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Power maps and epicyclic spaces

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Abstract

In this paper, we present the basic facts of the theory of epicyclic spaces for the first time considered by T. Goodwillie in an unpublished letter to F. Waldhausen. An epicyclic space is a space-valued, contravariant functor on a category $\tilde{\Lambda}$ which contains the cyclic category. Our results parallel basic facts of the theory of cyclic spaces established in [1] and [7]. We show that the geometric realization of an epicyclic space has an action of a monoid which is a semidirect product of S^1 and the multiplicative monoid of natural numbers. We also show that the homotopy colimit of an epicyclic space is homotopy equivalent to the bar construction for the monoid action. Finally, we give an explicit description of the homotopy type of the classifying space of the category $\tilde{\Lambda}$.

0. Introduction

It is well known that the free loop space $\Lambda Y = \mathrm{map}(S^1, Y)$ carries a lot of interesting structure. Our object in this paper is to investigate the piece of structure given by power maps $\varphi_k : \Lambda Y \to \Lambda Y$ which are induced on the free loop space by k-fold coverings of the circle $\omega_k : S^1 \to S^1$, $\omega_k(z) = z^k$. Despite their very simple definition, the power maps are of fundamental significance for the topology of the free loop space (cf. [2], where the power maps are shown to induce maps on real and equivariant cohomology of ΛM which correspond to Adams operations on the Hochschild and cyclic homology of the de Rham algebra of a 1-connected manifold M). The maps φ_k together with rotation of loops yield on ΛY a right action of a topological monoid $\mathcal{M}$. By definition $\mathcal{M} = \mathbb{N} \times S^1$ and multiplication in $\mathcal{M}$ is given by the formula:

$$(k_1, z_1)(k_2, z_2) = (k_1 k_2, z_1 z_2^{k_1}). \tag{0.1}$$

We define the action of $\mathcal{M}$ on ΛY by sending a pair $(\alpha, (k, z)) \in \Lambda Y \times \mathcal{M}$ to a loop whose value on $x \in S^1$ is $\alpha(zx^k)$. *If* Y is connected, with no loss of generality, one can assume that Y is the classifying space BG of a topological group G. Let $N_*^{\mathrm{cyc}} G$ be the

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cyclic nerve of G, i.e., the cyclic space with $N_n^{\mathrm{cyc}}G = G^{n+1}$, cf. [1]. It is a rudimentary fact of cyclic theory that there exists a homotopy equivalence $f: |N_*^{\mathrm{cyc}}G| \to \Lambda BG$ between the geometric realization of the cyclic set $N_*^{\mathrm{cyc}}G$ and the free loop space ΛBG, cf. [1, Proposition 1.20]. The map f is S^1-equivariant with respect to the canonical S^1-action on the geometric realization of the cyclic set $N_*^{\mathrm{cyc}}G$ and rotation of loops on ΛBG.

Theorem A(2) below shows that f is also equivariant with respect to the actions of the monoid $\mathcal{M}$: the $\mathcal{M}$-action on ΛBG considered above and an $\mathcal{M}$-action on $|N_*^{\mathrm{cyc}}G|$ induced by its cyclic structure and maps $p_n^k: N_n^{\mathrm{cyc}}G \to N_{k(n+1)-1}^{\mathrm{cyc}}G$ which repeat $(n+1)$-tuples k times, i.e., which send $(g_o, g_1, \ldots, g_n) \in G^{(n+1)}$ to the $k(n+1)$-tuple

$$(g_0, \ldots, g_n, g_0, \ldots, g_n, \ldots, g_0, \ldots, g_n) \in G^{k(n+1)}.$$

The collection of maps $\{p_*^k : k = 1, 2, \ldots\}$, while not simplicial maps, do define continuous maps on $|N_*^{\mathrm{cyc}}G|$.

It was an idea of T. Goodwillie to extend the cyclic category Λ to a category $\tilde{\Lambda}$ by adding to it some extra morphisms such that the cyclic set $N_*^{\mathrm{cyc}}G$ becomes a functor on $\tilde{\Lambda}^{\mathrm{op}}$ with the maps p_n^k in its range. In [8] Goodwillie describes the category $\tilde{\Lambda}$ which he calls the epicyclic category. Functors $\tilde{\Lambda}^{\mathrm{op}} \to \mathbf{Top}$ are called epicyclic spaces.

In this paper, using methods developed for the case of cyclic spaces in [1] and generalized in [7], we obtain basic facts on epicyclic spaces. Our first result is the following.

Theorem A. (1) *If X_* is an epicyclic space, then its geometric realization has a canonical, right action of the monoid $\mathcal{M}$.*

(2) *The homotopy equivalence $f: |N_*^{\mathrm{cyc}}G| \to \Lambda BG$ is $\mathcal{M}$-equivariant with respect to the $\mathcal{M}$-action induced on $|N_*^{\mathrm{cyc}}G|$ by the maps p_n^k and the $\mathcal{M}$-action on the free loop space ΛBG.*

(3) *The homotopy colimit over $\tilde{\Lambda}$ of an epicyclic space X_* is homotopy equivalent to the two-sided bar construction $B(|X_*|, \mathcal{M}, \mathrm{pt})$ for the $\mathcal{M}$-action of part* (1).

Recall that the realization of a cyclic space X_* with respect to the cyclic category Λ (i.e., the homotopy colimit of X_* over Λ) has the homotopy type of the Borel construction $ES^1 \times_{S^1} |X_*|$ for the S^1-action on $|X_*|$, cf. [1, Theorem 1.8] or [7, Theorem 5.12].

In case when X_* is the one-point epicyclic space, one obtains the following corollary.

Corollary. *The classifying spaces $B\tilde{\Lambda}$ and $B\mathcal{M}$ are homotopy equivalent.* □

In the final part of the paper, we describe the homotopy type of the classifying space $B\mathcal{M}$. The result is as follows.

Theorem B. *The fundamental group of $B\mathcal{M}$ is isomorphic to the multiplicative group of positive rational numbers. The universal covering of $B\mathcal{M}$ has the homotopy type of the*

Eilenberg–Mac Lane space $K(\mathbb{Q}, 2)$, *where* $\mathbb{Q}$ *is the additive group of rationals.* $\pi_1 B\mathcal{M}$ *acts on* $\pi_2 B\mathcal{M}$ *by multiplication of rational numbers.*

The Corollary and Theorem B determine the homotopy type of the classifying space $B\tilde{\Lambda}$. On the other hand (see Remark 3.5), the universal cover of $B\tilde{\Lambda}$ is homotopy equivalent to the orbit space of the classifying space $E\mathcal{F}$ for the family $\mathcal{F}$ of finite subgroups of S^1, cf. [6]. This fact suggests a link between epicyclic theory and equivariant homology theories modeled on the space $E\mathcal{F}$, see eg., [4, p. 25], we plan to address the matter elsewhere.

The organization of the paper is as follows. In Section 1 we introduce necessary notation and definitions. The following sections contain proofs of Theorems A and B.

1. Notation and definitions

Let Δ denote the simplicial category generated by faces $\delta^i = \delta^i_n : [n-1] \to [n]$ and degeneracies $\sigma^i = \sigma^i_n : [n+1] \to [n]$ satisfying the usual commutation relations. Let Λ_k denote the *k-cyclic category* with the same objects as Δ and containing Δ as a subcategory, but with additional morphisms $\tau_n = {}_k\tau_n : [n] \to [n]$ satisfying the additional relations

(1) ${}_k\tau_n \delta^i_n = \delta^{i-1}_n {}_k\tau_{n-1}$ if $i \geq 1$,

(2) ${}_k\tau_n \sigma^i_n = \sigma^{i-1}_n {}_k\tau_{n-1}$ if $i \geq 1$,

(3) $({}_k\tau_n)^{k(n+1)} = \mathrm{id}_{[n]}$.

When $k = 1$, Λ_1 is Connes' cyclic category denoted simply Λ. Let $P_k : \Lambda_{kl} \to \Lambda_l$ be the functor which is the identity of Δ and satisfying $P_k({}_{kl}\tau_n) = {}_l\tau_n$. We evidently have $P_k \circ P_l = P_{kl}$.

A *k-cyclic* space X_* is a contravariant functor from the category Λ_k to the category of spaces. It is completely specified by the spaces $X_n = X([n])$, face maps $d_i = X(\delta^i) : X_n \to X_{n-1}$, degeneracy maps $s_i = X(\sigma^i) : X_n \to X_{n+1}$, and cyclic maps $t_n = {}_k t_n = X({}_k\tau_n) : X_n \to X_n$. The face and degeneracy maps endow X_* with the structure of a simplicial space, while the cyclic maps specify actions by the cyclic groups $\mathbb{Z}/k(n+1)$ on the spaces X_n of n-simplices. These two structures are interrelated by the commutation relations

(1) $d_i t_n = t_{n-1} d_{i-1}$ if $i \geq 1$,

(2) $s_i t_n = t_{n+1} s_{i-1}$ if $i \geq 1$.

When we refer to the geometric realization of a k-cyclic space we mean the geometric realization of its underlying simplicial space.

The simplicial category Δ may be realized as a category of sets and functions by indentifying the object $[n]$ with the poset $\{0 < 1 < 2 < \cdots < n\}$. The morphisms of Δ can then be identified as order preserving morphisms between these posets. We then define the kth *subdivision functor* $\mathrm{Sd}_k : \Delta \to \Delta$ as follows (cf. [3]). One can identify the k-fold disjoint sum of posets $[n] \amalg [n] \amalg \cdots \amalg [n]$ (with block ordering) with the poset

$[(n+1)k-1]$. Thus we define

$$\mathrm{Sd}_k([n]) = [(n+1)k-1] \quad \text{on objects,}$$

$$\mathrm{Sd}_k(\alpha) = \alpha \amalg \alpha \amalg \cdots \amalg \alpha \quad \text{on morphisms.}$$

The subdivision functor can be extended to a functor $\mathrm{Sd}_{kl}: \Lambda_{kl} \to \Lambda_l$ by defining $\mathrm{Sd}_k({}_{kl}\tau_n) = {}_l\tau_{k(n+1)-1}$. It is obvious that $\mathrm{Sd}_k \circ \mathrm{Sd}_l = \mathrm{Sd}_{kl}$.

If X_* is a simplicial space, we regard it as a functor $\Delta^{\mathrm{op}} \to \mathbf{Top}$ and define the k-th subdivision $\mathrm{Sd}_k X_*$ to be the composite functor $X \circ \mathrm{Sd}_k$. If X_* is an l-cyclic space, then $\mathrm{Sd}_k X_*$ is a kl-cyclic space. Although X_* and $\mathrm{Sd}_k X_*$ are nonisomorphic simplicial spaces, their geometric realization are homeomorphic by a canonical homeomorphism $D_k: |\mathrm{Sd}_k X_*| \to |X_*|$ induced by the diagonal imbedding *diag*$: \Delta^n \to \Delta^{k(n+1)-1}$ given by

$$\mathrm{diag}_k(t) = \left(\frac{1}{k}t, \frac{1}{k}t, \ldots, \frac{1}{k}t\right),$$

where we view the affine simplex $\Delta^{k(n+1)-1}$ as the k-fold join of Δ^n.

Referring to [7] we can see that Λ_k is the category associated to the crossed simplicial group ${}_kC_*$ whose set of n-simplices is ${}_kC_n = \mathbb{Z}/k(n+1)$. Moreover it follows that ${}_kC_*$ can be identified with the kth subdivision $\mathrm{Sd}_k(C_*)$ of the cyclic crossed-simplicial group C_*. It follows from results of [7] that if X_* is an *l-cyclic* space, the actions

$$\mathbb{Z}/l(n+1) \times X_n \to X_n$$

induced by the cyclic operators ${}_kt_n$ give rise to an action μ of the crossed simplicial group $\mathrm{Sd}_l C_*$ on the simplicial space X_*. Moreover, upon passage to geometric realizations, we obtain an action by the topological group $\mathrm{Sd}_l C_*| \cong |C_*| \cong S^1$ on $|X_*|$.

The functor $P_k: \Lambda_{kl} - \Lambda_l$ defined above, corresponds to the homomorphism of crossed simplicial groups ${}_kP_*: \mathrm{Sd}_{kl}C_* \to Sd_l C_*$ given on n-simplices by reduction $\mathrm{mod}\, l(n+1)$, $\mathbb{Z}/kl(n+1) \to \mathbb{Z}/l(n+1)$. We have a commutative triangle

$$\begin{array}{ccc} |\mathrm{Sd}_{kl}C_*| & \xrightarrow{\;|{}_kP_*|\;} & |\mathrm{Sd}_l C_*| \\ & {\scriptstyle D_k}\searrow \quad \nearrow{\scriptstyle \omega_k} & \\ & |\mathrm{Sd}_l C_*| & \end{array} \tag{1.1}$$

where $\omega_k(z) = z^k$ on $|\mathrm{Sd}_l C_*| \cong S^1$. For an l-cyclic space X_*, we denote by μ_k the $|\mathrm{Sd}_{kl}C|$-action on $|X_*|$ given by the formula

$$\mu_k(z, x) = \mu(|{}_kP_*|(z), x). \tag{1.2}$$

Since the subdivision $\mathrm{Sd}_k X_*$ of an l-cyclic space X_* is a kl-cyclic $|\mathrm{Sd}_{kl}C_*|$ also acts on $|\mathrm{Sd}_k X_*|$. We denote the action by $\mathrm{Sd}_k\mu$. The naturality of the construction in [7] implies that the following diagram commutes:

$$\begin{array}{ccc} |\mathrm{Sd}_{kl}C_*| \times |\mathrm{Sd}_k X_*| & \xrightarrow{|\mathrm{Sd}_k \mu|} & |\mathrm{Sd}_k X_*| \\ {\scriptstyle D_k \times D_k}\downarrow & & \downarrow{\scriptstyle D_k} \\ |_l C_*| \times |X_*| & \xrightarrow{\mu} & |X_*| \end{array} \tag{1.3}$$

Following Goodwillie [8] we introduce the epicyclic category and the concept of epicyclic space.

Definition 1.1 (1) The epicyclic category $\tilde{\Lambda}$ is the small category generated by morphisms and relations of the cyclic category Λ together with new morphisms

$$\pi_n^k : [k(n+1)-1] \to [n], \qquad k, n \in \mathbb{N}, \quad k \geq 1$$

which are subject to the following relations,

(i) $\pi_n^1 = \mathrm{id}_{[n]}$, $\pi_n^l \pi_{l(n+1)-1}^k = \pi_n^{kl}$,

(ii) $\alpha \pi_m^k = \pi_n^k \mathrm{Sd}_k(\alpha)$, for $\alpha \in \mathrm{Hom}_\Delta([m],[n])$,

(iii) $\tau_n \pi_n^k = \pi_n^k \mathrm{Sd}_k(\tau_n)$.

Relations (ii) and (iii) amount to requiring that, for every $k \geq 1$, postcomposition with morphisms π_n^k in $\tilde{\Lambda}$ yields a natural transformation of functors up to which the following diagram commutes:

$$\begin{array}{ccc} \Lambda_k & \xrightarrow{\mathrm{Sd}_k} & \Lambda \\ {\scriptstyle P_k}\downarrow & \swarrow\!\!\!\!\swarrow & \downarrow{\scriptstyle i} \\ \Lambda & \xrightarrow{i} & \tilde{\Lambda} \end{array} \tag{1.4}$$

where i is the inclusion functor.

(2) An epicyclic space is a functor $X : \tilde{\Lambda}^{\mathrm{op}} \to \mathbf{Top}$, or equivalently a cyclic space X_* with additional continuous operators $\{ p_n^k = X_n(\pi_n^k) : X_n \to X_{k(n+1)-1} \}$ satisfying the following relations,

(i) $p_n^1 = \mathrm{id}_{X_n}$, $p_{l(n+1)-1}^k p_n^l = p_n^{kl}$,

(ii) $p_n^k d_i = \mathrm{Sd}_k(d_i) p_{n+1}^k$,

$p_n^k s_i = \mathrm{Sd}_k(s_i) p_{n-1}^k$,

(iii) $p_n^k t_n = t_{k(n+1)-1} p_n^k$.

Note that the maps $p_{l(n+1)-1}^k$ define simplicial maps $p_*^k : \mathrm{Sd}_{kl} X_* \to \mathrm{Sd}_l X_*$ which are maps of kl-cyclic spaces (cf. diagram (1.4)).

(3) A morphism of epicyclic spaces $f_* : X_* \to Y_*$ is a natural transformation of functors $X \Rightarrow Y$, and is given by continuous maps $f_n : X_n \to Y_n$ which commute with all the operators d_i, s_i, t_n, and p_n^k.

Remarks 1.2. Recall that in the cyclic category Λ every morphism can be uniquely factored as $\alpha \tau_m^r$, for some $\alpha \in \mathrm{Hom}_\Delta([m],[n])$. In $\tilde{\Lambda}$ every morphism can be factored as $\pi_n^k \lambda$ for some $\lambda \in \mathrm{Hom}_\Lambda([m],[k(n+1)-1])$. However, since by relation (iii) of Definition 1.1(1), we have $\pi_n^k \tau_{k(n+1)-1}^{n+1} = \pi_n^k$, the factorization is unique only up to a $\mathbb{Z}/k = \langle \tau_{k(n+1)-1}^{n+1} \rangle$-ambiguity. It follows that the simplicial set $\mathrm{Hom}_{\tilde{\Lambda}}(-,[n])$ can be identified with a disjoint union of simplicial sets

$$\coprod_{k \geq 1} \mathrm{Hom}_{\Lambda}(-, [k(n+1)-1])/\mathbb{Z}/k,$$

where $\mathbb{Z}/k$ acts $\mathrm{Hom}_{\Lambda}(-, [k(n+1)-1])$ by cyclicly permuting k blocks of $k(n+1)-1] = [n] \amalg [n] \cdots \amalg [n]$. Under geometric realization we obtain a homeomorphism:

$$|\mathrm{Hom}_{\tilde{\Lambda}}(-, [n])| = \coprod_{k \geq 1} S^1 \times_{\mathbb{Z}/k} \Delta^{k(n+1)-1}, \tag{1.5}$$

where the $\mathbb{Z}/k$-action is the restriction of the standard $\mathbb{Z}/m$-action ($m = k(n+1)-1$) on $S_1 \times \Delta^m$ given by

$$\tau_m(z, u_0, u_1, \ldots, u_m) = (ze^{-2\pi i u_0}, u_1, \ldots, u_m, u_0). \tag{1.6}$$

Example 1.3. The cyclic nerve $N_*^{\mathrm{cyc}}G$ of a group G together with the maps p_n^k from the Introduction is an epicyclic set, cf. [3]. More generally, the cyclic nerve of any small topological category is an epicyclic space, cf. [8].

Example 1.4. The total singular complex T_*X of a topological space X which has a right action $\tilde{\mu}$ of the monoid $\mathcal{M}$ from (0.1) is an epicyclic set. To check that T_*X extends to $\tilde{\Lambda}$ one has to define maps p_n^k which satisfy relations dual to (i), (ii) and (iii) from Definition 1.1(2). We give a formula for p_n^2 leaving the general case to the reader. Let $\sigma: \Delta^n \to X$ be a singular simplex and let $(s,t) \in \Delta^{2n+1} = \Delta^n * \Delta^n$. Denote by $|s|$ the sum of coordinates of s, $p_n^2(\sigma)$ is by definition a simplex $\Delta^{2n+1} \to X$ whose value at (s,t) equals $\tilde{\mu}(\sigma(s+t), (2, e^{2\pi i|s|}))$, where $(2, e^{2\pi i|s|}) \in \mathcal{M}$.

Example 1.5. Similarly to the categories Δ and Λ, the epicyclic category $\tilde{\Lambda}$ has "an affine realization" by which we mean a functor $\tilde{\Lambda} \to$ **Top** whose value of an object $[n]$ of $\tilde{\Lambda}$ is the affine n-simplex Δ^n. The value of the functor on the morphism π_n^k is an affine map $\Delta^{k(n+1)-1} \to \Delta^n$ which sends a point $(s_1, s_2, \ldots, s_k)$ in the simplex $\Delta^{k(n+1)-1}$, viewed as a k-fold join $\Delta^n * \Delta^n * \cdots * \Delta^n$, to $(s_1 + s_2 + \cdots + s_k)$ in Δ^n. In what follows, the affine map $\Delta^{k(n+1)-1} \to \Delta^n$ is also denoted by π_n^k.

Remark 1.6. A natural question arises. What is the homotopy type of the classifying space $B\tilde{\Lambda}$ of the epicyclic category $\tilde{\Lambda}$. To answer the question one can try to adapt the argument given in [1, Proposition 1.12] (see also [7, Proposition 5.8]) based on Quillen's Theorem B [10], which shows that $B\Lambda \simeq BS^1$. Let $\tilde{j}: \Delta \to \tilde{\Lambda}$ be the inclusion functor. Proceeding as in [1] one obtains:

$$B([n]\backslash\backslash \tilde{j}) \simeq \mathrm{Hom}_{\tilde{\Lambda}}(-, [n])| = \coprod_{k \geq 1} S^1 \times_{\mathbb{Z}/k} \Delta^{k(n+1)-1},$$

cf. Remark 1.2. (Here $[n]\backslash\backslash \tilde{j}$ denotes the overcategory at the object $[n]$ for the functor j.) However, for a morphism $\gamma = \pi_n^l \lambda$ from $\tilde{\Lambda}$, the transition map

$$B(\gamma\backslash\backslash \tilde{j}): B([m]\backslash\backslash \tilde{j}) \to B([n]\backslash\backslash \tilde{j})$$

induces multiplication by l on the fundamental group, i.e. it is not a homotopy equivalence for $l > 1$. Consequently, we can not use Quillen's Theorem B here. We return to the problem of determining the homotopy type of $B\tilde{\Lambda}$ in Section 3.

2. Proof of Theorem A

Proof of (1). For X_* an epicyclic space let us define the map $\bar{\varphi}_k : |X_*| \to |X_*|$ to be the composite

$$|X_*| \xrightarrow{|p^k_*|} |\mathrm{Sd}_k X_*| \xrightarrow{D_k} |X_*|, \tag{2.1}$$

where D_k is the homeomorphism from Section 1. We are going to show that the maps φ_k have the following equivariant properties:

(i) $\bar{\varphi}_k(z^k x) = z\bar{\varphi}_k(x)$, for $z \in |X_*|$,

(ii) $\bar{\varphi}_l \circ \bar{\varphi}_k = \bar{\varphi}_{lk}$.

Assuming (i) and (ii), one checks easily that the following formula gives a right action $\tilde{\mu}$ of the monoid $\mathscr{M}$ on $|X_*|$.

$$\tilde{\mu}(x,(k,z)) = \bar{\varphi}_k(z^{-1}x). \tag{2.2}$$

To complete the proof it is enough to show that (i) and (ii) hold. Since p^k_* is a map of k-cyclic spaces, the following diagram commutes,

$$\begin{array}{ccc} |\mathrm{Sd}_k C_*| \times |X_*| & \xrightarrow{\mu_k} & |X_*| \\ {\scriptstyle 1 \times |p^k_*|} \downarrow & & \downarrow {\scriptstyle |p^k_*|} \\ |\mathrm{Sd}_k C_*| \times |\mathrm{Sd}_k X_*| & \xrightarrow{\mathrm{Sd}_k \mu} & |\mathrm{Sd}_k X_*| \end{array} \tag{2.3}$$

where μ_k and $\mathrm{Sd}_k\mu$ are the actions from (1.2) and (1.3). The diagram (2.3) composed with diagram (1.3) gives the following commutative square,

$$\begin{array}{ccc} S^1 \times |X_*| & \xrightarrow{\mu_k} & |X_*| \\ {\scriptstyle 1 \times \bar{\varphi}_k} \downarrow & & \downarrow {\scriptstyle \bar{\varphi}_k} \\ S^1 \times |X_*| & \xrightarrow{\mu} & |X_*| \end{array} \tag{2.4}$$

where we have identified $|\mathrm{Sd}_k C_*|$ and $|C_*|$ with S^1. Commutativity of (2.4) implies property (i). Property (ii) is an immediate consequence of relation (i) from Definition 1.1(2). □

Proof of (2). Since the nerve $N^{\mathrm{cyc}}_* G$ is an epicyclic set (cf. Example 1.3), formula (2.2) in the proof of part (1) gives a right action of the monoid $\mathscr{M}$ on $|N^{\mathrm{cyc}}_* G|$. To complete the proof it is enough to show that for any $k \geq 1$ the following diagram commutes,

$$\begin{array}{ccc} |N^{\mathrm{cyc}}_* G| & \xrightarrow{f} & \Lambda BG \\ {\scriptstyle \bar{\varphi}_k} \downarrow & & \downarrow {\scriptstyle \varphi_k} \\ |N^{\mathrm{cyc}}_* G| & \xrightarrow{f} & \Lambda BG \end{array} \tag{2.5}$$

where $\bar{\varphi}_k$ is the map (2.1) induced by the maps $p_n^k : N_n^{\text{cyc}} G \to N_{k(n+1)-1}^{\text{cyc}} G$ from the Introduction, and φ_k is the kth power map on the free loop space ΛBG. The map f was defined in [1] as a composition which in the present notation looks as follows:

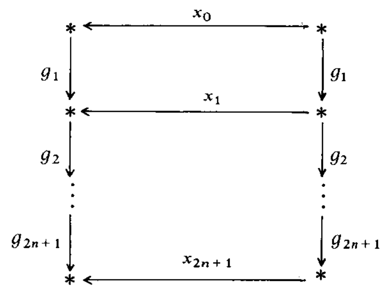
$$|N_*^{\text{cyc}} G| \to |N_* \text{Ad}\, G| \to |F(C_*, N_* G)| \to \Lambda BG. \tag{2.6}$$

(The first two maps are homeomorphisms, which identify the cyclic nerve of G with the nerve of the adjoint category of G and then with the simplicial function space $F(C_*, N_* G)$ which in [1] was denoted by $F_*(S_*^1, B_* G)$. The last map is the natural map from a simplicial mapping space to the topological mapping space.) Commutativity of (2.5) follows by a lengthy but straightforward diagram chase using formulas for f given in [1, Proof of Proposition 1.20] and the following lemma.

Lemma 2.1. *There exists a natural, simplicial isomorphism*

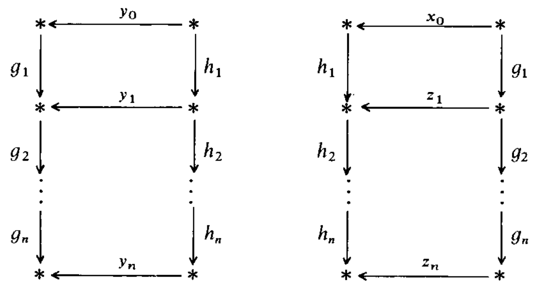
$$g : \text{Sd}_k F(C_*, N_* G) \to F(\text{Sd}_k C_*, N_* G).$$

Proof. To simplify notation we assume that $k = 2$. The general case follows in a similar way. First of all, as in [1], identify n-simplices of $\text{Sd}_2 F(C_*, N_* G)$ with diagrams:



(2.7)

where $g_i, x_i \in G$ and $x_i = g_i^{-1} x_{i-1} g_i$. Since the simplicial set $\text{Sd}_2 C_*$ has four non-degenerate simplices: two 0-dimensional and two 1-dimensional, we can identify n-simplices of $F(\text{Sd}_2 C_*, N_* \text{G})$ with pairs of diagrams,



(2.8)

where

$$y_i = h_i^{-1} y_{i-1} g_i, \quad z_i = g_i^{-1} z_{i-1} h_i, \qquad i = 1, 2, \ldots, n.$$

On n-simplices we define the map g by sending a diagram (2.7) to (2.8), where we put

$$z_i = g_{i+1}g_{i+2}\cdots g_{n+1+i},$$
$$y_i = x_{n+1+i}z_i^{-1},$$
$$h_i = g_{n+1+i}, \qquad i = 0, 1, 2, \ldots, n.$$

The inverse of g is given on n-simplices by sending a diagram (2.8) to (2.7), where we put

$$g_{n+1} = (g_1 g_2 \cdots g_n)^{-1} z_0,$$
$$g_{n+2} = h_1, g_{n+3} = h_2, \ldots, g_{2n+1} = h_n,$$
$$x_0 = z_0 y_0, x_1 = z_1 y_1, \ldots, x_n = z_n y_n,$$
$$x_{n+1} = y_0 z_0, x_{n+2} = y_1 z_1, \ldots, x_{2n+1} = y_n z_n. \qquad \square$$

Proof of (3). We adapt to the epicyclic case the proof of Theorem 5.12 from [7]. In the sequel, we use coends and homotopy colimits of space-valued functors, see e.g., [5, Section 1]. The homotopy colimit over $\tilde{\Lambda}^{\mathrm{op}}$ of an epicyclic space X_* will be denoted by hocolim X_*.

Definition 2.2. For a simplicial space X_*, let $\tilde{F}X_*$ be the epicyclic space defined by the following coend:

$$\tilde{F}X_* = \int_{[n]\in\Delta} X_n \times \mathrm{Hom}_{\tilde{\Lambda}}(-, [n]). \tag{2.2}$$

Let $\iota: X_* \to \tilde{F}X_*$ and $\nu: \tilde{F}(\tilde{F}X_*) \to \tilde{F}X_*$ be maps which are defined, respectively, by sending $x \in X_n$ to $[x, id_{[n]}] \in \tilde{F}X_n$ and by composition of morphisms in $\tilde{\Lambda}$. $\tilde{F}X_*$ is the free epicyclic space, on X_* in the sense that any simplicial map $X_* \to Y_*$, where Y_* is an epicyclic space, factors through ι. If X_* is epicyclic, let $\mathrm{ev}: \tilde{F}X_* \to X_*$ be the evaluation map: $\mathrm{ev}([x, \gamma]) = X(\gamma)(x)$. $\tilde{F}X_*$, ν and ι give rise to a monad $\tilde{F}$ on the category of simplicial spaces, cf. [9]. Let $B(\tilde{F}, \tilde{F}, X_*)$ denote the monadic two-sided bar construction whose space of n-simplices is the $(n+1)$-fold iterate $\tilde{F}(\cdots(\tilde{F}X_*)\cdots)$ with faces given by ev and the monadic multiplication ν and with degeneracies given by the monadic unit ι. Using standard methods, one can prove the following result (cf. [9, Proof of Theorem 9.10]).

Lemma 2.3. *If X_* is an epicyclic space, then the map* ev *induces a natural homotopy equivalence of homotopy colimits,*

$$\mathrm{hocolim}\,|B(\tilde{F}, \tilde{F}, X_*)| \to \mathrm{hocolim}\, X_*. \qquad \square$$

The proof of Theorem A(3) is based on the following two lemmas which are epicyclic versions of Theorem 5.3 and Proposition 5.11 from [7].

Lemma 2.4. *For a simplicial space X_*, there exists a natural homotopy equivalence,*

$$\rho_1 : |\tilde{F}X_*| \to |X_*| \times \mathcal{M}.$$

If X_* is epicyclic, then the following diagram commutes,

$$\begin{array}{ccc} |\tilde{F}X_*| & \xrightarrow{\rho_1} & |X_*| \times \mathcal{M} \\ & {\scriptstyle |\mathrm{ev}|}\searrow \quad \swarrow{\scriptstyle \tilde{\mu}} & \\ & |X_*| & \end{array} \tag{2.10}$$

where $\tilde{\mu}$ is the action constructed in Theorem A(1).

Lemma 2.5. *For a simplicial space X_*, there exists a natural homotopy equivalence*

$$\rho_2 : \operatorname{hocolim} \tilde{F}X_* \to |X_*|$$

such that the following diagram commutes,

$$\begin{array}{ccc} \operatorname{hocolim} \tilde{F}(\tilde{F}X_*) & \xrightarrow{\rho_2} & |\tilde{F}X_*| \\ {\scriptstyle \operatorname{hocolim} \nu}\downarrow & & \downarrow \\ \operatorname{hocolim} \tilde{F}X_* & \xrightarrow{\rho_2} & |X_*| \end{array} \tag{2.11}$$

where the vertical map on the right side is the composition of the map ρ_1 from Lemma 2.4 *and projection onto the first factor of $|X_*| \times \mathcal{M}$.*

Granting these lemmas for the moment, we see that

$$\begin{aligned} \operatorname{hocolim} X_* &\simeq \operatorname{hocolim} |B(\tilde{F}, \tilde{F}, X_*)| \quad &&\text{(by Lemma 2.3)} \\ &\cong |n \to \operatorname{hocolim} |\tilde{F}^{n+1} X_*|| \quad &&\text{(by commutativity of coends)} \\ &\simeq |n \to |\tilde{F}^n X_*|| \quad &&\text{(by Lemma 2.5)} \\ &\simeq |n \to |X_*| \times \mathcal{M}^n| \quad &&\text{(by Lemma 2.4)} \\ &= B(|X_*|, \mathcal{M}, \mathrm{pt}), \end{aligned}$$

which proves part (3) of Theorem A.

Since the homotopy colimit of the one-point epicyclic space is $B\tilde{\Lambda}$, Theorem A(3) implies the following equivalence.

Corollary 2.6. $B\tilde{\Lambda} \simeq B\mathcal{M}$.

Proof. $B\tilde{\Lambda} \cong \operatorname{hocolim} \mathrm{pt} \simeq B(\mathrm{pt}, \mathcal{M}, \mathrm{pt}) = BM.$ □

Proofs of Lemmas 2.4 and 2.5 (sketch). We construct maps ρ_1 and ρ_2, leaving to the reader to fill in details of the proofs.

Define ρ_1 to be the following map:

$$|\tilde{F}X_*| \xrightarrow{\alpha_1} |X_*| \times |\tilde{F}\mathrm{pt}| \xrightarrow{\alpha_2} |X_*| \times \mathcal{M}. \tag{2.12}$$

The map α_1 in (2.12) is given by the formula

$$\alpha_1([x,\gamma,t]) = ([x,\gamma(t)],\ [\mathrm{pt},\gamma,t]), \tag{2.13}$$

where $x \in X_n$, $\gamma: [m] \to [n]$ is a morphism in $\tilde{\Lambda}$ and $\gamma(t)$ denotes value on $t \in \Delta^m$ of the affine map $\Delta^m \to \Delta^n$ associated to γ, as in Example 1.4. In order to define α_2 notice that by Remark 1.2 we have

$$|\tilde{F}\mathrm{pt}| = |\mathrm{Hom}_{\tilde{\Lambda}}(-,[0])| = \coprod_{k \geq 1} S^1 \times_{\mathbb{Z}/k} \Delta^{k-1}.$$

Collapsing simplices Δ^{k-1} to barycenters gives a retraction of $|\tilde{F}\mathrm{pt}|$ onto $\amalg_{k \geq 1} S^1/\mathbb{Z}/k$.

The last space is homeomorphic to the monoid $\mathcal{M}$ via a map induced by k-fold covers ω_k of S^1 which together with the retraction gives a homotopy equivalence between $|\tilde{F}\mathrm{pt}|$ and $\mathcal{M}$. For the map α_2 in (2.12) we take the Cartesian product of the identity on $|X_*|$ and the equivalence $|\tilde{F}\mathrm{pt}| \to \mathcal{M}$.

Define ρ_2 to be the following composite:

$$\operatorname{hocolim} \tilde{F}X_* \xrightarrow{\beta_1} \int_{[n] \in \Delta} X_n \times B(\tilde{\Lambda}//[n]) \xrightarrow{\beta_2} |X_*|. \tag{2.14}$$

The map β_1 in (2.14) is a homeomorphism obtained by formal properties of homotopy colimits as in [7, Proof of Proposition 5.11]. Here, $\tilde{\Lambda}//[n]$ denotes the comma category whose objects are morphisms $\gamma: [m] \to [n]$ in $\tilde{\Lambda}$ and whose morphisms are commutative triangles:

$$\begin{array}{ccc} [m_1] & & \\ & \searrow^{\gamma_1} & \\ \alpha \downarrow & & [n] \\ & \nearrow_{\gamma_2} & \\ [m_2] & & \end{array}$$

where α is a morphism in Δ. For β_2 in (2.14) we take a map induced on coends by a cosimplicial equivalence:

$$\tilde{d}_n: B(\tilde{\Lambda}//[n]) \to \Delta^n$$

To define $\tilde{d}_n$ one uses the formula for the equivalence $\tilde{d}_n: B(\tilde{\Lambda}//[n]) \to \Delta^n$ from [7, Proof of Proposition 5.11] together with the affine maps $\pi_n^k: \Delta^{k(n+1)-1} \to \Delta^n$ from Example 1.5. It follows easily that the map $\beta_2 = \int 1 \times \tilde{d}_n$ is an equivalence, cf. [5, Proposition 1.13]. □

3. Proof of Theorem B

Let $\mathcal{U}$ be the multiplicative group of positive rational numbers. Note that $\mathcal{U}$ is the group completion of the monoid of natural numbers. The monoid $\mathcal{M}$ acts on $\mathcal{U}$ and $\mathbb{N}$ by projection onto first factor $\mathcal{M} \to \mathbb{N}$ and multiplication in $\mathcal{U}$. In the sequel, we use the two-sided bar construction for the actions, cf. [9]. We will show that the obvious projection $B(\mathrm{pt}, \mathcal{M}, \mathbb{N}) \to B(\mathrm{pt}, \mathcal{M}, \mathrm{pt})$ is the universal covering of $B\mathcal{M} = B(\mathrm{pt}, \mathcal{M}, \mathrm{pt})$.

Lemma 3.1. $B(\mathrm{pt}, \mathcal{M}, \mathbb{N}) \simeq B(\mathrm{pt}, \mathcal{M}, \mathcal{U})$.

Lemma 3.2. $B(\mathrm{pt}, \mathcal{M}, \mathcal{U}) \simeq K(\mathbb{Q}, 2)$.

Since $\mathcal{U}$ acts freely on the space $B(\mathrm{pt}, \mathcal{M}, \mathcal{U})$ and the orbit space of the action is $B\mathcal{M}$, it follows from the lemmas that the universal covering of $B\mathcal{M}$ is $K(\mathbb{Q}, 2)$. The statement in Theorem B about the action of $\pi_1 B\mathcal{M}$ on $\pi_2 B\mathcal{M}$ can be easily deduced from proofs given below. We leave this to the reader.

Proof of Lemma 3.1. First of all observe that $\mathcal{U} = \varinjlim \mathcal{U}_n$, where $\mathcal{U}_n = \mathbb{N}$ and the direct system is indexed by the poset of natural numbers ordered by divisability. Maps $\mathcal{U}_n \to \mathcal{U}_{nl}$ of the system are given by multiplication by l. Since $B(\mathrm{pt}, \mathcal{M}, \mathcal{U}) = \varinjlim B(\mathrm{pt}, \mathcal{M}, \mathcal{U}_n)$, it is enough to show that maps

$$B(\mathrm{pt}, \mathcal{M}, \mathcal{U}_n) \to B(\mathrm{pt}, \mathcal{M}, \mathcal{U}_{nl}) \tag{3.1}$$

induced by the system maps $\mathcal{U}_n \to \mathcal{U}_{nl}$ are equivalences. To achieve this we identify $B(\mathrm{pt}, \mathcal{M}, \mathcal{U}_n)$ with the classifying space of the translation category $\mathcal{C}$ whose objects are natural numbers and whose morphisms are $m \xrightarrow{(k,z)} mk$, where (k, z) is an element of $\mathcal{M}$. Composition in $\mathcal{C}$ is given by the multiplication (0.1), in $\mathcal{M}$. Under the identification the map (3.1) becomes BF_l, where $F_l : \mathcal{C} \to \mathcal{C}$ is the functor defined by

$$F_l(m) = ml, \qquad F_l(m \xrightarrow{(k,z)} mk) = (ml \xrightarrow{(k,z)} mkl).$$

Let us define another functor $G_l : \mathcal{C} \to \mathcal{C}$ by the formulas

$$G_l(m) = m, \qquad G_l(m \xrightarrow{(k,z)} mk) = (m \xrightarrow{(k,z^l)} mk).$$

The multiplication law in $\mathcal{M}$ gives the following commutative diagram in the category $\mathcal{C}$:

$$\begin{array}{ccc} m & \xrightarrow{(l,1)} & ml \\ {\scriptstyle (k,z)}\downarrow & & \downarrow{\scriptstyle (k,z^l)} \\ mk & \xrightarrow{(l,1)} & mkl \end{array} \tag{3.2}$$

For a fixed l and varying $M \in \mathbb{N}$, $(k, z) \in \mathcal{M}$, the diagrams (3.2) define a natural transformation from $\mathrm{Id}_{\mathcal{C}}$ to the functor $F_l \circ G_l = G_l \circ F_l$. It follows by [11, Proposition 2.1] that BG_l is the homotopy inverse of BF_l, i.e., the map (3.1) is an equivalence as required. □

Remark 3.3. The category $\mathcal{C}$ used in the proof of Lemma 3.1 is isomorphic to the orbit category $\mathcal{O}(\mathcal{F})$, cf. [6, p. 72], where $\mathcal{F}$ is the family of all finite subgroups of S^1. Objects of $\mathcal{O}(\mathcal{F})$ are orbits $S^1/\mathbb{Z}/m$, for $m \geq 1$. Morphisms in $\mathcal{O}(\mathcal{F})$ and S^1-maps of the orbits $S^1/\mathbb{Z}/m \to S^1/\mathbb{Z}/mk$. Using homeomorphisms $S^1/\mathbb{Z}/n \to S^1$ induced by

n-fold covers ω_n of S^1, one can identify a morphism in $\mathcal{O}(\mathscr{F})$ as above with a map $S^1 \to S^1$ given, for some $(k,z) \in \mathscr{M}$, by sending $x \in S^1$ to $zx^k \in S^1$. Then Corollary 2.6 and Lemma 3.1 imply that the classifying space $B\mathcal{O}(\mathscr{F})$ is homotopy equivalent to the universal cover of the classifying space $B\tilde{\Lambda}$. Note that $B\mathcal{O}(\mathscr{F}) \simeq E\mathscr{F}/S^1$, where $E\mathscr{F}$ is the classifying space for the family $\mathscr{F}$, cf. [6].

Proof of Lemma 3.2. By the proof of Lemma 3.1 $B(\mathrm{pt}, \mathscr{M}, \mathscr{U}) \simeq B\mathscr{C}$, where $\mathscr{C}$ is the same category as before. Note that $B\mathscr{C} = \varinjlim B\mathscr{C}_N$, where $\mathscr{C}_N$ is the full subcategory of $\mathscr{C}$ whose objects are numbers m which divide $N \in \mathbb{N}$. Maps of the direct system are Bj_N^M, where $j_N^M : \mathscr{C}_N \to \mathscr{C}_M$ denotes the obvious inclusion functior for N dividing M. We will prove the following two claims:

(1) $B\mathscr{C}_N \simeq K(\mathbb{Z}, 2)$,

(2) $\pi_2(Bj_N^M)$ is multiplication by M/N.

The lemma follows from the claims, since the limit of the direct system with spaces and maps as in (1) and (2) equals $K(\mathbb{Q}, 2)$.

Proof of (1) Let $\mathscr{D}_N$ be the full subcategory of the category $\mathscr{C}_N$ whose only object is the number N. Clearly, $B\mathscr{D}_N \simeq BS^1 = K(\mathbb{Z}, 2)$. We will show that $B\mathscr{D}_N$ is a deformation retract of the space $B\mathscr{C}_N$. Let $i_N : \mathscr{D}_N \to \mathscr{C}_N$ be the obvious inclusion and let $r_N : \mathscr{C}_N \to \mathscr{D}_N$ be the functor defined by the formulas

$$r_N(m) = N, \qquad r_N(m \xrightarrow{(k,z)} mk) = (N \xrightarrow{(1,z^l)} N),$$

where $l = N/mk$. Obviously $r_N \circ i_N = \mathrm{Id}_{\mathscr{D}_N}$, and to complete the proof it is enough to construct a homotopy between $\mathrm{Id}_{B\mathscr{C}_N}$ and $Bi_N \circ Br_N$. For a fixed $N \in \mathbb{N}$ and varying $m \in \mathbb{N}$, $(k,z) \in \mathscr{M}$, the following commutative diagrams in $\mathscr{C}_N$ define a natural transformation $\mathrm{Id}_{\mathscr{C}_N} \Rightarrow i_N \circ r_N$ which induces the required homotopy. (cf. [11]):

$$\begin{array}{ccc} m & \xrightarrow{(N/m,1)} & N \\ {\scriptstyle (k,z)}\downarrow & & \downarrow{\scriptstyle (1,z^{N/mk})} \\ mk & \xrightarrow{(N/mk,1)} & N \end{array} \tag{3.3}$$

Proof of (2). To identify the effect of the map $Bj_N^M : B\mathscr{C}_N \to B\mathscr{C}_M$ on the homotopy group of $B\mathscr{C}_N \simeq K(\mathbb{Z}, 2)$, one uses the following composition of maps:

$$B\mathscr{C}_N \xrightarrow{Bi_N} B\mathscr{C}_N \xrightarrow{Bj_N^M} B\mathscr{C}_M \xrightarrow{Br_M} B\mathscr{D}_M. \tag{3.4}$$

The maps $B_{i_N}$and B_{r_M} are equivalences from the proof of claim (2). It follows easily that (3.4) is a map of degree M/N on $K(\mathbb{Z}, 2)$. $\square$

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