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## Uniqueness of the multiplicative cyclotomic trace

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### ABSTRACT

Making use of the theory of noncommutative motives, we characterize the topological Dennis trace map as the unique *multiplicative* natural transformation from algebraic  $K$ -theory to topological Hochschild homology ( $THH$ ) and the cyclotomic trace map as the unique multiplicative lift through topological cyclic homology ( $TC$ ). Moreover, we prove that the space of all multiplicative structures on algebraic  $K$ -theory is contractible.

We also show that the algebraic  $K$ -theory functor from small stable  $\infty$ -categories to spectra is lax symmetric monoidal, which in particular implies that  $E_n$  ring spectra give rise to  $E_{n-1}$  ring algebraic  $K$ -theory spectra. Along the way, we develop a “multiplicative Morita theory”, establishing a symmetric monoidal equivalence between the  $\infty$ -category of small idempotent-complete stable  $\infty$ -categories and the Morita localization of the  $\infty$ -category of spectral categories.

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## 1. Introduction

Algebraic  $K$ -theory provides rich invariants of rings, schemes, and manifolds, encoding information reflecting arithmetic, geometry, and topology. The algebraic  $K$ -theory of a ring or scheme captures information about classical arithmetic invariants (e.g., the Picard and Brauer groups) [50]; for a manifold  $M$ , the algebraic  $K$ -theory of  $\Sigma_+^\infty \Omega M$  (Waldhausen’s  $A$ -theory) is closely related to stable pseudo-isotopy theory and  $B\text{Diff}(M)$ , the classifying space of the space of diffeomorphisms of  $M$  [54]. These seemingly disparate examples are unified by the perspective that algebraic  $K$ -theory is a functor of *stable categories*; e.g., a suitable enhancement of the derived category of a scheme, or the category of modules over a ring spectrum. This viewpoint was initiated by Thomason and Trobaugh [50]; see also [11].

For a commutative ring (or more generally a scheme)  $R$ , the derived category of  $R$  possesses the additional structure of a symmetric monoidal tensor product. Similarly, the category of modules over an  $E_\infty$  ring spectrum is symmetric monoidal. In this situation, the algebraic  $K$ -theory spectrum inherits the structure of an  $E_\infty$  ring spectrum [10,23,40,53]. These results have been important both as a source of structured ring spectrum models for geometric spectra (e.g., topological  $K$ -theory) as well as a source of input to calculation of algebraic  $K$ -groups.

The main computational tool for understanding algebraic  $K$ -theory of rings is the cyclotomic trace map  $K \rightarrow TC \rightarrow THH$  from algebraic  $K$ -theory to topological cyclic and Hochschild homology [14]; this map can be regarded as a spectrum-level enhancement of the Dennis trace  $K \rightarrow HC^- \rightarrow HH$ , and induces an equivalence between relative  $K$ -theory and relative  $TC$  [17,41] in many cases. Multiplicative structures play an essential role in these calculations, as both  $THH$  and  $TC$  are commutative ring spectra when applied to commutative rings and the cyclotomic trace is compatible with this multiplicative structure (e.g., see [29, §1] and [26, §6]).

However, the existing constructions of the multiplicative cyclotomic trace in the literature are very complicated and apply only in limited settings. Hence, it would be very useful to have a general characterization and construction of the multiplicative cyclotomic trace map in terms of a simple universal property. In this paper, making use of the

theory of noncommutative motives, we resolve this problem: Roughly speaking, the multiplicative topological Dennis trace is the unique multiplicative natural transformation from algebraic  $K$ -theory to  $THH$  (see [Theorem 1.11](#)), and the multiplicative cyclotomic trace is the unique lifting to  $TC$  (see [Theorem 1.15](#)).

**Statement of results.** Let  $\text{Cat}_\infty^{\text{perf}}$  denote the  $\infty$ -category of small idempotent-complete stable  $\infty$ -categories and exact functors. Recall from [[11](#), §6] that a functor  $E : \text{Cat}_\infty^{\text{perf}} \rightarrow \mathcal{D}$  with values in a stable presentable  $\infty$ -category  $\mathcal{D}$ , is called an *additive invariant* if it preserves filtered colimits and sends split-exact sequences of stable  $\infty$ -categories to cofiber sequences of spectra. When  $E$  moreover sends all exact sequences of stable  $\infty$ -categories to cofiber sequences, we say it is a *localizing invariant*. In [[11](#)] we produced stable presentable  $\infty$ -categories  $\mathcal{M}_{\text{add}}$  and  $\mathcal{M}_{\text{loc}}$  of noncommutative motives and functors

$$\mathcal{U}_{\text{add}} : \text{Cat}_\infty^{\text{perf}} \longrightarrow \mathcal{M}_{\text{add}} \quad \mathcal{U}_{\text{loc}} : \text{Cat}_\infty^{\text{perf}} \longrightarrow \mathcal{M}_{\text{loc}}$$

characterized by the following universal properties: given any stable presentable  $\infty$ -category  $\mathcal{D}$ , there are induced equivalences

$$(\mathcal{U}_{\text{add}})^* : \text{Fun}^{\text{L}}(\mathcal{M}_{\text{add}}, \mathcal{D}) \xrightarrow{\sim} \text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{D}) \tag{1.1}$$

$$(\mathcal{U}_{\text{loc}})^* : \text{Fun}^{\text{L}}(\mathcal{M}_{\text{loc}}, \mathcal{D}) \xrightarrow{\sim} \text{Fun}_{\text{loc}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{D}), \tag{1.2}$$

where the left-hand sides denote the  $\infty$ -categories of colimit-preserving functors and the right-hand sides the  $\infty$ -categories of additive and localizing invariants.

As stable  $\infty$ -categories,  $\mathcal{M}_{\text{add}}$  and  $\mathcal{M}_{\text{loc}}$  carry a natural enrichment in spectra; see [[11](#), §4]. In [[11](#)] we showed that the connective algebraic  $K$ -theory spectrum functor  $K(-)$  and the non-connective algebraic  $K$ -theory spectrum functor  $\mathbb{K}(-)$  become co-representable in  $\mathcal{M}_{\text{add}}$  and  $\mathcal{M}_{\text{loc}}$  respectively. More precisely, given any idempotent-complete small stable  $\infty$ -category  $\mathcal{A}$ , there are equivalences of spectra

$$\text{Map}(\mathcal{U}_{\text{add}}(\mathcal{S}_\infty^\omega), \mathcal{U}_{\text{add}}(\mathcal{A})) \simeq K(\mathcal{A}) \quad \text{Map}(\mathcal{U}_{\text{loc}}(\mathcal{S}_\infty^\omega), \mathcal{U}_{\text{loc}}(\mathcal{A})) \simeq \mathbb{K}(\mathcal{A}), \tag{1.3}$$

where  $\mathcal{S}_\infty$  denotes the stable  $\infty$ -category of spectra and  $\mathcal{S}_\infty^\omega$  the (essentially small) stable subcategory of compact objects in  $\mathcal{S}$  (see [Theorem 1.3](#) in [[11](#)]). Our first main result is the following:

**Theorem 1.4.** (*See [Theorem 5.8](#).*) *The  $\infty$ -categories  $\mathcal{M}_{\text{add}}$  and  $\mathcal{M}_{\text{loc}}$  carry natural symmetric monoidal structures making the functors  $\mathcal{U}_{\text{add}}$  and  $\mathcal{U}_{\text{loc}}$  symmetric monoidal. The tensor units are  $\mathcal{U}_{\text{add}}(\mathcal{S}_\infty^\omega)$  and  $\mathcal{U}_{\text{loc}}(\mathcal{S}_\infty^\omega)$  respectively.*

Here the symmetric monoidal structure is induced from the convolution symmetric monoidal structure on the  $\infty$ -category  $\text{Pre}((\text{Cat}_\infty^{\text{perf}})^\omega)$  of presheaves indexed on the

symmetric monoidal  $\infty$ -category of compact objects in  $\text{Cat}_\infty^{\text{perf}}$ . Similar considerations in the “dual” setting of covariant functors from  $\mathcal{M}_{\text{add}}$  and  $\mathcal{M}_{\text{loc}}$  to spectra give rise to symmetric monoidal structures on the  $\infty$ -categories of additive and localizing invariants:

**Theorem 1.5.** (See [Theorem 5.14](#).) *The  $\infty$ -categories of additive and localizing invariants,  $\text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)$  and  $\text{Fun}_{\text{loc}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)$ , are symmetric monoidal  $\infty$ -categories. That is, they are the underlying  $\infty$ -categories of symmetric monoidal presentable stable  $\infty$ -categories  $\text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)^\otimes$  and  $\text{Fun}_{\text{loc}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)^\otimes$ . The tensor units are the connective and non-connective algebraic  $K$ -theory functors  $K$  and  $\mathbb{K}$ .*

[Theorem 1.5](#) allows us to study  $E_n$  algebras and  $E_n$  maps in  $\text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)$  and  $\text{Fun}_{\text{loc}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)$ . Since algebraic  $K$ -theory is the tensor unit, a consequence of [Theorem 1.5](#) and the fact that the space of multiplicative maps out of the unit is contractible is the following strong uniqueness result:

**Corollary 1.6.** (See [Corollary 7.2](#).) *There exists a unique  $E_\infty$  algebra structure on  $K$ , viewed as an object of the symmetric monoidal  $\infty$ -category  $\text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)^\otimes$ . Furthermore, for any  $0 \leq n \leq \infty$  and any  $E_n$  algebra  $F$ , the space of  $E_n$  algebra maps from  $K$  to  $F$  is contractible. Analogous statements hold for  $\mathbb{K}$ .*

By combining [Theorem 1.5](#) with the recent work of Glasman [\[27\]](#), we obtain the following relation between  $E_\infty$  algebras and lax symmetric monoidal functors:

**Corollary 1.7.** *For any presentable symmetric monoidal  $\infty$ -category  $\mathcal{D}$ , there are equivalences of  $\infty$ -categories*

$$\begin{aligned} \text{Alg}_{/E_\infty}(\text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{D})) &\xrightarrow{\sim} \text{Fun}_{\text{add}}^{\text{lax}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{D}) \\ \text{Alg}_{/E_\infty}(\text{Fun}_{\text{loc}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{D})) &\xrightarrow{\sim} \text{Fun}_{\text{loc}}^{\text{lax}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{D}). \end{aligned}$$

This leads to the following sharpening of equivalences [\(1.1\)](#) and [\(1.2\)](#).

**Theorem 1.8.** *For any presentable symmetric monoidal  $\infty$ -category  $\mathcal{D}$ , there are equivalences of  $\infty$ -categories*

$$\begin{aligned} (\mathcal{U}_{\text{add}})^* : \text{Fun}^{\text{L,lax}}(\mathcal{M}_{\text{add}}, \mathcal{D}) &\xrightarrow{\sim} \text{Fun}_{\text{add}}^{\text{lax}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{D}) \\ (\mathcal{U}_{\text{loc}})^* : \text{Fun}^{\text{L,lax}}(\mathcal{M}_{\text{loc}}, \mathcal{D}) &\xrightarrow{\sim} \text{Fun}_{\text{loc}}^{\text{lax}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{D}), \end{aligned}$$

where the left-hand sides denote the  $\infty$ -category of lax symmetric monoidal colimit-preserving functors and the right-hand sides denote the  $\infty$ -categories of lax symmetric monoidal additive or localizing invariants, respectively.

In order to produce lax monoidal functors of  $\infty$ -categories  $\text{Cat}_\infty^{\text{perf}} \rightarrow \mathcal{S}_\infty$ , we develop a multiplicative version of Morita theory for spectral categories. In [\[11, §4\]](#) we proved

that the  $\infty$ -category  $\text{Cat}_\infty^{\text{perf}}$  of small stable idempotent-complete  $\infty$ -categories admits a model given by the localization of the category  $\text{Cat}_S$  of spectral categories (categories enriched in spectra) with respect to the class  $\mathcal{W}$  of Morita equivalences. Here recall that a spectral functor  $\mathcal{C} \rightarrow \mathcal{D}$  is a DK-equivalence if it induces weak equivalences  $\mathcal{C}(X, Y) \rightarrow \mathcal{D}(FX, FY)$  on all mapping spectra and is homotopically essentially surjective, and a Morita equivalence if it induces a DK-equivalence on the induced functor  $\text{Mod}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{D})$ . In particular, any (stable)  $\infty$ -category can be “rigidified” to a (stable) spectral category.

For our applications herein, we generalize this rigidification result to the multiplicative setting. To maintain homotopical control on the smash product of spectral categories, we use the notion of a flat spectral category, i.e., a spectral category such that tensoring with it preserves Morita equivalences. Since pointwise-cofibrant spectral categories are flat, and any flat spectral category is DK-equivalent to a pointwise-cofibrant spectral category, we will write  $\text{Cat}_S^{\text{flat}}$  for the full subcategory of pointwise-cofibrant spectral categories. Following [35], we write  $N$  for the nerve functor from categories (or, more generally, simplicial categories) to  $\infty$ -categories.

**Theorem 1.9.** (See [Theorem 4.6.](#)) *There is an equivalence of symmetric monoidal  $\infty$ -categories between  $(\text{Cat}_\infty^{\text{perf}})^\otimes$  and  $(N(\text{Cat}_S^{\text{flat}})[\mathcal{W}^{-1}])^\otimes$ , where  $\mathcal{W}$  denotes the class of Morita equivalences.*

Since the  $\infty$ -category associated to  $\text{Cat}_S^{\text{flat}}$  is a model for the  $\infty$ -category of spectral categories and Morita equivalences, we use  $(N(\text{Cat}_S^{\text{flat}})[\mathcal{W}^{-1}])^\otimes$  as a specific model of the symmetric monoidal  $\infty$ -category of spectral categories and Morita equivalences. Hence, [Theorem 1.9](#) allows us to describe multiplicative objects in  $\text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)$  and  $\text{Fun}_{\text{loc}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)$  as functors from spectral categories to spectra. Specifically, we prove in [Section 6](#) that suitable homotopical point-set lax symmetric monoidal functors from spectral categories to spectra give rise to  $E_\infty$  algebras in  $\text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)$  and  $\text{Fun}_{\text{loc}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)$ .

**Theorem 1.10.** (See [Theorem 6.3.](#)) *Let  $E$  be a lax symmetric monoidal functor from spectral categories to spectra. Further assume that  $E$  preserves Morita equivalences between flat spectral categories and that the induced functor  $\tilde{E} : \text{Cat}_\infty^{\text{perf}} \rightarrow \mathcal{S}_\infty$  is an additive invariant. Then  $\tilde{E}$  naturally extends to an  $E_\infty$  algebra object of  $\text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)^\otimes$ . The analogous results for localizing invariants hold.*

Our main example of a functor  $E$  satisfying the hypotheses of [Theorem 1.10](#) is topological Hochschild homology; see [Corollary 6.9](#).

**Applications.** Recall that one of the most interesting applications of the theory developed in [\[11\]](#) was the proof that the set of homotopy classes of natural transformations from  $K$  to  $THH$  is isomorphic to  $\mathbb{Z}$ , with the topological Dennis trace corresponding to  $1 \in \mathbb{Z}$ . The

following consequence of [Corollary 1.6](#) and [Theorem 1.10](#), which is our main application, is a significant sharpening of this result.

**Theorem 1.11.** (See [Theorem 7.3](#).) *The space of maps of  $E_\infty$  algebras from  $K$  to  $THH$  in  $\text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)^\otimes$  is contractible. Equivalently, the space of natural transformations of lax symmetric monoidal functors from  $K \rightarrow THH$  in  $\text{Fun}_{\text{add}}^{\text{lax}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)$  is contractible. The unique element of this space is the topological Dennis trace map. The analogous result holds for  $\mathbb{K}$ .*

Although  $TC$  is not an additive invariant (as it does not preserve filtered colimits), we can extend this result to an identification of the cyclotomic trace. Specifically,  $TC$  can be described as  $\text{holim}_n TC^n$ , where the  $TC^n$  are additive invariants, and this characterization gives rise to the following result.

**Theorem 1.12.** (See [Theorem 7.4](#).) *The space of maps of  $E_\infty$  algebras from  $K \rightarrow TC^n$  in  $\text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)^\otimes$  is contractible. Equivalently, the space of natural transformations of lax symmetric monoidal functors from  $K \rightarrow TC^n$  in  $\text{Fun}_{\text{add}}^{\text{lax}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)$  is contractible. The unique homotopy class of maps of  $E_\infty$  algebras in  $\text{Fun}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)^\otimes$  from  $K$  to  $TC$  that restrict to maps of  $E_\infty$  algebras  $K \rightarrow TC^n$  is the cyclotomic trace.*

Our second application, which generalizes results of [\[10,23\]](#), is the following:

**Theorem 1.13.** (See [Proposition 5.9](#) and [Corollary 5.10](#).) *The algebraic  $K$ -theory functors are lax symmetric monoidal as functors from  $\text{Cat}_\infty^{\text{perf}}$  to  $\mathcal{S}_\infty$ . In particular, for every  $E_n$  object  $\mathcal{A}$  in  $\text{Cat}_\infty^{\text{perf}}$ , with  $0 \leq n \leq \infty$ ,  $K(\mathcal{A})$  and  $\mathbb{K}(\mathcal{A})$  are  $E_n$  ring spectra.*

[Theorem 1.13](#), combined with Lurie’s proof [\[36, 8.1.2.6\]](#) of a conjecture of Mandell [\[37, 5.3\]](#), which asserts that an  $E_{n+1}$  algebra in spectra gives rise to an  $E_n$  category of compact modules in  $\text{Cat}_\infty^{\text{perf}}$ , implies that the  $K$ -theory of an  $E_{n+1}$  ring spectrum is an  $E_n$  ring spectrum. Our third application, which is a consequence of [Theorem 1.4](#) and equivalences [\(1.3\)](#), is the following:

**Corollary 1.14.** (See [Corollary 5.18](#).) *The symmetric monoidal homotopy categories  $\text{Ho}(\mathcal{M}_{\text{add}})$  and  $\text{Ho}(\mathcal{M}_{\text{loc}})$  are enriched over the symmetric monoidal homotopy category  $\text{Ho}(\text{Mod}_{A(*)})$  of  $A(*)$ -modules, where  $A(*) = K(\mathbb{S}) \simeq \mathbb{K}(\mathbb{S})$ .*

Our final application is a consistency result for multiplicative structures on algebraic  $K$ -theory. Associated to a spectral category  $\mathcal{C}$  is its pre-triangulated spectral category  $\text{Perf}(\mathcal{C})$  of (homotopically) compact  $\mathcal{C}$ -modules, which has a Waldhausen structure inherited from the projective model structure on  $\mathcal{C}$ -modules. There are now two possible constructions of algebraic  $K$ -theory landing in the  $\infty$ -category of spectra. This is depicted as follows:

$$\begin{array}{ccccc}
 \mathbf{N}(\mathbf{Cat}_{\mathcal{S}}) & \xrightarrow{\text{Perf}(-)} & \mathbf{N}(\mathbf{Cat}_{\mathcal{S}}) & \xrightarrow{K(-)} & \mathcal{S}_{\infty} \\
 & & \downarrow & & \parallel \\
 & & \mathbf{Cat}_{\infty}^{\text{perf}} & \xrightarrow{\mathcal{U}_{\text{add}}(-)} & \mathcal{M}_{\text{add}} \xrightarrow{\text{Map}(\mathcal{U}_{\text{add}}(\mathcal{S}_{\infty}^{\omega}, -))} & \mathcal{S}_{\infty}.
 \end{array}$$

In [11, §7] it was proved that these two approaches (as well as a third  $\infty$ -categorical version of the Waldhausen construction) are canonically equivalent. The following corollary of Corollary 1.6 promotes this equivalence to the multiplicative setting.

**Corollary 1.15.** (See Corollary 7.5.) *Let  $\mathcal{C}$  be a symmetric monoidal spectral category, and  $\text{Perf}(\mathcal{C})$  be the resulting symmetric monoidal category of compact modules. The two algebraic  $K$ -theory spectra described above are naturally equivalent as  $E_{\infty}$  algebras in the  $\infty$ -category of spectra. If  $\mathcal{C}$  is a monoidal spectral category, the resulting algebraic  $K$ -theory spectra are naturally equivalent as  $A_{\infty}$  algebras in the  $\infty$ -category of spectra. Analogous results hold for  $\mathbb{K}$ .*

Finally, we note that there has been recent work on the subject of multiplicative structures on an  $\infty$ -categorical model of algebraic  $K$ -theory due to Barwick [1]. Naturally, the basic results are broadly similar, but certain technical differences arise in both the method of proof and the approach to describing the input data for algebraic  $K$ -theory.

**Notations.** Throughout the article we will use the letter  $\mathcal{T}$  to denote the symmetric monoidal simplicial model category of simplicial sets and the letter  $\mathcal{S}$  for the symmetric monoidal simplicial model category of symmetric spectra of simplicial sets [33].

## 2. Background on spectral categories

Our work depends on a careful analysis of the interplay between different models of the homotopy theory of stable homotopy categories. In this section, we briefly review the details of the model given by spectral categories. Other references on spectral categories include [7, Section 2], [11, Section 2.1], [42, Appendix A], or [45, Section 2].

Recall that a *spectral category*  $\mathcal{A}$  is a category enriched in the category  $\mathcal{S}$  of symmetric spectra. Concretely, it consists of the following data:

- A class of objects  $\text{obj}(\mathcal{A})$ ;
- A symmetric spectrum  $\mathcal{A}(x, y)$  for each ordered pair of objects  $(x, y)$ ;
- Composition morphisms in  $\mathcal{S}$

$$\mathcal{A}(y, z) \wedge \mathcal{A}(x, y) \longrightarrow \mathcal{A}(x, z) \quad x, y \in \text{obj}(\mathcal{A})$$

satisfying the usual associativity condition;

- Unit morphisms  $\mathbb{S} \rightarrow \mathcal{A}(x, x)$ ,  $x \in \text{obj}(\mathcal{A})$ , satisfying the usual unit condition with respect to the above composition.

A spectral category is called *small* whenever its class of objects forms a (small) set. A *spectral functor*  $F : \mathcal{A} \rightarrow \mathcal{B}$  is a functor enriched over  $\mathcal{S}$ . Concretely, it consists of a map  $\text{obj}(\mathcal{A}) \rightarrow \text{obj}(\mathcal{B})$  and of morphisms in  $\mathcal{S}$

$$F(x, y) : \mathcal{A}(x, y) \longrightarrow \mathcal{B}(Fx, Fy) \quad x, y \in \text{obj}(\mathcal{A})$$

satisfying the usual unit and associativity conditions.

**Notation 2.1.** Let  $\text{Cat}_{\mathcal{S}}$  denote the category of small spectral categories.

Given a small spectral category  $\mathcal{A}$ , one can form a genuine category  $[\mathcal{A}]$  by keeping the same set of objects and by defining  $[\mathcal{A}](x, y)$  as the set of morphisms in the homotopy category  $\text{Ho}(\mathcal{S})$  from the sphere spectrum  $\mathbb{S}$  to  $\mathcal{A}(x, y)$ . This gives naturally rise to a well-defined functor

$$[-] : \text{Cat}_{\mathcal{S}} \longrightarrow \text{Cat}$$

with values in the category of small categories.

We now turn to the homotopy theory of spectral categories.

**Definition 2.2.** A spectral functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is called a *DK-equivalence* if:

- The morphisms in  $\mathcal{S}$

$$F(x, y) : \mathcal{A}(x, y) \longrightarrow \mathcal{B}(Fx, Fy) \quad x, y \in \text{obj}(\mathcal{A})$$

are stable equivalences of spectra;

- The induced functor  $[F] : [\mathcal{A}] \rightarrow [\mathcal{B}]$  is an equivalence of categories.

As proved in [45, Thm. 5.10],  $\text{Cat}_{\mathcal{S}}$  carries a (right proper) Quillen model structure whose weak equivalences are the DK-equivalences. Moreover, the criteria of [4, Thm. 2.5, Thm. 2.20] implies that this model structure is in fact cofibrantly-generated.

For the purposes of algebraic  $K$ -theory (and related functors), it is convenient to work with a weaker notion of equivalence than DK-equivalence. Given a spectral category  $\mathcal{A}$ , an  $\mathcal{A}$ -*module* is a spectral functor from the opposite spectral category  $\mathcal{A}^{\text{op}}$  (where  $\mathcal{A}^{\text{op}}(x, y) := \mathcal{A}(y, x)$ ) to the spectral category  $\mathcal{S}$  of symmetric spectra. Let us denote by  $\widehat{\mathcal{A}}$  the spectral category of  $\mathcal{A}$ -modules. As explained in [42, A.1.1],  $\widehat{\mathcal{A}}$  carries a (combinatorial) spectral model structure in which the weak equivalences are the point-wise stable equivalences and the fibrations are the pointwise fibrations. In what follows we will denote by  $\mathcal{D}(\mathcal{A})$  the *derived category* of  $\mathcal{A}$ , i.e., the homotopy category  $\text{Ho}(\widehat{\mathcal{A}})$

associated to this model structure. Note that one has a (fully faithful) spectral Yoneda embedding

$$\mathcal{A} \longrightarrow \widehat{\mathcal{A}} \quad z \mapsto \mathcal{A}(-, z).$$

Let  $\mathcal{D}_{\text{tri}}(\mathcal{A})$  denote the smallest triangulated subcategory of  $\mathcal{D}(\mathcal{A})$  containing the image of  $\mathcal{A}$  under the Yoneda embedding, and  $\mathcal{D}_{\text{perf}}(\mathcal{A})$  the smallest thick triangulated subcategory of  $\mathcal{D}(\mathcal{A})$  containing the image of  $\mathcal{A}$  under the Yoneda embedding.

Note that every spectral functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  gives rise to a restriction/extension Quillen adjunction

$$\begin{array}{ccc} & \widehat{\mathcal{B}} & \\ & \uparrow & \downarrow \\ F_! & & F^* \\ & \widehat{\mathcal{A}} & \end{array}$$

We hence obtain a total left-derived functor  $\mathbb{L}F_! : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{B})$  which restricts to  $\mathbb{L}F_! : \mathcal{D}_{\text{perf}}(\mathcal{A}) \rightarrow \mathcal{D}_{\text{perf}}(\mathcal{B})$  and furthermore to  $\mathbb{L}F_! : \mathcal{D}_{\text{tri}}(\mathcal{A}) \rightarrow \mathcal{D}_{\text{tri}}(\mathcal{B})$ .

**Definition 2.3.** A spectral functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is called:

- A *triangulated equivalence* if  $\mathbb{L}F_! : \mathcal{D}_{\text{tri}}(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}_{\text{tri}}(\mathcal{B})$  is an equivalence;
- A *Morita equivalence* if  $\mathbb{L}F_! : \mathcal{D}_{\text{perf}}(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}_{\text{perf}}(\mathcal{B})$  is an equivalence.

As we shall recall below, with these notions of equivalence spectral categories provide a point-set model of the homotopy theory of small stable homotopical categories. To this end, it is sometimes convenient to work with a variant model of spectral categories. Let  $\mathcal{M}$  denote the symmetric monoidal category of  $S$ -modules of Elmendorf, Kriz, Mandell and May [25] and  $\text{Cat}_{\mathcal{M}}$  denote the category of small categories enriched in  $\mathcal{M}$  and the spectral (enriched) functors. The notions of DK-equivalence, triangulated equivalence, and Morita equivalence generalize in the obvious fashion. Recall from [38, 3.7] that there is a Quillen equivalence  $(\mathbb{N} \circ \mathbb{P}, \mathbb{U} \circ \mathbb{N}^{\sharp})$  connecting  $\mathcal{S}$  and  $\mathcal{M}$ . The left adjoint is strong monoidal and the right adjoint is lax monoidal.

**Proposition 2.4.** *The induced adjoint pair of functors between  $\text{Cat}_{\mathcal{S}}$  and  $\text{Cat}_{\mathcal{M}}$  produces a transferred model structure on  $\text{Cat}_{\mathcal{M}}$  in which the weak equivalences are the DK-equivalences. This model structure is Quillen equivalent to the model structure on  $\text{Cat}_{\mathcal{S}}$ .*

**Proof.** We can apply standard transfer arguments to lift the DK-equivalence model structure on  $\text{Cat}_{\mathcal{S}}$  to a model structure on  $\text{Cat}_{\mathcal{M}}$  (e.g., see [3, §2.5]). We define weak equivalences and fibrations in  $\text{Cat}_{\mathcal{M}}$  to be maps which are weak equivalences and fibrations via  $\mathbb{U} \circ \mathbb{N}^{\sharp}$ . Since the right adjoint  $\mathbb{U} \circ \mathbb{N}^{\sharp}$  creates the weak equivalences, preserves

sequential colimits along closed inclusions, all objects in  $\text{Cat}_{\mathcal{M}}$  are fibrant, and there are functorial path objects (via the construction of [46, 4.1.1]), this specifies a transferred model structure on  $\text{Cat}_{\mathcal{M}}$ . Since  $\mathcal{M}$  and  $\mathcal{S}$  are Quillen equivalent, it is clear that the transferred model structure on  $\text{Cat}_{\mathcal{M}}$  is Quillen equivalent to  $\text{Cat}_{\mathcal{S}}$ .  $\square$

The category  $\text{Cat}_{\mathcal{M}}$  allows us to correct a small error in [11]. Specifically, we studied therein the construction  $\psi_{\text{perf}}$  that takes a small pointwise-cofibrant spectral category  $\mathcal{C}$  to the cofibrant, fibrant, homotopically compact objects in the projective model structure on  $\text{Mod}(\mathcal{C})$  [11, §4.1]. A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  induces a functor  $\text{Mod}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{D})$  via left Kan extension followed by fibrant replacement. Because of the appearance of fibrant replacement,  $\psi_{\text{perf}}$  is not a strict endofunctor on  $\text{Cat}_{\mathcal{S}}$ , although it does have suitable coherence to give rise to an  $\infty$ -functor. However, since all objects in  $\text{Cat}_{\mathcal{M}}$  and in the variant of  $\text{Mod}(\mathcal{C})$  in this setting are fibrant, the fact that the left Kan extension is a Quillen left adjoint (and so preserves cofibrant objects and hence cofibrant–fibrant objects) implies that  $\psi_{\text{perf}}$  is functorial in this setting. An analogous issue arises with the functoriality of the spectral envelope [11, 4.10], and can be corrected in the same fashion.

### 3. Background on $\infty$ -categories and $\infty$ -operads

The basic setting for our work is the theory of  $\infty$ -categories (and particularly stable  $\infty$ -categories), which provide a tractable way to handle the abstract homotopy theory of the “category of homotopical categories” as well as categories of homotopical functors. There are now many competing models of  $\infty$ -categories, including Rezk’s Segal spaces [47], the Segal categories [32,49] of Simpson and Tamsamani, the “quasicategories” (weak Kan complexes) introduced by Boardman and Vogt and studied by Joyal and Lurie [12,34,35], the homotopy theory of simplicial categories as studied by Dwyer, Kan and Bergner [20,5], and others, all of which are known to be equivalent (see [6] for a nice discussion of the situation). We have chosen to work in this paper with the theories of quasicategories and spectral categories. Our basic references for the former material are Lurie’s books [35,36]. In this section we give a brief review of certain essential foundational aspects of the theory of quasicategories and then review the theory of  $\infty$ -operads as we will apply it in the body of the paper, following [36, §2].

We begin by recalling the passage from categories with weak equivalences (e.g., model categories) to  $\infty$ -categories in the setting of quasicategories. One way to produce an  $\infty$ -category from a category  $\mathcal{C}$  with weak equivalences  $w\mathcal{C}$  is to take a fibrant replacement of the Dwyer–Kan simplicial localization of  $(\mathcal{C}, w\mathcal{C})$  and apply the coherent nerve functor  $N$ . In general, for a category  $\mathcal{C}$  with weak equivalences we will denote this process by  $N(L^H\mathcal{C})$ . For the purposes of studying the multiplicative structure induced by a monoidal product, it is convenient to use the repackaging of this approach to passing from a model category to an  $\infty$ -category described in [36, §1.3.4 and §4.1.3]. The construction of [36, Construction 4.1.3.1] produces from an  $\infty$ -category  $\mathcal{C}$  and a suitable

collection of weak equivalences  $\mathcal{W}$  an  $\infty$ -category  $\mathcal{C}[\mathcal{W}^{-1}]$ . In particular, given a model category  $\mathcal{D}$  which is not necessarily simplicial, the coherent nerve of the subcategory of cofibrant objects  $\mathcal{D}^c$  is an  $\infty$ -category and the  $\infty$ -category  $\mathbb{N}(\mathcal{D}^c)[\mathcal{W}^{-1}]$  is a version of the underlying  $\infty$ -category of  $\mathcal{D}$ .

**Proposition 3.1.** *Let  $\mathcal{C}$  be a combinatorial model category with weak equivalences  $\mathcal{W}$ . Then there is a categorical equivalence  $\mathbb{N}(\mathcal{C}^c)[\mathcal{W}^{-1}] \rightarrow \mathbb{N}(L^H\mathcal{C})$  induced by the inclusion  $\mathcal{C}^c \rightarrow L^H\mathcal{C}^c$ .*

**Proof.** This follows from the fact that the map  $\mathbb{N}(\mathcal{C}) \rightarrow \mathbb{N}(L^H\mathcal{C})$  exhibits  $\mathbb{N}(L^H\mathcal{C})$  as the  $\infty$ -category obtained from  $\mathbb{N}(\mathcal{C})$  by inverting the morphisms of  $\mathcal{W}$ , in the sense of [36, Definition 1.3.4.1]. To see this, recall from [16] that the combinatorial model category  $\mathcal{C}$  is Quillen equivalent to a combinatorial simplicial model category  $\mathcal{C}'$ . By [36, Lemma 1.3.4.21], this induces an equivalence of  $\infty$ -categories

$$\mathbb{N}(\mathcal{C}^c)[\mathcal{W}^{-1}] \longrightarrow \mathbb{N}((\mathcal{C}')^c)[\mathcal{W}^{-1}].$$

Next, [36, Theorem 1.3.4.20] implies that there is an equivalence of  $\infty$ -categories

$$\mathbb{N}((\mathcal{C}')^c)[\mathcal{W}^{-1}] \longrightarrow \mathbb{N}((\mathcal{C}')^\circ),$$

where  $(\mathcal{C}')^\circ$  denotes the full simplicial subcategory consisting of the cofibrant–fibrant objects in  $\mathcal{C}'$ . Thus, it suffices to compare  $\mathbb{N}(L^H\mathcal{D})$  and  $\mathbb{N}(\mathcal{D}^\circ)$  for a simplicial model category  $\mathcal{D}$ . By [21, 4.8] there is a natural zig-zag of equivalences of simplicial categories connecting  $\mathcal{D}^\circ$  and  $L^H\mathcal{D}$ , and so the result follows.  $\square$

Next, we briefly review the definitions of stable  $\infty$ -categories. An  $\infty$ -category that has finite colimits and limits is stable [36, 1.1.1.9] when pushout and pullback squares coincide [36, 1.1.3.4]; it is straightforward to check that a stable  $\infty$ -category has a triangulated structure on its homotopy category coming from the cofiber sequences [36, 1.1.2.13]. A functor between stable  $\infty$ -categories is exact when it preserves finite colimits [36, §1.1.4]. An exact functor is an equivalence when it induces an equivalence on the underlying (triangulated) homotopy categories. One of the basic attractive aspects of the theory of quasicategories is that the quasicategory of quasicategories is tractable. Herein, we are particularly interested in  $\text{Cat}_\infty^{\text{ex}}$  and  $\text{Cat}_\infty^{\text{perf}}$ , which are respectively the  $\infty$ -categories of small stable and idempotent-complete small stable  $\infty$ -categories (with the equivalences given as above).

We now turn to the theory of  $\infty$ -operads. Let  $\text{Fin}_*$  denote the category with objects the pointed sets  $\langle n \rangle = \{*, 1, 2, \dots, n\}$  and morphisms those functions which preserve the basepoints  $*$ . (Classically, this category is also denoted  $\Gamma^{\text{op}}$ .) Recall that  $\mathbb{N}$  will denote the nerve functor; the nerve of an ordinary category is a quasicategory [35, 1.1.5.5]. In mild abuse of notation, we will also use  $\mathbb{N}$  to denote the homotopy coherent nerve functor from simplicial categories to quasicategories [35, 1.1.2.6]. This is reasonable since

the homotopy coherent nerve of an ordinary category regarded as a discrete category coincides with the standard nerve [35, 1.1.5.8].

Before providing the definition of an  $\infty$ -operad, we recall two basic definitions. First, a map  $f : \langle m \rangle \rightarrow \langle n \rangle$  is inert if  $f^{-1}(i)$  has precisely one element for  $i \neq *$  [35, 2.1.1.8]. There are distinguished inert morphisms  $\rho^i : \langle n \rangle \rightarrow \langle 1 \rangle$  that take everything to the basepoint except  $i$ , which is taken to 1. Second, we need to recall the definition of  $p$ -coCartesian morphisms. Given an object  $x \in \mathcal{C}$ , let  $\mathcal{C}_{x/}$  denote the category of objects under  $x$ . Similarly, given a morphism  $f$  in an  $\infty$ -category  $\mathcal{C}$ , let  $\mathcal{C}_{f/}$  denote the category of morphisms under  $f$ . Then given a functor  $p : \mathcal{C} \rightarrow \mathcal{D}$ , a morphism  $f : x \rightarrow y$  in  $\mathcal{C}$  is  $p$ -coCartesian (lifting  $p(f) : px \rightarrow py$ ) if the natural map

$$\mathcal{C}_{f/} \longrightarrow \mathcal{C}_{x/} \times_{\mathcal{D}_{p(x)}} \mathcal{D}_{p(f)/}$$

is a trivial fibration of simplicial sets [35, §2.4.1]. A map  $p : \mathcal{C} \rightarrow \mathcal{D}$  is a coCartesian fibration if it is an inner fibration (has the right lifting property with respect to inner horn inclusions) and for each  $c \in \mathcal{C}$  and map  $pc \rightarrow d$ , there is a  $p$ -coCartesian edge  $c \rightarrow c'$  that lifts the given map; see [35, §2.4] for more details.

**Definition 3.2.** (See [36, Definition 2.1.1.10].) An  $\infty$ -operad is then an  $\infty$ -category  $\mathcal{O}^\otimes$  and a functor  $p : \mathcal{O}^\otimes \rightarrow \mathbf{N}(\mathbf{Fin}_*)$  satisfying the following conditions:

- (1) For every inert morphism  $f : \langle m \rangle \rightarrow \langle n \rangle$  in  $\mathbf{Fin}_*$  and every object  $C$  of  $\mathcal{O}_{\langle m \rangle}^\otimes$ , there exists a  $p$ -coCartesian morphism  $\tilde{f} : C \rightarrow C'$  in  $\mathcal{O}^\otimes$  lifting  $f$  and hence an induced functor  $f_! : \mathcal{O}_{\langle m \rangle}^\otimes \rightarrow \mathcal{O}_{\langle n \rangle}^\otimes$ .
- (2) Let  $C \in \mathcal{O}_{\langle m \rangle}^\otimes$  and  $C' \in \mathcal{O}_{\langle n \rangle}^\otimes$  be objects, let  $f : \langle m \rangle \rightarrow \langle n \rangle$  be a morphism in  $\mathbf{Fin}_*$ , and let  $\text{map}_{\mathcal{O}^\otimes}^f(C, C')$  denote the union of the components of  $\text{map}_{\mathcal{O}^\otimes}(C, C')$  which lie over  $f \in \text{hom}_{\mathbf{Fin}_*}(\langle m \rangle, \langle n \rangle)$ . Choose  $p$ -coCartesian morphisms  $C' \rightarrow C'_i$  lying over the inert morphisms  $\rho^i : \langle n \rangle \rightarrow \langle 1 \rangle$  for each  $1 \leq i \leq n$ . Then the induced map

$$\text{map}_{\mathcal{O}^\otimes}^f(C, C') \longrightarrow \prod_{1 \leq i \leq n} \text{map}_{\mathcal{O}^\otimes}^{\rho^i \circ f}(C, C'_i)$$

is a homotopy equivalence.

- (3) For every finite collection of objects  $C_1, \dots, C_n$  of  $\mathcal{O}_{\langle 1 \rangle}^\otimes$ , there exists an object  $C$  of  $\mathcal{O}_{\langle n \rangle}^\otimes$  and  $p$ -coCartesian morphisms  $C \rightarrow C_i$  covering  $\rho^i$ ,  $1 \leq i \leq n$ .

This is the generalization of the notion of a multicategory (colored operad); to obtain the generalization of an operad we restrict to  $\infty$ -operads equipped with an essentially surjective functor  $\Delta^0 \rightarrow p^{-1}(\langle 1 \rangle)$ . To make sense of this, note that  $p^{-1}(\langle 1 \rangle)$  should be thought of as the “underlying”  $\infty$ -category associated to  $\mathcal{O}^\otimes$ , which should contain only a single (equivalence class of) object if we’re interested studying the  $\infty$ -version of an ordinary operad.

More precisely, given a multicategory  $\mathcal{A}$ , we can construct a category  $\tilde{\mathcal{A}}$  as follows: the objects are finite sets of objects in  $\mathcal{A}$ , and morphisms from  $\{X_1, \dots, X_m\} \rightarrow \{Y_1, \dots, Y_n\}$  are specified by maps  $\langle m \rangle \rightarrow \langle n \rangle$  in  $\text{Fin}_*$  and a collection of morphisms  $\{\phi_j \in \mathcal{A}(\{X_i\}_{i \in \alpha^{-1}(j)}, Y_j)\}_{1 \leq j \leq n}$ . Composition is determined by the composition laws in the multicategory. There is a natural functor  $\tilde{\mathcal{A}} \rightarrow \text{Fin}_*$ ; [Definition 3.2](#) is modeled on this structure. It is straightforward to check that this construction, the “category of operations”, is functorial.

More generally, given a simplicial multicategory, there is an analogue of this construction which yields a simplicial category [[36, 2.1.1.22](#)], and applying the homotopy coherent nerve to the resulting category yields an  $\infty$ -operad provided that each morphism simplicial set of the multicategory is a Kan complex; consult [[36, Proposition 2.1.1.27](#)] for further discussion or see [[30, §2](#)]. This construction is sometimes referred to as the *operadic nerve* of the simplicial multicategory.

We now turn to some examples of interest. The identity map  $N(\text{Fin}_*) \rightarrow N(\text{Fin}_*)$  is an  $\infty$ -operad; this is the analogue of the  $E_\infty$  operad. More generally, for each  $0 \leq n \leq \infty$ , we can define a topological category  $\tilde{\mathbb{E}}[n]$  as follows (see also [[36, Definition 5.1.0.2](#)]). The objects of  $\tilde{\mathbb{E}}[n]$  are the objects of  $\text{Fin}_*$ . Morphisms consist of maps  $\alpha : \langle n \rangle \rightarrow \langle m \rangle$  along with for each non-basepoint  $j \in \langle m \rangle$  disjoint rectilinear embeddings  $(0, 1)^n \rightarrow (0, 1)^m$  (i.e., maps given by component-wise linear maps) for each element of  $\alpha^{-1}(j)$ . Composition is defined in the usual fashion. Using the singular complex functor to get a simplicial category and applying the homotopy coherent nerve, there is a natural functor  $N(\tilde{\mathbb{E}}[n]) \rightarrow N(\text{Fin}_*)$  which is an  $\infty$ -operad [[36, 5.1.0.3](#)]. We refer to the resulting  $\infty$ -operad as the  $E_n$  operad.

Given an  $\infty$ -operad  $q : \mathcal{O}^\otimes \rightarrow N(\text{Fin}_*)$  and a coCartesian fibration  $p : \mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$  from an  $\infty$ -category  $\mathcal{C}^\otimes$ , we say that  $p : \mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$  is an  $\mathcal{O}$ -monoidal  $\infty$ -category if the composite  $q \circ p : \mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes \rightarrow N(\text{Fin}_*)$  is an  $\infty$ -operad. Such a map  $p$  is called a *coCartesian fibration of  $\infty$ -operads*. In particular, a *symmetric monoidal  $\infty$ -category* is an  $\infty$ -operad  $\mathcal{C}^\otimes$  such that the structure map is a coCartesian fibration of  $\infty$ -operads (see [[36, Example 2.1.2.18](#)])

$$p : \mathcal{C}^\otimes \longrightarrow N(\text{Fin}_*).$$

The underlying  $\infty$ -category is obtained as the fiber  $\mathcal{C} = p^{-1}(\langle 1 \rangle)$ . More generally, the fiber over  $\langle n \rangle$  is equivalence to the  $n$ -fold product of  $\mathcal{C}$ . In abuse of terminology, we will say that an  $\infty$ -category  $\mathcal{C}$  is a symmetric monoidal  $\infty$ -category if it is equivalent to  $p^{-1}(\langle 1 \rangle)$  for some symmetric monoidal  $\infty$ -category  $\mathcal{C}^\otimes$ .

Recall from [[36, Example 2.1.1.5](#)] that given a symmetric monoidal category  $\mathcal{C}$ , there is an associated multicategory in which the multihomomorphisms are given by the formula

$$\text{hom}((X_1, \dots, X_n), Y) = \text{hom}(X_1 \otimes \dots \otimes X_n, Y).$$

Associated to this multicategory we can construct a category  $\mathcal{C}^\otimes \rightarrow \text{Fin}_*$  over  $\text{Fin}_*$  (see [36, Construction 2.1.1.7]) such that the coherent nerve  $N(\mathcal{C}^\otimes) \rightarrow N(\text{Fin}_*)$  exhibits  $N(\mathcal{C}^\otimes)$  as a symmetric monoidal  $\infty$ -category; see [36, Examples 2.1.1.21 and 2.1.2.22].

When  $\mathcal{C}$  is a symmetric monoidal model category, we can also obtain a symmetric monoidal  $\infty$ -category using the coherent nerve. Specifically, there is a symmetric monoidal  $\infty$ -category  $(N(\mathcal{C}^\circ)[\mathcal{W}^{-1}])^\otimes$  with underlying  $\infty$ -category  $N(\mathcal{C}^\circ)[\mathcal{W}^{-1}]$ ; see [36, Proposition 4.1.3.4 and Example 4.1.3.6]. If  $\mathcal{C}$  is simplicial, then there is an equivalence of symmetric monoidal  $\infty$ -categories between  $(N(\mathcal{C}^\circ)[\mathcal{W}^{-1}])^\otimes$  and the operadic nerve  $N^\otimes(\mathcal{C}^\circ)$  (cf. [36, Definition 2.1.1.23]) of  $\mathcal{C}^\circ$ , the full subcategory of cofibrant–fibrant objects of  $\mathcal{C}$  by [36, Corollary 4.1.3.16].

An  $\infty$ -operad map

$$f : \mathcal{O}^\otimes \longrightarrow \mathcal{O}'^\otimes$$

is a map of simplicial sets  $f : \mathcal{O}^\otimes \rightarrow \mathcal{O}'^\otimes$  over  $N(\text{Fin}_*)$  such that  $f$  takes inert morphisms in  $\mathcal{O}^\otimes$  to inert morphisms in  $\mathcal{O}'^\otimes$ . The  $\infty$ -category of  $\infty$ -operad maps is written  $\text{Alg}_{\mathcal{O}}(\mathcal{O}')$  and is defined to be the full subcategory of  $\text{Fun}_{N(\text{Fin}_*)}(\mathcal{O}^\otimes, \mathcal{O}'^\otimes)$  spanned by the  $\infty$ -operad maps. (Here  $\text{Fun}_{N(\text{Fin}_*)}$  denotes maps over  $N(\text{Fin}_*)$ .)

More generally, we can work over a fixed  $\infty$ -operad  $\mathcal{O}^\otimes$ . If  $p : \mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$  is an  $\infty$ -operad map such that  $p$  is also a categorical fibration (a *fibration of  $\infty$ -operads*) and  $\alpha : \mathcal{O}'^\otimes \rightarrow \mathcal{O}^\otimes$  is an arbitrary  $\infty$ -operad map, then  $\text{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$  is the full subcategory of  $\text{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)$  spanned by the  $\infty$ -operad maps; see [36, Definitions 2.1.2.7 and 2.1.3.1]. An object of  $\text{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$  is referred to as an  $\mathcal{O}'$ -algebra object of  $\mathcal{C}$  over  $\mathcal{O}$ . Note that the  $\infty$ -category  $\text{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$  is the fiber of the projection  $\text{Alg}_{\mathcal{O}'}(\mathcal{C}) \rightarrow \text{Alg}_{\mathcal{O}'}(\mathcal{O})$  over  $\alpha$ . Also, when  $\alpha : \mathcal{O}'^\otimes \rightarrow \mathcal{O}^\otimes$  is the identity, we write  $\text{Alg}_{/\mathcal{O}}(\mathcal{C})$  in place of  $\text{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ .

For example, the  $\infty$ -category of commutative algebras in a symmetric monoidal  $\infty$ -category  $\mathcal{C}^\otimes$  is given by suitable sections of the coCartesian fibration  $\mathcal{C}^\otimes \rightarrow N(\text{Fin}_*)$ . The data of such a section at  $\langle n \rangle$  is  $n$  copies of the underlying object (the section evaluated at  $\langle 1 \rangle$ ), and the lifting condition for  $p$ -coCartesian edges provides the multiplications. See [28, §4.2] for a nice discussion in detail.

**Definition 3.3.** Let  $\mathcal{O}^\otimes$  be an  $\infty$ -operad with a single object and let  $\mathcal{C}$  be an  $\mathcal{O}$ -monoidal  $\infty$ -category. There is a natural map from  $\text{Alg}_{/\mathcal{O}}(\mathcal{C}) \rightarrow \mathcal{C}$  induced by the image of the object in  $\mathcal{O}$ . Then the *space of  $\mathcal{O}$ -algebra structures* on an object  $X$ , denoted  $\text{Alg}_{/\mathcal{O}}(X)$ , is the largest Kan complex contained in the full subcategory of  $\text{Alg}_{/\mathcal{O}}(\mathcal{C})$  spanned by objects which project to  $X$  under this map.

Given an  $\infty$ -operad  $\mathcal{O}^\otimes$  and  $\mathcal{O}$ -monoidal  $\infty$ -categories  $p : \mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$  and  $q : \mathcal{D}^\otimes \rightarrow \mathcal{O}^\otimes$ , we have two associated categories of functors between them:

- (1) The  $\infty$ -category of  $\infty$ -operad maps  $\text{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$ , which should be thought of as the analogue of lax  $\mathcal{O}$ -monoidal functors; consult [36, Definition 2.1.2.7] for details.

- (2) The full subcategory  $\text{Fun}_{\mathcal{O}}^{\otimes}(\mathcal{C}, \mathcal{D})$  of  $\text{Alg}_{\mathcal{C}}(\mathcal{D})$  consisting of the  $\infty$ -operad maps over  $\mathcal{O}^{\otimes}$  which carry  $p$ -coCartesian morphisms to  $q$ -coCartesian morphisms, which should be regarded as the analogue of symmetric monoidal functors; consult [36, Definition 2.1.3.7] for details.

We now explain how to produce symmetric monoidal functors of  $\infty$ -categories from point-set data. We rely on the foundational work of [24].

**Lemma 3.4.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be symmetric monoidal categories and  $f : \mathcal{C} \rightarrow \mathcal{D}$  a lax symmetric monoidal functor. Then there is an induced  $\infty$ -operad map*

$$N(f) : N(\mathcal{C})^{\otimes} \longrightarrow N(\mathcal{D})^{\otimes},$$

*making  $N(-)^{\otimes}$  into a functor from the (nerve of the) category of symmetric monoidal categories and lax symmetric monoidal functors to the  $\infty$ -category of  $\infty$ -operads.*

**Proof.** Associated to any symmetric monoidal category is an underlying multicategory, and the work of [24, §3], particularly the proof of Theorem 1.1, implies that a lax map  $f : \mathcal{C} \rightarrow \mathcal{D}$  induces a map of multicategories. It is clear from [36, Remark 2.1.1.7] that a map of multicategories induces a functor  $f^{\otimes} : \mathcal{C}^{\otimes} \rightarrow \mathcal{D}^{\otimes}$  over the natural projections to  $\text{Fin}_*$ . Since  $f^{\otimes}$  came from a map of multicategories, on passage to the coherent nerve  $f^{\otimes}$  takes inert morphisms to inert morphisms and so induces a map of  $\infty$ -operads. The behavior of composites follows immediately from the functoriality of the passage from lax maps to multicategory maps.  $\square$

Next, we integrate this with the localization  $N(\mathcal{C})[W^{-1}]$ .

**Proposition 3.5.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be symmetric monoidal categories equipped with subcategories  $W_{\mathcal{C}} \subset \mathcal{C}$  and  $W_{\mathcal{D}} \subset \mathcal{D}$  of weak equivalences such that both collections of weak equivalences are closed under tensor with any fixed object. Let  $f : \mathcal{C} \rightarrow \mathcal{D}$  be a lax symmetric monoidal functor such that  $f$  preserves weak equivalences and  $f$  is weakly symmetric monoidal in the sense that the maps  $1_{\mathcal{D}} \rightarrow f(1_{\mathcal{C}})$  and  $f(c_1) \otimes f(c_2) \rightarrow f(c_1 \otimes c_2)$ , for all pairs of objects  $c_1$  and  $c_2$  of  $\mathcal{C}$ , are in  $W_{\mathcal{D}}$ . Then*

- (1) *there are symmetric monoidal  $\infty$ -categories  $N(\mathcal{C})[W_{\mathcal{C}}^{-1}]^{\otimes}$  and  $N(\mathcal{D})[W_{\mathcal{D}}^{-1}]^{\otimes}$  with underlying  $\infty$ -categories  $N(\mathcal{C})[W_{\mathcal{C}}^{-1}]$  and  $N(\mathcal{D})[W_{\mathcal{D}}^{-1}]$  respectively, and*
- (2) *there is a map of  $\infty$ -operads*

$$\tilde{f} : N(\mathcal{C})[W_{\mathcal{C}}^{-1}]^{\otimes} \longrightarrow N(\mathcal{D})[W_{\mathcal{D}}^{-1}]^{\otimes}$$

*such that on underlying  $\infty$ -categories  $\tilde{f}$  restricts to the functor induced by  $f$  via Lemma 3.4.*

**Proof.** By Lemma 3.4, we have a map

$$N(\mathcal{C})^\otimes \xrightarrow{N(f)} N(\mathcal{D})^\otimes,$$

and since by hypothesis the localization  $N(\mathcal{D}^\otimes) \rightarrow N(\mathcal{D}^\otimes)[W_{\mathcal{D}}^{-1}]$  is symmetric monoidal (it satisfies the criterion of [36, Proposition 4.1.3.4]), we have a lax symmetric monoidal functor

$$N(\mathcal{C})^\otimes \xrightarrow{N(f)} N(\mathcal{D})[W_{\mathcal{D}}^{-1}]^\otimes.$$

Furthermore,  $N(f)$  lies in the subcategory

$$\text{Fun}^\otimes(N(\mathcal{C}), N(\mathcal{D})[W_{\mathcal{D}}^{-1}]) \subseteq \text{Alg}_{N(\mathcal{C})}(N(\mathcal{D})[W_{\mathcal{D}}^{-1}])$$

of symmetric monoidal functors since each of the comparison maps (for each active map  $\mu : \langle n \rangle \rightarrow \langle m \rangle$  in  $\Gamma$ )

$$\mu!(F(\mathcal{A}_1, \dots, \mathcal{A}_n)) \longrightarrow F(\mu!(\mathcal{A}_1, \dots, \mathcal{A}_n))$$

is a weak equivalence in  $\mathcal{D}$ , and hence becomes an equivalence in  $N(\mathcal{D})[W_{\mathcal{D}}^{-1}]$  [36, Definition 1.3.4.1]. Finally, since  $f$  takes elements of  $W_{\mathcal{C}}$  to elements of  $W_{\mathcal{D}}$ , the result follows from [36, Proposition 4.1.3.4], which we summarize in the following commutative diagram:

$$\begin{array}{ccc} N(\mathcal{C})^\otimes & \xrightarrow{N(f)} & N(\mathcal{D})^\otimes \\ \downarrow & & \downarrow \\ N(\mathcal{C})[W_{\mathcal{C}}^{-1}]^\otimes & \xrightarrow{N(\bar{f})} & N(\mathcal{D})[W_{\mathcal{D}}^{-1}]^\otimes. \end{array} \quad \square$$

We also use a comparison between point-set  $E_\infty$  algebras in a symmetric monoidal simplicial model category with  $E_\infty$  algebras in the underlying  $\infty$ -category. We thank Jacob Lurie for suggesting this argument. (See also [30, §6.2] for progress towards this kind of result in the context of arbitrary multicategories rather than just operads.)

**Proposition 3.6.** *Let  $\mathcal{O}$  be a cofibrant simplicial  $E_\infty$  operad and  $\mathcal{C}$  a symmetric monoidal simplicial model category. Then there is an equivalence of  $\infty$ -categories*

$$N(\text{Alg}_{\mathcal{O}}(\mathcal{C})) [W^{-1}] \simeq \text{CAlg}(N(\mathcal{C})[W^{-1}]^\otimes).$$

**Proof.** First, there is an equivalence of symmetric monoidal  $\infty$ -categories between  $N(\mathcal{C})[W^{-1}]^\otimes$  and  $N(\mathcal{C}^\circ)^\otimes$ , where  $\mathcal{C}^\circ$  denotes the cofibrant–fibrant objects in  $\mathcal{C}$  [36,

**Variant 4.1.3.17.** By the functoriality of the operadic homotopy coherent nerve, we have a canonical map

$$N(\text{Alg}_{\mathcal{O}}(\mathcal{C}^\circ)) \rightarrow \text{Alg}_{N(\mathcal{O})^\otimes}(N(\mathcal{C}^\circ)^\otimes) \simeq \text{Alg}_{N(\mathcal{O})^\otimes}(N(\mathcal{C})[W^{-1}]^\otimes).$$

Since this map clearly preserves weak equivalences, it factors as a map

$$\gamma : N(\text{Alg}_{\mathcal{O}}(\mathcal{C}^\circ))[W^{-1}] \rightarrow \text{Alg}_{N(\mathcal{O})^\otimes}(N(\mathcal{C})[W^{-1}]^\otimes).$$

Since  $\mathcal{O}$  is cofibrant, the results of [48, §4] produce a  $J$ -semi model category structure on the category  $\text{Alg}_{\mathcal{O}}(\mathcal{C})$ . The proof of [36, Theorem 1.3.4.20] goes through in this context to show that the map

$$N(\text{Alg}_{\mathcal{O}}(\mathcal{C}^\circ))[W^{-1}] \rightarrow N(\text{Alg}_{\mathcal{O}}(\mathcal{C}))[W^{-1}]$$

is an equivalence of  $\infty$ -categories.

Finally, we follow the strategy of the proof of [36, Theorem 4.4.4.7] to prove that  $\gamma$  is an equivalence of  $\infty$ -categories. We have the commutative diagram

$$\begin{array}{ccc} N(\text{Alg}_{\mathcal{O}}(\mathcal{C}^\circ))[W^{-1}] & \xrightarrow{\gamma} & \text{Alg}_{N(\mathcal{O})^\otimes}(N(\mathcal{C})[W^{-1}]^\otimes) \\ & \searrow G & \swarrow G \\ & N(\mathcal{C})[W^{-1}] & \end{array}$$

and we will apply the  $\infty$ -categorical Barr–Beck theorem via [36, Corollary 6.2.2.14]. The verification of the required hypotheses proceeds exactly as in [36, 4.4.4.7] except for two conditions. First, we need to check that  $G : N(\text{Alg}_{\mathcal{O}}(\mathcal{C}^\circ))[W^{-1}] \rightarrow N(\mathcal{C})[W^{-1}]$  preserves geometric realizations of simplicial objects. Second, we need to show that the free  $\mathcal{O}$ -algebra functor  $\mathcal{C} \rightarrow \text{Alg}_{\mathcal{O}}(\mathcal{C}) \rightarrow \mathcal{C}$  on a cofibrant–fibrant object  $X$  is computed by the homotopy colimit  $\coprod_n (\mathcal{O}(n) \times X^{\otimes n})_{h\Sigma_n}$ . Both of these can be verified using the  $J$ -semi model structure on  $\mathcal{O}$ -algebras in  $\mathcal{C}$  and the fact that  $\mathcal{O}$  is a cofibrant  $E_\infty$  operad. For the first, the arguments for [25, §VII.3] apply since the free  $\mathcal{O}$ -algebra functor commutes with geometric realizations in  $\mathcal{C}$ . For the second, this is an immediate consequence of the fact that  $\mathcal{O}$  is a cofibrant  $E_\infty$  operad and so the derived functor of the free  $\mathcal{O}$ -algebra functor is a homotopy colimit of the desired form.  $\square$

We conclude this section with some remarks about the behavior of the unit object in monoidal  $\infty$ -categories. Let  $\mathbf{1}$  denote the unit object in an  $E_n$  monoidal  $\infty$ -category. Our main results rely on the following analogues of the standard fact that the unit is initial in a monoidal category; these  $\infty$ -categorical versions follow from [36, Proposition 3.2.1.8] (e.g., see [36, Corollary 3.2.1.9]).

**Lemma 3.7.** *Let  $\mathcal{C}^{\otimes}$  be an  $E_n$  monoidal  $\infty$ -category,  $n \geq 0$ . Then the  $\infty$ -category  $\text{Alg}_{/E_n}(\mathcal{C})$  has an initial object, and an object  $A$  of  $\text{Alg}_{/E_n}(\mathcal{C})$  is initial if and only if the unit map  $\mathbf{1} \rightarrow A$  is an equivalence in  $\mathcal{C}$ .*

**Corollary 3.8.** *Let  $\mathcal{C}^{\otimes}$  be an  $E_n$  monoidal  $\infty$ -category,  $n \geq 0$ . Then, the  $\infty$ -category  $\text{Alg}_{/E_n}(\mathbf{1})$  of  $E_n$  algebra structures on the unit object  $\mathbf{1}$  of  $\mathcal{C}$  is contractible and, for any other  $E_n$  algebra  $A$  of  $\mathcal{C}$ , the space of  $E_n$  algebra maps from  $\mathbf{1}$  to  $A$  is contractible.*

#### 4. Multiplicative Morita theory

The  $\infty$ -category  $\text{Cat}_{\infty}^{\text{perf}}$  of idempotent-complete small stable  $\infty$ -categories has a symmetric monoidal structure with product  $\otimes^{\vee}$  and the unit the  $\infty$ -category  $\mathcal{S}_{\infty}^{\omega}$  of compact objects in  $\mathcal{S}_{\infty}$  [36, §6.3.1]. Recall from [11, Thms. 4.22 and 4.23] that we have a description of  $\text{Cat}_{\infty}^{\text{perf}}$  as the accessible localization of the  $\infty$ -category  $\text{N}(\text{Cat}_{\mathcal{S}})[W^{-1}]$  along the Morita equivalences. The goal of this section is to promote this equivalence to an equivalence of symmetric monoidal  $\infty$ -categories, using the smash product of spectral categories; see Theorem 4.6.

The category  $\text{Cat}_{\mathcal{S}}$  has a closed symmetric monoidal product given by taking  $(\mathcal{C}, \mathcal{D})$  to the spectral category with objects  $\text{ob } \mathcal{C} \times \text{ob } \mathcal{D}$  and morphism spectra  $\mathcal{C}(c, c') \wedge \mathcal{D}(d, d')$ . However, the smash product of cofibrant spectral categories is not necessarily cofibrant, and consequently the model structure is not monoidal [45] (and see [51] for a discussion of this in the setting of DG-categories). This issue is one of the persistent technical difficulties in working with these models of  $\text{Cat}_{\infty}^{\text{perf}}$ .

To resolve this problem, we employ the notion of flat objects and functors (e.g., see [31, B.4]). Recall that a functor between model categories is flat if it preserves weak equivalences and colimits. An object  $X$  of a model category (whose underlying category is monoidal with respect to a tensor product  $\otimes$ ) is then said to be flat if the functor  $X \otimes (-)$  is a flat functor. Cofibrant objects in a monoidal model category are flat; in particular, cofibrant spectra are flat. The utility of this definition comes from the fact that the smash product of flat spectra computes the derived smash product.

We define a spectral category  $\mathcal{C}$  to be pointwise-cofibrant if each morphism spectrum  $\mathcal{C}(x, y)$  is a cofibrant spectrum. The following proposition summarizes the facts about pointwise-cofibrant spectral categories that we will need. Recall that a spectral category  $\mathcal{C}$  has an associated spectral category of perfect modules (equivalently, homotopically compact modules, or retracts of finite cell modules), and that a map of spectral categories  $f : \mathcal{C} \rightarrow \mathcal{D}$  is a *Morita equivalence* if  $f$  induces a DK-equivalence between spectral categories of perfect modules; see [11, §2] for details.

**Proposition 4.1.**

- (1) *Every spectral category is functorially Morita equivalent to a pointwise-cofibrant spectral category with the same objects.*

- (2) *The subcategory of pointwise-cofibrant spectral categories is closed under the smash product.*
- (3) *A pointwise-cofibrant spectral category is flat with respect to the smash product of spectral categories.*
- (4) *If  $\mathcal{C}$  and  $\mathcal{D}$  are pointwise-cofibrant spectral categories, the smash product  $\mathcal{C} \wedge \mathcal{D}$  computes the derived smash product  $\mathcal{C} \wedge^{\mathbb{L}} \mathcal{D}$ .*

**Proof.** By construction of the generating cofibrations (see [45, Def. 4.4]), there exists a cofibrant resolution functor  $Q(-)$  on  $\text{Cat}_{\mathcal{S}}$  such that for any spectral category  $\mathcal{C}$  the spectral functor  $Q(\mathcal{C}) \rightarrow \mathcal{C}$  induces the identity map on the set of objects. Item (1) follows then from [45, Prop. 4.18], which shows that every cofibrant spectral category is pointwise-cofibrant. Item (2) follows from the fact that the smash product of cofibrant spectra remains cofibrant. Item (4) follows from item (3).

Let  $\mathcal{C}$  be a pointwise-cofibrant spectral category. The functor  $- \wedge \mathcal{C} : \text{Cat}_{\mathcal{S}} \rightarrow \text{Cat}_{\mathcal{S}}$  clearly preserves colimits as the symmetric monoidal structure is closed. Hence, in order to prove item (3), it remains to show that if  $f : \mathcal{A} \rightarrow \mathcal{B}$  is a Morita equivalence (see [44, Def. 6.1]), then  $f \wedge \text{id} : \mathcal{A} \wedge \mathcal{C} \rightarrow \mathcal{B} \wedge \mathcal{C}$  is also a Morita equivalence.

If  $f : \mathcal{A} \rightarrow \mathcal{B}$  is a Morita equivalence then the induced map  $f_! : \text{Mod}(\mathcal{A}) \rightarrow \text{Mod}(\mathcal{B})$  is a DK-equivalence of spectral categories. Let  $\mathcal{C}$  be a pointwise-flat spectral category. We must show that

$$(f \wedge \text{id})_! : \text{Mod}(\mathcal{A} \wedge \mathcal{C}) \longrightarrow \text{Mod}(\mathcal{B} \wedge \mathcal{C})$$

is also a DK-equivalence, which is to say that it is homotopically fully faithful and essentially surjective. We begin with the former. Since any cofibrant  $\mathcal{A} \wedge \mathcal{C}$ -module is a retract of a cellular  $\mathcal{A} \wedge \mathcal{C}$ -module and  $(f \wedge \text{id})_!$  is a left Quillen functor which preserves representable modules, it suffices to check that  $f \wedge \text{id}$  itself is homotopically fully faithful. But this follows because  $\mathcal{C}$  is pointwise-cofibrant, so

$$(f \wedge \text{id})((a', c'), (a, c)) : (\mathcal{A} \wedge \mathcal{C})((a', c'), (a, c)) \longrightarrow (\mathcal{B} \wedge \mathcal{C})((fa', c'), (fa, c))$$

is a weak equivalence of spectra for any pair of objects  $(a, c)$  and  $(a', c')$  of  $\mathcal{A} \wedge \mathcal{C}$ . To verify essential surjectivity, it suffices to check that the  $\mathcal{B} \wedge \mathcal{C}$ -module  $\widehat{(b, c)}$  represented by the object  $(b, c) \in \mathcal{B} \wedge \mathcal{C}$  is equivalent to the image of an  $\mathcal{A} \wedge \mathcal{C}$ -module. Since  $f$  is a Morita equivalence, there exists a perfect  $\mathcal{A}$ -module  $M$  and a weak equivalence of  $\mathcal{B}$ -modules  $f_!M \simeq \widehat{b}$ . Thus

$$(f \wedge \text{id})_!(M \wedge \widehat{c})(b', c') \simeq M(b') \wedge \mathcal{C}(c', c) \simeq \mathcal{B}(b', b) \wedge \mathcal{C}(c', c) \simeq (f \wedge \text{id})_!(\widehat{(b, c)})(b', c'),$$

so that  $(f \wedge \text{id})_!$  is also homotopically essentially surjective.  $\square$

**Remark 4.2.** The use of pointwise-cofibrant spectral categories is analogous to the use of homotopically flat DG-modules in the differential graded setting. See for instance [15] for a similar development to the work of this section in that context.

Therefore, we use the subcategory  $\text{Cat}_{\mathcal{S}}^{\text{flat}}$  of pointwise-cofibrant spectral categories to produce a suitable symmetric monoidal model of the  $\infty$ -category of idempotent-complete small stable  $\infty$ -categories. The following lemma is a first consistency check:

**Lemma 4.3.** *Let  $\text{Cat}_{\mathcal{S}}^{\text{c}}$  denote the full subcategory of cofibrant objects in  $\text{Cat}_{\mathcal{S}}$ . The functor induced by cofibrant replacement  $\text{Cat}_{\mathcal{S}}^{\text{flat}} \rightarrow \text{Cat}_{\mathcal{S}}^{\text{c}}$  induces a categorical equivalence*

$$\mathbf{N}(\text{Cat}_{\mathcal{S}}^{\text{flat}})[\mathcal{W}^{-1}] \longrightarrow \mathbf{N}(\text{Cat}_{\mathcal{S}}^{\text{c}})[\mathcal{W}^{-1}].$$

**Proof.** Recall from the proof of Proposition 4.1 that the cofibrant replacement of a spectral category is pointwise-cofibrant. It is now clear that the inclusion and the functorial cofibrant replacement induce inverse equivalences.  $\square$

By combining Proposition 3.1 with [11, 4.22, 4.23] we obtain:

**Proposition 4.4.** *There is an equivalence of  $\infty$ -categories*

$$\text{Cat}_{\infty}^{\text{perf}} \simeq \mathbf{N}(\text{Cat}_{\mathcal{S}}^{\text{flat}})[\mathcal{W}^{-1}].$$

Furthermore, Proposition 4.1 implies that  $\text{Cat}_{\mathcal{S}}^{\text{flat}}$  is a symmetric monoidal category such that the smash product preserves equivalences. Thus we conclude from [36, Proposition 4.1.3.4 and Example 4.1.3.6] that we obtain a symmetric monoidal  $\infty$ -category  $(\mathbf{N}(\text{Cat}_{\mathcal{S}}^{\text{flat}})[\mathcal{W}^{-1}])^{\otimes}$  with underlying  $\infty$ -category  $\mathbf{N}(\text{Cat}_{\mathcal{S}}^{\text{flat}})[\mathcal{W}^{-1}]$ . We now upgrade the comparison of [11, 4.22, 4.23] to a comparison of symmetric monoidal  $\infty$ -categories.

To explain how to do this, we need to review some details about the construction of the symmetric monoidal structure on  $\text{Cat}_{\infty}^{\text{perf}}$ . Recall that this is determined by the symmetric monoidal structure on the  $\infty$ -category  $\text{Pr}_{\text{st}}^{\text{L}}$ , as follows. First, there is an equivalence between the  $\infty$ -category  $\text{Pr}_{\text{st},\omega}^{\text{L}}$  of compactly-generated stable presentable  $\infty$ -categories and idempotent-complete small stable  $\infty$ -categories which is realized by passage to compact objects (denoted  $(-)^{\omega}$ ) and the formation  $\text{Ind}(-)$  of the Ind-category [35, §5.5.7]. Next,  $\text{Pr}_{\text{st},\omega}^{\text{L}}$  inherits a symmetric monoidal structure from the structure on  $\text{Pr}_{\text{st}}^{\text{L}}$  [36, 6.3.7.11]. Unwinding this, for  $\mathcal{C}$  and  $\mathcal{D}$  in  $\text{Cat}_{\infty}^{\text{perf}}$ , the tensor product  $\mathcal{C} \otimes^{\vee} \mathcal{D}$  can be computed as  $(\text{Ind}(\mathcal{C}) \otimes \text{Ind}(\mathcal{D}))^{\omega}$ .

In turn, the symmetric monoidal structure on  $\text{Pr}_{\text{st}}^{\text{L}}$  can be obtained as a symmetric monoidal localization of the  $\infty$ -category  $\text{Pr}^{\text{L}}$  of presentable  $\infty$ -categories, as we now describe. The symmetric monoidal structure on  $\text{Pr}^{\text{L}}$  is induced from the monoidal structure on the  $\infty$ -category of  $\infty$ -categories [36, 6.3.1.14], and in  $\text{Pr}^{\text{L}}$  the  $\infty$ -category of spectra  $\mathcal{S}_{\infty}$  is an idempotent object [36, 6.3.2.18]. Therefore, the functor  $- \otimes \mathcal{S}_{\infty}$  in  $\text{Pr}^{\text{L}}$  defines

a symmetric monoidal localization on  $\text{Pr}^{\text{L}}$  such that the full subcategory of local objects is precisely  $\text{Pr}_{\text{st}}^{\text{L}}$  [36, 6.3.2.19].

Translating back to small  $\infty$ -categories, the symmetric monoidal structure on  $\text{Pr}^{\text{L}}$  is determined by the Cartesian monoidal structure on  $\text{Cat}_{\infty}$ . Furthermore, we can characterize this structure in terms of the category  $\text{Cat}_{\Delta}$  of small simplicial categories. Since  $\text{Cat}_{\Delta}$  has all homotopy limits, the associated  $\infty$ -category  $\text{N}(\text{Cat}_{\Delta})[W^{-1}]$  admits a (necessarily unique) Cartesian symmetric monoidal structure which we denote  $\text{N}(\text{Cat}_{\Delta})[W^{-1}]^{\otimes}$ ; see [36, Corollary 2.4.1.9]. The uniqueness implies that there is an equivalence of symmetric monoidal  $\infty$ -categories

$$\text{N}(\text{Cat}_{\Delta})[W^{-1}]^{\otimes} \simeq \text{Cat}_{\infty}^{\times}$$

in which  $\text{Cat}_{\infty}$  is endowed with the Cartesian symmetric monoidal structure.

Recall that  $\mathcal{S}_{\infty}$  is an idempotent commutative algebra object of  $\text{Pr}^{\text{L}}$  and that the presheaf functor  $\text{Pre} : \text{Cat}_{\infty} \rightarrow \text{Pr}_{\omega}^{\text{L}}$  (specifically, the covariant version in which the functor  $\text{Pre}(\mathcal{C}) \rightarrow \text{Pre}(\mathcal{D})$  induced by a functor  $\mathcal{C} \rightarrow \mathcal{D}$  is left adjoint to the restriction  $\text{Pre}(\mathcal{D}) \rightarrow \text{Pre}(\mathcal{C})$ ) is symmetric monoidal. We therefore obtain a symmetric monoidal functor

$$\text{Pre}_{\mathcal{S}_{\infty}} \simeq \text{Pre}(-) \otimes \mathcal{S}_{\infty} : \text{Cat}_{\infty} \longrightarrow \text{Pr}_{\text{st},\omega}^{\text{L}},$$

the functor which sends the small  $\infty$ -category  $\mathcal{C}$  to the compactly generated  $\infty$ -category of spectral presheaves on  $\mathcal{C}$ . Composing with the symmetric monoidal equivalence  $(-)^{\omega} : \text{Pr}_{\text{st},\omega}^{\text{L}} \rightarrow \text{Cat}_{\infty}^{\text{perf}}$ , we obtain a symmetric monoidal functor

$$\text{Pre}_{\mathcal{S}_{\infty}}(-)^{\omega} : \text{Cat}_{\infty} \longrightarrow \text{Cat}_{\infty}^{\text{perf}}.$$

**Lemma 4.5.** *There is a unique commutative algebra structure  $\text{Cat}_{\infty}^{\text{perf}\otimes}$  on  $\text{Cat}_{\infty}^{\text{perf}}$  in  $\text{Pr}^{\text{L}}$  such that*

$$\text{Pre}_{\mathcal{S}_{\infty}}(-)^{\omega} : \text{Cat}_{\infty} \longrightarrow \text{Cat}_{\infty}^{\text{perf}}$$

*extends to a map of commutative algebras  $\text{Cat}_{\infty}^{\times} \rightarrow \text{Cat}_{\infty}^{\text{perf}\otimes}$ .*

**Proof.** Recall that a commutative algebra in  $\text{Pr}^{\text{L}}$  is a symmetric monoidal  $\infty$ -category such that tensoring with a fixed object preserves colimits. As a consequence, it suffices to show that the functor  $\text{Pre}_{\mathcal{S}_{\infty}}(-)^{\omega}$  uniquely determines the tensor products of a collection of objects that generate  $\text{Cat}_{\infty}^{\text{perf}}$  under colimits.

As reviewed in Section 2,  $\text{Cat}_{\infty}^{\text{perf}}$  is equivalent to the  $\infty$ -category produced by taking the underlying  $\infty$ -category associated to the model category of spectral categories with weak equivalences the DK-equivalences and Bousfield localizing at the Morita equivalences. This model category is cofibrantly generated, with generating cofibrations and

acyclic cofibrations (described explicitly in [4, §2.16]) that have mapping spectra determined by the generating acyclic cofibrations and acyclic cofibrations of the category of symmetric spectra. In particular, the generating cofibrations and acyclic cofibrations have mapping spectra that are suspension spectra and therefore are in the image of  $\Sigma_+^\infty$  applied to the category of simplicial categories.

Finally, since any object in the Bousfield localization is weakly equivalent to a cellular object (generated as a filtered colimit of pushouts along generating cofibrations) in the DK-equivalence model structure, we see that any object of  $\text{Cat}_\infty^{\text{perf}}$  is weakly equivalent to a colimit of objects in the image of  $\text{Pre}_{\mathcal{S}_\infty}(-)^\omega$ . The result now follows from the fact that  $\text{Pre}_{\mathcal{S}_\infty}(-)^\omega$  is strong symmetric monoidal.  $\square$

We finally have all the tools needed for the proof of the multiplicative Morita theory result.

**Theorem 4.6** (Multiplicative Morita theory). *There is an equivalence of symmetric monoidal  $\infty$ -categories*

$$(\text{Cat}_\infty^{\text{perf}})^\otimes \simeq (\text{N}(\text{Cat}_\mathcal{S}^{\text{flat}})[\mathcal{W}^{-1}])^\otimes.$$

**Proof.** Consider the composite

$$\Phi : \text{Cat}_\Delta^f \xrightarrow{(-)_+} (\text{Cat}_\Delta^f)_* \xrightarrow{\Sigma_+^\infty} \text{Cat}_\mathcal{S}^{\text{flat}},$$

where  $\text{Cat}_\Delta^f$  denotes the full subcategory of fibrant simplicial categories and  $(\text{Cat}_\Delta)_*$  denotes the pointed simplicial categories (which means that all the mapping complexes have basepoints). The functor  $\Phi$  satisfies the requirements of Proposition 3.5, and so we have an induced map of symmetric monoidal  $\infty$ -categories

$$\text{Cat}_\infty^\times \simeq \text{N}(\text{Cat}_\Delta^f)[\mathcal{W}^{-1}]^\times \xrightarrow{\Sigma_+^\infty} \text{N}(\text{Cat}_\mathcal{S}^{\text{flat}})[\widetilde{\mathcal{W}}^{-1}]^\otimes,$$

where here  $\widetilde{\mathcal{W}}$  denotes the class of DK-equivalences of spectral categories. Next, by Proposition 4.1, composing with the symmetric monoidal localization at the Morita equivalences of spectral categories gives rise to a map of symmetric monoidal  $\infty$ -categories

$$\theta : \text{Cat}_\infty^\times \xrightarrow{\Sigma_+^\infty} \text{N}(\text{Cat}_\mathcal{S}^{\text{flat}})[\widetilde{\mathcal{W}}^{-1}]^\otimes \longrightarrow (\text{N}(\text{Cat}_\mathcal{S}^{\text{flat}})[\mathcal{W}^{-1}])^\otimes.$$

(Recall that we know from [11, 4.23] that this localization is in fact the Bousfield localization at a set of generating Morita equivalences.) By Proposition 4.4, we have that  $\text{N}(\text{Cat}_\mathcal{S}^{\text{flat}})[\mathcal{W}^{-1}]$  is equivalent to  $\text{Cat}_\infty^{\text{perf}}$ . Therefore, by the discussion preceding the theorem, in order to identify the symmetric monoidal structure on  $\text{N}(\text{Cat}_\mathcal{S}^{\text{flat}})[\mathcal{W}^{-1}]$  as a model of  $(\text{Cat}_\infty^{\text{perf}})^\otimes$ , it suffices to identify the composite  $\theta$  as  $\text{Pre}_{\mathcal{S}_\infty}(-)^\omega$ .

The comparison of [11, 4.22, 4.23] identifies the functor

$$\Psi : \text{Cat}_{\mathcal{M}}^{\text{flat}} \longrightarrow \text{Cat}_{\mathcal{M}}^{\text{flat}},$$

that takes a small pointwise-cofibrant spectral category  $\mathcal{C}$  to the cofibrant–fibrant homotopically compact objects in the projective model structure on  $\text{Mod}(\mathcal{C})$ , as the localization at the Morita equivalences. Therefore, ignoring the symmetric monoidal structure, the underlying functor of  $\theta$  can be described as the composite of  $\Phi$ , the equivalence between  $\text{Cat}_{\mathcal{S}}$  and  $\text{Cat}_{\mathcal{M}}$ , and  $\Psi$ . But now the identification of  $\theta$  is clear, since

$$N(\Psi(\Sigma_+^\infty \mathcal{C})) \simeq N(\text{Mod}(\Sigma_+^\infty \mathcal{C}))^\omega \simeq N(\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S}_\infty))^\omega$$

is precisely a model of  $\text{Fun}(N(\mathcal{C})^{\text{op}}, \mathcal{S}_\infty)^\omega \simeq \text{Pre}_{\mathcal{S}_\infty}(N(\mathcal{C}))^\omega$ .  $\square$

One immediate consequence of the preceding comparison result is that we can explicitly describe the mapping spectra in the tensor product of small stable  $\infty$ -categories in terms of the smash product of spectra. Specifically, let  $\mathcal{C}$  and  $\mathcal{D}$  be small stable idempotent complete  $\infty$ -categories and  $\tilde{\mathcal{C}}$  and  $\tilde{\mathcal{D}}$  cofibrant–fibrant pre-triangulated spectral categories lifting  $\mathcal{C}$  and  $\mathcal{D}$ . Then there is a natural equivalence

$$(\mathcal{C} \otimes^\vee \mathcal{D})((c, d), (c', d')) \simeq \tilde{\mathcal{C}}(c, c') \wedge \tilde{\mathcal{D}}(d, d').$$

### 5. The symmetric monoidal structure on noncommutative motives

In this section, we show that  $\mathcal{M}_{\text{add}}$  and  $\mathcal{M}_{\text{loc}}$  are symmetric monoidal  $\infty$ -categories and that the localization functors

$$\mathcal{U}_{\text{add}} : \text{Cat}_\infty^{\text{perf}} \longrightarrow \mathcal{M}_{\text{add}} \quad \text{and} \quad \mathcal{U}_{\text{loc}} : \text{Cat}_\infty^{\text{perf}} \longrightarrow \mathcal{M}_{\text{loc}}$$

are symmetric monoidal. This is an interesting result in its own right; for instance, it implies that  $\mathcal{M}_{\text{add}}$  is canonically enriched in  $A(*) = K(\mathbb{S})$ -modules (see [Corollary 5.18](#)). Herein, we use these results to compare symmetric monoidal structures on the  $\infty$ -categories of colimit-preserving functors from  $\mathcal{M}_{\text{add}}$  to  $\mathcal{S}_\infty$  and additive functors from  $\text{Cat}_\infty^{\text{perf}}$  to  $\mathcal{S}_\infty$  (and analogously in the localizing case).

Our approach is motivated by the following classical picture. If  $R$  is a spectrum, then we may view the associated cohomology theory as a (pre)sheaf of spectra  $\mathcal{T}^{\text{op}} \rightarrow \mathcal{S}_\infty$  on the  $\infty$ -category of spaces. If  $R$  has an  $E_\infty$  structure, then this functor is canonically lax symmetric monoidal via the external cup product pairing

$$\begin{array}{ccc} \text{Map}(\Sigma_+^\infty X, R) \wedge \text{Map}(\Sigma_+^\infty Y, R) & \longrightarrow & \text{Map}(\Sigma_+^\infty (X \times Y), R \wedge R) \\ & & \downarrow \\ & & \text{Map}(\Sigma_+^\infty (X \times Y), R). \end{array}$$

In fact, the lax symmetric monoidal structure on  $\text{Map}(\Sigma_+^\infty(-), R)$  is equivalent to a symmetric monoidal structure on  $R$  itself, as the latter may be recovered by restricting to the point. Here, we are studying the analogous picture in the setting of noncommutative motives.

We begin by fixing some notation. We will refer to commutative algebra objects of  $\text{Pr}_{\text{st}}^{\text{L}}$  as *stable presentable symmetric monoidal  $\infty$ -categories* and denote by  $\text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$  the  $\infty$ -category on these objects and morphisms the colimit-preserving symmetric monoidal functors. Note that this specifies a full subcategory of the more general class of (large) symmetric monoidal  $\infty$ -categories: an object of the latter is an object of the former if and only if the category is stable, presentable, and the monoidal product commutes with colimits in each variable; see [36, §6.3.2]. Also, recall that  $\text{Cat}_\infty^{\text{perf}}$  is itself presentable and that the monoidal product commutes with colimits in each variable.

Now, recall that  $\mathcal{U}_{\text{add}}$  is defined to be the composite of the Yoneda embedding

$$\phi : \text{Cat}_\infty^{\text{perf}} \longrightarrow \text{Pre}((\text{Cat}_\infty^{\text{perf}})^\omega)$$

(where the presheaves are restricted to the full subcategory  $(\text{Cat}_\infty^{\text{perf}})^\omega$  of  $\text{Cat}_\infty^{\text{perf}}$  consisting of the compact objects), followed by stabilization and then localization at a generating set  $\mathcal{E}$  of split-exact sequences. The functor  $\mathcal{U}_{\text{loc}}$  is defined analogously, with localization taken with respect to a generating set of all exact sequences. As such, our investigation of the symmetric monoidal structure involves assembling the analysis of each piece of the composite.

**Yoneda embedding and stabilization.** First, we observe that the symmetric monoidal structure on  $\text{Cat}_\infty^{\text{perf}}$  descends to the subcategory of compact objects. For this, we need a technical proposition.

**Proposition 5.1.** *Let  $\mathcal{A}$  be a compact object of  $\text{Cat}_\infty^{\text{perf}}$  and  $\{\mathcal{B}_i\}_{i \in I}$  a filtered diagram in  $\text{Cat}_\infty^{\text{perf}}$ . Then the map*

$$\text{colim}_{i \in I} \text{Fun}^{\text{ex}}(\mathcal{A}, \mathcal{B}_i) \longrightarrow \text{Fun}^{\text{ex}}(\mathcal{A}, \text{colim}_{i \in I} \mathcal{B}_i)$$

*is an equivalence of small stable idempotent-complete  $\infty$ -categories.*

**Proof.** The inclusion  $\text{Fun}^{\text{ex}}(\mathcal{A}, \mathcal{B}) \rightarrow \text{Fun}^{\text{ex}}(\mathcal{A} \otimes^\vee \mathcal{B}^{\text{op}}, \mathcal{S}_\infty)$  is the full subcategory on those  $f : \mathcal{A} \otimes^\vee \mathcal{B}^{\text{op}} \rightarrow \mathcal{S}_\infty$  which restrict to representable functors for each  $a$  in  $\mathcal{A}$ . This gives a commuting square

$$\begin{array}{ccc} \text{colim} \text{Fun}^{\text{ex}}(\mathcal{A}, \mathcal{B}_i) & \longrightarrow & \text{colim} \text{Fun}^{\text{ex}}(\mathcal{A} \otimes^\vee \mathcal{B}_i^{\text{op}}, \mathcal{S}_\infty), \\ \downarrow & & \downarrow \\ \text{Fun}^{\text{ex}}(\mathcal{A}, \text{colim} \mathcal{B}_i) & \longrightarrow & \text{Fun}^{\text{ex}}(\mathcal{A} \otimes^\vee \text{colim} \mathcal{B}_i^{\text{op}}, \mathcal{S}_\infty) \end{array}$$

where we use the fact that  $(\operatorname{colim} B_i)^{\operatorname{op}} \simeq \operatorname{colim} B_i^{\operatorname{op}}$ . The horizontal maps are fully faithful inclusions and the vertical map on the right is an equivalence since  $\operatorname{Fun}^{\operatorname{ex}}(\mathcal{A} \otimes^{\vee} (-)^{\operatorname{op}}, \mathcal{S}_{\infty})$  preserves filtered colimits. It follows that the vertical map on the left is fully faithful. Restricting to maximal subgroupoids (i.e., the mapping spaces), we see that for  $A$  compact the left-hand vertical map induces an equivalence (by definition), so it is also essentially surjective.  $\square$

This allows us to prove the following result.

**Proposition 5.2.** *The  $\infty$ -category of compact objects  $(\operatorname{Cat}_{\infty}^{\operatorname{perf}})^{\omega}$  in  $\operatorname{Cat}_{\infty}^{\operatorname{perf}}$  is a full symmetric monoidal subcategory.*

**Proof.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be compact small stable idempotent-complete  $\infty$ -categories. We must show that  $\mathcal{A} \otimes^{\vee} \mathcal{B}$  is compact. To this end, let  $\mathcal{C}_i$  be a filtered system of small stable idempotent-complete  $\infty$ -categories. By Proposition 5.1, we obtain the following sequence of equivalences

$$\begin{aligned} \operatorname{map}(\mathcal{A} \otimes^{\vee} \mathcal{B}, \operatorname{colim}_{i \in I} \mathcal{C}_i) &\simeq \operatorname{map}(\mathcal{A}, \operatorname{Fun}^{\operatorname{ex}}(\mathcal{B}, \operatorname{colim}_{i \in I} \mathcal{C}_i)) \\ &\simeq \operatorname{map}(\mathcal{A}, \operatorname{colim}_{i \in I} \operatorname{Fun}^{\operatorname{ex}}(\mathcal{B}, \mathcal{C}_i)) \\ &\simeq \operatorname{colim}_{i \in I} \operatorname{map}(\mathcal{A}, \operatorname{Fun}^{\operatorname{ex}}(\mathcal{B}, \mathcal{C}_i)) \simeq \operatorname{map}(\mathcal{A} \otimes^{\vee} \mathcal{B}, \operatorname{colim}_{i \in I} \mathcal{C}_i), \end{aligned}$$

which implies that  $\operatorname{map}(\mathcal{A} \otimes^{\vee} \mathcal{B}, -)$  commutes with filtered colimits. (Here we are denoting by  $\operatorname{map}(-, -)$  the derived simplicial mapping space.)  $\square$

Next, we use the fact that passage to presheaves and stabilization are both symmetric monoidal functors.

**Proposition 5.3.** *The stable presentable  $\infty$ -category*

$$\operatorname{Stab}(\operatorname{Pre}(\operatorname{Cat}_{\infty}^{\operatorname{perf}})^{\omega}) \simeq \operatorname{Fun}(((\operatorname{Cat}_{\infty}^{\operatorname{perf}})^{\omega})^{\operatorname{op}}, \mathcal{S}_{\infty})$$

*has a canonical presentable symmetric monoidal structure. This structure is compatible with the symmetric monoidal structure on  $\operatorname{Cat}_{\infty}^{\operatorname{perf}}$  in the sense that the functor*

$$\operatorname{Cat}_{\infty}^{\operatorname{perf}} \longrightarrow \operatorname{Stab}(\operatorname{Pre}(\operatorname{Cat}_{\infty}^{\operatorname{perf}})^{\omega})$$

*is symmetric monoidal.*

**Proof.** The  $\infty$ -category  $\operatorname{Pre}(\mathcal{C})$  admits a symmetric monoidal structure such that the Yoneda embedding  $\mathcal{C} \rightarrow \operatorname{Pre}(\mathcal{C})$  is a symmetric monoidal functor; see [36, Corollary 6.3.1.12]. Furthermore, the  $\infty$ -category  $\operatorname{Stab}(\mathcal{C})$  admits a symmetric monoidal structure such that the stabilization functor  $\mathcal{C} \rightarrow \operatorname{Stab}(\mathcal{C})$  is symmetric monoidal; see

[36, Example 6.3.1.22 and Proposition 6.3.2.18]. It follows that  $\text{Stab}(\text{Pre}(\text{Cat}_\infty^{\text{perf}})^\omega)$  is a stable presentable symmetric monoidal  $\infty$ -category. Finally, the functor

$$\text{Cat}_\infty^{\text{perf}} \longrightarrow \text{Stab}(\text{Pre}(\text{Cat}_\infty^{\text{perf}})^\omega)$$

is symmetric monoidal since we know that  $\text{Cat}_\infty^{\text{perf}}$  is compactly generated; it is generated under filtered colimits by  $(\text{Cat}_\infty^{\text{perf}})^\omega$  [11, 3.22] (and  $\text{Ind}(\mathcal{C})$  is a symmetric monoidal subcategory of  $\text{Pre}(\mathcal{C})$  [36, Proposition 6.3.1.10]).  $\square$

**Localization at a generating set.** Finally, in order to show that  $\mathcal{M}_{\text{add}}$  and  $\mathcal{M}_{\text{loc}}$  are symmetric monoidal, it will suffice to show that the localization at  $\mathcal{E}$  inherits the structure of a symmetric monoidal  $\infty$ -category. As a consequence of the argument, we will also show that the localization functor is symmetric monoidal. Recall that  $\mathcal{E}$  consists of the maps

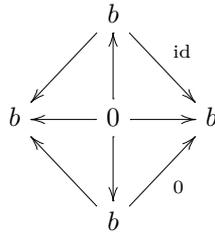
$$\widehat{\mathcal{B}}/\widehat{\mathcal{A}} \longrightarrow \widehat{\mathcal{C}}$$

in  $\text{Stab}(\text{Pre}(\text{Cat}_\infty^{\text{perf}})^\omega)$  associated to a generating set of split-exact sequences  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$  in  $(\text{Cat}_\infty^{\text{perf}})^\omega$ , where here  $\widehat{\mathcal{B}}/\widehat{\mathcal{A}}$  denotes the cofiber [11, §5].

Provided that we can show that the localization functor is compatible with the symmetric monoidal structure (in the sense of [36, Definition 2.2.1.6]), then [36, Proposition 2.2.1.9] will establish that the localization is symmetric monoidal. Recall that to show compatibility, by [36, Example 2.2.1.7] it suffices to show that for every local equivalence  $X \rightarrow Y$  and any object  $Z$ , the induced map  $X \otimes Z \rightarrow Y \otimes Z$  is a local equivalence. To do this, we characterize exact sequences of  $\infty$ -categories in terms of acyclics. (See also [7, §7] for similar identifications in the language of spectral categories.) Recall that an exact sequence is in particular a cofiber sequence [11, 5.9].

**Lemma 5.4.** *Let  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$  be a cofiber sequence of idempotent-complete small stable  $\infty$ -categories such that  $\mathcal{A} \rightarrow \mathcal{B}$  is fully faithful. Then  $\mathcal{A}$  is canonically equivalent to the fiber (over 0) of  $\mathcal{B} \rightarrow \mathcal{C}$ , i.e., the full subcategory of  $\mathcal{B}$  on the objects which are equivalent to  $*$  in  $\mathcal{C}$ .*

**Proof.** Write  $\mathcal{A}'$  for the fiber of  $\mathcal{B} \rightarrow \mathcal{C}$  and let  $\mathcal{A} \rightarrow \mathcal{A}'$  be the resulting map, which is necessarily fully faithful. To check that it is essentially surjective, it suffices to check on triangulated homotopy categories. Let  $b$  be an object of  $\mathcal{A}'$ . Recall that we can identify  $\text{Ho}(\mathcal{C}) \simeq \text{Ho}(\mathcal{B}/\mathcal{A})$  as the Verdier quotient of triangulated categories  $\text{Ho}(\mathcal{B})/\text{Ho}(\mathcal{A})$  [11, 5.13]. By the description of maps in the Verdier quotient, we can easily check that the identity map  $b \rightarrow b$  must be equal to the zero map  $b \rightarrow b$ . That is, there is a commutative diagram



which implies that the cofiber  $b$  of  $0 \rightarrow b$  must lie in  $\mathcal{A}$ .  $\square$

**Lemma 5.5.** *Let  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$  be an exact sequence of idempotent-complete small stable  $\infty$ -categories. Then, for any idempotent-complete small stable  $\infty$ -category  $\mathcal{D}$ , the sequence*

$$\mathcal{D} \otimes \mathcal{A} \longrightarrow \mathcal{D} \otimes \mathcal{B} \longrightarrow \mathcal{D} \otimes \mathcal{C}$$

is exact.

**Proof.** Since the tensor product on  $\text{Cat}_\infty^{\text{perf}}$  commutes with colimits, it is enough to show that  $\mathcal{D} \otimes \mathcal{A}$  is equivalent to the full subcategory  $\mathcal{F}$  of  $\mathcal{D} \otimes \mathcal{B}$  consisting of those objects which are sent to zero objects in  $\mathcal{D} \otimes \mathcal{C}$ . We first show that the functor  $\mathcal{D} \otimes \mathcal{A} \rightarrow \mathcal{D} \otimes \mathcal{B}$  is fully faithful by direct computation of the mapping spectra: given a pair of objects  $d, d'$  of  $\mathcal{D}$  and  $a, a'$  of  $\mathcal{A}$  with images  $b, b'$  in  $\mathcal{B}$ , we have that

$$\begin{aligned} \text{Map}((d, a), (d', a')) &\simeq \text{Map}(d, d') \wedge \text{Map}(a, a') \\ &\simeq \text{Map}(d, d') \wedge \text{Map}(b, b') \simeq \text{Map}((d, b), (d', b')). \end{aligned}$$

Since  $\mathcal{D} \otimes \mathcal{A}$  is stable, it follows that the inclusion  $\mathcal{D} \otimes \mathcal{A} \rightarrow \mathcal{D} \otimes \mathcal{B}$  is fully faithful on all objects which are retracts of finite colimits of objects of the form  $(d, a)$ ; since all objects of  $\mathcal{D} \otimes \mathcal{A}$  are of this form, we see that the inclusion is fully faithful. The fact that  $\mathcal{D} \otimes \mathcal{A}$  surjects onto  $\mathcal{F}$  now follows from [Lemma 5.4](#).  $\square$

**Proposition 5.6.** *The localization of  $\text{Stab}(\text{Pre}(\text{Cat}_\infty^{\text{perf}})^\omega)$  at  $\mathcal{E}$  is compatible with the symmetric monoidal structure given above.*

**Proof.** Since the  $\infty$ -category  $\text{Stab}(\text{Pre}(\text{Cat}_\infty^{\text{perf}})^\omega)$  is generated under filtered colimits by representables and these tensors commute with filtered colimits, it suffices to check that

$$\text{map}(\widehat{\mathcal{D}} \otimes (\widehat{\mathcal{C}}/(\widehat{\mathcal{B}}/\widehat{\mathcal{A}})), F) \simeq *$$

for all  $\mathcal{D}$  and split-exact sequences  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$  in  $\mathcal{E}$ . This follows because

$$\widehat{\mathcal{D}} \otimes (\widehat{\mathcal{C}}/(\widehat{\mathcal{B}}/\widehat{\mathcal{A}})) \simeq \widehat{\mathcal{D} \otimes \mathcal{C}}/(\widehat{\mathcal{D} \otimes \mathcal{B}}/\widehat{\mathcal{D} \otimes \mathcal{A}}),$$

and, by Lemma 5.5,  $\mathcal{D} \otimes \mathcal{A} \rightarrow \mathcal{D} \otimes \mathcal{B} \rightarrow \mathcal{D} \otimes \mathcal{C}$  is exact when  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$  is exact, and therefore split-exact when  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$  is split-exact.  $\square$

Similarly, we have the following result for localizing invariants:

**Proposition 5.7.** *The localization of  $\text{Stab}(\text{Pre}(\text{Cat}_\infty^{\text{perf}})^\omega)$  at the set generated by those objects of the form  $\widehat{\mathcal{C}}/(\widehat{\mathcal{B}}/\widehat{\mathcal{A}})$  for each (equivalence class of) exact sequence  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$  with  $\mathcal{B}$   $\kappa$ -compact is compatible with the symmetric monoidal structure given above.*

**Proof.** As in the proof of 5.6, it is enough to check that  $\text{map}(\widehat{\mathcal{D}} \otimes (\widehat{\mathcal{C}}/(\widehat{\mathcal{B}}/\widehat{\mathcal{A}})), F) \simeq *$  for all  $\mathcal{D}$  and exact sequences  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$  with  $\mathcal{B}$   $\kappa$ -compact. The result follows because

$$\widehat{\mathcal{D}} \otimes (\widehat{\mathcal{C}}/(\widehat{\mathcal{B}}/\widehat{\mathcal{A}})) \simeq \widehat{\mathcal{D} \otimes \mathcal{C}}/(\widehat{\mathcal{D} \otimes \mathcal{B}}/\widehat{\mathcal{D} \otimes \mathcal{A}}),$$

and  $\mathcal{D} \otimes \mathcal{A} \rightarrow \mathcal{D} \otimes \mathcal{B} \rightarrow \mathcal{D} \otimes \mathcal{C}$  is exact since  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$  is exact by Lemma 5.5.  $\square$

Summarizing, we have the following theorem:

**Theorem 5.8.** *The  $\infty$ -categories  $\mathcal{M}_{\text{add}}$  and  $\mathcal{M}_{\text{loc}}$  are endowed with natural symmetric monoidal structures making the functors  $\mathcal{U}_{\text{add}}$  and  $\mathcal{U}_{\text{loc}}$  symmetric monoidal. The tensor units are  $\mathcal{U}_{\text{add}}(\mathcal{S}_\infty^\omega)$  and  $\mathcal{U}_{\text{loc}}(\mathcal{S}_\infty^\omega)$  respectively.*

In particular, this implies that algebraic  $K$ -theory is a lax symmetric monoidal functor.

**Proposition 5.9.** *The functors*

$$K(-) = \text{Map}(\mathcal{U}_{\text{add}}(\mathcal{S}_\infty^\omega), \mathcal{U}_{\text{add}}(-)) : \text{Cat}_\infty^{\text{perf}} \longrightarrow \mathcal{S}_\infty$$

and

$$\mathbb{K}(-) = \text{Map}(\mathcal{U}_{\text{loc}}(\mathcal{S}_\infty^\omega), \mathcal{U}_{\text{loc}}(-)) : \text{Cat}_\infty^{\text{perf}} \longrightarrow \mathcal{S}_\infty$$

are lax symmetric monoidal.

**Proof.** We give the argument for  $K(-)$ ; the proof for  $\mathbb{K}(-)$  is the same. Our work in [11, §4] gave a construction for any stable  $\infty$ -category of the mapping spectrum functor  $\text{Map}_{\mathcal{C}}(X, -)$ . Since  $\mathcal{U}_{\text{add}}$  is lax symmetric monoidal, it will suffice to show that  $\text{Map}(\mathcal{U}_{\text{add}}(\mathcal{S}_\infty^\omega), -)$  is lax symmetric monoidal. First, observe that this functor preserves limits, and so by the adjoint functor theorem [35, 5.5.2.9] it has a left adjoint. The left adjoint can be described as follows. Since  $\mathcal{M}_{\text{add}}$  is a presentable stable  $\infty$ -category, it is tensored over  $\mathcal{S}_\infty$  (see [36, Remark 6.3.2.17]) in the sense of [36, Definition 4.2.1.19]. Therefore, the left adjoint is given by  $\mathcal{U}_{\text{add}}(\mathcal{S}_\infty^\omega) \otimes (-)$ . By definition, this left adjoint is symmetric monoidal. Then, [36, Corollary 8.3.2.7] implies (as in [36, Example 8.3.2.8]) that there exists a lax symmetric monoidal right adjoint extending  $\text{Map}(\mathcal{U}_{\text{add}}(\mathcal{S}_\infty^\omega), -)$ .  $\square$

This result in turn has the following corollary:

**Corollary 5.10.** *Let  $\mathcal{A}$  be an  $E_n$  object in the symmetric monoidal  $\infty$ -category  $\text{Cat}_\infty^{\text{perf}}$ ,  $0 \leq n \leq \infty$ . Then  $K(\mathcal{A})$  and  $\mathbb{K}(\mathcal{A})$  are  $E_n$  ring spectra.*

Since an  $E_n$  ring spectrum has an  $\infty$ -category of compact modules which is an  $E_{n-1}$  object in  $\text{Cat}_\infty^{\text{perf}}$  (see [36, Theorem 5.1.4.2]), we can specialize Corollary 5.10 to conclude that algebraic  $K$ -theory takes  $E_n$  ring spectra to  $E_{n-1}$  ring spectra.

**Corollary 5.11.** *Let  $R$  be an  $E_n$  ring spectrum. Then  $K(R)$  and  $\mathbb{K}(R)$  are  $E_{n-1}$  ring spectra.*

**Symmetric monoidal structure on additive and localizing invariants.** We now obtain “dual” symmetric monoidal structures on the functor  $\infty$ -categories of additive and localizing invariants.

**Proposition 5.12.** *Let  $\mathcal{C}$  be a small symmetric monoidal  $\infty$ -category. Then there is a natural equivalence*

$$\text{Fun}^L(\text{Stab}(\text{Pre}(\mathcal{C})), \mathcal{S}_\infty) \simeq \text{Stab}(\text{Pre}(\mathcal{C}^{\text{op}}))$$

*between the dual of  $\text{Stab}(\text{Pre}(\mathcal{C}))$ , the stable presentable symmetric monoidal  $\infty$ -category generated by  $\mathcal{C}$ , and the stable presentable symmetric monoidal  $\infty$ -category generated by  $\mathcal{C}^{\text{op}}$ .*

**Proof.** This follows from a calculation: If  $\text{Stab}(\text{Pre}(\mathcal{C})) = \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S}_\infty)$  is the presentable stable  $\infty$ -category freely generated by the small  $\infty$ -category  $\mathcal{C}$ , then

$$\text{Fun}^L(\text{Stab}(\text{Pre}(\mathcal{C})), \mathcal{S}_\infty) \simeq \text{Fun}^L(\text{Pre}(\mathcal{C}), \mathcal{S}_\infty) \simeq \text{Fun}(\mathcal{C}, \mathcal{S}_\infty) \simeq \text{Stab}(\text{Pre}(\mathcal{C}^{\text{op}}))$$

is a presentable stable  $\infty$ -category which is dual to  $\text{Stab}(\text{Pre}(\mathcal{C}))$  under the symmetric monoidal structure on  $\text{Pr}_{\text{st}}^L$ . (See [11, §3.3] for further discussion.)  $\square$

In particular, we have the following:

**Corollary 5.13.** *Let  $\mathcal{C}$  be a small symmetric monoidal  $\infty$ -category. Then the  $\infty$ -category  $\text{Fun}^L(\text{Stab}(\text{Pre}(\mathcal{C})), \mathcal{S}_\infty)$  is symmetric monoidal.*

**Proof.** If  $\mathcal{C}$  is a small symmetric monoidal  $\infty$ -category, then  $\mathcal{C}^{\text{op}}$  is also a small symmetric monoidal  $\infty$ -category. Therefore, [36, Corollary 6.3.1.12] shows that the Day convolution product endows  $\text{Pre}(\mathcal{C}^{\text{op}})$  with a canonical symmetric monoidal structure. Since stabilization is a symmetric monoidal functor, this implies that  $\text{Stab}(\text{Pre}(\mathcal{C}^{\text{op}}))$  is also symmetric monoidal.  $\square$

**Proposition 5.6** and **Corollary 5.13** now imply the  $\infty$ -categories  $\text{Fun}^L(\mathcal{M}_{\text{add}}, \mathcal{S}_\infty)$  and  $\text{Fun}^L(\mathcal{M}_{\text{loc}}, \mathcal{S}_\infty)$  are symmetric monoidal. Analogous results for the categories of additive and localizing invariants hold:

**Theorem 5.14.** *The  $\infty$ -categories  $\text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)$  and  $\text{Fun}_{\text{loc}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)$  are symmetric monoidal  $\infty$ -categories. The units are the connective and non-connective algebraic K-theory functors  $K(-)$  and  $\mathbb{K}(-)$ .*

**Proof.** The argument for **Corollary 5.13** implies that, for any infinite regular cardinal  $\kappa$ , the  $\infty$ -category of functors  $\text{Fun}((\text{Cat}_\infty^{\text{perf}})^\kappa, \mathcal{S}_\infty)$  is a symmetric monoidal  $\infty$ -category (again with respect to the convolution tensor product). It is straightforward to check (again using the criterion of [36, Proposition 2.2.1.6]) that this induces symmetric monoidal structures on the subcategories  $\text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)$  and  $\text{Fun}_{\text{loc}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)$ . Specifically, the analogues of the arguments for **Proposition 5.6** and **Proposition 5.7** hold with  $((\text{Cat}_\infty^{\text{perf}})^\kappa)^{\text{op}}$  in place of  $(\text{Cat}_\infty^{\text{perf}})^\kappa$ .  $\square$

Moreover, we have the following comparison result:

**Theorem 5.15.** *The functor  $\mathcal{U}_{\text{add}}$  induces a symmetric monoidal equivalence*

$$\text{Fun}^L(\mathcal{M}_{\text{add}}, \mathcal{S}_\infty)^\otimes \xrightarrow{\simeq} \text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)^\otimes.$$

*In particular,  $\mathcal{U}_{\text{add}}$  induces an equivalence of  $E_\infty$ -algebras in the two symmetric monoidal  $\infty$ -categories. Analogously, the functor  $\mathcal{U}_{\text{loc}}$  induces a symmetric monoidal equivalence*

$$\text{Fun}^L(\mathcal{M}_{\text{loc}}, \mathcal{S}_\infty)^\otimes \xrightarrow{\simeq} \text{Fun}_{\text{loc}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)^\otimes.$$

**Proof.** Tracing through the constructions of the symmetric monoidal structures and using the fact that  $\mathcal{U}_{\text{add}}$  is symmetric monoidal, we see that it induces a symmetric monoidal functor

$$\text{Fun}^L(\mathcal{M}_{\text{add}}, \mathcal{S}_\infty)^\otimes \longrightarrow \text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)^\otimes.$$

Since we know it induces an equivalence on the underlying categories, this implies it induces an equivalence of symmetric monoidal  $\infty$ -categories; see [36, Remark 2.1.3.8]. The argument for the localizing case is analogous.  $\square$

Using recent work of Glasman [27], we can rephrase the preceding result to obtain the following relation between  $E_\infty$ -algebras and lax symmetric monoidal functors. Specifically, the main result of [27] establishes that there is an equivalence

$$\text{Alg}_{/E_\infty}(\text{Fun}(\mathcal{C}, \mathcal{S}_\infty)^\otimes) \simeq \text{Fun}^{\text{lax}}(\mathcal{C}, \mathcal{S}_\infty),$$

for any small  $\infty$ -category  $\mathcal{C}$ , where the functor category is given the convolution symmetric monoidal structure. Combining this equivalence with the preceding result, we have the following:

**Corollary 5.16.** *There are equivalences of  $\infty$ -categories*

$$\begin{aligned} \text{Alg}_{/E_\infty}(\text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)) &\xrightarrow{\sim} \text{Fun}_{\text{add}}^{\text{lax}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty) \\ \text{Alg}_{/E_\infty}(\text{Fun}_{\text{loc}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)) &\xrightarrow{\sim} \text{Fun}_{\text{loc}}^{\text{lax}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty) \end{aligned}$$

and

$$\begin{aligned} \text{Alg}_{/E_\infty}(\text{Fun}^{\text{L}}(\mathcal{M}_{\text{add}}, \mathcal{D})) &\xrightarrow{\sim} \text{Fun}^{\text{L,lax}}(\mathcal{M}_{\text{add}}, \mathcal{D}) \\ \text{Alg}_{/E_\infty}(\text{Fun}^{\text{L}}(\mathcal{M}_{\text{loc}}, \mathcal{D})) &\xrightarrow{\sim} \text{Fun}^{\text{L,lax}}(\mathcal{M}_{\text{loc}}, \mathcal{D}) \end{aligned}$$

Corollary 5.16 and Theorem 5.15 leads to the following comparison.

**Theorem 5.17.** *For any presentable symmetric monoidal  $\infty$ -category  $\mathcal{D}$ , there are equivalences of  $\infty$ -categories*

$$\begin{aligned} (\mathcal{U}_{\text{add}})^* : \text{Fun}^{\text{L,lax}}(\mathcal{M}_{\text{add}}, \mathcal{D}) &\xrightarrow{\sim} \text{Fun}_{\text{add}}^{\text{lax}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{D}) \\ (\mathcal{U}_{\text{loc}})^* : \text{Fun}^{\text{L,lax}}(\mathcal{M}_{\text{loc}}, \mathcal{D}) &\xrightarrow{\sim} \text{Fun}_{\text{loc}}^{\text{lax}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{D}), \end{aligned}$$

where the left-hand sides denote the  $\infty$ -category of lax symmetric monoidal colimit-preserving functors and the right-hand sides denote the  $\infty$ -categories of lax symmetric monoidal additive or localizing invariants, respectively.

**The localizing subcategory generated by the unit.** One interesting application of the symmetric monoidal structures on  $\mathcal{M}_{\text{add}}$  and  $\mathcal{M}_{\text{loc}}$  is the fact that these imply these categories are enriched over the endomorphisms of the unit; i.e., algebraic  $K$ -theory spectra of  $\mathbb{S}$ . Specifically, the following result follows from Theorem 5.8 and from the equivalences (1.3).

**Corollary 5.18.** *The symmetric monoidal homotopy categories  $\text{Ho}(\mathcal{M}_{\text{add}})$  and  $\text{Ho}(\mathcal{M}_{\text{loc}})$  are enriched over the homotopy category  $\text{Ho}(A(*)\text{-Mod})$  of  $A(*)$ -modules.*

In particular, if  $\mathcal{A}$  and  $\mathcal{B}$  are small stable  $\infty$ -categories, then the mapping spectrum  $\text{Map}(\mathcal{U}_{\text{add}}(\mathcal{B}), \mathcal{U}_{\text{add}}(\mathcal{A}))$  is a module over

$$\text{Map}(\mathcal{U}_{\text{add}}(\mathcal{S}_\infty^\omega), \mathcal{U}_{\text{add}}(\mathcal{S}_\infty^\omega)) \simeq K(\mathbb{S}) = A(*).$$

Similarly, the mapping spectrum  $\text{Map}(\mathcal{U}_{\text{loc}}(\mathcal{B}), \mathcal{U}_{\text{loc}}(\mathcal{A}))$  is a module over

$$\text{Map}(\mathcal{U}_{\text{loc}}(\mathcal{S}_\infty^\omega), \mathcal{U}_{\text{loc}}(\mathcal{S}_\infty^\omega)) \simeq \mathbb{K}(\mathbb{S}) \simeq A(*).$$

**Remark 5.19.** In fact, it is possible to promote the enrichments of  $\text{Ho}(\mathcal{M}_{\text{add}})$  and  $\text{Ho}(\mathcal{M}_{\text{loc}})$  over  $\text{Ho}(A(*)\text{-Mod})$  to enrichments of  $\mathcal{M}_{\text{add}}$  and  $\mathcal{M}_{\text{loc}}$  over the  $\infty$ -category  $A(*)\text{-Mod}$  of  $A(*)$ -modules. This can be done directly, using the formalism of  $\infty$ -operads as in [36, Definition 4.2.1.28]. Briefly, the action of  $A(*)$  on the mapping spectra is given as follows. The endomorphism spectrum  $\text{End}(\mathbf{1})$  of the unit is an  $A_\infty$ -ring spectrum (even  $E_\infty$ ). For objects  $X$  and  $Y$ , the mapping spectrum  $F(X \otimes \mathbf{1}, Y)$  is equivalent to  $F(X, Y)$  and has an action of  $\text{End}(\mathbf{1})$  via the composite map

$$\text{End}(\mathbf{1}) \longrightarrow \text{End}(X \otimes \mathbf{1}) \longrightarrow F(X \otimes \mathbf{1}, Y).$$

In any symmetric monoidal stable  $\infty$ -category  $\mathcal{C}$ , we can consider the smallest stable subcategory  $\text{Loc}_{\mathcal{C}}(\mathbf{1})$  generated by the unit object  $\mathbf{1}$  which is closed under (not necessarily finite) direct sum; this is a lift of the localizing subcategory of the homotopy category generated by the image of the unit. If  $\mathcal{C}$  is generated by the unit, then this subcategory is actually all of  $\mathcal{C}$ ; in general, it is smaller.

Let  $F_{\mathcal{C}}(-, -)$  denote the mapping spectrum in  $\mathcal{C}$ . The endomorphism spectrum  $\text{End}_{\mathcal{C}}(\mathbf{1}) = F_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$  is a commutative ring spectrum, and there is a functor

$$F_{\mathcal{C}}(\mathbf{1}, -) : \mathcal{C} \longrightarrow \text{End}_{\mathcal{C}}(\mathbf{1})\text{-Mod}.$$

When  $\mathbf{1}$  is a compact object in  $\mathcal{C}$ , this functor induces an equivalence between the category of modules over  $\text{End}_{\mathcal{C}}(\mathbf{1})$  and  $\text{Loc}_{\mathcal{C}}(\mathbf{1})$  (see [22] for a nice discussion of this kind of “generalized Morita theory”). Moreover, there is an induced equivalence

$$F_{\mathcal{C}}(X, Y) \xrightarrow{\sim} F_{\text{End}_{\mathcal{C}}(\mathbf{1})}(F_{\mathcal{C}}(\mathbf{1}, X), F_{\mathcal{C}}(\mathbf{1}, Y))$$

for every  $X \in \text{Loc}_{\mathcal{C}}(\mathbf{1})$  and  $Y \in \mathcal{C}$ . One consequence of this is the following Ext spectral sequence (e.g., see [25, 4.1]):

**Corollary 5.20.** *Given objects  $X_1$  and  $X_2$  in  $\text{Loc}_{\mathcal{C}}(\mathbf{1})$ , we have a convergent spectral sequence*

$$\begin{aligned} E_2^{p,q} &= \text{Ext}_{\pi_{-*} \text{End}_{\mathcal{C}}(\mathbf{1})}^{p,q}(\pi_{-*} F_{\mathcal{C}}(\mathbf{1}, X_1), \pi_{-*} F_{\mathcal{C}}(\mathbf{1}, X_2)) \\ &\Rightarrow \pi_{-p-q} F_{\text{End}_{\mathcal{C}}(\mathbf{1})}(F_{\mathcal{C}}(\mathbf{1}, X_1), F_{\mathcal{C}}(\mathbf{1}, X_2)) \end{aligned}$$

and we can interpret both the  $E_2$  term and the target in terms of maps in  $\mathcal{C}$ .

Since  $\mathbb{K}(\mathbb{S}) \simeq K(\mathbb{S}) = A(*)$ , in our setting these localizing subcategories can be identified with the  $\infty$ -category of  $A(*)$ -modules.

### 6. THH and TC as multiplicative theories

In this section, we discuss a model for  $THH$  which is a multiplicative localizing invariant, i.e., an object of the  $\infty$ -category  $\text{Alg}_{/E_\infty}(\text{Fun}^L(\mathcal{M}_{\text{loc}}, \mathcal{S}_\infty)^\otimes)$ . We also explain

how to interpret the construction of  $TC$  in this context;  $TC$  itself is not an additive invariant as it does not preserve filtered colimits, but constituents of  $TC$  do yield additive invariants.

More generally, we explain the passage from lax symmetric monoidal functors from small spectral categories to spectra, such that the lax comparison maps are weak equivalences of spectra, to objects in  $\text{Alg}_{/E_\infty}(\text{Fun}((\text{Cat}_\infty^{\text{perf}})^\omega, \mathcal{S}_\infty)^\otimes)$ ; see [Theorem 6.3](#). If, in addition, the underlying functor is additive or localizing, then we shall see [Section 7](#) that these lax symmetric monoidal spectral-valued functors will admit unique multiplicative natural transformations of additive functors from  $K$  or localizing functors from  $\mathbb{K}$ . We will be especially interested in the case of  $THH$  and  $TC$ , which we will show admit concrete models as lax symmetric monoidal functors from small spectral categories to spectra.

We begin by producing a particular model of the convolution product on the  $\infty$ -category  $\text{Fun}((\text{Cat}_\infty^{\text{perf}})^\omega, \mathcal{S}_\infty)$ . Let  $\text{Cat}_\mathcal{S}^{\text{cpt}}$  denote a small full subcategory of  $\text{Cat}_\mathcal{S}$  which contains a representative of each weak equivalence class of compact objects, consists of pointwise-cofibrant objects, is closed under smash product, and is closed under the functorial cofibrant–fibrant replacement. Let  $\text{Pre}_\mathcal{S}((\text{Cat}_\mathcal{S}^{\text{cpt}})^{\text{op}})$  denote the category of presheaves of symmetric spectra (a.k.a. spectral presheaves) on  $(\text{Cat}_\mathcal{S}^{\text{cpt}})^{\text{op}}$ . This has a symmetric monoidal product given by the Day convolution.

**Proposition 6.1.** *With respect to the projective model structure and the convolution symmetric monoidal structure,  $\text{Pre}_\mathcal{S}((\text{Cat}_\mathcal{S}^{\text{cpt}})^{\text{op}})$  is a symmetric monoidal simplicial model category, as is the Bousfield localization of  $\text{Pre}_\mathcal{S}((\text{Cat}_\mathcal{S}^{\text{cpt}})^{\text{op}})$  at the Morita equivalences in  $(\text{Cat}_\mathcal{S}^{\text{cpt}})^{\text{op}}$ , which we denote by  $\text{Pre}_\mathcal{S}^{\text{Mor}}((\text{Cat}_\mathcal{S}^{\text{cpt}})^{\text{op}})$ . Moreover, the symmetric monoidal simplicial model structure on  $\text{Pre}_\mathcal{S}^{\text{Mor}}((\text{Cat}_\mathcal{S}^{\text{cpt}})^{\text{op}})$  induces an equivalence of symmetric monoidal  $\infty$ -categories*

$$(\text{N}(\text{Pre}_\mathcal{S}^{\text{Mor}}((\text{Cat}_\mathcal{S}^{\text{cpt}})^{\text{op}}))[\mathcal{W}^{-1}])^\otimes \simeq \text{Fun}((\text{Cat}_\infty^{\text{perf}})^\omega, \mathcal{S}_\infty)^\otimes.$$

**Proof.** The existence of the symmetric monoidal projective model structure on  $\text{Pre}_\mathcal{S}((\text{Cat}_\mathcal{S}^{\text{cpt}})^{\text{op}})$  is straightforward (e.g., see [\[18, 4.2\]](#)). The criterion of [\[15, 5.8\]](#) and the fact that the (opposite of the) Morita equivalences on  $\text{Cat}_\mathcal{S}^{\text{cpt}}$  are closed under the smash product imply that  $\text{Pre}_\mathcal{S}^{\text{Mor}}((\text{Cat}_\mathcal{S}^{\text{cpt}})^{\text{op}})$  is a symmetric monoidal model category. The fact that the  $\infty$ -categorical Yoneda embedding is fully-faithful [\[35, 5.1.3.1\]](#) implies that there is an equivalence

$$\text{N}(\text{Pre}_\mathcal{S}^{\text{Mor}}((\text{Cat}_\mathcal{S}^{\text{cpt}})^{\text{op}}))[\mathcal{W}^{-1}] \simeq \text{Fun}((\text{Cat}_\infty^{\text{perf}})^\omega, \mathcal{S}_\infty),$$

and the universal property of the  $\infty$ -categorical Day convolution (see [\[36, Corollary 6.3.1.12\]](#)) promotes the equivalence of [Theorem 4.6](#) to an equivalence of symmetric monoidal  $\infty$ -categories

$$(\text{N}(\text{Pre}_\mathcal{S}^{\text{Mor}}((\text{Cat}_\mathcal{S}^{\text{cpt}})^{\text{op}}))[\mathcal{W}^{-1}])^\otimes \simeq \text{Fun}((\text{Cat}_\infty^{\text{perf}})^\omega, \mathcal{S}_\infty)^\otimes. \quad \square$$

Although we do not know that the model structure on  $\text{Pre}_{\mathcal{S}}^{\text{Mor}}((\text{Cat}_{\mathcal{S}}^{\text{cpt}})^{\text{op}})$  satisfies the monoid axiom, nonetheless the results of [48, §4] produce a  $J$ -semi model category structure on the category of algebras over a cofibrant  $E_{\infty}$  operad  $\mathcal{O}$ . Such a structure suffices to maintain homotopical control over the underlying  $\infty$ -category. In particular, cofibrant–fibrant objects in the  $J$ -semi model category of  $\mathcal{O}$ -algebras forget to cofibrant–fibrant objects in the underlying category.

Next, recall the following comparison [39, 22.1]:

**Proposition 6.2.** *The category of commutative algebras in the presheaf category  $\text{Pre}_{\mathcal{S}}((\text{Cat}_{\mathcal{S}}^{\text{cpt}})^{\text{op}})$  equipped with the (symmetric monoidal) convolution product is equivalent to the category of lax symmetric monoidal functors  $\text{Cat}_{\mathcal{S}}^{\text{cpt}} \rightarrow \mathcal{S}$ .*

We now use Proposition 3.6 and the  $J$ -semi model structure on  $E_{\infty}$ -algebras in  $\text{Pre}_{\mathcal{S}}^{\text{Mor}}((\text{Cat}_{\mathcal{S}}^{\text{cpt}})^{\text{op}})$  to deduce the following theorem.

**Theorem 6.3.** *Let  $E$  be a lax symmetric monoidal functor from spectral categories to spectra. Assume that  $E$  preserves Morita equivalences of flat spectral categories and that the induced functor  $\tilde{E} : \text{Cat}_{\infty}^{\text{perf}} \rightarrow \mathcal{S}_{\infty}$  is an additive invariant. Then  $\tilde{E}$  naturally extends to an  $E_{\infty}$ -algebra object of  $\text{Fun}_{\text{add}}(\text{Cat}_{\infty}^{\text{perf}}, \mathcal{S}_{\infty})^{\otimes}$ . The analogous results for localizing invariants hold.*

**Proof.** By Proposition 6.2,  $E$ , when restricted to  $\text{Cat}_{\mathcal{S}}^{\text{cpt}}$ , yields a commutative algebra in  $\text{Pre}_{\mathcal{S}}((\text{Cat}_{\mathcal{S}}^{\text{cpt}})^{\text{op}})$ . Fixing a cofibrant–fibrant  $E_{\infty}$  operad  $\mathcal{O}$ , we can restrict along the map from  $\mathcal{O}$  to the terminal operad to produce an  $\mathcal{O}$ -algebra structure on  $E$ . After cofibrant–fibrant replacement in the  $J$ -semi model structure on  $\mathcal{O}$ -algebras in  $\text{Pre}_{\mathcal{S}}^{\text{Mor}}((\text{Cat}_{\mathcal{S}}^{\text{cpt}})^{\text{op}})$ , Proposition 3.6 now implies that  $E$  descends to an  $E_{\infty}$  algebra in the symmetric monoidal  $\infty$ -category  $\text{Fun}((\text{Cat}_{\infty}^{\text{perf}})^{\omega}, \mathcal{S}_{\infty})$  such that the underlying functor agrees with  $\tilde{E}$  up to equivalence.

According to the results of Section 5, the additive localization is symmetric monoidal, so we obtain a symmetric monoidal functor

$$\text{Fun}((\text{Cat}_{\infty}^{\text{perf}})^{\omega}, \mathcal{S}_{\infty})^{\otimes} \longrightarrow \text{Fun}_{\text{add}}((\text{Cat}_{\infty}^{\text{perf}})^{\omega}, \mathcal{S}_{\infty})^{\otimes} \simeq \text{Fun}^{\text{L}}(\mathcal{M}_{\text{add}}, \mathcal{S}_{\infty})^{\otimes}.$$

Since  $\tilde{E}$  is already an additive invariant, it comes from an  $E_{\infty}$  algebra in the symmetric monoidal  $\infty$ -category  $\text{Fun}_{\text{add}}((\text{Cat}_{\infty}^{\text{perf}})^{\omega}, \mathcal{S}_{\infty})^{\otimes}$  of additive functors from compact idempotent-complete small stable  $\infty$ -categories to spectra. But this latter symmetric monoidal  $\infty$ -category is (symmetric monoidal) equivalent to the symmetric monoidal  $\infty$ -category  $\text{Fun}^{\text{L}}(\mathcal{M}_{\text{add}}, \mathcal{S}_{\infty})^{\otimes}$  by Proposition 5.15, so we may also regard  $\tilde{E}$  as an  $E_{\infty}$  algebra here.  $\square$

**Remark 6.4.** Equivalently, Corollary 5.16 implies that a functor satisfying the conditions of the preceding theorem gives rise to a lax symmetric monoidal additive functor of  $\infty$ -categories.

We now apply this work in the context of the topological Hochschild homology of spectral categories, which can be constructed using a version of the Hochschild–Mitchell cyclic nerve [7, 3.1].

**Definition 6.5.** For a small spectral category  $\mathcal{C}$  let

$$THH_q(\mathcal{C}) = \bigvee \mathcal{C}(c_{q-1}, c_q) \wedge \cdots \wedge \mathcal{C}(c_0, c_1) \wedge \mathcal{C}(c_q, c_0),$$

where the sum is over the  $(q + 1)$ -tuples  $(c_0, \dots, c_q)$  of objects of  $\mathcal{C}$ . This becomes a simplicial object using the usual cyclic bar construction face and degeneracy maps. We will write  $THH(\mathcal{C})$  for the geometric realization.

We do not expect this construction to have the correct homotopy type unless the spectral category  $\mathcal{C}$  is pointwise-cofibrant.

**Lemma 6.6.** *The point-set functor  $THH$  from Definition 6.5 descends to a functor of  $\infty$ -categories*

$$THH : \text{Cat}_\infty^{\text{perf}} \longrightarrow \mathcal{S}_\infty.$$

**Proof.** Restricting to pointwise-cofibrant spectral categories, the result follows from the fact that  $THH$  takes Morita equivalences of spectral categories to weak equivalences [7, 5.9, 5.12].  $\square$

**Remark 6.7.** We can construct  $THH$  directly on the level of  $\infty$ -categories as follows. (This perspective is closely related to the view taken in [2], for instance.) A small stable  $\infty$ -category  $\mathcal{C}$  determines an exact functor

$$\text{Map}_{\mathcal{C}} : \mathcal{C}^{\text{op}} \otimes \mathcal{C} \longrightarrow \mathcal{S}_\infty$$

given by the morphism spectra in  $\mathcal{C}$  (cf. [11, §3]). Extending by (filtered) colimits results in a colimit preserving functor

$$\Theta_{\mathcal{C}} : \text{Fun}^{\text{ex}}(\mathcal{C}^{\text{op}} \otimes \mathcal{C}, \mathcal{S}_\infty) \simeq \text{Fun}^{\text{ex}}(\mathcal{C} \otimes \mathcal{C}^{\text{op}}, \mathcal{S}_\infty) \simeq \text{Ind}(\mathcal{C}^{\text{op}} \otimes \mathcal{C}) \longrightarrow \mathcal{S}_\infty$$

(note that we must use the canonical equivalence  $\mathcal{C}^{\text{op}} \otimes \mathcal{C} \simeq \mathcal{C} \otimes \mathcal{C}^{\text{op}}$  in order to obtain the first map in the above composite). One can check that the value of  $\Theta_{\mathcal{C}}$  on  $\text{Map}_{\mathcal{C}} \in \text{Fun}^{\text{ex}}(\mathcal{C}^{\text{op}} \otimes \mathcal{C}, \mathcal{S}_\infty)$  is precisely the spectrum  $THH(\mathcal{C})$ . (See [52, 5.2.3] for a discussion of this in the context of DG categories.)

For spectral categories  $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$ , the standard shuffle product maps induce  $\Sigma_n$ -equivariant equivalences

$$THH(\mathcal{C}_1) \wedge THH(\mathcal{C}_2) \wedge \cdots \wedge THH(\mathcal{C}_n) \longrightarrow THH(\mathcal{C}_1 \wedge \mathcal{C}_2 \wedge \cdots \wedge \mathcal{C}_n).$$

These maps are associative and unital, and therefore we deduce the following lemma.

**Lemma 6.8.** *The shuffle product maps make THH a lax symmetric monoidal functor from spectral categories to spectra.*

Applying [Theorem 6.3](#) we obtain the following corollary:

**Corollary 6.9.** *The functor THH yields an object of  $\text{Alg}_{/E_\infty}(\text{Fun}^L(\mathcal{M}_{\text{loc}}, \mathcal{S}_\infty)^\otimes)$  or equivalently an object of  $\text{Fun}_{\text{loc}}^{\text{lax}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)$ .*

**Proof.** This follows from [Lemma 6.6](#), the discussion above and the fact that THH descends to a localizing invariant: THH preserves filtered homotopy colimits and sends exact sequences of small stable  $\infty$ -categories to cofiber sequences of spectra [[7](#), [7.1](#)].  $\square$

**Remark 6.10.** Since the shuffle product maps are equivalences, [Proposition 3.5](#) implies that THH in fact descends to a symmetric monoidal functor  $\text{Cat}_\infty^{\text{perf}} \rightarrow \mathcal{S}_\infty$ .

Establishing the analogue of [Corollary 6.9](#) for topological cyclic homology (TC) is somewhat more complicated. For one thing, TC does not preserve filtered homotopy colimits of spectral categories, and so cannot give rise to an additive or localizing invariant. Moreover, the model of THH given in [Definition 6.5](#) is not the construction that is used to build TC, and so a different construction (and argument) is needed to see that TC is lax symmetric monoidal.

We now review a different model of THH of a spectral category which is adapted to the construction of TC and related invariants which do preserve filtered homotopy colimits. Our treatment is relatively rapid; we refer the interested reader to [[8](#), [§5](#)] and [[7](#), [§4](#)] for a more detailed discussion.

The key observation that leads to the construction of TC is the fact that the cyclic bar construction of [Definition 6.5](#) is not just a simplicial set, but in fact a cyclic set; the cyclic operator is given by “rotating” the smash factors. As a consequence, the geometric realization is a spectrum with an  $S^1$ -action. In fact, THH can be constructed as an  $S^1$ -equivariant spectrum equipped with additional structure that models the structure of the free loop space. Specifically, the  $p$ th root self-equivalences  $S^1/H \cong S^1$  (for finite  $H \subset S^1$ ) induces equivariant weak equivalences

$$\text{Map}(S^1, X)^H \longrightarrow \text{Map}(S^1, X)$$

for reasonable spaces  $X$ . We are going to give a model of THH as a cyclotomic spectrum, which is a spectrum-level version of this structure. Roughly speaking, a cyclotomic spectrum is equipped with compatible maps

$$\Phi^H THH(\mathcal{C}) \longrightarrow THH(\mathcal{C})$$

for finite  $H \subset S^1$ , where  $\Phi^H$  denotes the geometric fixed points of the  $S^1$ -spectrum.

We now fix a prime  $p$  and consider the subgroups  $H = C_{p^k}$  as  $k$  varies. The structure of a cyclotomic spectrum supplies the system of “categorical” fixed points  $\{THH(\mathcal{C})^{C_{p^{n-1}}}\}$  with a pair of maps

$$F, R : THH(\mathcal{C})^{C_{p^n}} \longrightarrow THH(\mathcal{C})^{C_{p^{n-1}}},$$

where  $F$  (the Frobenius) denotes the obvious inclusion of fixed points and  $R$  (the restriction) is a much less obvious map coming from the cyclotomic structure.

For convenience, we denote the fixed points to be

$$TR^n(\mathcal{C}) = THH(\mathcal{C})^{C_{p^{n-1}}},$$

the categorical fixed points with respect to the induced  $C_{p^{n-1}}$  action. We then define  $TC^n(\mathcal{C})$  to be the homotopy equalizer

$$\text{holim}_{F,R} TR^n(\mathcal{C}) \longrightarrow TR^{n-1}(\mathcal{C})$$

and we finally define

$$TC(\mathcal{C}) = \text{holim}_n TC^n(\mathcal{C}),$$

where we form the homotopy limit over the maps induced by the restriction  $R$ ; this definition is equivalent to the one originally given in [14].

The issue that arises with Definition 6.5 is that traditionally it has been difficult to obtain the correct equivariant homotopy type directly from the cyclic nerve and in particular build the restriction maps. Instead, we need to use a variant of Bökstedt’s original construction, which we now review. Let  $\mathcal{I}$  denote the category of finite ordered sets, with objects  $n = \{1, \dots, n\}$  and morphisms the injections. For a symmetric spectrum  $X$ , we will denote by  $X_n$  the  $n$ th space.

**Definition 6.11.** Let  $\mathcal{C}$  be a small spectral category and  $X$  a space. Define the functor  $\mathcal{G}(\mathcal{C}, X)_{n_0, \dots, n_q}$  from  $\mathcal{I}^{q+1}$  to spaces by the formula

$$\mathcal{G}(\mathcal{C}, X)_{n_0, \dots, n_q} = \Omega^{n_0 + \dots + n_1 + \dots + n_q} \left( \bigvee \mathcal{C}(c_{q-1}, c_q)_{n_q} \wedge \dots \wedge \mathcal{C}(c_0, c_1)_{n_1} \wedge \mathcal{C}(c_q, c_0)_{n_0} \right)$$

and define

$$THH(\mathcal{C}; X)_q = \text{hocolim}_{\bar{n} \in \mathcal{I}^{q+q}} \mathcal{G}(\mathcal{C}, X)_{\bar{n}}.$$

For fixed  $X$ , the spaces  $THH(\mathcal{C}; X)_q$  assemble into a cyclic space with degeneracy map induced by the unit and face maps induced by the composition in  $\mathcal{C}$ . This assignment is functorial in  $X$ , and so restricting to spheres

$$S^n \cong S^1 \wedge S^1 \wedge \dots \wedge S^1$$

defines a symmetric spectrum  $THH(\mathcal{C}) = \{THH(\mathcal{C}; S^n)\}$ . When  $\mathcal{C}$  is a point-wise cofibrant spectral category, an elaboration of the work of Shipley [43] shows that there is a natural isomorphism in the stable category between this model of  $THH$  and the cyclic nerve of Definition 6.5 [7, 3.5].

Now we fix a complete  $S^1$ -universe  $U$  (i.e., an infinite-dimensional real  $S^1$ -inner product space that contains each irreducible finite-dimensional representation infinitely many times). Evaluating at the representation spheres  $S^V$ , where  $V$  is a finite-dimensional real  $S^1$ -inner product space, the collection  $\{THH(\mathcal{C}; S^V)\}$  forms an orthogonal  $S^1$ -spectrum. Here each space is given the diagonal  $S^1$ -action from the cyclic structure as well as the action of  $S^1$  on  $S^V$ .

This realization of  $THH(\mathcal{C})$  as an orthogonal  $S^1$ -spectrum is a cyclotomic spectrum [7, §4], and therefore can be used to define  $TC^n$  and  $TC$ . It is possible to show that  $THH(-)$  is a lax symmetric monoidal functor regarded as a functor to cyclotomic spectra in orthogonal spectra. However, herein we employ a shortcut due to Hesselholt and Madsen in order to obtain the multiplicative properties we want. Specifically, one can write down directly the restriction and Frobenius maps as maps of non-equivariant symmetric spectra.

To capture the monoidal properties of these maps, we use a generalization of Bökstedt’s original construction of  $THH$  [29, §1.7]. Let  $P$  be a finite ordered set. Define

$$\mathcal{G}(\mathcal{C}, X)_{n_0, \dots, n_q}^P : (\mathcal{I}^P)^{q+1} \longrightarrow \mathcal{T}$$

to be the functor determined by composing  $\mathcal{G}(\mathcal{C}, X)$  with the functor  $\cup_P : \mathcal{I}^P \rightarrow \mathcal{I}$ . This construction is functorial in both  $X$  with respect to continuous maps and  $\mathcal{P}$  with respect to injective maps. For fixed  $P$ , as above we can define an orthogonal spectrum  $THH^P(\mathcal{C})$ . Bökstedt’s lemma about “good indexing categories” implies that  $THH^P(\mathcal{C})$  is equivalent to  $THH(\mathcal{C})$  [13, 1.6]. Moreover, the proof of [29, 1.7.1] extended to spectral categories as in [26, §6.2] gives rise to an  $S^1 \times \Sigma_m \times \Sigma_n$ -equivariant multiplication

$$THH^P(\mathcal{C}; X) \wedge THH^Q(\mathcal{D}; Y) \longrightarrow THH^P \amalg^Q(\mathcal{C} \wedge \mathcal{D}; X \wedge Y), \tag{6.12}$$

where  $|P| = m$  and  $|Q| = n$ .

Considering the collection of spaces  $\{THH^P(\mathcal{C}; S^n)\}$ , where  $|P| = n$ , we have a symmetric spectrum which is again equivalent to  $THH(\mathcal{C})$ . The multiplication map in Eq. (6.12) is associative and unital, and so we have the following result:

**Lemma 6.13.** *The construction  $\{THH^P(\mathcal{C}; S^n)\}$  gives rise to a lax symmetric monoidal functor from spectral categories to symmetric spectra.*

Moreover, one can directly construct the restriction and Frobenius maps  $R_r, F_r$  on the fixed points of this model. Furthermore, these maps are compatible with the product structure [29, 1.7.1]. Therefore, we have the following extension of [29, 3.6].

**Theorem 6.14.** *There are lax symmetric monoidal functors  $TC^n$  and  $TC$  from spectral categories to symmetric spectra.*

Although  $TC$  does not preserve filtered colimits and therefore cannot be a localizing invariant, using [11, 10.8] we do have the following analogue of Corollary 6.9.

**Corollary 6.15.** *For each  $n$ , the functor  $TC^n$  from spectral categories to spectra yields an object of  $\text{Alg}_{/E_\infty}(\text{Fun}^L(\mathcal{M}_{\text{loc}}, \mathcal{S}_\infty)^\otimes)$  or equivalently  $\text{Fun}_{\text{loc}}^{\text{lax}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)$ .*

We conclude this section by remarking on what we did not prove. Although we constructed  $THH$  and  $TC$  as lax symmetric monoidal functors from spectral categories to spectra, we did not construct point-set models of the topological Dennis trace or cyclotomic trace that are compatible with the multiplicative structure. To do so involves “mixing” Waldhausen’s  $S_\bullet$  construction with the construction of  $THH$  and  $TC$ , and handling multiplicative coherence is quite intricate in this framework. Although this can be carried out using the coherence machinery of [9] (see also [19, §V.4] for discussion of this approach to multiplicative coherence), a distinct advantage of our framework herein is that we will obtain existence and uniqueness results for these multiplicative trace maps without an explicit model.

### 7. Uniqueness results

In this section, we apply the framework we have developed to deduce various uniqueness results for multiplicative structures on algebraic  $K$ -theory and apply the work of Section 6 to deduce uniqueness and existence results for multiplicative natural transformations out of algebraic  $K$ -theory. In particular, our work gives universal constructions of the topological Dennis trace  $K \rightarrow THH$  and cyclotomic trace  $K \rightarrow TC$ .

Using equivalences (1.1) and (1.2), Theorem 1.5 implies that  $\text{Fun}^L(\mathcal{M}_{\text{add}}, \mathcal{S}_\infty)$  and  $\text{Fun}^L(\mathcal{M}_{\text{loc}}, \mathcal{S}_\infty)$  are symmetric monoidal  $\infty$ -categories with tensor units the respective algebraic  $K$ -theory functors. Hence, Corollary 3.8 immediately implies the following result:

**Proposition 7.1.**

- (1) *For any  $n \geq 0$ , the space  $\text{Alg}_{/E_n}(\text{Map}(\mathcal{U}_{\text{add}}(\mathcal{S}_\infty^\omega), -))$  of  $E_n$  monoidal structures on the algebraic  $K$ -theory functor is contractible.*
- (2) *For any  $n \geq 0$ , the space of maps in  $\text{Alg}_{/E_n}(\text{Fun}^L(\mathcal{M}_{\text{add}}, \mathcal{S}_\infty))$  with source  $\text{Map}(\mathcal{U}_{\text{add}}(\mathcal{S}_\infty^\omega), -)$  is contractible.*

*Analogous results hold in the localizing case.*

In particular, we obtain the following corollary:

**Corollary 7.2.** *There exists a unique  $E_\infty$  algebra structure on the  $K$ -theory functor, viewed as an object of the symmetric monoidal  $\infty$ -category  $\text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)^\otimes$ . Furthermore, for any  $0 \leq n \leq \infty$  and any  $E_n$  algebra  $F$ , the space of  $E_n$  algebra maps from  $K$  to  $F$  is contractible. Analogous statements hold for  $\mathbb{K}$ .*

Coupled with [Theorem 1.10](#), this specializes into our main application:

**Theorem 7.3.** *The space of maps of  $E_\infty$ -algebras in  $\text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)^\otimes$  from  $K$ -theory to  $\text{THH}$  is contractible. Equivalently, the space of lax symmetric monoidal additive functors from  $K$ -theory to  $\text{THH}$  is contractible. The unique element is the topological Dennis trace map.*

We identify the unique element as the topological Dennis trace map using the main classification result of [\[11, 10.6\]](#). Specifically, the image of this element under the forgetful functor to natural transformations of additive functors is the unit in the set  $\mathbb{Z}$  of homotopy classes of natural transformations of additive functors  $K \rightarrow \text{THH}$ . Note that our results in fact give a construction of the multiplicative trace in the generality of stable  $\infty$ -categories.

We can deduce analogous results about  $TC$  even though  $TC$  is not itself a localizing invariant. As in [\[11, 10.11\]](#) we use the fact that  $TC$  can be described as  $\text{holim}_n TC^n(-)$ . As such, any natural transformation of functors  $K \rightarrow TC$  is equivalent to the data of compatible maps to each  $TC^n$ . [Corollary 6.15](#) now implies that our uniqueness results extend to natural transformations which arise from transformations  $K \rightarrow TC^n$ . Since the cyclotomic trace is of this form, we deduce the desired uniqueness result.

**Theorem 7.4.** *For each  $n$ , the space of  $E_\infty$  algebra maps from  $K$  to  $TC^n$  in  $\text{Fun}_{\text{add}}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)^\otimes$  is contractible. Equivalently, the space of lax symmetric monoidal additive functors from  $K$  to  $TC^n$  is contractible. The unique homotopy class of maps of  $E_\infty$  algebras in  $\text{Fun}(\text{Cat}_\infty^{\text{perf}}, \mathcal{S}_\infty)^\otimes$  from  $K$  to  $TC$  that restrict to maps of  $E_\infty$  algebras  $K \rightarrow TC^n$  is the multiplicative cyclotomic trace.*

Another application of [Corollary 7.2](#) is the following uniqueness result:

**Corollary 7.5.** *Let  $\mathcal{C}$  be a symmetric monoidal spectral category, and  $\text{Perf}(\mathcal{C})$  be the resulting symmetric monoidal category of compact modules. There is a unique  $E_\infty$  algebra structure on  $K(\mathcal{C})$  in the  $\infty$ -category of spectra. Similarly, if  $\mathcal{C}$  is a monoidal spectral category, there is a unique  $A_\infty$  structure on  $K(\mathcal{C})$  in the  $\infty$ -category of spectra. Analogous results hold for  $\mathbb{K}$ .*

**Proof.** Since  $K$ -theory has a unique structure as a lax symmetric monoidal functor from  $\text{Cat}_\infty^{\text{perf}}$  to  $\mathcal{S}_\infty$ , in particular when evaluated at any point there is a unique  $E_\infty$  or  $A_\infty$  structure on the resulting spectrum.  $\square$

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