iota^* z = \Delta^* z = 0$. Since $\iota^* z = 0$, z is a finite sum of the form
$$\sum_{\alpha,\varepsilon,i} \lambda_{\alpha,\varepsilon,i} b^\varepsilon u^i \mathcal{P}_\pi x_\alpha.$$
Since $\Delta^* z = 0$, we have

$$(20)\quad 0 = \sum_{\alpha,\varepsilon,i} \lambda_{\alpha,\varepsilon,i} b^\varepsilon u^i P_\pi x_\alpha$$
$$= \sum (-1)^{j+m|\alpha|(|\alpha|-1)/2}(m!)^{|\alpha|}\lambda_{\alpha,\varepsilon,i} u^{i+m|\alpha|-2jm} b^\varepsilon [P^j x_\alpha + (-1)^{|\alpha|}(\beta P^j x_\alpha) bu^{-1}]$$

by equation (4) of section 2. Now let K be $\max\{i+m|\alpha| \mid \lambda_{\alpha,\varepsilon,i} \neq 0\}$ and let S be the set of triples (α,ε,i) with $\lambda_{\alpha,\varepsilon,i}\neq 0$ and $i+m|\alpha| = K$. Then the coefficient of u^k in line (20) is
$$\sum_{(\alpha,\varepsilon,i)\in S} (-1)^{m|\alpha|(|\alpha|-1)/2}(m!)^{|\alpha|}\lambda_{\alpha,\varepsilon,i} b^\varepsilon x_\alpha$$
since all other terms in line (20) involve smaller powers of u. But this is a contradiction since the x_α are linearly independent.

§4. Atiyah's power operations in K-theory

In this section we show that the power operations in KU and KO defined by Atiyah [17] give H_∞^d structures for these spectra which agree with those constructed in VII §7. We shall work with complex K-theory, but everything is similar for KO.

We begin by recalling the definition of Atiyah's operations. Let G be a finite group. If Y is a G-space let $\mathrm{Vect}_G Y$ be the set of isomorphism classes of equivariant vector bundles over Y; we write Vect Y for the case where G is the trivial group. If Y is a free G-space there is a natural bijection

$$\mathrm{Vect}_G Y \cong \mathrm{Vect}(Y/G)$$

(see [18, 1.6.1]). If Y is any G-space we write Λ for the composite

$$\mathrm{Vect}_G Y \to \mathrm{Vect}_G(EG \times Y) \to \mathrm{Vect}(EG \times_G Y),$$

where the first map is induced by the projection $EG \times Y \to Y$. The map Λ is additive and hence if Y is a finite G-complex we obtain a map

$$K_G Y \to K(EG \times_G Y)$$

which will also be denoted by Λ. Now if X is a finite nonequivariant complex and we let Σ_j act on X^j by permuting the factors then the j-fold tensor power gives a map

$$\bar{\mathcal{P}}_j : \mathrm{Vect}\ X \to \mathrm{Vect}_{\Sigma_j} X^j \to K_{\Sigma_j} X^j$$

which however is not additive. In order to extend it to virtual bundles and to the relative case we must use the "difference construction" [94, Proposition 3.1]. Let (Y,B) be a G-pair and consider the set of complexes

$$0 \leftarrow E_0 \xleftarrow{d_1} E_1 \leftarrow \cdots \xleftarrow{d_n} E_n \leftarrow 0$$

of G-vector bundles E_i over Y which are acyclic over B. We write $\mathcal{D}_G(Y,B)$ for the set of isomorphism classes of such complexes. Two elements E_* and E'_* of $\mathcal{D}_G(Y,B)$ are homotopic, denoted $E_* \simeq E'_*$, if there is an element H_* $\mathcal{D}_G(Y \times I, B \times I)$ (with G acting trivially on I) which restricts to E_* and E'_* at the two ends. We say that E_* and E'_* are equivalent, written $E_* \sim E'_*$, if there are complexes F_* and F'_* which are acyclic on Y such that

$$E_* \oplus F_* \simeq E'_* \oplus F'_*.$$

It is shown in [94, appendix] that there is a natural epimorphism

$$\Gamma : \mathcal{D}_G(Y,B) \to K_G(Y,B)$$

which induces a bijection from the equivalence classes in $\mathcal{D}_G(Y,B)$ to $K_G(Y,B)$. If B is empty Γ is easy to describe: it takes E_* to $\sum (-1)^i E_i$. Γ is additive and multiplicative if we define addition and multiplication in $\mathcal{D}_G$ to be the direct sum and tensor product of complexes. Now if (X,A) is any pair of finite CW complexes the j-fold tensor product of complexes give a map

$$\mathcal{D}(X,A) \to \mathcal{D}_{\Sigma_j}((X,A)^j).$$

If E_* and E'_* in $\mathcal{D}(X,A)$ are homotopic by a homotopy H_* then the restriction of $H_*^{\otimes j}$ along the diagonal map

$$(X,A)^j \times I \to (X,A)^j \times I^j$$

gives a homotopy between $E^{\otimes j}$ and $(E'_*)^{\otimes j}$. If F_* is acyclic on X then the inclusion $(E_*)^{\otimes j} \to (E_* \oplus F_*)^{\otimes j}$ is Σ_j-equivariantly split and is a homology equivalence by the Künneth theorem, so that $E_*^{\otimes j} \sim (E_* \oplus F_*)^{\otimes j}$. It follows that the j-fold tensor product preserves equivalence and we can pass to equivalence classes to obtain a map

$$\overline{\mathcal{P}}_j : K(X,A) \to K_G((X,A)^j).$$

Letting A be the basepoint * of X we write $\mathcal{P}_j$ for the composite

$$\tilde{K}X = K(X,*) \longrightarrow K_{\Sigma_j}((X,*)^j) \xrightarrow{\Lambda} K(E\Sigma_j \times_{\Sigma_j} (X,*)^j) = \tilde{K}D_jX.$$

We can extend $\mathcal{P}_j$ to all even dimensions by letting it take the Bott element $b \in \tilde{K}^{-2}(S^0)$ to b^j. It is easy to see that the $\mathcal{P}_j$ are consistent in the sense of 1.3, so by 1.2 and 1.3 we have

<u>Theorem 4.1</u>. KU (resp. KO) has a unique H_∞^2 (resp. H_∞^8) ring structure for which the power operations are those defined by Atiyah.

We shall see in Section 6 that the H_∞^8 structure on KO extends to an H_∞^4 structure. Our next result answers an obvious question.

<u>Proposition 4.2</u>. The structures on KO and KU given by 4.1 are the same as those given by VII.7.2.

For the proof we need a lemma.

<u>Lemma 4.3</u>. Let X be a based space and let $\lambda : X^+ \to X$ be the based map which is the identity on X. Then

$$(D_j\lambda)^* : \tilde{F}^*D_jX \to \tilde{F}^*(D_j(X^+))$$

is a split monomorphism for any theory F.

<u>Proof of 4.3</u>. If $\nu : \Sigma^\infty X \to \Sigma^\infty X^+$ is the map given in the proof of 3.3 then $(D_j\nu)^*(D_j\lambda)^* = (D_j(\Sigma^\infty\lambda \circ \nu))^* = 1$.

<u>Proof of 4.2</u>. Let $\mathcal{P}_j$ be Atiyah's power operation and let $\mathcal{P}'_j$ be that given by VII.7.2. By VII.7.7 we have

$$\mathcal{P}'_j(\Sigma^2 b) = (\mathcal{P}'_j\Sigma^2 1)\cdot b^j$$

while by 1.1(iii) we have

$$\mathcal{P}'_j(\Sigma^2 b) = (\mathcal{P}'_j\Sigma^2 1)\cdot\mathcal{P}'_j b.$$

Since $\mathcal{P}'_j\Sigma^2 1$ is an orientation for the Thom complex $D_j S^2$ this implies $\mathcal{P}'_j b = b^j = \mathcal{P}_j b$. It therefore suffices by 1.3 to show that $\mathcal{P}_j$ and $\mathcal{P}'_j$ are equal on $\tilde{K}X$ for any finite complex X, and by 4.3 it suffices to show that they agree on $\tilde{K}(X^+) = KX$. They do agree on Vect X by [71, VIII.1.2]. But any element x of KX can be written in the form V-W with V,W ε Vect X, and we have

$$\mathcal{P}_j V = \mathcal{P}_j(x + W) = \mathcal{P}_j x + \mathcal{P}_j W + \sum_{i=1}^{j-1} \tau_{i,j-i}[(\mathcal{P}_i x)(\mathcal{P}_{j-i} W)]$$

by 1.1(vi), and similarly for $\mathcal{P}'_j$. Hence

$$\mathcal{P}_j x = \mathcal{P}_j V - \mathcal{P}_j W - \sum_{i=1}^{j-1} \tau_{i,j-i}[(\mathcal{P}_i x)(\mathcal{P}_{j-i} W)]$$

and similarly for $\mathcal{P}'_j$. We therefore have $\mathcal{P}_j x = \mathcal{P}'_j x$ by induction on j.

By analogy with Section 2 we now ask what operations in K-theory can be obtained from the internal power operation

$$P_\pi : KX \to K(X \times B\pi)$$

The structure of $K(B\pi)$ has been determined by Atiyah [16]: $\tilde{K}(B\pi)$ is a $\hat{Z}_p$-module and the composite

$$IR(\pi) \otimes \hat{Z}_p \xrightarrow{\Lambda \otimes 1} \tilde{K}(B\pi) \otimes \hat{Z}_p \longrightarrow \tilde{K}(B\pi)$$

is an isomorphism, where $IR(\pi)$ is the augmentation ideal. If ρ is the automorphism group of π then the invariant subgroup $\tilde{K}(B\pi)^\rho$ is generated by $\Lambda(N-p)$, where N is the regular representation of π. Atiyah also shows that $K^1 B\pi = 0$. In particular, $K^* B\pi$ is flat over $K^*(pt)$ and we obtain a Künneth isomorphism

$$KX \otimes K(B\pi) \cong K(X \times B\pi)$$

for finite complexes X. Since P_π is the restriction of P_p we see that P_π actually lands in the invariant subring $KX \otimes K(B\pi)^\rho$. We can therefore define operations

$$\varphi^p : KX \to KX$$

and

$$\theta^p : KX \to KX \times \hat{Z}_p$$

by the equation

(1) $$P_\pi x = \varphi^p x \otimes 1 + \theta^p x \otimes \Lambda(N - P).$$

By 1.4(i) we have

(2) $\varphi^p x = x^p.$

Atiyah proves the relation

(3) $p\theta^p x = x^p - \psi^p x$

in [17]. Since the representation N of π is induced from the trivial representation of the trivial group we have $\Lambda(N) = \tau_\pi 1$ and therefore (1), (2) and (3) give

(4) $P_\pi x = \psi^p x \otimes 1 + \theta^p x \otimes \tau_\pi 1,$

an equation which will be used in §7.

We can in fact lift θ^p to KX by using the equivariant internal operation $\overline{P}_\pi$. This is the composite

$$KX \xrightarrow{\overline{\mathcal{P}}_\pi} K_\pi(X^p) \xrightarrow{\Delta^*} K_\pi X,$$

where Δ is the diagonal map from X with its trivial π-action to X^p with its permutation action. Clearly $P_\pi = \Lambda \circ \overline{P}_\pi$. Since π acts trivially on X, we have $K_\pi X \cong KX \otimes R\pi$. The ρ-invariant subring of $R\pi$ is generated by 1 and N-p, so we may define $\theta^p x$ as an element of KX by the equation

$$\overline{P}_\pi x = x^p \otimes 1 + \theta^p x \otimes (N - p).$$

The operation $\overline{P}_\pi$ satisfies the obvious analog of 1.4 and one can use its properties to obtain additivity and multiplicity formulas for θ^p and ψ^p (using equation (3) as the definition of ψ^p). One can also obtain the G-equivariant Adams operations in this way by starting with a G-complex X and constructing operations

$$\overline{\mathcal{P}}_j : K_G X \to K_{\Sigma_j \int G} X^j$$

exactly as before. The reader is referred to [34] for details.

§5. tom Dieck's operations in cobordism

In [31], tom Dieck constructed "Steenrod operations" (power operations in our terminology) for the cobordism spectra associated to the classical groups. In this section we use these operations to give H_∞^d structures for these spectra. A wider class of cobordism spectra will be investigated by Lewis in the sequel, and he will show that they have not just H_∞ but E_∞ structures. His results do not quite include those of this section, however, since his methods do not give the "d-structure" (i.e., the Σ_j-orientations) for the classical spectra.

Throughout this section we write G for any of the classical groups O, SO, $Spin^c$, U, SU, Sp or Spin. Let d = 1,2,2,2,4,4,4 respectively. We depart somewhat from standard notation (in this section only) by writing $G(i)$ for the group which acts on R^{di}. Let p_i be the universal $G(i)$-vector bundle over $BG(i)$, let $S(p_i)$ be its fibrewise one-point compactification, and let $T(p_i)$ be the Thom complex obtained by collapsing the points at ∞. We shall always identify principal $G(i)$-bundles with free $G(i)$-spaces, so that the principal bundle associated to p_i is $EG(i)$. If q is any $G(i)$-vector bundle with principle bundle Q, there is a bundle map $F:q \to p_i$ and induced maps $S(F):S(q) \to S(p_i)$ and $T(f):T(q) \to T(p_i)$. If F' is another such map we shall need to know that $T(F')$ is homotopic to $T(F)$ (of course this is well-known for the maps of base spaces induced by F and F'). Now F has the form $\tilde{F} \times_{G(i)} \mathbf{R}^{di}$ for some $G(i)$-map $\tilde{F}:Q \to EG(i)$ and $S(F)$ is equal to $\tilde{F} \times_{G(i)} S^{di}$, and similarly for F' and $S(F')$. It is shown in [32] that there is at most one $G(i)$-equivariant homotopy class of $G(i)$-maps from any $G(i)$-space into $EG(i)$, so it follows that $S(F)$ is homotopic to $S(F')$ by a homotopy preserving the base points in each fibre, and hence $T(F) \simeq T(F')$ as required.

Now we define the Thom prespectrum TG by letting $(TG)_{di} = T(p_i)$ with

$$\sigma:\Sigma^d T(p_i) \to T(p_{i+1})$$

induced by any bundle map from $p_i \oplus R^d$ to p_{i+1}. We wish to show that TG is an H_∞^d ring prespectrum. For this we need some bundle theoretic observations.

Let p be a $G(i)$-vector bundle over X with associated principal bundle P. Then $E\Sigma_j \times_{\Sigma_j} p^j$ is a vector bundle over $E\Sigma_j \times_{\Sigma_j} X^j$; we wish to give it a canonical $G(ij)$-bundle structure. Let $H = G(i)^j$. Then p^j is an H bundle over X^j with principal bundle P^j, and Σ_j acts on everything on the left. However, its action on P^j does not commute with the right H-action (P^j is not a "Σ_j-equivariant principal H-bundle"). Instead we have $\sigma(ph) = (\sigma p)(\sigma h)$ for $\sigma \in \Sigma_j$, $p \in P^j$, $h \in H$. Now let $Q = P^j \times_h G(ij)$. This is a principal $G(ij)$-bundle over X^j with associated vector bundle p^j. Because of our choice of d the permutation action of Σ_j on $(\mathbf{R}^{di})^j$ lifts to a homomorphism $\Sigma_j \to G(ij)$ denoted $\sigma \longmapsto \bar{\sigma}$, and we have $\sigma(h) = \bar{\sigma}h\bar{\sigma}^{-1}$ for all $h \in H$. We define a left action of Σ_j on Q by $\sigma(p,g) = (\sigma p,\overline{\sigma g})$; it is easy to check that this action is well-defined and that it commutes with the right action of $G(ij)$. Thus Q is a Σ_j-equivariant principal $G(ij)$-bundle and hence so is its pullback $E\Sigma_j \times Q$ to $E\Sigma_j \times_{\Sigma_j} X^j$. Since Σ_j acts freely on $E\Sigma_j \times Q$ and commutes with $G(ij)$ we can divide out by its action to get a principal $G(ij)$-bundle $E\Sigma_j \times_{\Sigma_j} Q$ over $E\Sigma_j \times_{\Sigma_j} X^j$. The reader can check that the associated vector bundle is $E\Sigma_j \times_{\Sigma_j} p^j$.

Since $T(E\Sigma_j \times_{\Sigma_j} p^j)$ is naturally homeomorphic to $D_j T(p)$ we obtain maps

$$\zeta_{j,i}:D_j(TG)_{di} \cong T(E\Sigma_j \times_{\Sigma_j} p_i^j) \longrightarrow T(p_{ij}) = (TG)_{dij}$$

for all $i,j \geq 0$. The diagrams of Definition VII.5.1 commute since in each case the two composites are induced by bundle maps into a universal bundle. Thus we have shown

Proposition 5.1. The maps $\zeta_{j,i}$ are an H_∞^d structure for TG.

Now define MG = Z(TG). Every G(i)-vector bundle q has a canonical Thom class in this theory represented by the map

$$T(q) \longrightarrow T(p_i) \xrightarrow{\kappa} (MG)_{di}$$

At this point we need some $\lim^1$ information.

Lemma 5.2. All of the pairs (TG,MG'), (TG,KU), (TG,KO), (TG,ku) and (TG,kO) are $\lim^1$-free.

Proof. First consider (TG,MG'). The pair (TU,MU) is clearly $\lim^1$-free since the spectral sequence $E_r(TU_{2i};MU)$ collapses for dimensional reasons. For each other choice of G and G' there are maps $f:MU \to MG'$ and $g:TG \to TU$ satisfying the hypotheses of VII.4.4, hence each pair (TG,MG') is $\lim^1$-free. A similar argument gives the remaining cases.

Corollary 5.3. MG is an H_∞^d ring spectrum.

On the other hand, it was shown in [71,IV§2] that MG has an E_∞ ring structure. Such structures always determine H_∞ structures, as mentioned in I§4; see [Equiv, VII§2] for the details. Let $\xi_j^E : D_j MG \to MG$ be the structural maps obtained in this way and let ξ_j^H be those obtained from 5.1 and 5.3. As one would expect, the two structures agree:

Proposition 5.4. For each j, $\xi_j^E \simeq \xi_j^H$.

Proof. We use the notations and Definitions of VII§8. Fix i and let $a = a_i$. It suffices to show that the elements z_i^H and z_i^E in cobordism represented by the composites

$$T(\eta_i) \wedge_{\Sigma_j} T(p_i)^{(j)} \xrightarrow{\kappa} (D_j MG)_a \xrightarrow{(\xi_j^E)_a} (MG)_a$$

and

$$T(\eta_i) \wedge_{\Sigma_j} T(p_i)^{(j)} \xrightarrow{\kappa} (D_j MG)_a \xrightarrow{(\xi_j^E)_a} (MG)_a$$

are equal. An inspection of the proofs of [71, IV.2.2] and [Equiv. VII.2.4] shows that the second composite is induced by a bundle map from $\eta_i \oplus (p_i)^j$ into the universal bundle p_a, hence z_i^E is the canonical Thom class in $MG^a(T(\eta_i) \wedge_{\Sigma_j} T(p_i)^{(j)})$. On the other hand by Proposition VII.8.1 there is a relative Thom isomorphism

$$\Psi:(MG)^a(T(\eta_i)\wedge_{\Sigma_j}(T(p_i)^{(j)})\longrightarrow(MG)^{a+dij}(\Sigma^a T(E\Sigma_j\times_{\Sigma_j}(p_i)^j))$$

which takes z_i^H to the canonical Thom class in the target group. Since the canonical Thom class of a Whitney sum is the product of the Thom classes, the relative Thom isomorphism Ψ takes the Thom class of $T(\eta_i)\wedge_{\Sigma_j}(T(p_i)^{(j)})$ to that of $\Sigma^a T(E\Sigma_j\times_{\Sigma_j}(p_i)^j)$. Thus $\Psi z_i^H=\Psi z_i^E$ and the result follows.

We conclude this section with a discussion of cobordism operations related to P_π. The situation in unoriented cobordism is quite simple: there is a Künneth isomorphism

$$MO^*(X\times BZ_2)\cong(MO^*X)[[\chi]]$$

where χ is the MO^* Euler class of the Hopf bundle, and we can define operations

$$R^i:MO^qX\to MO^{q+i}X$$

for $i\in Z$ by the equation

$$P_2x=\sum_i(R^ix)\chi^{q-i}.$$

One can prove various properties of the R^i exactly as in §2 (see [31, §15]).

To deal with the case of complex cobordism we need some formal-groups notation. Let $F(x,y)$ be the formal group of MU and let $[n](x)$ be the power series defined inductively for $n\geq 0$ by $[1](x)=x$ and $[n+1](x)=F([n](x),x)$. There is a Künneth theorem due to Landweber [49]:

$$MU^*(X\times B\pi)\cong(MU^*X)[[u]]/[p](u),$$

where u is the Euler class of a nontrivial irreducible <u>complex</u> representation of π. The power series $[p](u)$ has leading term pu but is not divisible by p, so that in particular $MU^*B\pi$ is torsion free. We cannot continue as in the unoriented case since the power series $[p](u)$ and the ring $MU^*B\pi$ admit no simple descriptions. There is however a relation between P_π and the Landweber-Novikov operations s_α which is due to Quillen and was used by him to give a proof of the structure theorem for π_*MU. Let $a_j(x)$ for $j\geq 1$ be the coefficient of y^j in the power series $\prod_{i=1}^{p-1}F([i](x),y)$. For a multi-index $\alpha=(\alpha_1,\ldots,\alpha_k)$ let $a(x)^\alpha=a_1(x)^{\alpha_1}\cdots a_k(x)^{\alpha_k}$. Define $\chi\in MU^{2p-2}B\pi$ by the equation $\chi\cdot\Sigma^2 1=P_\pi\Sigma^2 1$; thus χ is the Euler class of the complex reduced regular representation.

Proposition 5.5. For any finite complex X there is an integer $m \geq 0$ such that the equation

$$(1) \qquad (P_\pi x)\chi^{m-q} = \sum_{|\alpha| \leq m} (S_\alpha x)a(u)^\alpha \chi^{m-|\alpha|}$$

holds for all $x \in MU^{2q}X$.

For the proof see [93] or [11]. There is a similar relation between P_π and s_α in the unoriented case. Since the right side of equation (1) is additive in x we have

Corollary 5.6. $(P_\pi)(x+y) - P_\pi x - P_\pi y)\cdot\chi^m = 0$ for large m.

§6. The Atiyah-Bott-Shapiro orientation.

It is well-known that the KU and KO orientations constructed by Atiyah, Bott and Shapiro in [19] give rise to ring maps

$$\phi^U : MSpin^c \to KU$$

and

$$\phi^O : MSpin \to KO$$

In this section we shall prove

Theorem 6.1. ϕ^U is an H^2_∞ ring map and ϕ^O is an H^8_∞ ring map.

Remark 6.2. MSpin actually has an H^4_∞ structure, as shown in §5. By combining 6.1 with VII.6.2 we see that the H^8_∞ structures for KO and kO constructed in §4 and in VII§7 extend to H^4_∞ structures.

We shall give the proof of 6.1 only for ϕ^O, which will henceforth be denoted by ϕ; the remaining case is similar. If p is a Spin(8i)-vector bundle we denote its Atiyah-Bott-Shapiro orientation in KO(T(p)) by $\mu(p)$.

First we translate 6.1 to a bundle-theoretic statement. As usual, let p_{8i} be the universal Spin(8i)-vector bundle. If $X \subset BSpin(8i)$ is any finite complex, we obtain an orientation class

$$\mu(p_{8i}|X) \quad \widetilde{KO}(T(p_{8i}|X)).$$

These classes are consistent as X varies, hence by 5.2 and VII.4.2 they determine a unique class in $\widetilde{KO}(TSpin_{8i})$ which is represented by a map

$$\mu_i : TSpin_{8i} \to BO \times Z.$$

The sequence $\{\mu_i\}$ is a map of prespectra, and ϕ is defined to be $Z\{\mu_i\}$ (see VII§1). The multiplicative property [19, 11.1 and 11.3] of the Atiyah-Bott-Shapiro orientation implies at once that $\{\mu_i\}$ is a ring map, and hence so is ϕ by 5.2 and VII.2.3. Similarly, Theorem 6.1 is a consequence of the following property of μ.

Proposition 6.3. If p is any Spin(8i)-vector bundle then

$$\mu(E\Sigma_j \times_{\Sigma_j} p^j) = \mathcal{P}_j \mu(p),$$

where $\mathcal{P}_j$ is the power operation defined in §4.

In the terminology of §1, Proposition 6.3 says that $\mathcal{P}_j$ satisfies tom Dieck's axiom P4. tom Dieck gives a simple proof of the analogous statement for the KU-orientation of complex bundles in [31, §12].

For the proof of 6.3 we need to recall several technical facts from [19]. The first is the "shrinking" construction in $\mathcal{D}(D,Y)$. Let

$$E_*: \qquad 0 \leftarrow E_0 \xleftarrow{d_1} E_1 \leftarrow \cdots \xleftarrow{d_n} E_n \leftarrow 0$$

be a complex of real vector bundles over X which is acyclic over Y. Choose Euclidean metrics in each E_i and let $\delta_i: E_{i-1} \to E_i$ be the adjoint of d_i with respect to the chosen metrics. Let

$$s(E_*): \qquad 0 \leftarrow s(E)_0 \xleftarrow{D} s(E)_1 \leftarrow 0$$

be the complex with $s(E)_0 = \bigoplus_{i \text{ even}} E_i$, $s(E)_1 = \bigoplus_{i \text{ odd}} E_i$, and differential

$$D(e_1, e_3, \ldots) = (d_1e_1, \delta_2e_1 + d_3e_3, \delta_4e_3 + d_5e_5, \ldots)$$

Then s(E) is in $\mathcal{D}(X,Y)$ and it defines the same element in KO(X,Y) that E does (see [19, p.22]). The same construction works G-equivariantly provided that the chosen Euclidean metrics are G-invariant.

Next we need the Clifford algebra C_i. By definition, C_i is the quotient of the tensor algebra $T(\mathbf{R}^i)$ by the ideal generated by the set $\{x \otimes x - \|x\|^2 \cdot 1 \mid x \in \mathbf{R}^i\}$. The grading on $T(\mathbf{R}^i)$ gives C_i a Z_2-grading by even and odd degrees and we will write $\boxtimes$ for the Z_2-graded tensor product of two Z_2-graded objects. By a module M over C_i we mean a Z_2-graded real vector space with a map

$$C_i \boxtimes M \to M$$

satisfying the usual properties. Equivalently, such a structure is given by two maps

$$\mathbf{R}^i \otimes M^0 \to M^1$$

and $$\mathbf{R}^i \otimes M^1 \to M^0,$$

each denoted by $x \otimes m \longmapsto xm$, such that

(1) $$x(xm) = -\|x\|^2 m$$

for all x,m. In particular, the latter description shows that if M is a C_i-module and N is a C_j-module then $M \boxtimes N$ is a C_{i+j}-module with

$$(x \oplus y)(m \otimes n) = xm \otimes n + (-1)^{|m|} x \otimes yn$$

for all $x \in \mathbf{R}^i$, $y \in \mathbf{R}^j$, $m \in M$, $n \in N$. If M is any module over C_i we can define a complex

$$E(M): \qquad 0 \longleftarrow E_0(M) \xleftarrow{d} E_1(M) \longleftarrow 0$$

of real vector bundles over R^i by letting $E_0(M) = R^i \times M^0$, $E_1(M) = R^i \times M^1$, and $d(x,m) = (x,xm)$. Equation (1) shows that this is acyclic except at 0, and in particular it defines an element of $KO(D^i,S^{i-1})$.

We can now define two complexes over $(R^i)^j$, namely $E(M^{\boxtimes j})$ and the external tensor product $E(M)^{\otimes j}$. The first has length 2 and the second has length j+1. We need to be able to compare them.

<u>Lemma 6.4</u>. The inner product in $E(M)^{\otimes j}$ can be chosen so that $s(E(M)^{\otimes j})$ is isomorphic to $E(M^{\boxtimes j})$.

<u>Proof</u>. It is shown in [19,p. 25] that one can choose inner products in M^0 and M^1 so that the adjoint of $x:M^1 \to M^0$ is $-x:M^0 \to M^1$ for each $x \in \mathbf{R}^i$. We define an inner product in $M^{\otimes j}$ by

$$\langle m_1 \otimes \cdots \otimes m_j\, ,\, m_1' \otimes \cdots \otimes m_j' \rangle = \langle m_1, m_1' \rangle \cdots \langle m_j, m_j' \rangle$$

with the understanding that $\langle m,m' \rangle = 0$ if $|m| \neq |m'|$. Then $s(E(M)^{\times j})$ and $E(M^{\boxtimes j})$ clearly involve the same two bundles, but they have different differentials, say d and d'. The definition of the shrinking construction gives

$$d(x,m_1 \otimes \cdots \otimes m_j) =$$
$$\sum_{k=1}^{j} (-1)^{|m_1|+\cdots+|m_i|-1} (x,m_1 \otimes \cdots \otimes m_{i-1} \otimes x_i m_i \otimes m_{i-1} \otimes \cdots \otimes m_j)$$

if $x = x_1 \oplus \cdots \oplus x_j \in (R^i)^j$, while the definition of $M^{\boxtimes j}$ as a C_{ij}-module gives

$$d'(x,m_1 \otimes \cdots \otimes m_j) =$$

$$\sum_{k=1}^{j} (-1)^{|m_1|+\cdots+|m_{i-1}|}(x,m_1 \otimes \cdots \otimes m_{i-1} \otimes x_i m_i \otimes m_{i+1} \otimes \cdots \otimes m_j).$$

The required isomorphism is given by taking $(x,m_1 \otimes \cdots \otimes m_j)$ to itself if $|m_1| + \cdots + |m_j|$ is congruent to 0 or 1 mod 4 and to its negative in the remaining cases.

Next we recall that Spin(i) is a subgroup of the group of units of C_i (in fact this is the definition of Spin(i) in [19, p.8]) and that the resulting conjugation action on $R^i \subset C_i$ agrees with its usual action on R^i. We can therefore define an action of Spin(i) on E(M) through automorphisms by $g(x,m) = (gxg^{-1},gm)$. Now if P is a principal Spin(i)-bundle over X with associated vector bundle $p:V \to X$ we can define a complex E(M,P) over $V = P \times_{Spin(i)} \mathbf{R}^i$ by

$$E(M,P) = P \times_{Spin(i)} E(M).$$

This complex defines an element of $\mathcal{D}(BV,SV)$ and hence of $\widetilde{KO}(T(p))$. If P is a G-equivariant principal bundle for some G (i.e., G acts from the left on P and commutes with the right action of Spin(i)) then E(M,P) has a left G-action and defines an element of $\widetilde{KO}_G(T(p))$. If G acts freely on P we can divide out by its action , and it is easy to see that the quotient complex E(M,P)/G is just E(M,P/G).

Atiyah, Bott and Shapiro specify a module λ over C_8 for which $E(\lambda)$ represents the Bott element in $\widetilde{KO}(S^8)$ (see [19, p.15]), and if P is a principal Spin(8i)-bundle they define $\mu(p) \in \widetilde{KO}(T(p))$ to be the element represented by $E(\lambda^{\boxtimes i},P)$.

From now on we fix i, P and p and denote $\lambda^{\boxtimes i}$ by M. Let $q = p^j$ with its permutation action by Σ_j and let Q be the associated Σ_j-equivariant Spin(8ij)-bundle as defined in Section 5. To prove 6.3 it suffices to show that $E(M^{\boxtimes j},Q)$ and the external tensor product $E(M,P)^{\otimes j}$ define the same element of $\widetilde{KO}_{\Sigma_j}(T(q))$. We can describe these complexes more simply: the first is

$$P^j \times_{Spin(8i)^j} E(M^{\boxtimes j})$$

and the second is

$$P^j \times_{Spin(8i)^j} (E(M)^{\otimes j});$$

in each case Σ_j acts through permutations of <u>both</u> factors. Now it is shown in [19, p. 25] that the inner products on M^0 and M^1 used in the proof of Lemma 6.4 can be chosen to be invariant under Spin(8i), hence the inner product on $E(M)^{\otimes j}$ used in the proof of that lemma is invariant under both $(Spin(8i))^j$ and Σ_j, and so is the isomorphism $s(E(M)^{\otimes j}) \cong E(M^{\boxtimes j})$. It follows that $s(E(M,P)^{\otimes j})$ is isomorphic to $E(M^{\boxtimes j},Q)$ as required.

§7. p-local H_∞ ring maps.

In this section we make some general observations about p-local H_∞ ring maps and apply them to show that the Adams operations are H_∞ ring maps and that the Adams summand of $KU_{(p)}$ is an H_∞^2 ring spectrum. We also obtain a sufficient condition for BP to be an H_∞^2 ring spectrum.

Throughout this section we let p be a fixed prime and let $\pi \subset \Sigma_p$ be generated by a p-cycle.

Lemma 7.1. Let F be a p-local spectrum and let Y be any spectrum. The map

$$\beta^*: F^*(D_{jp}Y) \to F^*(D_j D_\pi Y)$$

is split monic, and if j is prime to p the map

$$\alpha^*: F^* D_j Y \to F^*(Y \wedge D_{j-1}Y)$$

is split monic.

Proof. The subgroup $\Sigma_j \int \pi$ of Σ_{jp} has index prime to p, and hence the composite

$$H^*(\Sigma_{jp};M) \longrightarrow H^*(\Sigma_j \int \Sigma_p;M) \xrightarrow{\tau} H^*(\Sigma_{jp};M)$$

is an isomorphism for any p-local Σ_{jp}-module M. Thus

$$F^* D_{jp} Y \xrightarrow{\iota^*} F^* D_{\Sigma_j \int \pi} Y$$

is split monic by I.2.4. The result for β^* follows since β factors as

$$D_j D_\pi Y \simeq D_{\Sigma_j \int \pi} Y \xrightarrow{\iota} D_{jp} Y$$

and the result for α^* is similar.

As an application, we have

Proposition 7.2. Let E and F be H_∞^d ring spectra with power operations $\mathcal{P}_j$ and $\mathcal{P}'_j$. Suppose that F is p-local. Let $f: E \to F$ be a ring map such that the equation

$$(1) \qquad f_* \circ \mathcal{P}_p = \mathcal{P}'_p \circ f_*$$

holds on $E^{di}Y$ for all $i \in Z$ and all spectra Y. Then f is an H_∞^d ring map.

Proof. We shall show that $f_* \circ \mathcal{P}_j = \mathcal{P}'_j \circ f_*$ for all j by induction on j. This is trivial for j = 1 since $\mathcal{P}_1$ is the identity. Suppose it is true for all k < j. If j is prime to p we have $\alpha^* f_*\mathcal{P}_j y = (f_* y)(f_*\mathcal{P}_{j-1}y)$ and $\alpha^*\mathcal{P}'_j f_* y = (f_* y)(\mathcal{P}'_{j-1}f_* y)$. If j has the form kp we have $\beta^* f_*\mathcal{P}_j y = f_*\mathcal{P}_k\mathcal{P}_\pi x$ and $\beta^*\mathcal{P}'_j f_* x = \mathcal{P}'_k\mathcal{P}'_\pi f_* x$. In either case the result follows from 7.1 and the inductive hypothesis.

Under the usual $\lim^1$ hypotheses, it suffices to check equation (1) for spaces of for finite CW complexes. However, for actual calcualtions it is much easier to deal with the internal operation P_π than with $\mathcal{P}_\pi$. Our next result allows us to reduce to this case when we are dealing with spectra like KU or MU.

Proposition 7.3. Let F be a p-local spectrum such that $\pi_* F$ is free over $Z_{(p)}$ in even dimensions and zero in odd dimensions. Let X be a space such that $H_*(X;Z)$ is free abelian in even dimensions and zero in odd dimensions. Suppose that X and F have finite type. Then the map

$$\iota^* \oplus \Delta^* : \tilde{F}^* D_\pi X \to \tilde{F}^* X^{(p)} \oplus \tilde{F}^*(X \wedge B\pi^+)$$

is monic.

Proof. First let $F = HZ_{(p)}$. The Bockstein on $\tilde{H}^*(D_\pi X; Z_p)$ is given by II.5.5 and it follows that $E_2 = E_\infty$ in the Bockstein spectral sequence. Thus $\tilde{H}^*(D_\pi X; Z_{(p)})$ is a direct sum of copies of $Z_{(p)}$ and Z_p, so it suffices to show that the maps $(\iota^* \oplus \Delta^*) \otimes \mathbf{Q}$ and $(\iota^* \oplus \Delta^*) \otimes Z_p$ are monic. For the first we observe that $\iota^* \otimes \mathbf{Q}$ is a split injection by a simple transfer argument. For the second we use 3.5 and the universal coefficient theorem. This completes the proof for $F = HZ_{(p)}$. For the general case, we observe that $\iota^* \oplus \Delta^*$ induces a monomorphism on E_2 of the Atiyah-Hirzebruch spectral sequence and that the spectral sequences for $X^{(p)}$ and $X \wedge B\pi^+$ collapse for dimensional reasons.

Our first application is to the Adams operation

$$\psi^k : KU_{(p)} \to KU_{(p)}$$

with k prime to p. This is well-known to be a ring map.

Theorem 7.4. If Y is any spectrum and $y \in KU^{2n}Y$ then $\psi^k\mathcal{P}_j y = k^{-jn}\mathcal{P}_j(k^n\psi^k y)$. In particular, ψ^k is an H_∞ ring map but not an H_∞^2 ring map.

Proof. Let $\mathcal{P}'_j y = k^{-jn}\mathcal{P}_j k^n y$ for y $K^{2n}Y$. We must show $\psi^k\mathcal{P}_j = \mathcal{P}'_j\psi^k$. The $\mathcal{P}'_j$ are consistent in the sense of 1.2 and thus define another H_∞^2 structure on $KU_{(p)}$ (which agrees with the standard H_∞ structure but has different Σ_j-orientations). By 7.2 it suffices to show $\psi^k\mathcal{P}_p = \mathcal{P}'_p\psi^k$, and by 1.3 it suffices to show this for finite complexes. Since ψ^k is a ring map we clearly have $\iota^*\psi^k\mathcal{P}_p = \iota^*\mathcal{P}'_p\psi^k$, so by 7.3 it suffices to show

(2) $$\psi^k P_\pi x = P'_\pi \psi^k x$$

for all $x \in K^{2n}X$ whenever X is a finite complex. If x is the Bott element b then $\psi^k b = kb$ and $P_\pi b = b^p$ so (2) is satisfied in this case. Thus we may assume n = 0. Since ψ^k is a stable map it commutes with the transfer, and thus (2) will follow from equation (4) of section 4 once we show that ψ^k commutes with θ^p. It suffices to show this for the universal case BU × Z, and since K(BU × Z) is torsion free it suffices to show that ψ^k commutes with $p\theta_p$. But this is immediate from equation (3) of Section 4.

Next we recall the Adams idempotents

$$E_a : KU_{(p)} \to KU_{(p)}, \quad a \in Z_{p-1}$$

defined in [5, Lecture 4]. These idempotents split off pieces of $KU_{(p)}$ which we shall denote by $L_0, \ldots, L_{p-2}$. Thus the idempotent E_a factors into a projection map and an inclusion map:

$$KU_{(p)} \xrightarrow{r_a} L_a \xrightarrow{s_a} KU_{(p)}$$

with $r_a s_a = 1$. Since $\sum_{a \in Z_{p-1}} E_a = 1$ we have $KU_{(p)} \simeq L_0 \vee \cdots \vee L_{p-2}$. The E_a satisfy the formulas $E_0 1 = 1$,

(3) $$E_a b^n = \begin{cases} 0 & \text{if } n \not\equiv a \bmod p-1 \\ b^n & \text{otherwise} \end{cases}$$

and

(4) $$E_a(xy) = \sum (E_{a'}x)(E_{a-a'}y).$$

In particular, the image of E_0 is a subring of K^*X and hence L_0 has a unique structure for which s_0 is a ring map. On the other hand, (3) implies that the kernel of E_0 is not an ideal and hence there is no ring structure on L_0 for which r_0 is a ring map.

<u>Proposition 7.5</u>. L_0 has a unique H_∞^2 ring structure for which s_0 is an H_∞^2 ring map.

<u>Proof</u>. We must show that $\mathcal{P}_j$ takes the image of E_0 to itself, i.e., that the equation

(5) $$E_0 \mathcal{P}_j E_0 y = \mathcal{P}_j E_0 y$$

holds on $K^{2n}Y$ for every $n \in Z$ and every spectrum Y.

Let ch be the Chern character and let X be a finite complex. We have $ch(\psi^p E_a x) = ch(E_a \psi^p x)$ for all $a \in Z_{p-1}$ and all $x \in KX$ by [5, p.84-85] and [1, 5.1(vi)]. Hence $\psi^p E_a = E_a \psi^p$ by [5, Lemma 4 of lecture 4]. As in the proof of 7.4 it follows that $E_a \theta^p = \theta^p E_a$ and that $E_a P_\pi x = P_\pi E_a x$ for all $x \in KX$. Now let $n \in Z$ and let a be the class of n in Z_{p-1}. Then we have

$$\begin{aligned} E_0 P_\pi E_0(b^n x) &= E_0 P_\pi (b^n E_{-a} x) = E_0(b^{pn} P_\pi E_{-a} x) \\ &= b^{pn} E_{-a} P_\pi E_{-a} x = b^{pn} P_\pi E_{-a} x \\ &= P_\pi (b^n E_{-a} x) = P_\pi E_0 (b^n x) \end{aligned}$$

for all $x \in KX$. As in the proof of 7.4 it follows that (5) holds on the space level with $\mathcal{P}_j$ replaced by $\mathcal{P}_\pi$. Since both sides of (5) are stable in the sense of 1.2 and 1.3, it follows that (5) holds on the spectrum level with $\mathcal{P}_j$ replaced by $\mathcal{P}_\pi$. The rest of the proof is an induction on j just like that in the proof of 7.2. We give the inductive step when j has the form kp:

$$\begin{aligned} \beta^* E_0 \mathcal{P}_j E_0 y &= E_0 \beta^* \mathcal{P}_j E_0 y = E_0 \mathcal{P}_k \mathcal{P}_\pi E_0 y \\ &= E_0 \mathcal{P}_k (E_0 \mathcal{P}_\pi E_0 y) = (\mathcal{P}_k E_0) \mathcal{P}_\pi E_0 y \quad \text{by inductive hypothesis} \\ &= \mathcal{P}_k \mathcal{P}_\pi E_0 y = \beta^* \mathcal{P}_j E_0 y\ , \end{aligned}$$

so that (5) holds in this case by 7.1. The remaining case is similar.

It would obviously be desirable to have an analog of 7.5 for BP. In this case the Quillen idempotent ε factors into a projection and an inclusion

$$MU_{(p)} \xrightarrow{r} BP \xrightarrow{s} MU_{(p)}$$

which are both ring maps. We could therefore attempt to factor the operations $\mathcal{P}_j$ either through the inclusion (as in the proof of 7.5) or through the projection (or both). The proof of 7.5 shows that the $\mathcal{P}_j$ factor through s_* if and only if the following equation holds for all finite complexes X and all $x \in MU^{2i}X$.

$$\varepsilon P_\pi \varepsilon x = P_\pi \varepsilon x. \tag{6}$$

Similarly, the $\mathcal{P}_j$ factor through r_* if and only if the equation

$$\varepsilon P_\pi \varepsilon x = \varepsilon P_\pi x \tag{7}$$

holds. In either case the resulting structural maps on BP would be the composites

$$\xi_j' : D_j BP \xrightarrow{D_j s} D_j MU \xrightarrow{\xi_j} MU \xrightarrow{r} BP.$$

The point is that, while these maps $\xi_j^!$ clearly satisfy the first and third diagrams of Definition I.4.3, the diagram involving β is much harder to verify and equations (6) and (7) give two sufficient conditions for it to commute. We conclude this section by giving some weaker sufficient conditions.

Lemma 7.6. Equation (6) or (7) holds in general if it does when x is the Euler class $v \in MU^2CP^\infty$ of the Hopf bundle over CP^∞.

Proof. Suppose $\varepsilon P_\pi \varepsilon v = \varepsilon P_\pi v$. Since ε is a ring map we have $\varepsilon \mathcal{P}_\pi \varepsilon v = \varepsilon \mathcal{P}_\pi v$ by 7.3 (with $X = CP^\infty$). Now $\varepsilon \mathcal{P}_\pi \varepsilon$ and $\varepsilon \mathcal{P}_\pi$ both satisfy tom Dieck's axioms P1, P2, and P3, so Theorem 11.2 of [31] implies that they are equal, hence $\varepsilon P_\pi \varepsilon = \varepsilon P_\pi$ for all spaces as required. The other case is similar.

Next we need some notation. Let $f(x) = \frac{[p](x)}{x} \in MU^*[[x]]$ where $[p](x)$ is the power series defined at the end of Section 5. Let $[p]'(x) \in BP^*[[x]]$ be $r_*[p](x)$ and let $f'(x) = r_*f(x)$. Let $u' \in BP^*B\pi$ be r_*u, so that u' is the BP-Euler class of a nontrivial complex irreducible representation of π. Landweber's Künneth theorem for $MU^*(X \times B\pi)$ given in Section 5 implies

$$BP^*(X \times B\pi) \cong (BP^*X)[[u']]/[p'](u')$$

Lemma 7.7. Equation (7) holds for all X if and only if equation

$$(8) \qquad r_*P_\pi \varepsilon [CP^n] = r_*P_\pi[CP^n] \mod f'(u')$$

holds in $BP^*B\pi$ for all $n \geq 0$.

Proof. Assume that (8) holds. We shall show that $r_*P_\pi \varepsilon v = r_*P_\pi v$, where v is as in 7.6. Let M^*X denote the even-dimensional part of $MU^*_{(p)}X$ and let P be the composite

$$M^*X \xrightarrow{P_\pi} M^*B\pi \cong (M^*X)[[u]]/[p](u) \longrightarrow (M^*X)[[u]]/f(u).$$

If M^*X has no p-torsion then, since f(x) has constant term p, u is not a zero-divisor in $M^*(X)[[u]]/f(u)$. The element χ of Corollary 5.6 has leading term $(p-1)!u^{p-1}$, hence χ is also not a zero divisor. Thus 5.6 implies that P is additive for such X. It is also multiplicative by 1.4(iii). In particular we have a ring homomorphism

$$P:M^*(pt) \to M^*(pt))[[u]]/f(u).$$

Since the elements $[CP^n]$ generate $M^*(pt) \otimes Q$ as a ring and since $MU^*(B\pi)$ is torsion free, equation (8) implies

(9) $\quad r_*P_\pi \varepsilon x = r_*P_\pi x \mod f'(u')$

for all $x \in MU^*(pt)$.

Now let $\varepsilon v = \sum_{i=1}^{\infty} b_i v^i$. Since ε is an idempotent we have $b_1 = 1$ and $\varepsilon b_i = 0$ for $i \geq 2$. Hence (9) gives

$$r_*P_\pi b_i = 0 \mod f'(u')$$

for all $i \geq 2$. Now the ring homomorphism

$$P: M^*(CP^\infty) \to M^*(CP^\infty \times B\pi) \cong M^*[[v,u]]/f(u)$$

is continuous with respect to the usual filtrations by [31, Theorem 5.1] and hence we have

$$r_*P_\pi \varepsilon v \equiv r_*P_\pi \sum_{i=1}^{\infty} b_i v \equiv \sum_{i=1}^{\infty} (r_*P_\pi b_i)(r_*P_\pi v)^i \equiv r_*P_\pi v \mod f'(u').$$

Finally, we observe that the map

$$BP^*(CP^\infty \times B\pi) \cong BP^*[[v',u']]/[p]'(u') \to BP^*[[v',u']]/u' \oplus BP^*[[v',u']]/f'(u')$$

is monic since u' and $f'(u')$ are relatively prime. We have shown that $r_*(P_\pi \varepsilon v - P_\pi v)$ goes to zero in the second summand, so we need only show that it goes to zero in the first. But the map

$$BP^*(CP^\infty \times B\pi) \to BP^*[[v',u']]/u' \cong BP^*[[v']]$$

can be identified with the restriction

$$(1 \times \iota)^*: BP^*(CP^\infty \times B\pi) \to BP^*CP^\infty$$

and the result follows since

$$(1 \times \iota)^* r_*(P_\pi \varepsilon v - P_\pi v) = r_*((\varepsilon v)^p - v^p) = (r_*v)^p - (r_*v)^p = 0.$$

We can now use Quillen's formula 5.5 to give a very explicit equation which is equivalent to (7).

<u>Corollary 7.8</u>. Equation (7) holds for all X if and only if the element

$$\sum_{|\alpha| \leq n} (c_\alpha, b^{-n-1}) r_*[CP^{n-|\alpha|}] r_*(a(u)^\alpha)(r_*\chi)^{n-|\alpha|}$$

of $BP^*B\pi$ is zero for each n not of the form p^k-1. Here the (c_α, b^{-n-1}) are certain numerical coefficients defined in [6, Theorem 4.1 of part I].

Proof. This is immediate from 5.5, 7.7, and [6, Theorems I.4.1 and II.15.2].

There is no obvious reason for the elements specified in 7.8 to be zero. If they were zero, it would be evidence of a rather deep connection between P_π and ε. The author's opinion is that there is no such deep connection and that neither equation (7) nor equation (6) holds in general.