CHAPTER V

THE HOMOTOPY GROUPS OF H_∞ RING SPECTRA

By Robert R. Bruner

§1. Explicit homotopy operations and relations

This section contains statements of our results on homotopy operations as well as some applications of these results. The proofs depend on material in §2 and will be given in §3.

Note that, aside from the computations in π_*S at the end of this section, all the results here apply to the homotopy of any H_∞ ring spectrum Y. Let $\xi: D_pY \to Y$ denote the structure map.

The order of results in this section is:

relation to other operations,

particular operations and relations,

Cartan formulas,

computations in π_*S,

remarks.

In order not to interrupt the main flow of ideas, we have deferred a number of remarks until the end of the section.

Throughout this section let $E_r(X,Y)$ be the ordinary mod p Adams spectral sequence converging to $[X,Y]_*$, and let $E_r(S,\mathcal{D})$ be the spectral sequence of IV §6 based on ordinary mod p homology. Let $\mathcal{D}$ be the sequence

$$\mathcal{D} = \{D_pS^n \cdots \leftarrow D_p^iS^n \leftarrow \cdots \leftarrow D_p^1S_n \leftarrow D_p^0S^n\}.$$

From the spectral sequence $E_r(S,\mathcal{D})$ we obtain an isomorphism between an associated graded of $\pi_*D_pS^n$ and $E_\infty(S,\mathcal{D})$:

$$E^0(\pi_*D_pS^n) \cong E_\infty(S,\mathcal{D}).$$

Write $E^0(\alpha)$ for the image in $E_\infty^{s,*}(S,\mathcal{D})$ of an element $\alpha \in \pi_*D_pS^n$ of filtration s. By IV.7.5, $E_2(S,\mathcal{D})$ is free over $E_2(S,S)$ on generators e_i corresponding to the cells of D_pS^n. By 2.9 below, a more convenient basis over $E_2(S,S)$ is given by the elements

$$\beta^{\varepsilon}P^j = (-1)^j \nu(n) e_{jq-\varepsilon-n(p-1)}$$

where $\varepsilon = 0$ or 1 ($\varepsilon = 0$ if $p = 2$), $q = 2(p-1)$ ($q = 1$ if $p = 2$), $jq-\varepsilon \geq n(p-1)$ and ν is the function defined in IV.2.4 ($\nu = 1$ if $p = 2$). Thus, $E^0(\alpha)$ can be written as a linear combination of the $\beta^{\varepsilon}P^j$ with coefficients in $E_2(S,S)$. Recall the operation $\alpha^*: \pi_n Y \to \pi_N Y$ associated to each element $\alpha \in \pi_N D_p S^n$.

Relation of the α^* to other operations

Proposition 1.1. If $\iota: S^{np} \to D_p S^n$ is the natural map then $\iota^*(x) = x^p$ and

$$E^0(\iota) = \begin{cases} P^n & p = 2 \\ P^j & p > 2 \text{ and } n = 2j \\ 0 & p > 2 \text{ and } n \text{ odd} \end{cases}$$

Propoosition 1.2. Let $h: \pi_* \to H_*$ be the Hurewicz homomorphism. If $E^0(\alpha) = \beta^{\varepsilon}P^j$ then $h \circ \alpha^* = \beta^{\varepsilon}Q^j \circ h$, where $\beta^{\varepsilon}Q^j$ is the Dyer-Lashof operation defined in III.1.

If $E^0(\alpha) = \sum a_{j,\varepsilon}\beta^{\varepsilon}P^j$, with each $a_{j,\varepsilon} \in E_2(S,S)$ and $\overline{x} \in E_2(S,Y)$, we let $E^0(\alpha)(\overline{x}) = \sum a_{j,\varepsilon}\beta^{\varepsilon}P^j(\overline{x})$.

Proposition 1.3. (Kahn, Milgram) If $x \in \pi_n Y$ is detected by $\overline{x} \in E_2(S,Y)$, then $\alpha^*(x)$ is detected by $E^0(\alpha)(\overline{x})$.

To see the relation to Toda brackets, suppose we have compressed α into the np+i skeleton $D_p^i S^n$ and that it projects to $\tilde{a}$ on the top cell S^{np+i}. Let $D_p^{i-1}(x) = D_p(x)|D_p^{i-1}S^n$ and let $c_i \in \pi_{np+i-1}D_p^{i-1}S^n$ be the attaching map of the np+i cell.

Proposition 1.4. $\alpha^*(x) \in \langle \tilde{a}, c_i, \xi D_p^{i-1}(x) \rangle$. The set of all such $\alpha^*(x)$ is a coset of $\xi D_p^{i-1}(x) \circ \pi_N D_p^{i-1}S^n$.

Note: We will frequently find further that $E^0(\alpha) = a\beta^{\varepsilon}P^j$ where $i = jq-\varepsilon-n(p-1)$ and $(-1)^j\nu(n)a$ detects $\tilde{a}$. Then

$$E^0(\alpha)(\overline{x}) = E^0(\alpha^*(\overline{x})) = a\beta^{\varepsilon}P^j(\overline{x}),$$

so that α^* is detected by Toda brackets in essentially the same fashion as by Steenrod operations in $E_2(S,Y)$.

Particular operations and relations

Hereafter, if $\theta \in E_\infty(S,\mathcal{D})$ and $x \in \pi_n Y$, let $\theta(x) = \{\alpha^*(x) | E^0(\alpha) = \theta\}$. Clearly, the indeterminacy in $\theta(x)$, defined to be

$$\mathrm{Ind}(\theta(x)) = \{\alpha^*(x) - \beta^*(x) | E^0(\alpha) = \theta = E^0(\beta)\},$$

is the set of values of all homotopy operations on x whose corresponding element in $E_\infty(S,\mathcal{D})$ has higher filtration than does θ.

Proposition 1.5 (Kahn, Milgram): The following are equivalent:

(i) $\beta^\varepsilon P^j$ acts on $\pi_n Y$

(ii) $e_i \in E_\infty(S,\mathcal{D})$, $i = jq-\varepsilon-n(p-1)$

(iii) $D_p^i S^n$ is reducible

(iv) if $p = 2$ then $n \equiv -i-1 \quad (2^{\phi(i)})$;
if $p > 2$ then $\varepsilon = 0$ and $n = 2j$,
or $\varepsilon = 1$ and $j \equiv 0 \quad (p^{\psi(i)})$.

The functions ϕ and ψ are defined in 2.5 and 2.11 below.

Definition 1.6. If $p = 2$, let $\beta_0 = 2$, $\beta_1 = \eta$, $\beta_2 = \nu$ and let β_j be a generator of Im J in dimension $8a+2^b-1$, where $j = 4a+b$ and $0 \le b \le 3$. If $p > 2$, let $\alpha_0 = p$, and let α_j be a generator of Im J in dimension $jq-1$.

Theorem 1.7 (Toda, Barratt, Mahowald, Cooley): Let $p = 2$. If $x \in \pi_n Y$ and $j = 4a+b$, $0 \le b \le 3$, then

$$\beta_j \circ x^2 = 0 \quad \text{if} \quad n \equiv 2^j - 8a - 2^b - 1 \quad (2^{j+1})$$

and $\beta_j \circ P^{n+1}(x) = \alpha x^2$ for some $\alpha \in \pi_{8a+2^b}S$ if $n \equiv 0 \ (2)$ and $n \equiv 2^j - 8a - 2^b - 2 \ (2^{j+1})$.

Theorem 1.8. Let $p > 2$ and $x \in \pi_n Y$. Let $\varepsilon_p(a)$ denote the exponent of p in the prime factorization of a. If $n = 2k-1$ then

$$\alpha_j \circ \beta P^k x = 0 \qquad \text{if } j = 0$$
$$\text{or } j > 0 \text{ and } \varepsilon_p(k+j) = j-1.$$

If $n = 2k$ then

$$\alpha_j \circ \beta P^{k+1} x = \alpha x^p \quad \text{for some } \alpha \ \pi_{(j+1)q-2}S$$
$$\text{if } j = 0$$
$$\text{or } j > 0 \text{ and } \varepsilon_p(k+j+1) = j-1.$$

Theorem 1.9. The operations listed in Tables 1.1 and 1.3 exist on π_n and satisfy the relations listed in Tables 1.2 and 1.4. In Tables 1.1 and 1.3 the columns labelled "indeterminacy" list generators for the indeterminacy of each operation, and the columns labelled "τ_{p*}" list the values of

$$\tau_{p*}:\pi_N D_p S^n \to \pi_N S^{np} \cong \pi_{N-np}S$$

thereby indicating the deviation from additivity of the given operation (by IV.7.4).

TABLE 1.1

Operations on π_n for $p > 2$

n	operations	indeterminacy	τ_{p*}
$n = 2k-1$	βP^k	0	0
	$h_0 P^k$	0	0
	$g_1 P^k$	0	0
$n = 2k-1$, $k \equiv -1$ (p)	βP^{k+1}	$h_0 P^k$	0
$n = 2k-1$, $k \equiv -2$ (p)	$h_0 \beta P^{k+1}$	$\alpha_1 \beta P^k$	0
	βP^{k+2}	$g_1 P^k$ and $h_0 P^{k+1}$ (if it exists)	0
$n = 2k$	P^k	0	p!
	βP^{k+1}	$\alpha_1 P^k$	multiple of α_1
	$h_0 P^{k+1}$	$\alpha_2 P^k$	multiple of α_2
$n = 2k$, $k \equiv -2$ (p)	βP^{k+2}	$h_0 P^{k+1}$ and $\alpha_2 P^k$	multiple of α_2

TABLE 1.2

Relations among operations on π_n for $p > 2$

$\underline{n}$	$\underline{\text{relations}}$
$n = 2k-1$	$p\beta P^k = ph_0P^k = pg_1P^k = 0$ $(k+1)\alpha_1\beta P^k = 0$
$n = 2k-1$ $k \equiv -1\ (p)$	$p\beta P^{k+1} = -h_0P^k$ $\alpha_1\beta P^{k+1} \equiv 0 \mod \alpha_2\beta P^k$
$n = 2k-1$ $k \equiv -2\ (p)$	$ph_0\beta P^{k+1} \equiv 0 \mod \alpha_2\beta P^k$
$n = 2k$	$k\alpha_1P^k = p\beta P^{k+1}$ $(k+2)\alpha_1\beta P^{k+1} = 0$
$n = 2k$ $k \equiv -2\ (p)$	$p\beta P^{k+2} \equiv -h_0P^{k+1} \mod \alpha_2P^k$

TABLE 1.3

Operations on π_n for $p = 2$

n	operations	indeterminacy	τ_{2*}
$n \equiv 0$ (4)	P^n	$2P^n$	2
	P^{n+1}	ηP^n	η
	P^{n+3}	$2P^{n+3}, \nu P^n$	multiple of ν
	$h_1 P^{n+2}$	$2h_1 P^{n+2}, \nu P^n$	multiple of ν
$n \equiv 1$ (4)	P^n	0	0
	$h_1 P^{n+1}$	$\eta^2 P_n$	0 or η^2
	P^{n+2}	$2P^{n+2}$	0 or η^2
	$h_1 P^{n+5}$	$2h_1 P^{n+5}$, $\nu^2 P^n$	0 or ν^2
	$h_1^2 P^{n+4}$	$2h_1^2 P^{n+4}$, $\nu^2 P^n$	0 or ν^2
	$h_1^3 P^{n+3}$	$2h_1^3 P^{n+3}$, $\nu^2 P^n$	0 or ν^2
$n \equiv 1$ (8)	P^{n+6}	$2P^{n+6}$	0 or ν^2
$n \equiv 2$ (4)	P^n	$2P^n$	2
	P^{n+1}	ηP^n	0
	$h_1 P^{n+4}$	$2h_1 P^{n+4}$	0
	$h_1^2 P^{n+3}$	$2h_1^2 P^{n+3}$	0
	$h_1^3 P^{n+2}$	$2h_1^3 P^{n+2}$	0
	$h_2 P^{n+3}$	$\nu^2 P^n$	0 or ν^2
$n \equiv 2$ (8)	P^{n+5}	$2P^{n+5}$	0
$n \equiv 3$ (4)	P^n	0	0
	$h_1 P^{n+1}$	0	0 or η^2
	$h_1 P^{n+3}$	$2h_1 P^{n+3}$	0
	$h_1^2 P^{n+2}$	$\eta^2 h_1 P^{n+1}$	0
	$h_2 P^{n+2}$	0	0
$n \equiv 3$ (8)	P^{n+4}	$2P^{n+4}$	0

TABLE 1.4

Relations among operations for $p = 2$

$n \equiv 0\ (4)$	$2P^{n+1} = 0$	
	$2h_1 P^{n+2} = \eta^2 P^{n+1}$	
$n \equiv 0\ (8)$	$2P^{n+3} = h_1 P^{n+2}$	
	$\eta P^{n+3} = 0$	
	$2\nu P^{n+3} = \nu h_1 P^{n+2} = 0$	
$n \equiv 4\ (8)$	$2P^{n+3} = h_1 P^{n+2} + \nu P^n$	
	$\eta P^{n+3} = \nu P^{n+1}$	
	$\nu h_1 P^{n+2} = \nu^2 P^n$	
$n \equiv 1\ (4)$	$2P^n = 0$	$2h_1 P^{n+5} = h_1^2 P^{n+4}$
	$2h_1 P^{n+1} = \eta^2 P^n$	$2h_1^2 P^{n+4} = h_1^3 P^{n+3}$
	$2P^{n+2} = h_1 P^{n+1}$	$2h_1^3 P^{n+3} = 0$
	$\eta h_1 P^{n+1} = 0$	
$n \equiv 1\ (8)$	$\eta P^{n+2} = 0$	
	$2\nu P^{n+2} = 0$	
	$2P^{n+6} = h_1 P^{n+5}$	
$n \equiv 5\ (8)$	$\eta P^{n+2} = \nu P^n$	
	$\nu P^{n+2} = 0$	
$n \equiv 2\ (4)$	$2P^{n+1} = \eta P^n$	$2h_1 P^{n+4} = h_1^2 P^{n+3}$
	$\eta P^{n+1} = 0$	$2h_1^2 P^{n+3} = h_1^3 P^{n+2}$
	$4\nu P^n = 0$	$2h_1^3 P^{n+2} = 0$
$n \equiv 2\ (8)$	$2P^{n+5} = h_1 P^{n+4}$	
	$\eta P^{n+5} = h_2 P^{n+3}$	
$n \equiv 6\ (8)$	$\nu P^{n+1} = 0$	
	$2h_2 P^{n+3} = \nu^2 P^n$	
	$\eta h_1 P^{n+4} \equiv 0 \mod \nu^2 P^n$	

$n \equiv 3\ (4)$	$2P^n = 0$	$2h_1P^{n+3} = h_1^2P^{n+2}$
	$\eta P^n = 0$	$2h_1^2P^{n+2} = \eta^2 h_1 P^{n+1}$
	$2h_1P^{n+1} = 0$	$2h_2P^{n+2} = 0$
$n \equiv 3\ (8)$	$2P^{n+4} = h_1P^{n+3}$	
	$\eta P^{n+4} = h_2P^{n+2}$	
	$\eta h_2P^{n+2} = \nu^2P^n$	
$n \equiv 7\ (8)$	$\nu P^n = 0$	
	$\eta h_1P^{n+3} = 0$	
	$\eta h_2P^{n+2} = 0$	

Cartan Formulas

For later computations we need the Cartan formulas for the first operation above the p^{th} power.

Proposition 1.10. Let $p = 2$, $x \in \pi_n Y$, $y \in \pi_m Y$. Assume n+m is even. Then

$$P^{n+m+1}(xy) = \begin{cases} P^{n+1}(x)y^2 + x^2P^{m+1}(y) + c_{n,m}\eta x^2y^2 & n \equiv m \equiv 0\ (2) \\ S_{n,m}(x,y) & n \equiv 3\ (4) \text{ or } m \equiv 3\ (4) \\ S_{n,m}(x,y) + c_{n,m}\eta x^2y^2 & n \equiv m \equiv 1\ (4) \end{cases}$$

where $S_{n,m}:\pi_n \times \pi_m \to \pi_{2(n+m)+1}$ is an operation such that

$$E^0(S_{n,m}) = P^nP^{m+1} + P^{n+1}P^m$$

and

$$2S_{n,m}(x,y) = \begin{cases} \eta x^2y^2 & n \equiv m \equiv 1\ (4) \\ 0 & n \equiv 3\ (4) \text{ or } m \equiv 3\ (4)\ , \end{cases}$$

and where $c_{n,m}$ is an integer depending only on n and m.

Proposition 1.11. Let $p > 2$, $x \in \pi_n Y$ and $y \in \pi_m Y$. Then

(i) if $n = 2j$ and $m = 2k$,

$$\beta P^{j+k+1}(xy) = \beta P^{j+1}(x)y^p + x^p\beta P^{k+1}(y) + d_{n,m}\alpha_1 x^p y^p$$

where $d_{n,m}$ is an integer depending only on n and m.

(ii) if $n = 2j$ and $m = 2k-1$,

$$\beta P^{j+k}(xy) = x^p\beta P^k(y)$$

(iii) if $n = 2j-1$ and $m = 2k-1$,

$$\beta P^{j+k}(xy) = S_{j,k}(x,y)$$

where $S_{j,k} : \pi_{2j-1}Y \times \pi_{2k-1}Y \to \pi_{2(j+k)p-3}Y$ is an operation such that $E^0(S_{j,k}) = \beta P^j \cdot P^k + P^j \cdot \beta P^k$ and $pS_{j,k}(x,y) = 0$.

Computations

Our final results contain extensions to all H_∞ ring spectra of classical results about π_*S due to Toda, Barratt, Mahowald, Gray and Milgram, as well as some low dimensional calculations at the prime 2.

Let $\doteq$ denote equality up to multiplication by a unit.

Proposition 1.12. If $p = 2$ then $P^1(2) = \eta$.

Proposition 1.13. If $p > 2$ then $\beta P^1(p) \doteq \alpha_1$ and $\beta P^{p-1}(\alpha_1) = \beta_1$.

Combined with the Cartan formulas 1.10 and 1.11, these yield the following results.

Proposition 1.14. Let $x \in \pi_n Y$ and $n = 2j$. If $p = 2$ then $P^{n+1}(2x) = \eta x^2$. If $p > 2$ then $\beta P^{j+1}(px) \doteq \alpha_1 x^p$ and $\beta P^{j+p-1}(\alpha_1 x) = \beta_1 x^p$. The indeterminacy of each is 0.

Corollary 1.15. Let $x \in \pi_n Y$. If $p = 2$, $n \not\equiv 1$ (4) and $2x = 0$, then $\eta x^2 = 0$. If $p > 2$ and $px = 0$ then $\alpha_1 x^p = 0$. If $p > 2$ and $\alpha_1 x = 0$ then $\beta_1 x^p = 0$. In particular, $\alpha_1 \beta_1^p = 0$.

In the next proposition, the statement "$aP^j(x) = y$ mod A" means that A is the indeterminacy of aP^j when applied to x. If the indeterminacy is not mentioned, it is 0.

Proposition 1.16. The following hold in π_*S localized at 2.

(i) $P^1(\eta) = \eta^2$

(ii) $P^3(\nu) = \nu^2$, $h_1P^4(\nu) = \eta\sigma$ or $\bar{\nu}$, $h_1^2P^5(\nu) = 0$.

(iii) P^3, h_1P^4, h_1P^6, $h_1^2P^5$, and h_2P^5 annihilate 2ν and 4ν.

(iv) P^6, P^7, $h_1^2P^9$, and $h_1^3P^8$ annihilate ν^2.

(v) $P^7(\sigma) = \sigma^2$, $h_1P^8(\sigma) = \eta^*$ or $\eta^* + \eta\rho$,

$$h_1P^{10}(\sigma) = \nu^* \bmod \langle 2\nu^*\rangle + \langle\eta\bar{\mu}\rangle,$$

$$h_1^2P^9(\sigma) = 2\nu^* \bmod \langle 4\nu^*\rangle + \langle\eta\bar{\mu}\rangle.$$

(vi) $P^7(2\sigma) = 0$, $h_1P^8(2\sigma) = 0$, $h_2P^9(2\sigma) = 0$,

$$h_1P^{10}(2\sigma) = 2\nu^* \bmod \langle 4\nu^*\rangle + \langle\eta\bar{\mu}\rangle,$$

$$h_1^2P^9(2\sigma) = 4\nu^* \bmod \langle\eta\bar{\mu}\rangle.$$

(vii) P^7, h_1P^8, $h_1^2P^9$ and h_2P^9 annihilate 4σ,

$$h_1P^{10}(4\sigma) = 4\nu^* \mod \langle\eta\bar{\mu}\rangle .$$

Remarks: These are listed by the result to which they refer.

(1.4): The indeterminacy of the Toda bracket $\langle\tilde{a}, c_i, \xi D_p^{i-1}(x)\rangle$ in Proposition 1.4 is $\xi D_p^{i-1}(x) \circ \pi_N D_p^{i-1}S^n + (\pi_{np+i}Y) \circ \tilde{a}$, while the indeterminacy of $\alpha^*(x)$ is only $\xi D_p^{i-1}(x) \circ \pi_N D_p^{i-1}S^n$. This reflects the fact that $\alpha^*(x)$ uses the canonical null homotopy $D_p^i(x)$ of $D_p^{i-1}(x) \circ c_i$, whereas the Toda bracket allows any null homotopy of $\xi D_p^{i-1}(x) \circ c_i$.

(1.8): Since π_{pq-2} is the first nonzero homotopy group of S in a dimension congruent to -2 mod q, we get

$$\alpha_j \beta P^{k+1}x = 0$$

for $j < p-1$ satisfying the hypotheses of (1.8).

(1.9): (i) In the range of dimensions listed, the operations and relations given in Tables 1.1 through 1.4 generate all the operations and relations over π_*S. For examples, when $n \equiv 0$ (4) and $p = 2$:

(a) ηP^n and η^2P^n are nonzero operations because the relations listed do not force them to be 0;

(b) the relation $4h_1P^{n+2} = 0$ follows from the listed relation

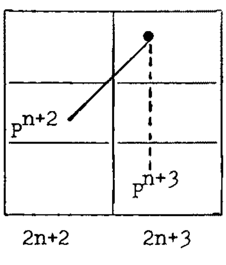
$$2h_1P^{n+2} = \eta^2P^{n+1},$$

and is therefore omitted;

(c) the redundant operation h_1P^{n+2} is included because the relation

$$2P^{n+3} = h_1P^{n+2}$$

which makes it redundant reflects a universally hidden extension:



$h_0P^{n+3} = 0$ in E_∞ and $2P^{n+3}x$ is detected by $h_1P^{n+2}x$.

(ii) The operations of degree n+3 for $n \equiv 0$ (4) and p = 2 are particularly interesting. If $n \equiv 0$ (8) then by [59] $\pi_{2n+3}D_2S^n = Z_8 \oplus Z_8$. It is generated by νP^n and P^{n+3} with relations

$$2P^{n+3} = h_1P^{n+2}$$

and

$$4P^{n+3} = 2h_1P^{n+2} = \eta^2P^{n+1}.$$

If $n \equiv 4$ (8) then [59] gives $\pi_{2n+3}D_2S^n = Z_4 \oplus Z_{16}$ and it is generated by h_1P^{n+2} (of order 4) and P^{n+3} (of order 16) with relations

$$2h_1P^{n+2} = \eta^2P^{n+1}$$
$$2P^{n+3} = h_1P^{n+2} + \nu P^n$$
$$4P^{n+3} = \eta^2P^{n+1} + 2\nu P^n$$
$$8P^{n+3} = 4\nu P^n.$$

(iii) Entries in the τ_{p*} column such as "0 or η^2" indicate that we have not calculated τ_{p*}. Such entries simply list the elements of π_*S in the relevant dimension. Even this limited information is useful in Proposition 1.16.

<u>(1.10) and (1.11)</u>: Let $\psi: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be the diagonal of the Steenrod algebra ($\psi(P^n) = \sum P^i \otimes P^{n-i}$). If

$$E^0(\alpha) = \sum a_iA_i, \quad a_i \in E_2(S,S), \quad A_i \in \mathcal{A}$$

then

$$E^0(\delta_*(\alpha)) = \sum a_i\psi(A_i).$$

This defines $\delta_*(\alpha)$ and, hence, the formula for $\alpha^*(xy)$, modulo higher filtration in $E_\infty(S,\mathcal{D})$.

<u>(1.15)</u>: This proof that $\alpha_1\beta_1^p = 0$ differs from Toda's in that Toda views the product in π_*S as composition and studies $D_p(S^n \cup_p e^{n+1})$ while we view it as the smash product and study $D_pS^n \wedge D_pS^m$. Toda shows that

$$D_p(S^n \cup_p e^{n+1}) \supset S^{np} \cup_{\alpha_1} e^{np+q}$$

and

$$D_p(S^n \cup_{\alpha_1} e^{n+q}) \supset S^{np} \cup_{\beta_1} e^{np+pq-1}.$$

Thus, if px = 0 or $\alpha_1x = 0$ then $\alpha_1x^p = 0$ or $\beta_1x^p = 0$, respectively. The proof given in 1.15 uses the values of the operations on p and α_1, rather than the structure of D_p of their cofibers.

Segal [49] saw that the Cartan formula for homotopy operations should provide a proof that $\alpha_1\beta_1^p = 0$, but his explicit formulas were incorrect.

There is still another proof that $\alpha_1\beta_1^p = 0$ which uses virtually none of the machinery of homotopy operations, but does require that we have calculated enough of π_*S to know that the p^2q-3 stem is either 0 or Z_p. Given this, the relation

$$-\alpha_1\beta_1^p = p\beta P^{p^2-p}(\beta_1)$$

from Table 1.2 implies that $\alpha_1\beta_1^p = 0$.

Remark 1.17: This is a quick survey of results on homotopy operations which are not included here. Toda [106] shows thAt the extended powers propagate several relations. For example, if $\langle\alpha_1,p,x\rangle = 0$ then $\beta_s x^p = 0$ mod α_1 for $1 < s < p$. As corollaries he shows that $\beta_2\beta_1^p = 0$ and the β_s are nilpotent, foreshadowing Nishida's proof, a few years later, that all positive dimensional elements of π_*S are nilpotent.

Gray [36] obtained results similar to 1.15 using homotopy operations which are associativity or commutativity obstructions for ring spectra.

Oka and Toda [92] have extensive information on the cell structure of $D_p(S^n \cup_p e^{n+1})$ which they use, in particular, to show that $\gamma_1 \neq 0$.

Milgram [80] also uses extended powers $D_2(S^n \cup_{2^i} e^{n+1})$ to define homotopy operations which can be iterated to yield infinite families of elements in π_*S, presumably related to the elements detected by K-theory.

Cooley, in his thesis [30], uses extended powers to compute some Toda brackets and to derive 1.7 as well as the relation $\epsilon x^2 = 0$ if $x \in \pi_n$, $n \equiv 2,3,7$ (8), which is not in 1.7.

Milgram [79 and 81] computes the Coker J part of the operations on $\pi_8 S$ and $\pi_9 S$ using Steenrod operations in $E_2(S,S)$.

§2. Extended powers of spheres

In this section we collect the results on extended powers of spheres which are needed to prove the results of §1. They will also be essential to our results on differentials in the next chapter. First, we recall the values of the K and J groups of lens spaces. Then, we identify the spectra $D_\pi^i S^n$, π cyclic, as the suspension spectra of stunted lens spaces and determine when they are stably reducible or coreducible. Also, we show that, after localizing at p, $D_p S^n$ is a wedge summand of $D_\pi S^n$, which gives a simple cell structure to $D_p S^n$.

Throughout this section, let p be a prime, let $\pi \subset \Sigma_p$ be the p-Sylow subgroup generated by the p-cycle $(1\ 2\cdots p)$, and let W^k be the k-skeleton of a contractible π or Σ_p free CW complex W. (Definitions 2.1 and 2.7 provide the π free CW complexes which we shall use most frequently.)

The results for p = 2 are analogous to the results for odd primes, but are sufficiently simpler that we state them separately. We begin with odd primes.

Definition 2.1. Let $p > 2$ and let $\rho = \exp(2\pi i/p)$. Let π act on the unit sphere $S^{2k+1} \subset \mathbb{C}^{k+1}$ by letting a generator of π send (z_i) to (ρz_i). Let

$$\tilde{L}^{2k+1} = S^{2k+1}/\pi,$$

$$\tilde{L}^{2k} = \{[z_0,\ldots,z_k] \in \tilde{L}^{2k+1} \mid z_k^p \text{ is real and } \geq 0\},$$

and

$$\tilde{L}_n^{n+k} = \tilde{L}^{n+k}/\tilde{L}^{n-1},$$

where $[z_0,\ldots,z_k]$ denotes the equivalence class of $(z_0,\ldots,z_k)$ and $\tilde{L}^{2k-1}$ is embedded in $\tilde{L}^{2k}$ by setting $z_k = 0$. We call $\tilde{L}_n^{n+k}$ a stunted lens space.

Each representation of π on $\mathbb{C}^{k+1}$ without trivial subrepresentations yields a free π action on S^{2k+1} and a corresponding lens space S^{2k+1}/π. Since they are all stably equivalent we have simply chosen our favorite. Note, however, that the others reappear briefly in the proof of Proposition 2.4.

It is easy to see that $\tilde{L}^n - \tilde{L}^{n-1}$ is an open n cell. Thus $\tilde{L}_n^{n+k}$ has one cell in each dimension between n and n+k inclusive. Note that $\tilde{L}_1^n = \tilde{L}^n$ and $\tilde{L}_0^n = (\tilde{L}^n)^+$, the union of $\tilde{L}^n$ and a disjoint basepoint.

Since $\tilde{L}^\infty = S^\infty/\pi$ is a $K(\pi,1)$, $H^*(\tilde{L}^\infty;Z_p) = E\{x\} \otimes P\{\beta x\}$, with $|x| = 1$, and the Steenrod operations are specified by

$$P^i(x^\varepsilon(\beta x)^j) = \binom{j}{i}x^\varepsilon(\beta x)^{j+i(p-1)}.$$

The isomorphisms

$$H^i\tilde{L}_n^{n+k} \longrightarrow H^i\tilde{L}^{n+k} \longleftarrow H^i\tilde{L}^\infty$$

for $n \leq i \leq n+k$ then determine $H^*\tilde{L}_n^{n+k}$ as an $\mathcal{A}_p$ module.

Definition 2.2. Let $p > 2$ and let π act on $\mathbb{C}$ by multiplication by ρ. Let $\xi \in KU(\tilde{L}^{2k+1})$ be the bundle

$$S^{2k+1} \times_\pi \mathbb{C} \longrightarrow S^{2k+1} \times_\pi \{0\} = \tilde{L}^{2k+1},$$

let $\zeta_i = r(\xi^i) \in KO(\tilde{L}^{2k+1})$ where $r: KU \to KO$ forgets complex structure, let

$\zeta = J(\zeta_1) \in J(\tilde{L}^{2k+1})$, and let $\sigma = \xi - 1_{\mathbf{C}} \in \widetilde{KU}(\tilde{L}^{2k+1})$. Let ξ, ζ_i, ζ and σ also denote the restrictions of these elements to $\tilde{L}^{2k}$.

We collect some results from [47], [48] and [58] in the following theorem.

<u>Theorem 2.3</u>. Let $\tilde{L}^{2k} \to \tilde{L}^{2k+1}$ be the inclusion and let $\langle x \rangle$ denote the cyclic group generated by x.

(i) $i^*: KU(\tilde{L}^{2k+1}) \to KU(\tilde{L}^{2k})$ is an isomorphism and

$$\widetilde{KU}(\tilde{L}^{2k}) = \langle\sigma\rangle \oplus \langle\sigma^2\rangle \oplus \cdots \oplus \langle\sigma^{p-1}\rangle$$

(ii) $r: \widetilde{KU}(\tilde{L}^{2k}) \to \widetilde{KO}(\tilde{L}^{2k})$ is an epimorphism,

$$\widetilde{KO}(\tilde{L}^{2k+1}) = \widetilde{KO}(\tilde{L}^{2k}) \oplus \widetilde{KO}(S^{2k+1}),$$

and i^* is projection onto the first summand under this isomorphism.

(iii) $\tilde{J}(\tilde{L}^{2k}) = \langle Jr(\sigma)\rangle = \langle \zeta - 2\rangle$ and has order $p^{[k/(p-1)]}$,

$$\tilde{J}(\tilde{L}^{2k+1}) = \tilde{J}(\tilde{L}^{2k}) \oplus \tilde{J}(S^{2k+1})$$

and i^* is projection onto the first summand under this isomorphism. Also, $J(\zeta_i) = \zeta$ for $i = 1,2,\ldots,p-1$.

<u>Proof</u>. This is all in [47], [48] and [58] except $J(\zeta_i) = \zeta$, which follows from the Adams conjecture:

$$J(\zeta_i) = Jr\xi^i = Jr\psi^i\xi = Jr\xi = \zeta. \quad //$$

The extended powers $D^k_\pi S^n$ are suspension spectra of Thom spaces of complex bundles over $\tilde{L}^k = W^k/\pi$. Thus Theorem 2.3 ensures us that the following theorem (proved in [81]) identifies all such spectra. Note that its proof does not require p to be a prime.

<u>Theorem 2.4</u>. If $s \geq 0$, the Thom complex of $r + s\zeta$ over $\tilde{L}^k$ satisfies

$$\Sigma^\infty T(r + s\zeta) = \Sigma^\infty \Sigma^r \tilde{L}^{2s+k}_{2s}.$$

<u>Proof</u>. The contribution of the trivial r dimensional fibration is obvious and may be ignored. We will actually prove a much more precise result. If α is an n-dimensional representation of π, we let $R^n(\alpha)$ and $S^{n-1}(\alpha)$ denote R^n and S^{n-1} with π action given by α. If the action is free on S^{n-1} we obtain a closed manifold $L(\alpha) = S^{n-1}(\alpha)/\pi$. If α and β are two such representations of dimension n and k respectively, let $\alpha|L(\beta)$ be the bundle

$$S^{k-1}(\beta) \times_\pi R^n(\alpha) \longrightarrow S^{k-1}(\beta) \times_\pi \{0\} = L(\beta).$$

We claim that there is a homeomorphism

$$T(\alpha|L(\beta)) \cong L(\beta \oplus a)/L(\alpha),$$

where $L(\alpha)$ is embedded in $L(\beta \oplus a)$ as the last n coordinates. This will imply Theorem 2.4 for odd k (since $L(\beta)$ is odd dimensional, p being odd). The even case will follow by removing the top cell on each side, since the homeomorphism will be cellular if we give the Thom complex $T(\alpha|L(\beta))$ the natural cell structure compatible with that of $L(\beta)$.

To establish the claim, let $f: S^{k-1}(\beta) \times_\pi R^n(\alpha) \to S^{n+k-1}(\beta \oplus \alpha)/\pi$ be induced by the natural inclusion $S^{k-1}(\beta) \times R^n(\alpha) \to R^{n+k}(\beta \oplus \alpha) - \{0\}$ followed by the radial retraction $R^{n+k} - \{0\} \to S^{n+k-1}$. It is easy to check that f is one-to-one and maps onto everything except the copy of $L(\alpha)$ embedded as the last n coordinates. Just as easily, one sees that f sends the zero section of $\alpha|L(\beta)$ to the embedding of $L(\beta)$ as the first k coordinates. It follows that $\alpha|L(\beta)$ is the normal bundle of this embedding $L(\beta) \to L(\beta \oplus \alpha)$ and that its Thom complex is $L(\beta \oplus \alpha)/L(\alpha)$. //

The fact that $\zeta \in J(\tilde{L}^k)$ has finite order enables us to define stunted lens spectra in positive and negative dimensions.

Definition 2.5. Let $\psi(k) = [k/2(p-1)]$. If n is any integer, $\varepsilon = 0$ or 1, and $k \geq \varepsilon$, let

$$\tilde{L}^{2n+k}_{2n+\varepsilon} = \Sigma^{2(n-r)}\Sigma^{\infty}\tilde{L}^{2r+k}_{2r+\varepsilon}$$

for $r \equiv n \ (p^{\psi(k)})$ such that $r \geq 0$.

The following result shows that the spectrum $\tilde{L}^{n+k}_n$ is well-defined up to equivalence in $\overline{h}\mathcal{S}$. Recall that an n-dimensional complex X is reducible if $X/X^{n-1} \simeq S^n$ and the projection $X \to S^n$ has a right inverse. Dually, an (n-1)-connected complex X is coreducible if $X^n \simeq S^n$ and the inclusion $S^n \to X$ has a left inverse. Let $W = S^\infty$, let $q: W \to \tilde{L}^\infty$ be the quotient map and let $W^k = q^{-1}(\tilde{L}^k)$. Then we may define $D^k_\pi X = W^k \ltimes_\pi X^{(p)}$.

Theorem 2.6. Let S^n be the p-local n-sphere spectrum. Then

(i) $D^k_\pi S^n \simeq \Sigma^n \tilde{L}^{n(p-1)+k}_{n(p-1)}$.

(ii) $\tilde{L}^{2n+k}_{2n}$ is coreducible iff $n \equiv 0 \ (p^{\psi(k)})$, while $\tilde{L}^{2n+k}_{2n+1}$ is coreducible iff $k = 1$.

(iii) If $\varepsilon = 0$ or 1, $k \geq \varepsilon$ and $n \equiv r \ (p^{\psi(k)})$ then

$$\tilde{L}^{2n+k}_{2n+\varepsilon} \simeq \Sigma^{2(n-r)} \tilde{L}^{2r+k}_{2r+\varepsilon} .$$

(iv) $\tilde{L}^{b}_{a}$ and $\tilde{L}^{-a-1}_{-b-1}$ are (-1) dual spectra.

(v) If $\varepsilon = 0$ or 1 and $k \geq \varepsilon$ then $\tilde{L}^{2n+k}_{2n+\varepsilon}$ is reducible iff either $k = \varepsilon$ or k is odd and $2n+k+1 \equiv 0 \ (p^{\psi(k)})$.

<u>Proof</u>. If $n \geq 0$ then $D^k_\pi S^n = W^k \times_\pi S^{n(p)} = \Sigma^\infty T(n\gamma_k)$, where γ_k is the restriction to $\tilde{L}^k$ of the bundle over $\tilde{L}^\infty = B\pi$ induced by the regular representation of π. Since $\gamma_k = 1 + \zeta_1 + \cdots + \zeta_m$, $J(n\gamma_k) = n + nm\zeta$ (where $2m = p-1$). By Theorem 2.4,

$$\Sigma^\infty T(n\gamma_k) = \Sigma^n \tilde{L}^{n(p-1)+k}_{n(p-1)} .$$

If $n < 0$ then, by [Equiv, VI.5.3 and 5.4]

$$W^k \ltimes_\pi S^{n(p)} = W^k \ltimes_\pi (\Sigma^n S)^{(p)} \simeq \Sigma^{-N}\Sigma^\infty T(N + n\gamma_k)$$

for sufficiently large N, and since $J(n\gamma_k) = n + nm\zeta$, we find that

$$W^k \ltimes_\pi S^{n(p)} \simeq \Sigma^{-N}\Sigma^\infty T(N+n + nm\zeta) \simeq \Sigma^n \tilde{L}^{n(p-1)+k}_{n(p-1)}$$

by Definition 2.5 and Theorems 2.4 and 2.3.(iii). This proves (i).

By Theorem 2.4, $\tilde{L}^{2n+k}_{2n} = \Sigma^\infty T(n\zeta|\tilde{L}^k)$. By [15], $\Sigma^\infty T(n\zeta)$ is coreducible if and only if $\mathfrak{J}(n\zeta) = 0$, so the first half of (ii) follows by Theorem 2.3.(iii). For the second part of (ii) we need only note that the Bockstein is nonzero on H^{2n+1} if $k > 1$.

To prove (iii), note that $\mathfrak{J}(n\zeta) = \mathfrak{J}(r\zeta)$ if $n \equiv r \ (p^{\psi(k)})$ by Theorem 2.3.(iii).

To prove (iv), first consider $\tilde{L}^{2n+k}_{2n}$ with k odd. By Theorem 2.4, $\tilde{L}^{2n+k}_{2n} = \Sigma^\infty T(n\zeta|\tilde{L}^k)$. Since k is odd, $\tilde{L}^k$ is a closed manifold. By considering the fibration

$$S^1 \to \tilde{L}^k \to CP^{[k/2]},$$

we see that the tangent bundle of $\tilde{L}^k$ is $([k/2] + 1)\zeta - 1$. Atiyah's duality theorem [15, Theorem 3.3] implies that the (-1) dual of $\tilde{L}^{2n+k}_{2n}$ is $\Sigma^\infty T(1 - (n+[k/2] + 1)\zeta) = \tilde{L}^{-2n-1}_{-2n-k-1}$. To prove (iv) for the other three possible combinations of odd or even top and bottom cells, we use the duality between inclusion of the bottom cell of a complex and projection onto the top cell of its dual.

Finally, (v) follows from (ii) and (iv) by the duality between reductions and coreductions. //

Now we present the analogs of 2.1 through 2.6 for D_pS^n instead of $D_\pi S^n$. Since the transfer splits D_pS^n off as a wedge summand of $D_\pi S^n$, we can use this as a short-cut to the results we need. Let $X_{(p)}$ denote the p-localization of a spectrum or space X. The following result is proved in [7].

Proposition 2.7. There is a CW spectrum L with one cell in each nonnegative dimension congruent to 0 or -1 modulo 2(p-1), such that $L \simeq (\Sigma^\infty B\Sigma_p^+)_{(p)}$.

Definition 2.8. Let L^k be the k-skeleton of L and let $L_n^{n+k} = L^{n+k}/L^{n-1}$ if $n > 0$. If $n < 0$, $\varepsilon = 0$ or 1, and $k \geq \varepsilon$, let $L_{2n+\varepsilon}^{2n+k} = \Sigma^{2(n-r)}L_{2r+\varepsilon}^{2r+k}$ for $r \equiv n\ (p^{\psi(k)})$ such that $r \geq 0$.

Note that n and k are not uniquely determined by L_n^{n+k} as they are by $\tilde{L}_n^{n+k}$. For example, $L_1^q = L_2^q = \cdots = L_{q-1}^q$, where $q = 2(p-1)$, since L has no cells in dimensions $1,2,\ldots,q-2$.

Theorem 2.9. Let S^n be the p-local n-sphere spectrum and let $q = 2(p-1)$. Then

(i) $D_pS^{2j} \simeq \Sigma^{2j}L_{jq}^\infty$ and $D_pS^{2j-1} \simeq \Sigma^{2j-1}L_{jq-1}^\infty$. The maps $D_\pi S^n \to D_pS^n$ and $\tilde{L}_n^{n+k} \to L_n^{n+k}$ induced by the inclusion $\pi \subset \Sigma_p$ are projections onto wedge summands.

(ii) L_{jq}^{jq+k} is coreducible iff $j \equiv 0\ (p^{\psi(k)})$, while L_{jq-1}^{jq+k} is coreducible iff $k = -1$.

(iii) If $\varepsilon = 0$ or 1 and $i \equiv j\ (p^{\psi(k+2\varepsilon)})$ then

$$L_{jq-\varepsilon}^{jq+k} \simeq \Sigma^{(j-i)q}L_{iq-\varepsilon}^{iq+k}.$$

(iv) If ε and δ are 0 or 1 then $L_{jq-\varepsilon}^{iq-\delta}$ is (-1) dual to $L_{-iq+\delta-1}^{-jq+\varepsilon-1}$.

(v) If $\varepsilon = 0$ or 1 then $L_{jq-\varepsilon}^{jq+k}$ has a reducible jq+k cell iff either $k = \varepsilon = 0$ or $k = iq-1$ and $i+j \equiv 0\ (p^{i+\varepsilon-1})$.

Note: Part (i) shows that bottom dimensions of the form $jq-\varepsilon$, $\varepsilon = 0$ or 1, are more natural in this context than $jq+\varepsilon$. This accounts for the exponent $\psi(k+2\varepsilon)$ in (iii), where $\psi(k)$ might be expected.

Proof. By the remark preceding the theorem, the first statement in (i) can be abbreviated to $D_pS^n = \Sigma^n L^\infty_{n(p-1)}$. The transfer $(\Sigma^\infty B\Sigma_p)_{(p)} \to \Sigma^\infty B\pi$ splits off L^∞ and $L^\infty_{n(p-1)}$ as wedge summands of $\tilde{L}^\infty$ and $\tilde{L}^\infty_{n(p-1)}$ respectively. Similarly, the transfer splits off D_pS^n as a wedge summand of $D_\pi S^n$. The maps

$$D_pS^n \xrightarrow{t_1} D_\pi S^n \simeq \Sigma^n \tilde{L}^\infty_{n(p-1)} \xrightarrow{i_1} \Sigma^n L^\infty_{n(p-1)}$$

and

$$\Sigma^n L^\infty_{n(p-1)} \xrightarrow{t_2} \Sigma^n \tilde{L}^\infty_{n(p-1)} \simeq D_\pi S^n \xrightarrow{i_2} D_pS^n ,$$

where t_1 and t_2 are transfers, and i_1 and i_2 are induced by the inclusion $\pi \subset \Sigma_p$ are inverse equivalences because their composites induce isomorphisms in mod p homology. This proves (i). Now (ii)-(v) follows from 2.6 and (i). //

The preceding theorem does not assert that $W^k \ltimes_{\Sigma_p} S^{n(p)} \simeq \Sigma^n L^{n(p-1)+k}_{n(p-1)}$ where W^k is the k-skeleton of a contractible free Σ_p space, because this is not true. In general, $W^k \ltimes_{\Sigma_p} S^{n(p)}$ will have homology in dimension np+k which goes to 0 in D_pS^n and in $\Sigma^n L^{n(p-1)+k}_{n(p-1)}$. Since we are only interested in homology which is nonzero in D_pS^n, $\Sigma^n L^{n(p-1)+k}_{n(p-1)}$ is more useful to us than is $W^k \ltimes_{\Sigma_p} S^{n(p)}$.
<u>Therefore we will let</u> $D^k_pS^n = \Sigma^n L^{n(p-1)+k}_{n(p-1)}$, <u>rather than</u> $W^k \ltimes_{\Sigma_p} S^{n(p)}$.

The preceding theorem also shows that we may ignore the distinction between L^{n+k}_n and $\tilde{L}^{n+k}_n$ without harm. We used $\tilde{L}^{n+k}_n$ and $D_\pi S^n$ as a stepping stone to information about D_pS^n because J theory only gives information about coreducibility of Thom complexes, and we need Atiyah's S-duality theorem to convert this to information about reducibility. The S-duality theorem of Atiyah only applies to Thom complexes of bundles over manifolds so cannot be used on bundles over the skeleta of $B\Sigma_p$, or over the even skeleta of $B\pi$. Conveniently, the odd skeleta of $B\pi$ are manifolds (if we use a lens space for $B\pi$). To obtain analogous information about D_rS^n for nonprime r, a similar technique works. First, we split D_rS^n off of $D_\tau S^n$ using the transfer, where $\tau \subset \Sigma_r$ is a p-Sylow subgroup. Then the structure of τ (a Cartesian product of iterated wreath products of π) suggests manifolds mapping to $B\tau$ which we can use just as the odd skeleta of $B\pi$ are used here.

We now turn to the analogs of 2.1 through 2.6 for p = 2.

Definition 2.10. Let $n \geq 0$, let $\pi = \Sigma_2$ act antipodally on S^n and let

$$P^n = S^n/\pi$$

$$\text{and} \quad P_n^{n+k} = P^{n+k}/P^{n-1} .$$

We call P_n^{n+k} a stunted projective space. Let ξ in $KO(P^n)$ be the canonical real line bundle and let $\lambda = \xi - 1 \in KO(P^n)$.

Remarks. (1) If $p = 2$ we will agree to let L^n and $\tilde{L}^n$ mean P^n and let L_n^{n+k} and $\tilde{L}_n^{n+k}$ mean P_n^{n+k} so that uniform statements of results for all primes can be given. The P^n and P_n^{n+k} notation will still appear frequently because many of the results are not the same for even and odd primes.

(2) It is easy to see that $P^n - P^{n-1}$ is an open n-cell so that P_n^{n+k} has one cell in each dimension between n and n+k inclusive. Since $P^\infty = S^\infty/Z_2$ is a $K(Z_2,1)$, $H^*(P^\infty;Z_2) = P\{x\}$ with $|x| = 1$ and

$$Sq^i x^j = \binom{j}{i} x^{i+j}.$$

The isomorphisms

$$H^i P_n^{n+k} \to H^i P^{n+k} \leftarrow H^i P^\infty$$

for $n \leq i \leq n+k$ thus determine $H^* P_n^{n+k}$ as an $\mathcal{A}_2$ module.

Theorem 2.11. Let $\phi(n)$ be the number of integers j congruent to 0,1,2, or 4 mod 8 such that $0 < j \leq n$. Then $\widetilde{KO}(P^n) = \langle\lambda\rangle$ and has order $2^{\phi(n)}$. Furthermore,

$$J: KO(P^n) \to J(P^n)$$

is an isomorphism.

Proof. $KO(P^n)$ is computed in [1]. The computations there and the Adams conjecture imply the last statement. //

Theorem 2.12. If $s \geq 0$ the Thom complex of $r+s\xi$ over P^n satisfies

$$\Sigma^\infty T(r + s\xi) = \Sigma^\infty P_s^{s+n}.$$

Proof. The proof of Proposition 2.4 can easily be adapted to prove this as well.

As for odd primes, we can now define stunted projective spectra starting and ending in any positive or negative dimensions.

Definition 2.13. For $k \geq 0$ and any n let

$$P_n^{n+k} = \Sigma^{n-r}\Sigma^\infty P_r^{r+k}$$

for any $r \equiv n\ (2^{\phi(k)})$, $r \geq 0$.

The following result shows that P_n^{n+k} is well defined up to equivalence in $\overline{h}\mathcal{A}$. Let S^k have the antipodal action of π. We define $D_2^k X = S^k \ltimes_\pi X^{(2)}$.

Theorem 2.14. Let S^n be the 2-local n-sphere spectrum. Then

(i) $D_2^k S^n \simeq \Sigma^n P_n^{n+k}$

(ii) P_n^{n+k} is coreducible if and only if $n \equiv 0\ (2^{\phi(k)})$

(iii) If $n \equiv m\ (2^{\phi(k)})$ then $P_n^{n+k} \simeq \Sigma^{n-m} P_m^{m+k}$

(iv) P_a^b and P_{-b-1}^{-a-1} are (-1) dual spectra

(v) P_n^{n+k} is reducible if and only if $n+k+1 \equiv 0\ (2^{\phi(k)})$.

Proof (i) follows for $n \geq 0$ from Theorem 2.12 once we observe that the regular representation γ_k is $1 + \xi$. For $n < 0$ we have

$$D_2^k S^n = D_2^k(\Sigma^n S) \simeq \Sigma^{-N}\Sigma^\infty T(N + n\gamma_k)$$

by VI.5.3 and VI.5.4 of [Equiv], for sufficiently large N. Hence $D_2^k S^n \simeq \Sigma^n P_n^{n+k}$ for $n < 0$ also, again by 2.12.

Parts (ii) through (v) follow exactly as in 2.6. In (iv) we use the fact that P^n is a closed manifold with tangent bundle $(n+1)\xi - 1$. //

The last results in this section identify the top dimensional component of any attaching map of $D_p S^n$ by combining Theorems 2.6 and 2.14 with Milnor's result on Thom complexes of sphere bundles over suspensions. First we must define the maps under consideration. As in §1, $q = 2(p-1)$ and $\varepsilon = 0$ or 1 ($q = 1$ and $\varepsilon = 0$ if $p = 2$).

Definition 2.15. Define a function v_p by

$$v_p(n) = \max\{v \mid L_{n-v+1}^n \text{ is reducible}\}.$$

Let $v = v_p(n)$ and define $a_p(n) \in \pi_{v-1}S$ to be Σ^{v-n} of the composite

$$S^{n-1} \longrightarrow L^{n-v} \longrightarrow S^{n-v}$$

in which the first map is a lift of the ataching map of the n cell and the second is projection onto the top cell of L^{n-v}.

The indeterminacy in the definition of $a_p(n)$ is the kernel of the homomorphism induced on π_{n-1} by the inclusion of the bottom cell of L^{n-1}_{n-v} .

We will often omit the subscript p for typographical simplicity. The notations v and a are intended to be mnemonic: v stands for "vector field number" and a stands for "attaching map". Actually, v is not quite the vector field number as defined by Adams [1]; $v_2(n)$ is $\rho(n-1)$ in Adams' notation. The function v_p tells us how far we can compress each of the attaching maps of L^∞. The attaching map of the n cell factors through L^{n-v} if and only if L^n_{n-v+1} is reducible. Thus, it factors through L^{n-v} but not through L^{n-v-1}, where $v = v_p(n)$. By the definition of $v_p(n)$, $a_p(n)$ is nonzero. We obtain a good hold on v_p and a_p from the following two lemmas. Let $\varepsilon_p(j)$ be the exponent of p in the prime factorization of j.

<u>Proposition 2.16</u>. If $p > 2$ then, with $q = 2(p-1)$,

$$v_p(jq-\varepsilon) = \begin{cases} 1 & \varepsilon = 0 \\ q(1+\varepsilon_p(j)) & \varepsilon = 1 . \end{cases}$$

If $p = 2$ then $v_2(j) = 8a + 2^b$, where $\varepsilon_2(j+1) = 4a + b$ and $0 \leq b \leq 3$.

<u>Proposition 2.17</u>. If $v_p(n) = 1$ then $a_p(n)$ is the map of degree p. If $v_p(n) > 1$ then $a_p(n) \otimes 1$ generates $\mathrm{Im}\, J \otimes Z_{(p)}$ in dimension $v_p(n)-1$.

<u>Proof of 2.16</u>. Theorem 2.14.(v) shows that $v_2(j)$ is the maximum s such that $\varepsilon_2(j+1) = \phi(s-1)$. The formula for $v_2(j)$ follows easily from this. Theorem 2.9.(v) shows that if $p > 2$ then $v_p(jq) = 1$ while $v_p(jq-1)$ is the maximum s such that $\varepsilon_p(jq) = \psi(s-1)$. The formula for $v_p(jq-\varepsilon)$ follows immediately. //

<u>Proof of 2.17</u>. Let $n = jq-\varepsilon$, $v = v_p(n)$ and $a = a_p(n)$. We wish to construct a map of cofiber sequences

$$\begin{array}{ccccccc} S^{n-1} & \longrightarrow & L^{n-1}_{n-v} & \longrightarrow & L^{n}_{n-v} & \longrightarrow & S^n \\ \| & & \uparrow b & & \uparrow & & \| \\ S^{n-1} & \xrightarrow{a} & S^{n-v} & \longrightarrow & Ca & \longrightarrow & S^n \end{array}$$

where $Ca = S^{n-v} \cup_a e^n$, b is the inclusion of the bottom cell, and $a \otimes 1$ generates $\mathrm{Im}\, J \otimes Z_{(p)}$. By S-duality and Theorems 2.9.(iv) and 2.14.(iv), it is equivalent to construct a map of cofiber sequences

(*)
$$\begin{array}{ccccccc} S^{-n} & \longleftarrow & L^{v-n-1}_{-n} & \longleftarrow & L^{v-n-1}_{-n-1} & \longleftarrow & S^{-n-1} \\ \| & & \downarrow b^* & & \downarrow & & \| \\ S^{-n} & \xleftarrow{a} & S^{v-n-1} & \longleftarrow & Ca & \longleftarrow & S^{-n-1} \end{array}$$

in which b^* is the collapse onto the top cell and a is as before. The lemma is trivial when $v = 1$ so we may assume $v > 1$ and hence, that n is odd. Let γ be the bundle $-(n+1)\xi$ if $p = 2$ and $-j(p-1)\zeta$ if $p > 2$ over L^v. Then $L^{v-n-1}_{-n-1} = T(\gamma)$. By the definition of v, γ is trivial over L^{v-1} but not over L^v. This implies $\gamma = \pi^*\nu$ where $\pi: L^v \to L^v/L^{v-1} = S^v$ is the collapsing map and $0 \neq \nu$ $KO(S^v)$. By [85], $T(\nu)$ has attaching map $J(\nu)$. Thus, the inclusions of the fiber S^{-n-1} into $T(\gamma)$ and $T(\nu)$ induce a map (*) of cofiber sequences with $a = J(\nu)$. Since v is greater than 1, it is even when $p > 2$ by 3.2. Thus, 2.3.(iii) and 2.9.(i) when $p > 2$, and 2.11 when $p = 2$, imply that the kernel of $\tilde{J}(L^v) \to \tilde{J}(L^{v-1})$ is Z_p. Hence $\tilde{J}(\gamma)$ generates it, being nonzero. Since $\pi^*(a) = \tilde{J}(\gamma)$, $a \in \tilde{J}(S^v)$ must generate $\tilde{J}(S^v) \otimes Z_{(p)}$. //

In the notation of 1.6, Propositions 2.16 and 2.17 are summarized by the equations

$$a_2(j) \doteq \beta_{\varepsilon_2(j+1)}$$

$$a_p(jq) = p$$

and

$$a_p(jq-1) = \alpha_{1+\varepsilon_p(j)}$$

where $\doteq$ denotes equality up to multiplication by a unit of $Z_{(p)}$.

§3. Proofs for section 1 and other calculations

This section primarily consists of proofs of results of §1 with the additional necessary results (3.1-3.4) interspersed. Note, however, that the spectral sequence charts in Figures 3.1 to 3.9 can be very useful in conjunction with Theorem 1.10 since they show where in the Adams spectral sequence the elements detecting the results of homotopy operations must lie.

Proof of 1.1. $\iota^*(x) = x^p$ by IV.7.3.(iii). Clearly, $E^0(\iota) = e_0 \otimes \iota_n^p = e_0$, so the second statement is immediate from the definition:

$$\beta^\varepsilon P^j = (-1)^j \nu(n) e_{jq-\varepsilon-n(p-1)}.$$

Proof of 1.2. Recall from III §1 that the homology operations are defined by

$$Q^j x = \xi_*(e_{j-n} \otimes x^2) \quad \text{if } p = 2,$$

and

$$\beta^\varepsilon Q^j x = \xi_*((-1)^j \nu(n) e_{jq-\varepsilon-n(p-1)} \otimes x^p) \quad \text{if } p > 2.$$

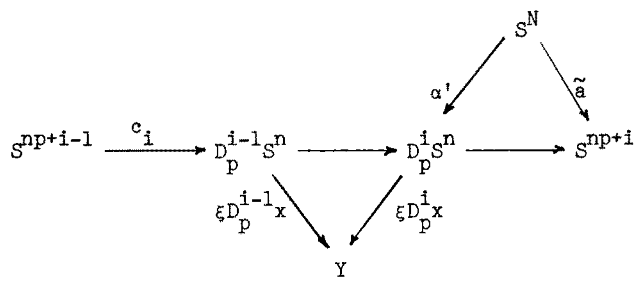
To prove 1.2 we simply calculate. If $p = 2$ and $E^0(\alpha) = P^j$ then

$$\begin{aligned} h\alpha^*(x) &= [\alpha^*(x)]_*(\iota_N) \\ &= \xi_* D_p(x)_* \alpha_*(\iota_N) \\ &= \xi_* D_p(x)_*(e_{j-n} \otimes \iota_n^2) \\ &= \xi_*(e_{j-n} \otimes h(x)^2) \\ &= Q^j h(x). \end{aligned}$$

The proof is essentially the same when $p > 2$. //

Proof of 1.3. This is just the naturality of the spectral sequence $E_r(S,\mathcal{B})$. //

Proof of 1.4. Consider the following commutative diagram, in which the row is the cofiber sequence of c_i and α' is a lift of α to $D_p^i S^n$.



Clearly $\alpha^*(x) = \xi D_p(x)\alpha = \xi D_p^i(x)\alpha'$ and this lies in the Toda bracket $\langle \tilde{a}, c_i, \xi D_p^{i-1}(x)\rangle$. If α and β both lift to $D_p^i S^n$ and project to $\tilde{a}$ on S^{np+i}, then $\alpha - \beta$ lifts to $D_p^{i-1}S^n$ so that $\alpha^*(x) - \beta^*(x)$ is in $\xi D_p^{i-1}(x) \circ \pi_N D_p^{i-1} S^n$. Conversely, if $\gamma \in \pi_N D_p^{i-1} S^n$ then $\alpha + \gamma$ also lifts to $D_p^i S^n$ and projects to α' on S^{np+i}. //

Proof of 1.5. By definition, $\beta^\epsilon P^j$ is defined on π_n if and only if e_i is a permanent cycle in $E_\infty(S,\mathcal{B})$. Thus (i) and (ii) are equivalent. Let $\mathcal{B}_i$ be $\mathcal{B}$ truncated at the np+i cell. The map of spectral sequences $E_r(S,\mathcal{B}_i) \to E_r(S,S^{np+i})$ induced by the projection $D_p^i S^n \to S^{np+i}$ sends e_i to the identity map of S^{np+i}. If $D_p^i S^n$ is reducible then there is a map back which splits $E_r(S,S^{np+i})$ off $E_r(S,\mathcal{B}_i)$, forcing e_i to be a permanent cycle. Conversely, if e_i is a permanent cycle then any map detecting it will be a reduction. Thus (ii) and (iii) are equivalent. Finally, (iii) and (iv) are equivalent by Theorems 2.6.(v), 2.9.(v) and 2.14.(v). //

Proof of 1.7. To show $\beta_j \circ x^2 = 0$, where $\beta_j \in \pi_{v-1}S$, we need only show that P_{n+1}^{n+v} is reducible and P_n^{n+v} is not, since this implies that the n+v cell is attached only to the n cell of P_n^{n+v}, and Proposition 2.17 implies that the attaching map is a generator of Im J in $\pi_{v-1}S$. If $j = 4a + b$ then $v = 8a + 2^b$, so 2.14.(v) implies that n must satisfy

$$n + 8a + 2^b \equiv -1 \quad (2^j)$$

$$\text{and} \qquad n + 8a + 2^b \not\equiv -1 \ (2^{j+1}).$$

To show $\beta_j \circ P^{n+1}x$ is a multiple of x^2, we must show that P_{n+1}^{n+v+1} is not reducible, but P_{n+2}^{n+v+1} is reducible, for then the top cell will be attached to the cells carrying x^2 and $P^{n+1}x$. The rest of the proof is the same as in the first case. //

Proof of 1.8. To show that $\alpha_j \circ \beta P^k x = 0$, for $x \in \pi_n Y$ and $n = 2k-1$, is trivial when $j = 0$. Simply note that L_{kq-1}^{kq} is a mod p Moore spectrum. When $j > 0$ we must show $L_{kq}^{(k+j)q-1}$ is reducible, while $L_{kq-1}^{(k+j)q-1}$ is not. By 2.9.(v) we need $k+j \equiv 0\ (p^{j-1})$ but $k+j \not\equiv 0\ (p^j)$.

When $n = 2k$, the relation $\alpha_j \circ \beta P^{k+1}x = \alpha \circ x^p$ for some α is also trivial when $j = 0$. We need only note that L_{kq+q-1}^{kq+q} is a mod p Moore spectrum. For $j > 0$, we must show that $L_{kq+q}^{(k+j)q+q-1}$ is reducible, but $L_{kq+q-1}^{(k+j)q+q-1}$ is not. By 2.9.(v) we must have $k+j+1 \equiv 0\ (p^{j-1})$ but $k+j+1 \not\equiv 0\ (p^j)$. //

When $n = 2k$, if we try to show $\alpha_j \circ x^p = 0$ by this technique we find we must assume $k+j \equiv 0\ (p^{j-1})$ and $k+j \not\equiv 0\ (p^{j-1})$, so that no information is available.

Before we compute the first few homotopy groups of $D_p S^n$ (and hence the first few homotopy operations), we describe the attaching maps of the first few cells. Exact definitions of the maps used in the following proposition can be found in the proof.

Proposition 3.1. Let $p = 2$.

(i) If $n \equiv 1\ (4)$ then $P_n^{n+3} \simeq S^n \cup_2 e^{n+1} \vee S^{n+2} \cup_{\tilde{\eta}+2} e^{n+3}$

(ii) If $n \equiv 2\ (4)$ then $P_n^{n+3} \simeq S^n \vee S^{n+1} \cup_{\eta+2} e^{n+2} \cup_\eta e^{n+3}$

(iii) If $n \equiv 3\ (4)$ then $P_n^{n+3} \simeq S^n \cup_2 e^{n+1} \cup_\eta e^{n+2} \cup_2 e^{n+3}$

(iv) If $n \equiv 0\ (4)$ then $P_n^{n+3} \simeq S^n \vee S^{n+1} \cup_2 e^{n+2} \vee S^{n+3}$.

Proof. Much of the structure of P_n^{n+3} is determined by Sq^1 and Sq^2 in $H^*P_n^{n+3}$. We will assume this information and fill in the rest. Suppose $n \equiv 0$ (4). Then 2.14 implies P_n^{n+3} is both reducible and coreducible, so only the middle two cells are attached. When $n \equiv 1$ (4), collapsing the bottom cell of the previous case yields $P_n^{n+2} \simeq S^n \cup_2 e^{n+1} \vee S^{n+2}$. Computing Sq^1 and Sq^2 shows e^{n+3} is attached to S^{n+2} by a map of degree 2, and is attached to the Moore spectrum by a map which projects to η on S^{n+1}. This projection induces an epimorphism

$$Z_4 = \pi_{n+2}(S^n \cup_2 e^{n+1}) \longrightarrow \pi_{n+2}S^{n+1} = Z_2.$$

Therefore, the attaching map is a generator $\tilde{\eta}$ of $\pi_{n+2}(S^n \cup_2 e^{n+1})$.

When $n \equiv 2$ (4), we start with $P_n^{n+2} \simeq S^n \vee S^{n+1} \cup_{\eta+2} e^{n+2}$. The long exact homotopy sequence of $S^n \vee S^{n+1} \to P_n^{n+2}$ shows that the inclusion $S^{n+1} \to P_n^{n+2}$ induces an isomorphism on π_{n+2}. Since Sq^2 is nonzero on $H^{n+1}P_n^{n+3}$, the n+3 cell is attached by the map

$$S^{n+2} \xrightarrow{\ \eta\ } S^{n+1} \longrightarrow P_n^{n+2},$$

which we also call η.

Finally, when $n \equiv 3$ (4), we start with $P_n^{n+2} \simeq S^n \cup_2 e^{n+1} \cup_\eta e^{n+2}$. The map $P_n^{n+2} \to S^{n+1} \vee S^{n+2}$ which collapses the bottom cell, induces on π_{n+2} a monomorphism

$$\pi_{n+2}P_n^{n+2} = Z_2 \oplus Z \rightarrowtail Z_2 \oplus Z = \pi_{n+2}S^{n+1} \oplus \pi_{n+2}S^{n+2}$$

which sends (a,b) to (a,2b). Computing Sq^1 and Sq^2 shows that the attaching map of the n+3 cell is $(0,1) \in \pi_{n+2}P_n^{n+2}$, which projects to the map of degree 2 on S^{n+2}. We simply call this map 2. //

Proposition 3.2. Let $p > 2$.

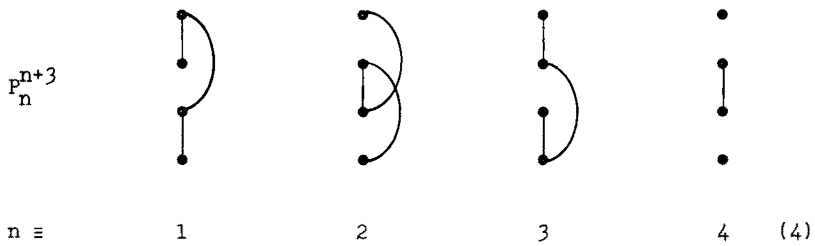
(1) $$L_{jq}^{jq+2q-1} \simeq S^{jq} \vee S^{jq+q-1} \bigcup_{-j\alpha_1+p} e^{jq+q} \bigcup_{-(j+2)\alpha_1} e^{jq+2q-1}.$$

(2) $$L_{jq-1}^{jq+q} \simeq S^{jq-1} \cup_p e^{jq} \bigcup_{-(j+1)\alpha_1} e^{jq+q-1} \bigcup_{-j\alpha_1+p} e^{jq+q}.$$

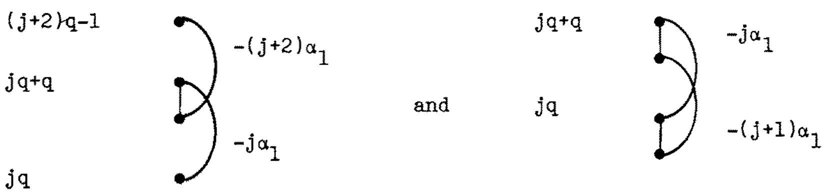
Proof. Recall that the first three nonzero homotopy groups of S localized at p are $\pi_0 = Z$, $\pi_{q-1} = Z_p$ generated by α_1, and $\pi_{2q-1} = Z_p$ generated by α_2. Thus $L_{jq}^{jq+q-1} = S^{jq} \vee S^{jq+q-1}$ is the only possibility. Computing β and P^1 in $H^*L_{jq}^{jq+q}$ shows that $L_{jq}^{jq+q} \simeq S^{jq} \vee S^{jq+q-1} \bigcup_{-j\alpha_1+p} e^{jq+q}$. Finally, the long exact homotopy sequence of $S^{jq} \vee S^{jq+q-1} \to L_{jq}^{jq+q}$ shows that the inclusion of S^{jq+q-1} induces an isomorphism of $\pi_{jq+2q-2}$. Thus the attaching map of the jq+2q-1 cell factors through S^{jq+q-1} and is determined to be $-(j+2)\alpha_1$ by computing P^1.

Collapsing the bottom cell and redefining j we find that $L^{jq+q-1}_{jq-1} \simeq S^{jq-1} \cup_p e^{jq} \underset{-(j+1)\alpha_1}{\cup} e^{jq+q-1}$. The long exact homotopy sequence of $S^{jq-1} \to L^{jq+q-1}_{jq-1}$ shows that the attaching map of the jq+q cell is determined by its projections onto S^{jq} and S^{jq+q-1}. Computing P^1 and β shows these to be $-j\alpha_1$ and p respectively. //

Diagrams of the cohomology with Sq^1 and Sq^2 or β and P^1 indicated are convenient mnemonic devices. For p = 2 we have

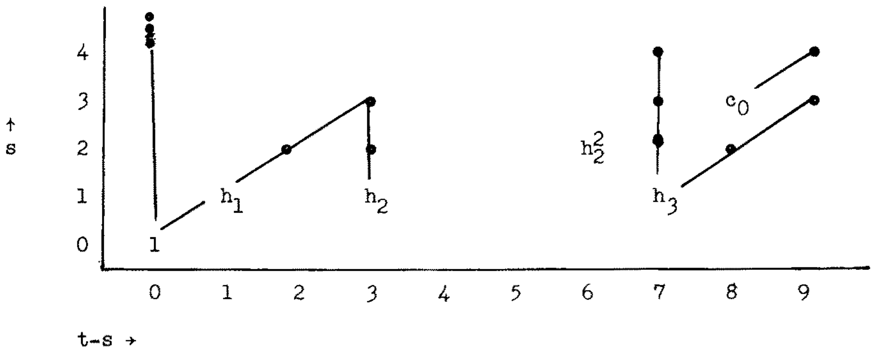


For p > 2, we have



We can also think of these diagrams as indicating cells by dots and attaching maps by lines, and this is how we have labelled the diagrams for p > 2.

The spectral sequence $E_r(S, \mathcal{P})$ will enable us to glean a maximal amount of information from Propositions 3.1 and 3.2. We begin with p = 2. Recall, from [66], the initial segment of the HZ_2 Adams spectral sequence for π_*S.



Vertical lines represent multiplication by h_0, detecting the map of degree 2, and diagonals represent multiplication by h_1, detecting η. We shall only use the first 8 stems ($t-s \leq 8$). Let $\mathcal{P}$ be the sequence

$$P_n^{n+8} \longleftarrow P_n^{n+7} \longleftarrow \cdots \longleftarrow P_n^{n+1} \longleftarrow P_n^n .$$

(Omitting the Σ^n from $D_2^i S^n = \Sigma^n P_n^{n+i}$ means a class in $E_r(S,\mathcal{P})$ will have stem degree equal to the amount by which the corresponding homotopy operation raises degrees.)

Proposition V.7.5 says that $E_2(S,\mathcal{P})$ is free over $E_2(S,S)$ on generators in each degree from n to n+k. Write x(i) for the element of $E_2(S,\mathcal{P})$ which is $x \in E_2(S,S)$ in the i summand, if $i \geq n$. Let x(i) mean 0 if $i < n$.

<u>Theorem 3.3</u>. In $E_r(S,\mathcal{P})$, for $t-s \leq 6$,

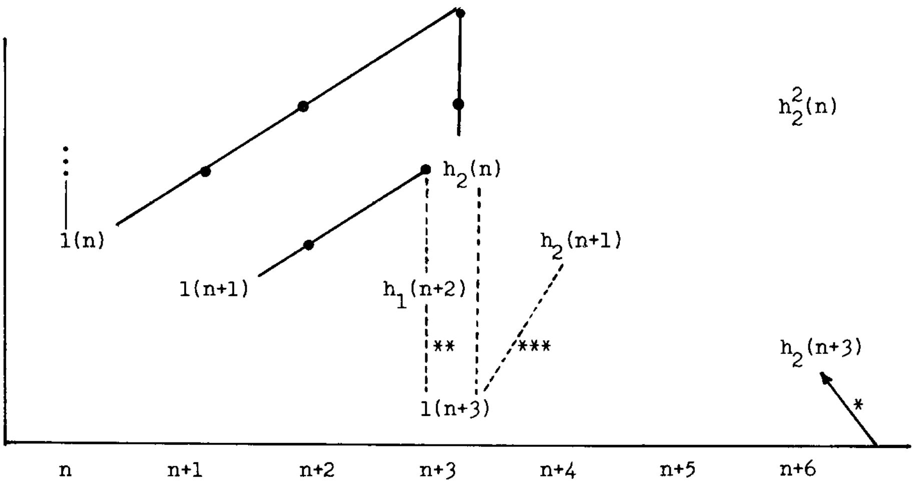
$$d_2x(i) = h_0x(i-1) \quad \text{if } i \equiv 0 \quad (2),$$

$$d_3x(i) = h_1x(i-2) \quad \text{if } i \equiv 0,1 \quad (4),$$

and $$d_5x(i) = h_2x(i-4) \quad \text{if } i \equiv 0,1,2,3 \quad (8).$$

In the same range, $E_\infty(S,\mathcal{P})$ is given by Figures 3.1 through 3.4.

<u>Note</u>: Dotted vertical lines indicate "hidden extensions". That is, they represent multiplications by 2 which cause an increase of more than 1 in filtration. Similarly, dotted diagonals indicate the effect of multiplication by η when this causes an increase in filtration of more than 1. See the proof of 1.9 for their derivation.

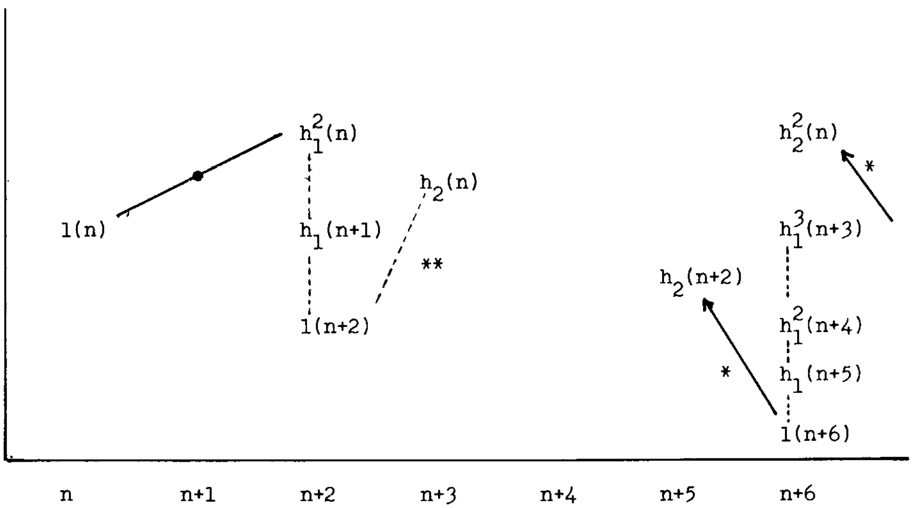


$n \equiv 0\ (4)$ *) hit by $d_5(l(n+7))$ iff $n \equiv 4\ (8)$

**) 2 times $l(n+3)$ is $h_1(n+2)$ if $n \equiv 0\ (8)$ and it is "$h_1(n+2) + h_2(n)$" if $n \equiv 4\ (8)$

***) if $n \equiv 4\ (8)$

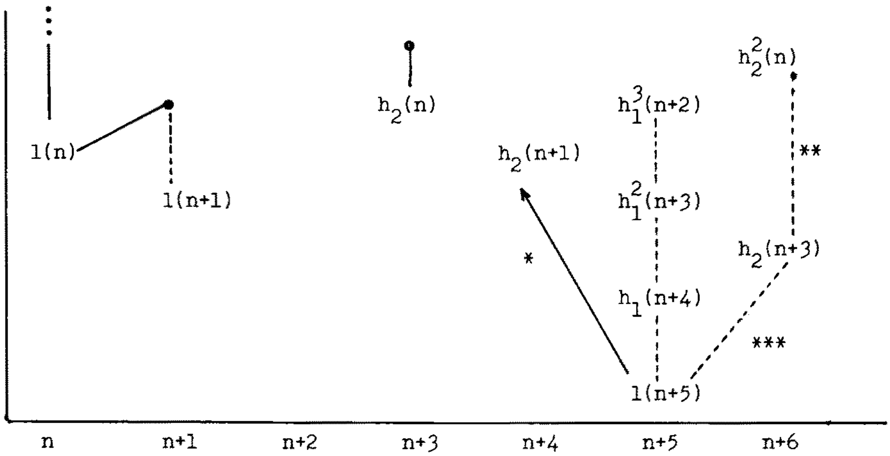
Figure 3.1



$n \equiv 1\ (4)$ *) differential iff $n \equiv 5\ (8)$

**) if $n \equiv 5\ (8)$

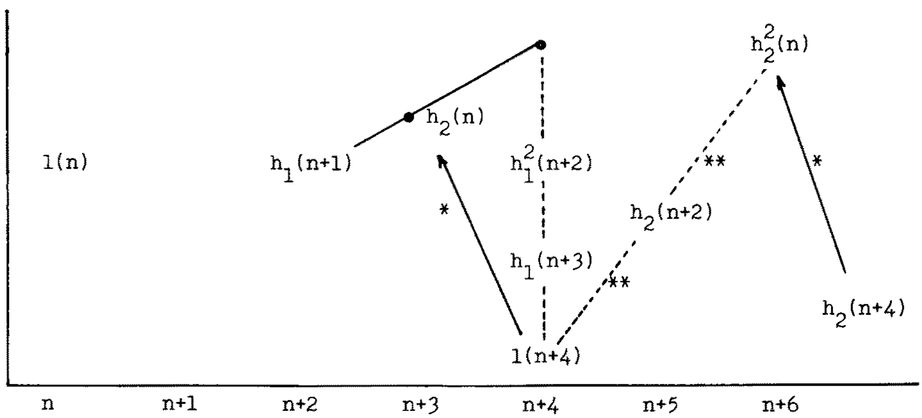
Figure 3.2



$n \equiv 2\ (4)$ *) differential iff $n \equiv 6\ (8)$

**) if $n \equiv 6\ (8)$

Figure 3.3   ***) if $n \equiv 2\ (8)$



$n \equiv 3\ (4)$ *) differential if $n \equiv 7\ (8)$

**) if $n \equiv 3\ (8)$

Figure 3.4

<u>Proof of 3.3</u>: The differentials listed correspond to attaching maps which can be detected by Sq^1, Sq^2 and Sq^4, and they hold in the spectral sequences for $\mathcal{P}'$, $\mathcal{P}''$ and $\mathcal{P}'''$ below

$\mathcal{P}'$ $$S^{i-1} \cup_2 e^i \longleftarrow S^{i-1} \longleftarrow *$$

$\mathcal{P}''$ $$S^{i-2} \cup_\eta e^i \longleftarrow S^{i-2} \equiv S^{i-2} \longleftarrow *$$

$\mathcal{P}'''$ $$S^{i-4} \cup_\nu e^i \longleftarrow S^{i-4} \equiv S^{i-4} \equiv S^{i-4} \equiv S^{i-4} \longleftarrow *$$

The differential $d_2x(i) = h_0x(i-1)$ if $i \equiv 0$ (2) is immediate, since $l(i) \notin E_2$ and by dimensional considerations $d_2l(i) = h_0(i-1)$ is the only possible d_2 on $l(i)$. The module structure over $E_2(S,S)$ now gives $d_2x(i) = h_0x(i-1)$.

The d_3 differential is slightly more complicated. There are two cases. If $i \equiv 1$ (4) then the i cell is not attached to the i-1 cell, but is attached to the i-2 cell by η; $d_3l(i) = h_1(i-2)$ follows as for d_2, and this implies $d_3x(i) = h_1x(i-2)$. If $i \equiv 0$ (4) then $l(i) \notin E_3$ since $d_2l(i) = h_0(i-1)$. However, the map of spectral sequences induced by $\mathfrak{C} \to \mathfrak{P}''$

$$\begin{array}{ccccccc}
\mathfrak{C} & S^{i-2} \vee S^{i-1} \cup_{\eta+2} e^i & \longleftarrow & S^{i-2} \vee S^{i-1} & \longleftarrow & S^{i-2} \\
\downarrow & \downarrow & & \downarrow & & \| \\
\mathfrak{P}'' & S^{i-2} \cup_{\eta} e^i & \longleftarrow & S^{i-2} & = & S^{i-2}
\end{array}$$

shows that elements of $E_3(S,\mathfrak{C})$ must satisfy $d_3x(i) = h_1x(i-2) + k$ where k is the kernel of $E_3(S,\mathfrak{C}) \to E_3(S, \mathfrak{P}'')$, that is, k must have the form $y(i-1)$. By inspection k must be 0 in the dimensions considered. Now, by truncating $\mathfrak{P}$ at the i cell, then collapsing the i-3 skeleton we can compare $E_3(S,\mathfrak{P})$ to $E_3(S,\mathfrak{C})$. Again we have $d_3x(i) = h_1x(i-2) + k$, where k is now a sum of elements coming from the i-3 cell or below. The first possibility is when $n \equiv 0$ (4). We must decide between $d_3h_1(n+4) = h_1{}^2(n+2)$ and $d_3h_1(n+4) = h_1{}^2(n+2) + h_2(n+1)$. Let P^n, P^{n+1}, h_1P^{n+2}, and P^{n+3} denote elements detected by $l(n)$, $l(n+1)$, $h_1(n+2)$, and $l(n+3)$, respectively. Comparing with Mahowald's calculations [59], we find that $2 \circ P^{n+3} = h_1P^{n+2}$ or $h_1P^{n+2} + \nu \circ P^n$, depending on n mod 8. Composing with η yields $\eta \circ h_1P^{n+2} = 0$. But if $d_3h_1(n+4)$ were $h_1^2(n+2) + h_2(n+1)$ we would have $\eta \circ h_1P^{n+2} = \nu \circ P^{n+1}$. Therefore we must have $d_3h_1(n+4) = h_1{}^2(n+2)$. The same argument, with minor variations, finishes all the d_3 differentials.

Finally, the d_5 differentials follow by similar comparisons with $E_5(S, \mathfrak{P}''')$. In all but one case, there is nothing in filtrations less than or equal to the filtration of $h_2x(i-4)$ so the comparison with $E_5(S,\mathfrak{P}''')$ is sufficient. The one remaining case is when $n \equiv 1$ (4). Here $h_1^3(n+3)$ lies between $h_2(n+4)$ and $h_2^2(n)$. Since the n+4 cell is not attached to the n+3 cell, the $d_5h_2(n+4) = h_2^2(n)$ is right here also.

There are no further possible differentials by inspection. The hidden extensions here are all evident from Mahowald's computation in [59] of the Adams spectral sequence of P_n^∞. //

<u>Note</u>. The spectral sequence $E_r(S,\mathfrak{P})$ has far more hidden extensions than $E_r(S,P_n^\infty)$ since the cells are spread apart in $E_r(S, \mathfrak{P})$ whereas they all occur in the same filtration in $E_r(S,P_n^\infty)$. By IV.7.6, the same hidden extensions occur among the elements generated by the $\beta^\varepsilon P^j x$ for a fixed x.

Proof of 1.9 when $p = 2$: A permanent cycle $x(i)$ corresponds to an operation xP^i. Thus, Table 1.3 is simply a list of the elements of $E_\infty(S, \mathcal{B})$, omitting most of those which are multiples by elements of π_*S of other elements of $E_\infty(S, \mathcal{B})$. The indeterminacy of an operation consists of those elements in the same stem and higher filtration, so it too can be read off Figures 3.1 through 3.4. With the exception of $\tau_{2*}(P^n)$ and $\tau_{2*}(P^{n+1})$, the values of τ_{2*} listed are the only elements of π_*S in the relevant dimension. Since $\pi_{2n}D_2S^n = Z_2$ when n is odd, $\tau_{2*}(P^n) = 0$ in this case. When n is even, $\iota:S^{2n} \to D_2S^n$ induces an isomorphism of π_{2n}. By II.1.10, the composite $\iota\tau_2:D_2S^n \to D_2S^n$ is multiplication by 2 on $H_{2n} \cong \pi_{2n}$. Thus $\tau_{2*}(P^n) = 2$. To calculate $\tau_{2*}(P^{n+1})$, first suppose $n \equiv 2$ (4). By Theorem 3.3, $\pi_{2n+2}D_2S^n = 0$. Therefore, $\eta P^{n+1} = 0$ and hence $\eta\tau_{2*}(P^{n+1}) = 0$. This forces $\tau_{2*}(P^{n+1})$ to be 0, not η. When $n \equiv 0$ (4), Theorem 3.3 gives $\pi_{2n+1}D_2S^n = Z_2 \oplus Z_2$ with generators P^{n+1} and ηP^n. By II.2.8, $\tau_{2*}(P^{n+1})$ is not zero and hence must be η.

Determining the relations in Table 1.4 amounts to determining the π_*S module structure of $\pi_*D_2S^n$. The indeterminacy of the operations in Table 1.3 induces a similar indeterminacy in the relations of Table 1.4. The relations are to be interpreted as asserting equality modulo the sum of the indeterminacies of the two sides. Thus, in order to prove that they hold, we need only show that they hold for some choice of representatives. The E_∞ terms in Theorem 3.3 force the following thirteen relations:

$2P^n = 0$	$n \equiv 1,3$ (4)
$\eta h_1 P^{n+1} = 0$	$n \equiv 1$ (4)
$2\nu P^{n+2} = 0$	$n \equiv 1$ (8)
$\nu P^{n+2} = 0$	$n \equiv 5$ (8)
$4\nu P^n = 0$	$n \equiv 2$ (4)
$\eta P^{n+1} = 0$	$n \equiv 2$ (4)
$\nu P^{n+1} = 0$	$n \equiv 6$ (8)
$\eta P^n = 0$, $2h_1P^{n+1} = 0$, $2h_2P^{n+2} = 0$	$n \equiv 3$ (4)
$\nu P^n = 0$, $\eta h_1 P^{n+3} = 0$, $\eta h_2 P^{n+2} = 0$	$n \equiv 7$ (8)

Another eighteen relations follow by considering the attaching maps given in Proposition 3.1, the spectral sequences in Theorem 3.3 and the reducibility and coreducibility given in Theorem 2.14. These are

$$\left.\begin{aligned} 2P^{n+1} &= 0 \\ 2h_1P^{n+2} &= \eta^2P^{n+1} \end{aligned}\right\} \quad n \equiv 0\ (4)$$

$$\left.\begin{aligned} \eta P^{n+3} &= 0 \\ 2P^{n+3} &= h_1P^{n+2} \\ 2\nu P^{n+3} &= \nu h_1P^{n+2} = 0 \end{aligned}\right\} \quad n \equiv 0\ (8)$$

$$\left.\begin{aligned} 2P^{n+3} &= h_1P^{n+2} + \nu P^n \\ \eta P^{n+3} &= \nu P^{n+1} \\ \nu h_1P^{n+2} &= \nu^2P^n \end{aligned}\right\} \quad n \equiv 4\ (8)$$

$$2P^{n+2} = h_1P^{n+1} \qquad n \equiv 1\ (4)$$

$$\left.\begin{aligned} \eta P^{n+2} &= 0 \\ 2P^{n+6} &= h_1P^{n+5} \end{aligned}\right\} \quad n \equiv 1\ (8)$$

$$\eta P^{n+2} = \nu P^n \qquad n \equiv 5\ (8)$$

$$2P^{n+1} = \eta P^n \qquad n \equiv 2\ (4)$$

$$\left.\begin{aligned} 2P^{n+5} &= h_1P^{n+4} \\ \eta P^{n+5} &= h_2P^{n+3} \end{aligned}\right\} \quad n \equiv 2\ (8)$$

$$\eta h_1P^{n+4} \equiv 0 \bmod \nu^2P^n \qquad n \equiv 6\ (8)$$

$$\left.\begin{aligned} 2P^{n+4} &= h_1P^{n+3} \\ \eta P^{n+4} &= h_2P^{n+2} \end{aligned}\right\} \quad n \equiv 3\ (8)$$

For example, when $n \equiv 0\ (8)$, the attaching map of the n+4 cell gives $2P^{n+3} = h_1P^{n+2}$. Then $2\nu P^{n+3} = \nu h_1P^{n+2}$ must be either 0 or ν^2P^n by the E_∞ term in Figure 3.1. But P_n^{n+7} is coreducible, so ν^2P^n is impossible. Similarly, when $n \equiv 4\ (8)$, the attaching map of the n+4 cell gives $2P^{n+3} = h_1P^{n+2} + \nu P^n$. (Note that, since P_n^{n+3} is coreducible, νP^n need not be considered a part of the indeterminacy of $2P^{n+3}$ or h_1P^{n+2}.) Thus $2\nu P^{n+3} = \nu h_1P^{n+2} + \nu^2P^n$. But νP^{n+3} is either 0 or ν^2P^n by the E_∞ term in Figure 3.1. Thus $2\nu P^{n+3} = 0$ and hence $\nu h_1P^{n+2} = -\nu^2P^n = \nu^2P^n$.

Four more relations come from the fact that $\pi_{n+2}(S^n \cup_2 e^{n+1}) \cong Z_4$, so that the composite of 2 and a map which projects to η on S^{n+1}, lifts to η^2 on S^n. These are

$$\left.\begin{aligned} 2h_1P^{n+1} &= \eta^2P^n \\ 2h_1P^{n+5} &= h_1^2P^{n+4} \end{aligned}\right\} \quad n \equiv 1\ (4)$$

$$2h_1P^{n+4} = h_1^2P^{n+3} \qquad n \equiv 2\ (4)$$

$$2h_1P^{n+3} = h_1^2P^{n+2} \qquad n \equiv 3\ (4)$$

The relations

$$\begin{array}{ll} 2h_1^2P^{n+4} = h_1^2P^{n+3} & \\ 2h_1^3P^{n+3} = 0 & n \equiv 1 \ (4) \\ 2h_1^2P^{n+3} = h_1^3P^{n+2} & n \equiv 2 \ (4) \\ 2h_1^2P^{n+2} = h_1^3P^{n+1} & n \equiv 3 \ (4) \\ \eta h_2 P^{n+2} = \nu^2 P^n & n \equiv 3 \ (8) \end{array}$$

are the only possibilities consistent with Mahowald's calculations [59] (note that these are not hidden extensions in his spectral sequence).

Finally, the relation $2h_2P^{n+3} = \nu^2P^n$ when $n \equiv 6$ (8) follows by comparison with the spectral sequence for the cofiber of the inclusion $P_{n+1}^{n+2} \to P_n^{n+4}$. In the cofiber, $2P^{n+3} = \nu P^n$ is obvious from the attaching maps. //

Now consider the odd primary case. Recall, from [55], that, in degrees less than pq-2, the HZ_p Adams spectral sequence has elements

$$\begin{array}{lll} & a_0^i \in E_2^{i,i} & \text{detecting } p^i,\ i = 0,1,2,\ldots, \\ & h_0 \in E_2^{1,q} & \text{detecting } \alpha_1 \in \pi_{q-1}, \\ \text{and} & g_{i-1} \in E_2^{i,iq+i-1} & \text{detecting } \alpha_i \in \pi_{iq-1},\ \text{for } 2 \le i \le p. \end{array}$$

Let $\mathfrak{P}$ be the sequence

$$L_{n(p-1)}^{n(p-1)+ps} \longleftarrow L_{n(p-1)}^{n(p-1)+ps-1} \longleftarrow \cdots \longleftarrow L_{n(p-1)}^{n(p-1)+1} \longleftarrow L_{n(p-1)}^{n(p-1)}$$

Since $L_{n(p-1)}^{\infty}$ has cells only in dimensions n(p-1) and greater which are congruent to 0 or -1 mod q, $E_2(S, \mathfrak{P})$ is free over $E_2(S,S)$ on generators in those degrees. Write $x(j,\varepsilon)$ for the element of $E_2(S, \mathfrak{P})$ which is x $E_2(S,S)$ in the $jq-\varepsilon$ summand, if $jq-\varepsilon \geq n(p-1)$. We agree to let $x(j,\varepsilon) = 0$ if $jq-\varepsilon < n(p-1)$.

<u>Theorem 3.4</u>. In $E_r(S, \mathfrak{P})$, $d_2(x(j,0)) = a_0x(j,1)$ and

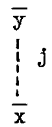
$$d_{2p-1}(x(j,1)) = -jh_0x(j-1,1).$$

In low dimensions $E_{2p}(S, \mathfrak{P})$ is given by Figures 3.5 through 3.9.

<u>Notes</u>: (1) The dotted arrows to the left represent possible d_{2p} differentials which we have not computed. This is why the theorem only claims to give $E_{2p}(S, \mathfrak{P})$. The indicated d_{2p} is the only possible remaining differential in the range listed. This is true for dimensional reasons except when $n = 2k-1$ and $k \equiv -2$ (p). Here the

possibility that $d_{4p-2}(1(k+2,1))$ is nonzero is excluded by the fact that $L_{kq-1}^{(k+2)q-1}$ is reducible when $k \equiv -2$ (p) by Theorem 2.9.(v).

(2) Dashed vertical lines represent hidden extensions. Precisely, if x and y are detected by $\bar{x}$ and $\bar{y}$, the notation



means that $px \equiv jy$ modulo higher filtrations. Of course, if j is 0 this means the extension is trivial. We replace j by a question mark if we have not settled the extension.

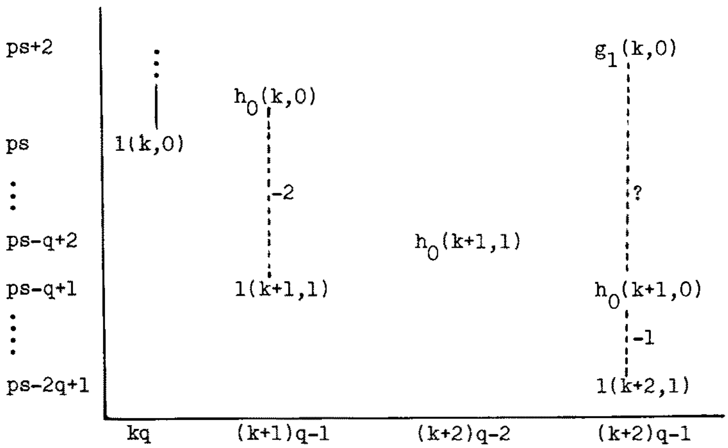


Figure 3.5 $n = 2k$, $k \equiv -2$ (p)

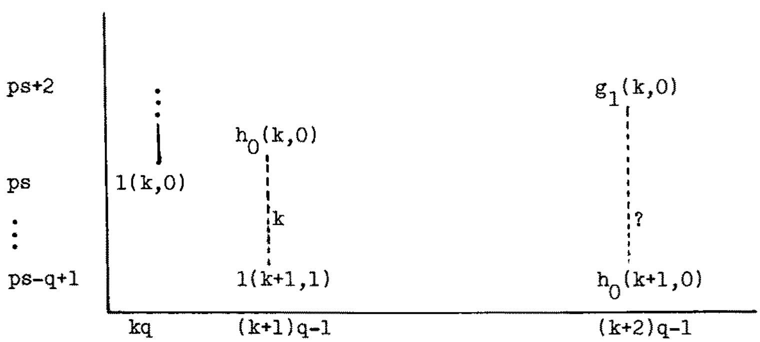


Figure 3.6 $n = 2k$, $k \not\equiv -2$ (p)

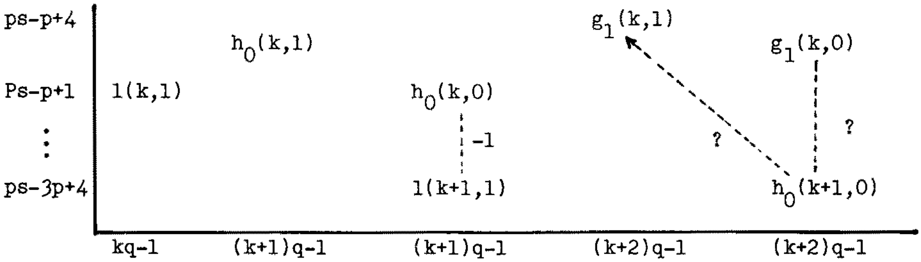


Figure 3.7 $n = 2k-1, \; k \equiv -1 \; (p)$

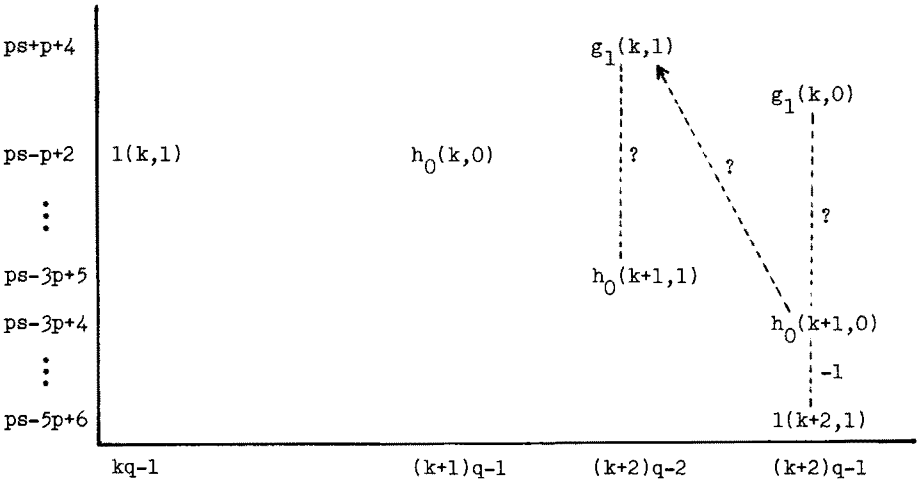


Figure 3.8 $n = 2k-1, \; k \equiv -2 \; (p)$

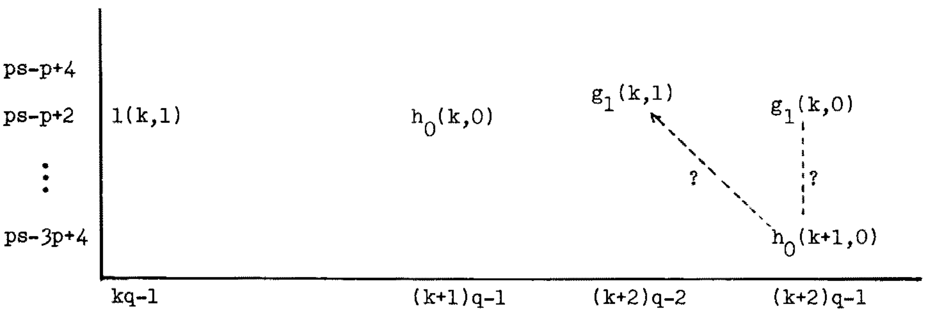


Figure 3.9 $n = 2k-1, \; k \not\equiv -1 \text{ or } -2 \; (p)$

Proof of 3.4. The differentials follow from the attaching maps in Proposition 3.2 just as 3.3 follows from 3.1. Applying them gives the values of $E_{2p}(S,\mathcal{D})$ listed in Figures 3.5 through 3.9. The indicated hidden extensions all come from the attaching maps of the even cells of $L^{\infty}_{n(p-1)}$. //

Proof of 1.9 when $p > 2$: A permanent cycle $x(j,\varepsilon)$ corresponds to a homotopy operation $x\beta^{\varepsilon}P^{j}$. Thus Table 1.1 is a list of those elements in Figures 3.5 through 3.9 which must be permanent cycles by Theorem 3.4. The indeterminacy is obtained from Figures 3.4 through 3.9 as for $p = 2$. The values of τ_{p*} listed are the only elements of $\pi_* S$ in the relevant dimensions, except for $\tau_{p*}(P^k) = p!$, which follows from II.1.10.

The relations in Table 1.2 are all determined by the attaching maps from Proposition 3.2. //

Proof of 1.10. By IV. 7.3.(v), to determine $P^{n+m+1}(xy)$ we must calculate the image of $P^{n+m+1} \in \pi_{2(n+m)+1} D_2 S^{n+m}$ under $\delta_* : \pi_* D_2 S^{n+m} \to \pi_*(D_2 S^n \wedge D_2 S^m)$. We need only consider

$$P^{n+m+2}_{n+m} \to P^{n+2}_{n} \wedge P^{m+2}_{m}$$

for dimensional reasons. If $\mathcal{D}_{n,m}$ is the skeletal filtration of $P^{n+2}_{n} \wedge P^{m+2}_{m}$, then $E_2(S, \mathcal{D}_{n,m})$ is generated over $E_2(S,S)$ by elements $1(j,k)$ with $n \le j \le n+2$ and $m \le k \le m+2$ corresponding to the cells of P^{n+2}_{n} and P^{m+2}_{m} in an obvious fashion. The attaching maps of P^{n+2}_{n} and P^{m+2}_{m} determine the differentials in low dimensions from which we get $E_{\infty}(S, \mathcal{D}_{n,m})$. The extension questions in $\pi_{2(n+m)+1}$ are also determined by P^{n+2}_{n} and P^{m+2}_{m} when $n \equiv m \equiv 0$ (2). When $n \equiv m \equiv 1$ (2) we need the fact that the top cell of the smash product of two mod 2 Moore spaces is attached to the bottom cell by η, to settle the extension question. We conclude that if $n \equiv m \equiv 0$ (2) then $\pi_{2(n+m)+1}$ is generated by $P^{n+1}P^m$, P^nP^{m+1}, and ηP^nP^m with relations

$$2P^{n+1}P^{m} = \begin{cases} 0 & n \equiv 0 \ (4) \\ \eta P^nP^m & n \equiv 2 \ (4) \end{cases}$$

and

$$2P^{n}P^{m+1} = \begin{cases} 0 & m \equiv 0 \ (4) \\ \eta P^nP^m & m \equiv 2 \ (4) \ . \end{cases}$$

If $n \equiv m \equiv 1$ (2) then $\pi_{2(n+m)+1}$ is generated by an element we call $S_{n,m}$ which is detected by $1(n+1,m) + 1(n,m+1)$ with the relation

$$2S_{n,m} = \begin{cases} 0 & n \equiv 3 \text{ or } m \equiv 3 \ (4) \\ \eta P^nP^m & n \equiv m \equiv 1 \ (4) \ . \end{cases}$$

From the image of $S_{n,m}$ in $E_\infty(S, \quad_{n,m})$ we can see that

$$E^0(S_{n,m}) = P^{n+1}P^m + P^nP^{m+1}.$$

Finally $\delta_*(P^{n+m+1})$ is determined modulo the kernel of the Hurewicz homomorphism by commutativity of the following diagram, in which the isomorphisms are Thom isomorphisms

$$\begin{array}{ccc}
\pi_*D_2S^{n+m} & \xrightarrow{\delta_*} & \pi_*D_2S^n \wedge D_2S^m \\
\downarrow h & & \downarrow h \\
H_*D_2S^{n+m} & \xrightarrow{\delta_*} & H_*D_2S^n \wedge D_2S^m \\
\wr\| & & \wr\| \\
H_*BZ_2 & \xrightarrow{\Delta_*} & H_*(BZ_2 \times BZ_2)
\end{array}$$

Since ηP^nP^m generates the kernel of the Hurewicz homomorphism we are done. //

Proof of 1.11. The commutative diagram above shows that the Hurewicz homomorphism must map the Cartan formula for a homotopy operation into the Cartan formula for its Hurewicz image. Case (i), $n = 2j$ and $m = 2k$, follows by an argument formally identical to, but easier than, the proof of 1.10 when $n \equiv m \equiv 0$ (2). Case (ii) is immediate from the homology Cartan formula because in this case we're in the Hurewicz dimension. Case (iii) follows just as in the proof of 1.10 when $n \equiv m \equiv 3$ (4). //

Proof of 1.12. In $E_2(S,S)$, $Sq^1(h_0) = h_1$ by [3]. Therefore, $P^1(2) = \eta$. //

Proof of 1.13. By definition $\beta P^1(p)$ is a unit times the composite

$$S^{2p-3} \xrightarrow{\beta P^1} D_pS \xrightarrow{D_p(p)} D_pS \xrightarrow{\xi} S,$$

where βP^1 is the inclusion of the 2p-3 cell. By II.1.8, $D_p(p) \equiv \iota_p\tau_p$ mod p, and by II.2.8, $\tau_p \circ \beta P^1 \neq 0$. Since $\xi\iota_p = 1$, the composite and hence $\beta P^1(p)$ are nonzero. The fact that $\beta P^{p-1}(\alpha_1) = \beta_1$ follows from the fact that in the Adams spectral sequence, $\beta P^{p-1}(h_0) = b_1^1$ using the notation of [66]. The latter can be computed directly from the definition of βP^{p-1} using the definitions

$$h_0 = [\xi_1], \quad b_1^1 = \sum_{i+j = p-1} \frac{1}{p}(i,j)[\xi_1^{pi}|\xi_1^{pj}]$$

in the bar construction. Alternatively, we may refer to Liulevicius' computation [55, pp. 26, 30] using [66, II-6.6] to translate it into our notation. //

Proof of 1.14. This is now immediate:

$$P^{n+1}(2x) = P^1(2)x^2 + 4P^{n+1}(x) + 4c_{0n}\eta x^2$$
$$= \eta x^2$$

since $2P^{n+1}(x)$ is either 0 or ηx^2 by 1.10. Similarly,

$$\beta P^{j+1}(px) = \beta P^1(p)x^p + p^p \beta P^{j+1}(x) + d_{0n}\alpha_1 p^p x^p$$
$$= \beta P^1(p)x^p + jp^{p-1}\alpha_1 P^j(x)$$
$$\doteq \alpha_1 x^p$$

since $p\beta P^{j+1}(x) = j\alpha_1 P^j(x)$. Finally $\beta P^{j+p-1}(\alpha_1 x) = x^p \beta P^{p-1}(\alpha_1) = x^p \beta_1$. The indeterminacy is always zero because where it is not automatically zero it is $4\eta x^2$ or $p^p \alpha_1 x^p$. //

Proof of 1.15. If $p = 2$ then $\eta x^2 = 0$ by Theorem 1.10 when $n \equiv 3$ (4) (even if $2x \neq 0$) while $0 = P^{n+1}(2x) = \eta x^2$ by Proposition 1.14 when $n \equiv 0$ (2). If $p > 2$ then $x^p = 0$ if n is odd, while if $n = 2j$, Proposition 1.14 implies that $0 = \beta P^{j+1}(px) \doteq \alpha_1 x^p$ and $0 = \beta P^{j+p-1}(\alpha_1 x) = \beta_1 x^p$. When $x = \beta_1$ the second of these formulas is $\alpha_1 \beta_1^p = 0$. //

Proof of 1.16. Several of the computations follow from $P^n(x) = x^2$ if $x \in \pi_n$, others from $\pi_4 = \pi_5 = \pi_{12} = \pi_{13} = 0$. Similarly, several indeterminacies are zero from Theorem 1.10 or because they lie in filtrations which are 0. We will prove the remainder of the results.

Since $P^4(h_2) = h_3$, $h_1 P^4(\nu)$ is detected by $h_1 h_3$ so is either $\eta\sigma$ or $\bar{\nu}$. By 1.10, $h_1^2 P^5(\nu) = 2h_1 P^6(\nu) = 0$ since $2\pi_{10} = 0$. Similarly, $h_1 P^4(2\nu) = 0$ by calculating Steenrod operations in Ext. Since $\tau_{2*}(h_1 P^6) = 0$, we get $h_1 P^6(2\nu) = 2h_1 P^6(\nu) = 0$, and since $\tau_{2*}(h_2 P^5) = 0$, we get $h_2 P^5(2\nu) = 2h_2 P^5(\nu) = 0$. By 1.10, $h_1^2 P^5(2\nu) = 2h_1 P^6(2\nu) = 0$ also. The operations on 4ν can all be calculated from the additivity rule $\alpha^*(4\nu) = 2\alpha^*(2\nu) + \tau_{2*}(\alpha)(2\nu)^2 = 2\alpha^*(2\nu)$.

Since $2\pi_{17} = 0$, the relations $h_1^2 P^9(\nu^2) = 2h_1 P^{10}(\nu^2)$ and $h_1^3 P^8(\nu^2) = 2h_1^2 P^9(\nu^2)$ force these elements to be 0 mod 0.

Since $P^8(h_3) = h_4$, $h_1 P^8(\sigma)$ is detected by $h_1 h_4$ so must be η^* or $\eta^* + \eta\rho$. Since $2h_1^2 P^9 = \eta^2 h_1 P^8$ and $\eta^2 h_1 P^8(\sigma)$ is detected by $h_1^3 h_4 = h_0^2 h_2 h_4$, it follows that $h_1^2 P^9(\sigma)$ is detected by $h_0 h_2 h_4$. Since $2h_1 P^{10} = h_1^2 P^9$ it follows that $h_1 P^{10}(\sigma)$ is detected by h_2h_4. Thus $h_1P^{10}(\sigma) = \nu^* \text{ or } \nu^* + \eta\bar{\mu}$ modulo $\langle 2\nu^* \rangle$, which is its indeterminacy, and similarly for $h_1^2P^9(\sigma)$.

Since $P^7(2\sigma) = 4\sigma^2 = 0$, we have

$$h_1P^8(2\sigma) = 2h_1P^8(\sigma) + \begin{Bmatrix} 0 \\ \text{or} \\ \eta^2 \end{Bmatrix} \sigma^2 = 0 + 0 = 0.$$

The remaining operations are additive except for

$$h_1P^8(4\sigma) = 2h_1P^8(2\sigma) + \begin{Bmatrix} 0 \\ \text{or} \\ \eta^2 \end{Bmatrix} 4\sigma^2 = 0 + 0 = 0 . \qquad //$$