ON THE EO-ORIENTABILITY OF VECTOR BUNDLES

P. BHATTACHARYA AND H. CHATHAM

Abstract. We study the orientability of vector bundles with respect to a family of cohomology theories called EO-theories. The EO-theories are higher height analogues of real K-theory KO. For each EO-theory, we prove that the direct sum of i copies of any vector bundle is EO-orientable for some specific integer i. Using the splitting principal, we reduce to the case of the canonical line bundle over $\mathbb{CP}^\infty$. Our method involves understanding the action of an order p subgroup of the Morava stabilizer group on the Morava E-theory of $\mathbb{CP}^\infty$. Our calculations have another application: We determine the homotopy type of the S^1-Tate spectrum associated to the trivial action of S^1 on all EO-theories.

1. Introduction

Let KO and KU respectively denote real and complex K-theory. It is a standard fact that every complex vector bundle is KU-orientable. However, the Hopf bundle γ over S^2 is not KO-orientable: we can see directly that the Thom isomorphism (Equation 2.4) fails. In other words,

$$\mathrm{KO} \wedge \mathrm{Th}(\gamma) \not\simeq \mathrm{KO} \wedge \mathrm{S}^2_+.$$

Indeed, the Thom space of γ is equivalent to $\mathbb{CP}^2$ and $\mathrm{KO}\wedge\mathbb{CP}^2 \simeq \mathrm{KU}$. On the other hand, $\mathrm{KO} \wedge \mathrm{S}^2_+ \simeq \mathrm{KO} \vee \Sigma^2\mathrm{KO}$. The Thom isomorphism fails in this case because the attaching map of the 4-cell in $\mathbb{CP}^2$ is the first Hopf element $\eta \in \pi_3(\mathrm{S}^2)$ which is detected in the Hurewicz image of KO. However, $\gamma^{\oplus 2}$ is KO-orientable. In fact, $\xi^{\oplus 2}$ is KO-orientable for every complex vector bundle ξ (see Corollary 2.30).

Definition 1.1. Let R be a homotopy commutative ring spectrum and let ξ be a vector bundle. The R-*orientation order* of ξ is

$$\Theta(\mathrm{R}, \xi) \coloneqq \min \{\, d > 0 \ :\ \xi^{\oplus d} \text{ is R-orientable} \,\}.$$

If no such integer exists we say $\Theta(\mathrm{R}, \xi) = \infty$. Suppose that $\mathbb{D}$ is a division algebra over $\mathbb{R}$ (one of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$). The $\mathbb{D}$-*orientation order* of R is

$$\Theta(\mathrm{R}, \mathbb{D}) \coloneqq \max \{\, \Theta(\mathrm{R}, \xi) \ :\ \xi \text{ is a } \mathbb{D}\text{-vector bundle} \,\}.$$

Remark 1.2. If ξ is a $\mathbb{R}$-vector bundle, then $\xi\oplus\xi$ can be given a complex structure. Thus, $\Theta(\mathrm{R}, \mathbb{R})$ divides $2 \cdot \Theta(\mathrm{R}, \mathbb{C})$.

Example 1.3. As claimed above, $\Theta(\mathrm{KO}, \mathbb{C}) = 2$. Tilman Bauer [Bau, Lemma 2.1] showed that $\Theta(\mathrm{TMF}, \mathbb{C}) = 24$[1].

[1]Although [Bau] remains unpublished, this result is widely accepted.

The EO-theories are higher height analogues of KO. Associated to each finite height formal group Γ over a perfect field $\mathbb{F}$ of characteristic p, there is an associated Morava E-theory E_Γ with an action of the Morava Stabilizer group $\mathrm{Aut}(\Gamma)$. If C_p is a subgroup of $\mathrm{Aut}(\Gamma)$ then we let EO_Γ be the homotopy fixed point spectrum $\mathrm{EO}_\Gamma \coloneqq \mathrm{E}^{\mathrm{hC}_p}$. The homotopy type of EO_Γ depends on Γ and the conjugacy class of the subgroup $\mathrm{C}_p \subseteq \mathrm{Aut}(\Gamma)$. Consider the case when $p = 2$ and $\widehat{\mathbb{G}}_m$ is the multiplicative formal group over $\mathbb{F}_2$. There is a standard action of C_2 on $\widehat{\mathbb{G}}_m$, and there are identifications $\mathrm{E}_{\widehat{\mathbb{G}}_m} = \mathrm{KU}_2^\wedge$ and $\mathrm{EO}_{\widehat{\mathbb{G}}_m} = \mathrm{KO}_2^\wedge$.

It is of interest to understand the homotopy groups of EO-theories. More generally it is of interest to understand $\pi_*(\mathrm{E}_\Gamma^{\mathrm{hG}})$ whenever $\mathrm{G} \subset \mathrm{Aut}(\Gamma)$ is a finite group acting on a formal group. For instance, the Kervaire problem [HHR11] is a famous application. However, the homotopy groups of EO_Γ remain unknown except when $p = 2$ or Γ has height $p-1$.

In this paper, we find an upper bound on $\Theta(\mathrm{EO}_\Gamma, \mathbb{C})$:

Main Theorem 1.4. *Let p be a prime, let k be a positive integer, and set $n = (p-1)k$. Let Γ be a height n formal group over a perfect field of characteristic p. Let $\mathrm{C}_p \hookrightarrow \mathrm{Aut}(\Gamma)$ be any embedding and EO_Γ denote the corresponding homotopy fixed point spectrum $\mathrm{E}_\Gamma^{\mathrm{hC}_p}$. Then*

$$\Theta(\mathrm{EO}_\Gamma, \mathbb{C}) \quad \textit{divides} \quad p^{p^k-1}.$$

We rely on a splitting principal argument to show that

$$\Theta(\mathrm{EO}_\Gamma, \mathbb{C}) = \Theta(\mathrm{EO}_\Gamma, \gamma_1),$$

where γ_1 is the tautological complex line bundle over $\mathbb{CP}^\infty$ (see Corollary 2.30). Then we prove Main Theorem 1.4 by directly analyzing the homotopy type of $\mathrm{EO}_\Gamma \wedge \mathbb{CP}^\infty$.

Quillen proved [Ada74, Part II, Lemma 4.6] that an R-orientation of γ_1 is equivalent data to a homotopy ring map $\mathrm{MU} \to \mathrm{R}$. We prove a similarly flavored result. Let $\mathrm{MU}[n]$ be the Thom spectrum of the map

$$\varphi_n \colon \mathrm{BU} \to \mathrm{BU}$$

given by multiplication by n in the additive $\mathbb{E}_\infty$-structure. Theorem 2.28 relates R-orientations of $\gamma_1^{\oplus n}$ to $\mathbb{A}_\infty$-ring maps $\mathrm{MU}[n] \to \mathrm{R}$. Using Main Theorem 1.4 and Theorem 2.28 we prove:

Corollary 1.5. *There exists an $\mathbb{A}_\infty$-ring map $\mathrm{MU}[n] \to \mathrm{EO}_\Gamma$ whenever p^{p^k-1} divides n.*

The ring $\pi_*\mathrm{EO}_\Gamma$ is unknown in most of the cases of study so we cannot prove Main Theorem 1.4 using a direct approach. We can however compute the C_p-action on ${\mathrm{K}_\Gamma}^*\mathbb{CP}^\infty$, where K_Γ is the Morava K-theory associated to Γ (see Notation 3.5). We show that ${\mathrm{K}_\Gamma}^*\mathbb{CP}^\infty$ admits a "Free $\oplus$ Finite" decomposition as a C_p-module. Using techniques such as relative Adams spectral sequence we obtain the following EO_Γ-module splitting:

Main Theorem 1.6. *When $n = (p-1)k$, there exists a splitting*

$$\mathrm{EO}_\Gamma \wedge \mathbb{CP}^\infty_+ \simeq \mathcal{M} \vee \mathcal{F}$$

in the category of EO_Γ*-modules, where* $\mathcal{F} \simeq \bigvee_{\mathbb{N}} \mathrm{E}_\Gamma$ *is a wedge of Morava* E*-theories and* $\mathcal{M}$ *is a compact* EO_Γ*-module.*

In fact, we show that the splitting map for the compact summand $\mathcal{M}$ in Main Theorem 1.6 factors through

$$\mathrm{EO} \wedge \mathbb{CP}_+^{p^{k+1}-p^k-1}.$$

This leads us to the proof of Main Theorem 1.4. Main Theorem 1.6 has another application:

Main Theorem 1.7. *Let* S^1 *act trivially on* EO_Γ*. Then there is an equivalence of* K_Γ*-local objects*

$$\mathrm{EO}_\Gamma^{\mathrm{tS}^1} \simeq \prod_{-\infty<k<\infty} \mathrm{E}_\Gamma.$$

Remark 1.8. Main Theorem 1.6 is a generalization of the splitting [GM95, §15]

$$\mathrm{KO} \wedge \mathbb{CP}_+^\infty \simeq \mathrm{KO} \vee \bigvee_{k\geq 1} \Sigma^{4k-2}\mathrm{KU} \tag{1.9}$$

and Main Theorem 1.7 is a generalization of the fact that

$$\mathrm{KO}^{\mathrm{tS}^1} \simeq \lim_{i\to-\infty} \prod_{i<k<\infty} \mathrm{KU}$$

Remark 1.10. The second author studied EO_Γ-orientations in his previous work when Γ has height $p-1$. He proved [Cha19, Corollary 1.6] that in this case

$$\Theta(\mathrm{EO}, \mathbb{C}) = p.$$

Thus, when $n = p-1$ our bound is not sharp.

Remark 1.11. At $p = 2$, Kitchloo, Lorman and Wilson [KLW16, KW15] studied a similar problem. There is a C_2-action on height n Johnson-Wilson theory $\mathrm{E}(n)$. The C_2 fixed points are commonly called real Johnson-Wilson theory, denoted $\mathrm{ER}(n)$. In [KLW16, KW15], the authors use genuine C_2-equivariant homotopy theory to describe $\mathrm{ER}(n)_*\mathrm{BO}$. They deduce that $\Theta(\mathrm{ER}(n), \mathbb{R}) = 2^{n+1}$ and so that $\Theta(\mathrm{ER}(n), \mathbb{C}) = 2^n$. Hahn and Shi [HS17] proved that there is an unstructured ring map $\mathrm{ER}(n) \to \mathrm{EO}_\Gamma$, where Γ is any height n formal group over a perfect field of characteristic 2. Combining these two facts it follows that when $p = 2$ and Γ has height n, $\Theta(\mathrm{EO}_\Gamma, \mathbb{C}) = 2^n$. Thus, when $p = 2$ our bound on $\Theta(\mathrm{EO}_\Gamma, \mathbb{C})$ is not sharp.

Motivated by Remark 1.10 and Remark 1.11, we conjecture:

Conjecture 1.12. *When the formal group* Γ *has height* $n = (p-1)k$*,*

$$\Theta(\mathrm{EO}_\Gamma, \mathbb{C}) = p^k.$$

Our methodology: A technical overview

Let G be a finite group acts faithfully on a power series ring $k[\![x]\!]$ over a field k. The action of G on $k[\![x]\!]$ restricts to an action of G on $k(\!(x)\!)$, resulting in a Galois extension $k(\!(x)\!)$ over the fixed ring $k(\!(x)\!)^{\mathrm{G}}$. Thus, there is a G-module isomorphism

$$f\colon k(\!(x)\!)^{\mathrm{G}}[\mathrm{G}] \to k(\!(x)\!),$$

where $k(\!(x)\!)^{\mathrm{G}}[\mathrm{G}]$ denotes the group ring. Thus, the restriction

$$\hat{f}\colon k[\![x]\!]^{\mathrm{G}}[\mathrm{G}] \to k[\![x]\!]$$

is injective and has a finite cokernel. This applies, for instance, in the case that G is a finite subgroup of $\mathrm{Aut}(\Gamma)$ acting on $\mathrm{K}^0_\Gamma(\mathbb{CP}^\infty)$ (compare Corollary 3.29). The fact that the discussion above holds for any finite group G suggests that one might be able to generalize Main Theorem 1.4 by obtaining explicit bounds on $\Theta(\mathrm{E}^{\mathrm{hG}}, \mathbb{C})$ for G an arbitrary finite subgroup of $\mathrm{Aut}(\Gamma)$. Also note that

$$\lim_n \mathrm{K}^0_\Gamma(\mathbb{CP}^\infty_{-n}) \cong k(\!(x)\!)$$

is a free G-module. This suggests that there might be an equivalence

$$(\mathrm{E}^{\mathrm{hG}}_\Gamma)^{\mathrm{tS}^1} = \lim_n \mathrm{EO}_\Gamma \wedge \mathbb{CP}^\infty_{-n} \simeq \mathrm{E}(\!(x)\!)$$

generalizing Main Theorem 1.7 to arbitrary finite subgroups $\mathrm{G} \subset \mathrm{Aut}(\Gamma)$.

Let

$$\mathrm{N} = \prod_{g\in G} g(x)$$

be the norm element. There is an isomorphism $k[\![x]\!]^{\mathrm{G}} \cong k[\![\mathrm{N}]\!]$ [O+73, Theorem 1]. Multiplication by the norm N gives a C_p-equivariant K-theory Thom isomorphism

$$\begin{array}{ccc} \mathrm{K}^*\mathbb{CP}^\infty_+ & \longrightarrow & \mathrm{K}^*\mathbb{CP}^\infty_{|G|} \\ \wr\| & & \wr\| \\ k[\![x]\!] & \xrightarrow{\cdot \mathrm{N}} & x^{|G|}k[\![x]\!]. \end{array}$$

There is a relative Adams spectral sequence for the map $\mathrm{EO}_\Gamma \to \mathrm{K}_\Gamma$ and a C_p-equivariant K-theory Thom isomorphism induces an isomorphism of relative Adams E_2-pages. However, N is not necessarily a permanent cycle in the relative Adams spectral sequence so the isomorphism of E_2-pages need not respect differentials. We can see that some power of N is a permanent cycle: If $d_r(\mathrm{N}^i) \neq 0$ then the Leibniz rule implies that $d_r(\mathrm{N}^{pi}) = 0$. So for each i the element N^{p^i} survives longer than $\mathrm{N}^{p^{i-1}}$. Devinatz [Dev05, Proposition 2.5] showed that the relative Adams spectral sequence for every EO_Γ-module has a uniform horizontal vanishing line, but he does not give any estimate for where that line is. It follows that some power of N must survive and this power gives an EO_Γ-Thom isomorphism.

If we knew that the horizontal vanishing line in the homotopy fixed point spectral sequence for EO_Γ were at t, we would deduce that $\mathrm{N}^{p^{t-1}}$ survives and so that $\Theta(\mathrm{EO}_\Gamma, \gamma_1)$ divides p^t. Alternatively, if we knew $\pi_*\mathrm{EO}_\Gamma$, we could study the Atiyah-Hirzebruch spectral sequence (AHSS) for $\mathrm{EO}^*_\Gamma\mathbb{CP}^\infty_c$ and prove that an appropriate Thom class is a nonzero permanent cycle (see Lemma 2.10). These direct approaches seem not to work except perhaps when the height of Γ is $p-1$.

Our approach is to produce a topological splitting of $\mathrm{EO}_\Gamma \wedge \mathbb{CP}^\infty$ (Main Theorem 1.6) and then manipulate this splitting to prove the appropriate Thom isomorphism.

We produce our splitting of $\mathrm{EO}_\Gamma \wedge \mathbb{CP}^\infty$ by studying the C_p-action on the $\mathrm{E}_{\Gamma *}\mathbb{CP}^\infty$. The C_p-action on $\mathrm{E}_{\Gamma *}\mathbb{CP}^\infty$ can be understood by relating it to the coaction of the Steenrod algebra on $\mathrm{H}_*\mathbb{CP}^\infty$. Let $\mathcal{P}_*$ denote the even subalgebra of the dual Steenrod algebra. The zigzag

$$\begin{array}{ccc} \mathrm{BP}_*\mathrm{BP} & \longrightarrow & \mathcal{P}_* \\ \downarrow & & \\ \mathrm{K}_{\Gamma *}\mathrm{E}_\Gamma = \mathrm{Maps}(\mathrm{Aut}(\Gamma), \mathrm{K}_{\Gamma *}) & & \end{array} \tag{1.13}$$

enables us to relate the action of $\mathrm{Aut}(\Gamma)$ on $\mathrm{K}_{\Gamma *}\mathrm{X}$ to the $\mathcal{P}_*$-coaction on $\mathrm{H}_*\mathrm{X}$. We put an appropriate filtration on $\mathrm{K}_*[\mathrm{C}_p]$ so that the action of $\mathrm{K}_*[\mathrm{C}_p]$ on $\mathrm{K}_{\Gamma *}\mathbb{CP}^\infty$ is filtered with respect to AHSS-filtration. We show that the action of $\operatorname{gr}\mathrm{K}_*[\mathrm{C}_p]$ on $\operatorname{gr}\mathrm{K}_{\Gamma *}\mathbb{CP}^\infty$ is entirely determined by the coaction of the quotient $\mathcal{B}_*$ of $\mathcal{P}_*$ on $\mathrm{H}_*\mathbb{CP}^\infty$. This is now completely explicit, and we see directly that the coaction of $\mathcal{B}_*$ on $\mathrm{H}_*\mathbb{CP}^\infty$ has a "Free$\oplus$Finite" decomposition. We deduce that $\mathrm{K}_{\Gamma *}\mathbb{CP}^\infty$ also has a "Free $\oplus$ Finite" decomposition. Then using a relative Adams spectral sequence argument, we lift the splitting of $\mathrm{K}_{\Gamma *}\mathbb{CP}^\infty$ to a splitting of $\mathrm{EO}_\Gamma \wedge \mathbb{CP}^\infty$.

We use this splitting to prove that

$$\Theta(\mathrm{EO}_\Gamma, \gamma_1) = \Theta(\mathrm{EO}_\Gamma, \gamma_1^{\hat{\beta}-1})$$

for some specific integer $\hat{\beta}$ depending on the characteristic and the height of Γ. At this point, we know that $\Theta(\mathrm{EO}_\Gamma, \gamma_1^{\hat{\beta}-1})$ divides $\Theta(\mathbb{S}_{(p)}, \gamma_1^{\hat{\beta}-1})$ and the latter quantity was computed by Atiyah and Todd [AT60] and so this will complete our proof of Main Theorem 1.4.

There is a purely algebraic interpretation of our approach to computing the C_p-action on $\mathrm{E}_*\mathbb{CP}^\infty$. A related discussion appears in [Pet18, pages 26 –30]. There is an isomorphism between $\mathrm{K}_{\Gamma *}\mathbb{CP}^\infty$ and the Hopf algebra $\mathrm{Dist}(\Gamma)$ of distributions on Γ. The Hopf algebra $\mathrm{Maps}^{\mathrm{cts}}(\mathrm{Aut}(\Gamma), \mathrm{K}_\Gamma)$ is the algebra $\mathcal{O}_{\mathrm{AutSch}^*(\Gamma)}$ of functions on the strict automorphism scheme of Γ. Taking distributions on the action map $\mathrm{AutSch}^*(\Gamma) \times \Gamma \to \Gamma$ and passing to an adjoint map gives a coaction $\mathrm{Dist}(\Gamma) \to \mathcal{O}_{\mathrm{AutSch}^*(\Gamma)} \otimes \mathrm{Dist}(\Gamma)$. This fits into a commutative square:

$$\begin{array}{ccc} \mathrm{K}_{\Gamma *}\mathrm{E}_\Gamma & \longrightarrow & \mathrm{K}_{\Gamma *}\mathrm{E}_\Gamma \otimes \mathrm{K}_{\Gamma *}\mathbb{CP}^\infty_+ \\ \wr\| & & \wr\| \\ \mathrm{Dist}(\Gamma) & \longrightarrow & \mathcal{O}_{\mathrm{AutSch}^*(\Gamma)} \otimes \mathrm{Dist}(\Gamma) \end{array}$$

We wish to compute the $\mathcal{O}_{\mathrm{AutSch}^*(\Gamma)}$-coaction on $\mathrm{Dist}(\Gamma)$. The AHSS-filtration on $\mathrm{K}_{\Gamma *}\mathbb{CP}^\infty_+$ is the same as the Poincaré-Birkhoff-Witt filtration on Γ. The associated graded of Γ under the PBW-filtration is the additive formal group $\widehat{\mathbb{G}}_a$. The action of $\mathrm{AutSch}^*(\Gamma)$ on Γ is filtered, and the associated graded of $\mathrm{AutSch}^*(\Gamma)$ is a closed subring scheme of $\mathrm{AutSch}^*(\widehat{\mathbb{G}}_a)$. We can directly compute the action of $\mathrm{AutSch}^*(\widehat{\mathbb{G}}_a)$ on $\mathrm{Dist}(\widehat{\mathbb{G}}_a)$. To finish relating this story to homotopy theory, we observe that there are isomorphisms $\mathcal{O}_{\mathrm{AutSch}^*(\widehat{\mathbb{G}}_a)} \cong \mathcal{P}_*$ and $\mathrm{Dist}(\widehat{\mathbb{G}}_a) \cong \mathrm{H}_*\mathbb{CP}^\infty_+$

and there is a commutative diagram

$$\begin{tikzcd} \mathrm{H}_*\mathbb{CP}^\infty_+ \arrow[r] \arrow[d, "\cong"] & \mathcal{P}_* \otimes \mathrm{H}_*\mathbb{CP}^\infty_+ \arrow[d, "\cong"] \\ \mathrm{Dist}(\widehat{\mathbb{G}}_a) \arrow[r] & \mathcal{O}_{\mathrm{AutSch}^*(\widehat{\mathbb{G}}_a)} \otimes \mathrm{Dist}(\widehat{\mathbb{G}}_a). \end{tikzcd}$$

1.1. Organization of the paper

In Section 2 we review classical concepts of Orientation Theory and their connection with Chromatic Homotopy Theory. We prove that $\Theta(\mathrm{EO}, \mathbb{C}) = \Theta(\mathrm{EO}, \gamma_1)$ (the splitting principle) and we prove a generalization of Quillen's theorem: If $\Theta(\mathrm{R}, \mathbb{C}) = \delta$ then there is a map $\mathrm{MU}[\delta] \to \mathrm{R}$ (Theorem 2.28).

In Section 3, we show that $\mathrm{K}_{\Gamma *}\mathbb{CP}^\infty \cong \mathrm{Free} \oplus \mathrm{Finite}$ as a C_p-module.

In Section 4, we use a relative Adams spectral sequence to lift the "Free $\oplus$ Finite" splitting of $\mathrm{K}_{\Gamma *}(\mathbb{CP}^\infty)$ to a "Free$\oplus$Finite" splitting of $\mathrm{EO}_\Gamma \wedge \mathbb{CP}^\infty$ (Main Theorem 1.6). We see from this that $\Theta_{\mathrm{EO}}(\gamma_1) = \Theta_{\mathrm{EO}}(\gamma_1^{p^{k+1}-p^k-1})$ which proves the Main Theorem 1.4. Finally, we prove Main Theorem 1.7 by showing that an appropriate $\lim^1$ term is trivial.

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2. Background

In this section, we review Orientation Theory, Chromatic Homotopy Theory, and the relationship between them. At the end, we review and extend the well-known result due to Quillen [Qui69] that a complex orientation of a homotopy commutative spectrum R is equivalent data to a homotopy ring map $\mathrm{MU} \to \mathrm{R}$. Our general proof restricted to the known case is significantly different than the original result (see [Ada74, Pages 31–52]).

We begin with a review of Orientation Theory.

2.1. Thom spaces and Thom spectra

Suppose that ξ is a vector bundle over a base space B with total space E_ξ. The Thom space of ξ, written $\mathrm{Th}(\xi)$, is the cofiber of the projection map

$$\mathrm{Th}(\xi) \coloneqq \mathrm{Cofiber}(\mathrm{E}_\xi^\times \to \mathrm{B}),$$

where $\mathrm{E}_\xi^\times$ denotes the complement of the zero section. Although B is not a pointed space, the cofiber of an unpointed map has a canonical base point so the Thom space $\mathrm{Th}(\xi)$ is a pointed space. When $\xi = \underline{\mathbb{R}}^r$ is a trivial bundle of rank r then

$$\mathrm{Th}(\xi) \simeq \Sigma^r \mathrm{B}_+.$$

For this reason, we think of the Thom space $\mathrm{Th}(\xi)$ as a twisted suspension of B_+. The key example for us is:

Example 2.1. The Thom space $\mathrm{Th}\big((\gamma_1^n)^{\oplus k}\big)$ is equivalent to

$$\mathbb{CP}_k^{n+k} \coloneqq \mathrm{Cofiber}(\mathbb{CP}_+^{k-1} \to \mathbb{CP}_+^{n+k})$$

where k is a positive integer and $1 \le n \le \infty$.

If ξ is a virtual vector bundle, then ξ has no Thom space but it does have a Thom spectrum, denoted $\mathrm{M}(\xi)$. For example, when k is negative, the spectrum $\mathbb{CP}_k^{n+k}$ is defined as the Thom spectrum of the virtual bundle $(\gamma_1^n)^{\oplus k}$. A construction of the Thom spectrum can be found in [ABG+14, Definition 3.12]. Here we summarize that construction.

We will ensure that

$$\mathrm{M}(\xi \oplus \underline{\mathbb{R}}^r) = \Sigma^r \mathrm{M}(\xi) \tag{2.2}$$

so we can give a definition in the case that ξ is zero dimensional and use (2.2) in other cases. If R is a ring spectrum, the connected components $\pi_0\mathrm{R}$ form a ring. The space of units $\mathrm{GL}_1(\mathrm{R})$ is the union of the connected components of $\Omega^\infty(\mathrm{R})$ corresponding to elements of $\pi_0(\mathrm{R})^\times$. If R is an $\mathbb{A}_\infty$-ring spectrum then $\mathrm{GL}_1(\mathrm{R})$ is an $\mathbb{A}_\infty$-space and can be delooped once to obtain a classifying space $\mathrm{BGL}_1(\mathrm{R})$. Let ξ be a zero-dimensional virtual vector bundle and let $f_\xi \colon \mathrm{B} \to \mathrm{BO}$ be the classifying map of ξ. The composition

$$\mathrm{B} \xrightarrow{f_\xi} \mathrm{BO} \xrightarrow{\mathrm{J}} \mathrm{BGL}_1(\mathbb{S}) \xrightarrow{\mathrm{B}(\iota)} \mathrm{BGL}_1(\mathrm{R})$$

classifies a principal $\mathrm{GL}_1(\mathrm{R})$-bundle $\mathrm{P}(\xi, \mathrm{R})$ which is a right $\mathrm{GL}_1(\mathrm{R})$-space. The Thom spectrum is defined as the derived smash product

$$\mathrm{M}(\xi, \mathrm{R}) \coloneqq \mathrm{P}(\xi, \mathrm{R})_+ \underset{\mathrm{GL}_1(\mathrm{R})_+}{\wedge} \mathrm{R}.$$

Let $\mathrm{M}(\xi) \coloneqq \mathrm{M}(\xi, \mathbb{S})$. There is a natural weak equivalence

$$\mathrm{M}(\xi, \mathbb{S}) \wedge \mathrm{R} \to \mathrm{M}(\xi, \mathrm{R}).$$

Example 2.3. If $\xi \cong \widetilde{\xi} - \underline{\mathbb{R}}^r$, where $\widetilde{\xi}$ is a genuine vector bundle, then

$$\mathrm{M}(\xi) \simeq \Sigma^{\infty - r}\, \mathrm{Th}(\widetilde{\xi}).$$

2.2. Orientations of vector bundles with respect to a generalized cohomology theory

Suppose that ξ is a zero-dimensional virtual bundle over a base B and that R is an $\mathbb{A}_\infty$-ring spectrum. We say ξ is R-orientable if the principal $\mathrm{GL}_1(\mathrm{R})$-bundle $\mathrm{P}(\xi,\mathrm{R})$ is trivializable. A trivialization of $\mathrm{P}(\xi,\mathrm{R})$ is equivalent data to a null homotopy of the classifying map

$$\mathrm{B}(\iota)\circ \mathrm{J}\circ f_\xi\colon \mathrm{B}\to \mathrm{BGL}_1(\mathrm{R}).$$

Such a null homotopy leads to an equivalence of R-module spectra

$$\mathrm{M}(\xi)\wedge \mathrm{R}\simeq \mathrm{M}(\xi,\mathrm{R})\simeq \mathrm{B}_+\wedge \mathrm{R}, \tag{2.4}$$

which we call the homological Thom isomorphism. Dually, an R-orientation for ξ leads to an equivalence of $\mathrm{R}^{\mathrm{B}_+}$-modules

$$\mathrm{R}^{\mathrm{B}_+}\simeq \mathrm{R}^{\mathrm{M}(\xi)},$$

which we call the cohomological Thom isomorphism. On π_0, the cohomological Thom isomorphism sends a map $f\colon \Sigma^\infty \mathrm{B}_+\to \mathrm{R}$ to the composition

$$\mathrm{M}(\xi)\xrightarrow{\mathrm{id}_\mathrm{R}} \mathrm{M}(\xi)\wedge \mathrm{R}\xrightarrow{\simeq} \mathrm{B}_+\wedge \mathrm{R}\xrightarrow{f\wedge \mathrm{id}_\mathrm{R}} \mathrm{R}\wedge R\xrightarrow{\mu_\mathrm{R}} \mathrm{R}. \tag{2.5}$$

When f is the composition $\Sigma^\infty \mathrm{B}_+\to \mathbb{S}\to \mathrm{R}$ of the collapse map with the unit of R, we refer to the corresponding element of $\mathrm{R}^d(\mathrm{M}(\xi))$ as an R-*Thom class*.

We now define an R-orientation of a vector bundle in the more general case when R is a homotopy commutative ring spectrum which does not necessarily have an $\mathbb{A}_\infty$-structure. If R is admits $\mathbb{A}_\infty$-ring structure, this definition agrees with the previous definition.

Definition 2.6. Let R be a homotopy commutative ring spectrum. An R-*Thom class* of a real rank d virtual vector bundle ξ over a base space B is an element $u_\mathrm{R}\in \mathrm{R}^d(\mathrm{M}(\xi))$ such that for every $b\in \mathrm{B}$ the class

$$(u_\mathrm{R})|_b\in \mathrm{R}^d\mathrm{M}(\xi_b)\cong \mathrm{R}^d\mathrm{S}^d\cong \pi_0\mathrm{R}$$

is the unit map $\iota_\mathrm{R}\in \pi_0(\mathrm{R})$. If ξ has at least one R-Thom class, we say that ξ is R-*orientable*. An R-orientation of ξ is a choice of an R-Thom class of ξ.

Example 2.7. Every vector bundle is $\mathrm{H}\mathbb{F}_2$-orientable and has a unique $\mathrm{H}\mathbb{F}_2$-orientation. A vector bundle ξ is $\mathrm{H}\mathbb{Z}$-orientable if and only if it is orientable in the classical sense i.e., the first Steifel-Whitney class $w_1(\xi)$ vanishes. Every $\mathrm{H}\mathbb{Z}$-orientable vector bundle ξ has exactly two distinct $\mathrm{H}\mathbb{Z}$-orientations.

If ξ has an R-Thom class u_R, taking homotopy groups on both sides of the cohomological Thom isomorphism leads to an R_*-module isomorphism

$$(-)\cup u_\mathrm{R}\colon \mathrm{R}^*(\mathrm{B}_+)\xrightarrow{\sim} \mathrm{R}^{*+d}(\mathrm{M}(\xi)) \tag{2.8}$$

Recalling the intuition that the Thom space $\mathrm{Th}(\xi)$ is a twisted suspension of B_+, we see that a vector bundle ξ of dimension d is R-orientable exactly when R cannot distinguish between the twisted suspension $\mathrm{Th}(\xi)$ and the untwisted suspension $\mathrm{Th}(\underline{\mathbb{R}}^d)\simeq \Sigma^d\mathrm{B}_+$.

Remark 2.9. If a virtual vector bundle ξ is R-oriented and there is a homotopy ring map $f\colon \mathrm{R} \to \mathrm{S}$ then ξ is S-oriented too: Pushing forward the R-Thom class $u_{\mathrm{R}} \in \mathrm{R}^*(\mathrm{M}(\xi))$ along f leads to an S-Thom class

$$\mathrm{M}(\xi) \xrightarrow{u_{\mathrm{R}}} \mathrm{R} \xrightarrow{f} \mathrm{S}.$$

Since the sphere spectrum $\mathbb{S}$ is initial among homotopy commutative ring spectra, an $\mathbb{S}$-orientation of a bundle ξ leads to a natural R-orientation for any homotopy commutative ring spectrum R. Similarly, if ξ is $\mathrm{H}\mathbb{Z}$-oriented then it is HA-oriented for every ring A.

Lemma 2.10. *Let ξ be an $\mathrm{H}\mathbb{Z}$-oriented real d-dimensional virtual vector bundle over a connected space* B *with Thom class $u_{\mathbb{Z}}$. Let* R *be a homotopy commutative ring spectrum. Then ξ is* R*-orientable if and only if the* HR_0 *Thom class*

$$u_{\mathrm{R}_0} \in \mathrm{H}^d(\mathrm{M}(\xi);\mathrm{R}_0)$$

is a permanent cycle in the AHSS

$$\mathrm{E}_2^{s,t} \coloneqq \mathrm{H}^s(\mathrm{M}(\xi);\mathrm{R}_t) \Rightarrow \mathrm{R}^{s-t}(\mathrm{M}(\xi)). \tag{2.11}$$

Proof. Suppose that u_{R_0} is a permanent cycle. Let $u_{\mathrm{R}} \in \mathrm{R}^d(\mathrm{M}(\xi))$ be an element detected by u_{R_0}. We will show that u_{R} is an R-Thom class for ξ. For any $b \in \mathrm{B}$, there is an associated map $\iota\colon \mathrm{S}^d \to \mathrm{M}(\xi)$. This map is independent of b up to homotopy because B is connected. The map ι induces a map ι^* of AHSS from (2.11) to

$$\mathrm{E}_2^{s,t} \coloneqq \mathrm{H}^s(\mathrm{S}^d;\mathrm{R}_t) \Rightarrow \mathrm{R}^{s-t}(\mathrm{S}^d) \tag{2.12}$$

which is concentrated in a single filtration and collapses at the E_2 page. We claim $\iota^*(u_{\mathrm{R}})$ represents the class

$$\mathrm{id}_{\mathrm{R}} \in \mathrm{R}_0 \cong \mathrm{R}^d(\mathrm{S}^d).$$

To see this, let $\tau_{\geq 0}\mathrm{R}$ denote the connective cover of R and consider the zigzag of ring maps

$$\mathrm{HR}_0 \longleftarrow \tau_{\geq 0}\mathrm{R} \longrightarrow \mathrm{R}. \tag{2.13}$$

which induce injections in π_0. The claim follows by comparing images of u_{R_0} in the Atiyah-Hirzebruch spectral sequences computing $\mathrm{HR}_0^*\mathrm{M}(\xi)$, $\tau_{\geq 0}\mathrm{R}^*\mathrm{M}(\xi)$ and $\mathrm{R}^*(\mathrm{M}(\xi))$ along the zigzag (2.13). We deduce that u_{R} is an R-Thom class for ξ, and hence that ξ is R-orientable.

Conversely, suppose that ξ is R-orientable. Using the R-Thom isomorphism we see that the composite

$$\mathrm{S}^d \wedge \mathrm{R} \xrightarrow{\ \iota\ } \mathrm{M}(\xi) \wedge \mathrm{R} \xrightarrow{\mathrm{Thom}} \Sigma^d \mathrm{B}_+ \wedge \mathrm{R} \longrightarrow \Sigma^d \mathrm{R}$$

represents the identity map. This means that $\Sigma^d\mathrm{R}$ is a summand of $\mathrm{M}(\xi) \wedge \mathrm{R}$. We deduce that every element of the d line $\mathrm{E}_2^{d,*}$ in (2.11) is a permanent cycle. In particular, u_{R_0} is a permanent cycle. ∎

2.3. James Periodicity and the Spherical Orientation Order of γ_1^k

Let $\mathbb{D}$ be a division algebra over $\mathbb{R}$. There are exactly three division algebras over $\mathbb{R}$: $\mathbb{D}$ is one of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $|\mathbb{D}|$ denote the dimension of $\mathbb{D}$ as an $\mathbb{R}$-vector space dimension. Let $\mathbb{DP}^k$ denote the corresponding projective space of lines in $\mathbb{D}^{k+1}$. In this subsection, we recall the classical problem of determining $\Theta(\mathbb{S}, \gamma_1^k(\mathbb{D}))$, where $\gamma_1^k(\mathbb{D})$ is the canonical $\mathbb{D}$-line bundle over $\mathbb{DP}^k$.

The Stiefel manifold $\mathrm{V}_k(\mathbb{D}^n)$ is the space of orthonormal k-frames in $\mathbb{D}^n$. There is a natural forgetful map

$$f_{\mathbb{D}}(n,k)\colon \mathrm{V}_k(\mathbb{D}^n) \to \mathrm{V}_1(\mathbb{D}^n) \simeq \mathrm{S}^{n|\mathbb{D}|-1}.$$

A section of $f_{\mathbb{D}}(n,k)$ exists if and only if there exists a set of $k-1$ everywhere linearly independent $\mathbb{D}$-vector fields on $\mathrm{S}^{n|\mathbb{D}|-1}$. Determining for which integers n and k the map $f_{\mathbb{R}}(n,k)$ has a section is equivalent to the classical vector fields on spheres problem. In 1958, I. M. James showed that:

Theorem 2.14 ([Jam58, Theorem 1.3 and Theorem 1.4])**.** *If $\mathbb{D} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ is a division algebra and k is a positive integer, there exists a positive integer n such that $f_{\mathbb{D}}(n,k)$ admits a section. Let $d(\mathbb{D},k)$ be the minimum such n.*

- *(i) For $\mathbb{D} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, $f_{\mathbb{D}}(n,k)$ admits a section if n is a positive multiple of $d(\mathbb{D},k)$.*
- *(ii) Conversely, if $\mathbb{D} \in \{\mathbb{C}, \mathbb{H}\}$ then $f_{\mathbb{D}}(n,k)$ admits a section only if n is a positive multiple of $d(\mathbb{R},k)$.*

(James also proved a more complicated converse in the case that $\mathbb{D} = \mathbb{R}$.)

Theorem 2.15 ([Jam59])**.** *Suppose that $\mathbb{D} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and that $f_{\mathbb{D}}(n,k)$ has a section. Let $\gamma_1^k(\mathbb{D})$ be the tautological line bundle over $\mathbb{DP}^k$. Then $n\gamma_1^k(\mathbb{D})$ is spherically trivial.*

Corollary 2.16. *The $\mathbb{S}$-orientation order of γ_1^k is $\Theta(\mathbb{S}, \gamma_1^k) = d(\mathbb{C},k)$.*

Remark 2.17. From Corollary 2.16, we see that $\Theta(\mathbb{S}_{(p)}, \gamma_1^k) = p^{\nu_p(d(\mathbb{C},k))}$.

Using the equivalence $\mathrm{Th}(n\gamma_1^k(\mathbb{D})) \simeq \mathbb{DP}_n^{k+n}$ one recovers the following result:

Corollary 2.18 (James Periodicity)**.** *Let k be a positive integer and suppose $d(\mathbb{R},k)$ divides n. Then there is an equivalence*

$$\Sigma^\infty \mathbb{DP}_n^{k+n} \simeq \Sigma^{n|\mathbb{D}|}\Sigma^\infty \mathbb{DP}_+^k.$$

James posed the problem of computing the quantity $d(\mathbb{D},k)$. In 1942, Hurwitz and Radon [Eck42] proved an upper bound on $d(\mathbb{R},k)$ by studying Clifford algebras. Atiyah and Todd [AT60] proved an upper bound on $d(\mathbb{C},k)$ in 1959 using the Chern character and characteristic series. In 1961, Adams [Ada62] proved a lower bound on $d(\mathbb{R},k)$ that is equal to the upper bound of [Eck42] thereby solved the vector fields on spheres problem. In 1965, Adams and Walker [AW65] proved a lower bound on $d(\mathbb{C},k)$ equal to the upper bound of [AT60], completing the computation of $d(\mathbb{C},k)$. In 1973, Sigrist and Suter [SS73] calculated $d(\mathbb{H},k)$. Feder and Gitler [FG77] found an exact characterization of when there is an equivalence $\Sigma^\infty \mathbb{D}\mathbb{P}^{b+l}_{a+l} \simeq \Sigma^{l|\mathbb{D}|}\Sigma^\infty\mathbb{D}\mathbb{P}^b_a$ as in Corollary 2.18 when $\mathbb{D} \in \{\mathbb{C}, \mathbb{H}\}$.

We will use the formula for $d(\mathbb{C}, k)$ so we quote it here:

Formula 2.19 ([AT60], [AW65])**.** For each prime p, the p-adic valuation of $d(\mathbb{C}, k)$ is given by

$$\Theta(\mathbb{S}_{(p)}, \gamma_1^k) = \nu_p(d(\mathbb{C}, k)) = \begin{cases} \max\left\{r + \nu_p(r) \ : \ 1 \le r \le \left\lfloor \frac{k}{p-1} \right\rfloor\right\} & \text{if } p \le k+1 \\ 0 & \text{if } p > k+1 \end{cases}$$

2.4. Complex Orientable Ring Spectra and Chromatic Homotopy Theory

In this subsection, we review the notion of a complex orientable ring spectrum and the connection to Chromatic Homotopy Theory.

A complex orientation of a homotopy commutative ring spectrum R is a choice of an R-Thom class for the tautological complex line bundle γ_1. There is an equivalence $\mathrm{M}(\gamma_1 - 1) = \Sigma^{-2}\Sigma^\infty\mathbb{C}\mathbb{P}^\infty$, which leads to the standard definition of a complex orientation:

Definition 2.20. A *complex orientation* of a homotopy commutative ring spectrum R is a map

$$u_{\mathrm{R}} \colon \Sigma^{-2}\Sigma^\infty\mathbb{C}\mathbb{P}^\infty \to \mathrm{R}$$

such that the following diagram is homotopy commutative:

$$\begin{tikzcd} \mathbb{S} \arrow[r, "1_{\mathrm{R}}"] \arrow[d] & \mathrm{R} \\ \Sigma^{-2}\Sigma^\infty\mathbb{C}\mathbb{P}^\infty \arrow[ur, "u_{\mathrm{R}}"'] & \end{tikzcd}$$

Proposition 2.21. *A complex orientation of* R *determines an* R*-orientation of every complex vector bundle* ξ*.*

This theorem is a version of the splitting principal. Applying Proposition 2.21 to the universal bundle over BU, we get a unital map $\mathrm{MU} \to \mathrm{R}$. The map $\mathrm{MU} \to \mathrm{R}$ is an R-Thom class for the tautological zero-dimensional virtual bundle classified by the identity map $\mathrm{id} \colon \mathrm{BU} \to \mathrm{BU}$. In fact:

Theorem 2.22. *The unital map* $\mathrm{MU} \to \mathrm{R}$ *determined by the* R*-orientation for the universal zero-dimensional virtual vector bundle is a homotopy ring map.*

If R is a complex orientable ring spectrum, the multiplication map $\mu \colon \mathbb{C}\mathbb{P}^\infty \times \mathbb{C}\mathbb{P}^\infty \to \mathbb{C}\mathbb{P}^\infty$ induces a map

$$\mathrm{R}^*\mathbb{C}\mathbb{P}^\infty_+ \to \mathrm{R}^*(\mathbb{C}\mathbb{P}^\infty_+ \wedge \mathbb{C}\mathbb{P}^\infty_+)$$

which gives a formal group over $\pi_*\mathrm{R}$. A choice of complex orientation of a ring spectrum R determines isomorphisms

$$\mathrm{R}^*\mathbb{C}\mathbb{P}^\infty_+ \cong \mathrm{R}_*[\![x]\!]$$
$$\mathrm{R}^*(\mathbb{C}\mathbb{P}^\infty_+ \wedge \mathbb{C}\mathbb{P}^\infty_+) \cong \mathrm{R}_*[\![x, y]\!]$$

and corresponds to a choice of coordinate on the formal group $\mathrm{R}^*\mathbb{CP}^\infty_+$.

There is a canonical complex orientation of MU represented by the identity map $\mathrm{MU} \to \mathrm{MU}$. Quillen [Ada74, Theorem 8.2] observed that the formal group law associated to this complex orientation on MU is the universal formal group law. If R is any other complex oriented ring spectrum, then there is a homotopy ring map $\mathrm{MU} \to \mathrm{R}$ such that the corresponding ring map

$$\pi_*\mathrm{MU} \to \pi_*\mathrm{R}$$

classifies the formal group law F_R over $\pi_*\mathrm{R}$.

This leads to many realization problems, for example: If S is a graded ring and Γ is a formal group law over S, when is there a complex oriented ring spectrum R such that $\pi_*\mathrm{R} \cong \mathrm{S}$ and $\mathrm{R}^*\mathbb{CP}^\infty_+ \cong \Gamma$? If such a ring spectrum R exists, can it be given an $\mathbb{E}_n$-ring structure? If R is a complex oriented $\mathbb{E}_n$-ring spectrum, can the homotopy ring map $\mathrm{MU} \to \mathrm{R}$ be made into an $\mathbb{E}_n$-map?

Answers to some such questions exist. Most classically, Landweber [Lan76] proved that if (S, F) is a "flat" formal group then there is a complex oriented homotopy commutative ring spectrum R with $(\pi_*\mathrm{R}, \mathrm{R}^*\mathbb{CP}^\infty_+) \cong (\mathrm{S}, \mathcal{O}_\mathrm{F})$. Goerss, Hopkins, and Miller proved a much stronger result when Γ is the universal deformation of a formal group over a perfect field.

Let FmlGp be the category where objects are pairs (k, Γ) a perfect field k of characteristic p and a formal group Γ of finite height over k and where morphisms $(k, \Gamma) \to (k', \Gamma')$ are pairs (f, ϕ) where $f\colon k \to k'$ is a ring homomorphism and $\phi\colon f^*(\Gamma) \to \Gamma'$ is an isomorphism of formal groups. Lubin and Tate [LT66] constructed a formal group Γ^{LT} called *the universal deformation of* Γ over a ring R^{LT} called *Lubin-Tate ring of* Γ. The formal group Γ^{LT} is flat, so the Landweber exact functor theorem says that it is the formal group associated to a complex orientable homotopy commutative ring spectrum.

Goerss, Hopkins, and Miller proved:

Theorem 2.23 (Goerss-Hopkins-Miller [GH04])**.** *There is a functor*

$$\begin{aligned} \mathrm{FmlGp} &\longrightarrow \mathbb{E}_\infty\text{–Rings} \\ (k, \Gamma) &\longmapsto \mathrm{E}_\Gamma \end{aligned}$$

such that E_Γ *is complex orientable and there is a canonical isomorphism of formal groups*

$$(\pi_*\mathrm{E}_\Gamma, \mathrm{E}_\Gamma^*\mathbb{CP}^\infty_+) \cong (\mathrm{R}^{\mathrm{LT}}, \mathcal{O}_{\Gamma^{\mathrm{LT}}})$$

where $\mathcal{O}_{\Gamma^{\mathrm{LT}}}$ *is the ring of functions on* Γ^{LT}.

Remark 2.24. Lurie [Lur18] has produced an improvement of Theorem 2.23 that works for families of formal groups that can be embedded into a p-divisible group in a sufficiently nice manner. His theory applies to the various versions of tmf for example. Lurie's theory does not apply to MU. It is an open question to determine whether or not there is an $\mathbb{E}_\infty$-ring map from MU to any Morava E-theory of height greater than two.

By Theorem 2.23 there is an action of $\mathrm{Aut}(\Gamma)$ on E_Γ by $\mathbb{E}_\infty$-maps.

Example 2.25. If we choose $\Gamma = \widehat{\mathbb{G}}_m$ the multiplicative formal group over $\mathbb{F}_p$ then $\mathrm{E}_{\widehat{\mathbb{G}}_m}$ is completed complex K theory

$$\mathrm{E}_{\widehat{\mathbb{G}}_m} \simeq \mathrm{KU}_p^\wedge$$

and $\mathrm{Aut}(\widehat{\mathbb{G}}_m) \cong (\mathbb{Z}_p)^\times$. In particular, $\widehat{\mathbb{G}}_m$ has an involution $x \mapsto x^{-1}$ which generates a $\mathrm{C}_2 \subseteq \mathrm{Aut}(\widehat{\mathbb{G}}_m)$.

$$\mathrm{E}_{\widehat{\mathbb{G}}_m}^{\mathrm{hC}_2} \simeq \mathrm{KO}_p^\wedge.$$

Although $\mathrm{KO}_2^\wedge$ is not complex orientable, it is complex 2-orientable as defined in Section 2.5.

2.5. Complex n-Orientations and the Splitting Principal

We say that a homotopy commutative ring spectrum R is *complex n-orientable* if the sum of n copies of the canonical line bundle $n\gamma_1$ over $\mathbb{CP}^\infty$ is R-orientable. Equivalently, R is complex n-orientable if n divides $\Theta(\mathrm{R}, \gamma_1)$. A complex n-orientation of R is a choice of R-Thom class for $n\gamma_1 - n$.

There is an equivalence $\mathrm{M}(n\gamma_1 - n) = \Sigma^{-2n}\mathbb{CP}_n^\infty$. For comparison to the standard definition of a complex orientable ring (Definition 2.20), we give an analogous definition:

Definition 2.26. A *complex n-orientation* of a homotopy commutative ring spectrum R is a map $f\colon \Sigma^{-2n}\mathbb{CP}_n^\infty \to \mathrm{R}$ such that the diagram

$$\begin{tikzcd} \mathbb{S} \arrow[r, "1_{\mathrm{R}}"] \arrow[d] & \mathrm{R} \\ \Sigma^{-2n}\mathbb{CP}_n^\infty \arrow[ur, "f"'] & \end{tikzcd}$$

is homotopy commutative.

Let $\mathrm{MU}[n]$ be the Thom spectrum of the multiplication by n map $\varphi_n\colon \mathrm{BU} \to \mathrm{BU}$. Note that ϕ_n is an infinite loop map with respect the additive infinite loop structure on BU. In other words, the following diagram commutes:

$$\begin{tikzcd} \mathrm{BU} \arrow[r, "\varphi_n"] \arrow[d, phantom, "\wr|"] & \mathrm{BU} \arrow[d, phantom, "\wr|"] \\ \Omega^\infty\Sigma^2\mathrm{ku} \arrow[r, "\Omega^\infty(\cdot n)"] & \Omega^\infty\Sigma^2\mathrm{ku} \end{tikzcd}$$

Thus φ_n is an $\mathbb{E}_\infty$-map and $\mathrm{MU}[n]$ is an $\mathbb{E}_\infty$-ring spectrum.

Remark 2.27. If $f_\xi\colon X \to \mathrm{BU}$ is the classifying map of a virtual vector bundle ξ, then $\varphi_n \circ f_\xi$ classifies the bundle $\xi^{\oplus n}$.

Using the $\mathbb{E}_\infty$-structure on BU, the classifying map $f_{\gamma_1}\colon \mathbb{CP}^\infty \to \mathrm{BU}$ extends to an $\mathbb{E}_\infty$-map $\tilde{f}_{\gamma_1}\colon \mathrm{Q}\mathbb{CP}^\infty \to \mathrm{BU}$. Applying [Mil85, Corollary D], we deduce that $\Sigma^\infty\Sigma\mathbb{CP}^\infty$ is a split summand of $\Sigma^\infty\mathrm{SU}$ (we set $k = 1$ in the statement of Corollary D). Thus we get an $\mathbb{A}_\infty$-map

$$\tau\colon \mathrm{BU} \simeq \Omega\mathrm{SU} \to \Omega\mathrm{Q}\Sigma\mathbb{CP}^\infty \simeq \mathrm{Q}\mathbb{CP}^\infty$$

which splits ρ. This leads us to the following theorem.

Theorem 2.28. *If* R *is an* $\mathbb{E}_\infty$*-ring spectrum, a complex* n*-orientation of* R *determines an* $\mathbb{A}_\infty$*-ring map* $\mathrm{MU}[n] \to \mathrm{R}$.

Proof. A complex n-orientation of R is equivalent data to a null homotopy of the composite map

$$\mathbb{CP}^\infty \xrightarrow{f_{\gamma_1}} \mathrm{BU} \xrightarrow{\varphi_n} \mathrm{BU} \xrightarrow{\mathrm{J}} \mathrm{BGL}_1(\mathbb{S}) \xrightarrow{\iota} \mathrm{BGL}_1(\mathrm{R}) \tag{2.29}$$

Using the $\mathbb{E}_\infty$-structure on BU we extend γ_1 to an $\mathbb{E}_\infty$-map $\tilde{f}_{\gamma_1}\colon \mathrm{Q}\mathbb{CP}^\infty \to \mathrm{BU}$. Using the $\mathbb{E}_\infty$-structure on the map $\mathrm{BU} \to \mathrm{BGL}_1(\mathrm{R})$, the null homotopy of the map (2.29) extends to a null homotopy of the map

$$\mathrm{Q}\mathbb{CP}^\infty \xrightarrow{\tilde{f}_{\gamma_1}} \mathrm{BU} \xrightarrow{\varphi_n} \mathrm{BU} \xrightarrow{\mathrm{J}} \mathrm{BGL}_1(\mathbb{S}) \xrightarrow{\iota} \mathrm{BGL}_1(\mathrm{R})$$

in the category of $\mathbb{E}_\infty$-spaces. Let $\tau\colon \mathrm{BU} \to \mathrm{Q}\mathbb{CP}^\infty$ denote the splitting map. We have

$$\mathrm{J} \circ \varphi_n \circ \tilde{f}_{\gamma_1} \circ \tau \simeq \mathrm{J} \circ \varphi_n.$$

Since τ is an $\mathbb{A}_\infty$-map and all of the other maps are $\mathbb{E}_\infty$-maps,

$$\iota \circ \mathrm{J} \circ \varphi_n \circ \tilde{f}_{\gamma_1} \circ \tau$$

is null homotopic in the category of $\mathbb{A}_\infty$-spaces. The result follows. ☐

Corollary 2.30 (The splitting principle)**.** *If* R *is an* $\mathbb{E}_\infty$*-ring spectrum then*

$$\Theta(\mathrm{R}, \xi) \textit{ divides } \Theta(\mathrm{R}, \gamma_1)$$

for any virtual complex vector bundle.

Proof. By Remark 2.27, the classifying map of $\xi^{\oplus n}$ factors through φ_n. By Theorem 2.28, it follows that $\xi^{\oplus n}$ is R-orientable. ☐

Remark 2.31. Theorem 2.22 does not require the assumption that R is an $\mathbb{E}_\infty$-ring. The proof of Theorem 2.22 relies on directly computing $\mathrm{R}^*(\mathrm{BU}(n))$ and $\mathrm{R}^*(\mathrm{MU}(n))$ which is possible when R is complex orientable. It is unclear to us how to generalize the classical proof.

Question 2.32. If R is a homotopy commutative ring which is complex n-orientable, is there a homotopy ring map $\mathrm{MU}[n] \to \mathrm{R}$?

3. The C_p-action on the Morava K-theory of $\mathbb{CP}^\infty$

For any Morava E-theory, there is an action of the associated Morava Stabilizer group on E by $\mathbb{E}_\infty$-ring maps. If p is a prime and $p-1$ divides n, then the Morava Stabilizer group of height n at the prime p contains a subgroup of order p, call it C_p. Our goal in this section is to understand the action of C_p on the E-homology of $\mathbb{CP}^\infty$ modulo the maximal ideal of E_*. In other words, we analyze the action of C_p on the Morava K-homology of $\mathbb{CP}^\infty$. We do so by resorting to the zigzag of (1.13) and the knowledge of the action of the Steenrod algebra on $\mathrm{H}_*\mathbb{CP}^\infty$. We observe a "Free $\oplus$ Finite" decomposition of $\mathrm{K}_*\mathbb{CP}^\infty$ as a $\mathrm{K}_*[\mathrm{C}_p]$-module (Corollary 3.29).

We begin with a general discussion about formal groups, the associated Morava E-theories, and the Morava Stabilizer groups. Let $\mathbb{F}$ be a perfect field of characteristic p and let Γ be a formal group of finite height n over $\mathbb{F}$. Let $\mathrm{Aut}(\Gamma)$ denote the automorphism group of Γ. Theorem 2.23 gives us an associated $\mathbb{E}_\infty$-ring spectrum E_Γ which is called a Morava E-theory. Theorem 2.23 also gives an action of $\mathrm{Aut}(\Gamma)$ on E_Γ by $\mathbb{E}_\infty$-maps. Let C_p denote the finite group of order p and fix a faithful action $\iota\colon \mathrm{C}_p \hookrightarrow \mathrm{Aut}(\Gamma)$ of C_p on Γ. Let

$$\mathrm{EO}_\Gamma \coloneqq \mathrm{E}^{\mathrm{hC}_p}.$$

The ring spectrum EO_Γ depends on isomorphism type of (Γ, ι) in the category of formal groups with C_p-action. In particular, two actions ι and ι' yield isomorphic EO-theories if and only if ι and ι' are conjugate to each other.

We first study the question: For which formal groups Γ does $\mathrm{Aut}(\Gamma)$ contains a cyclic subgroup of order p? See [Buj12] for a complete characterization of the finite subgroups of the Morava stabilizer group. Let n be the height of Γ. Let $\widetilde{\mathbb{F}} = \mathbb{F}_{p^n} \cdot \mathbb{F}$ be the compositum and let $\widetilde{\Gamma}$ be the base change of Γ to $\widetilde{\mathbb{F}}$. The ring $\mathrm{Aut}(\widetilde{\Gamma})$ has a convenient explicit description. The ring $\mathrm{End}(\widetilde{\Gamma})$ has a natural valuation map

$$\nu\colon\ \mathrm{End}(\widetilde{\Gamma}) \to \mathbb{Q} \tag{3.1}$$

given as follows: If $f \in \mathrm{End}(\widetilde{\Gamma})$ and x is a coordinate on $\widetilde{\Gamma}$ then $f(x) \in \mathbb{F}[\![x]\!]$ is a power series. In this case $f'(x)\,dx$ is an invariant 1-form on $\widetilde{\Gamma}$, so by standard Lie theory if $f'(0) = 0$ then $f' \equiv 0$. Thus, if the leading term of $f(x)$ vanishes, then f factors through the Frobenius ϕ. If $f \neq 0$, then repeating this factorization as many times as possible we can write $f(x) = g(\phi^v(x))$ where $g'(0) \neq 0$. The implicit function theorem says that g is an isomorphism. Setting $\nu(f) = v$ gives a well-defined valuation on $\mathrm{End}(\widetilde{\Gamma})$. This makes $\mathrm{End}(\widetilde{\Gamma})$ into a noncommutative valuation ring. The image of ν in (3.1) is $\frac{1}{n}\mathbb{Z}$. Inverting p in $\mathrm{End}(\widetilde{\Gamma})$ gives a division algebra, call it $\mathbb{D}(\widetilde{\Gamma})$. In this case $\mathrm{End}(\widetilde{\Gamma})$ is the ring of integers

$$\mathrm{End}(\widetilde{\Gamma}) = \mathcal{O}_{\mathbb{D}(\widetilde{\Gamma})},$$

where

$$\mathcal{O}_{\mathbb{D}(\widetilde{\Gamma})} \coloneqq \{\, f \in \mathbb{D}(\widetilde{\Gamma}) \ :\ \nu(f) \geq 0 \,\}.$$

Then $\mathrm{Aut}(\widetilde{\Gamma}) = \mathrm{End}(\widetilde{\Gamma})^\times$ is the ring of integral units in $\mathbb{D}(\widetilde{\Gamma})$:

$$\mathrm{Aut}(\widetilde{\Gamma}) \cong \mathcal{O}^\times_{\mathbb{D}(\widetilde{\Gamma})}, \tag{3.2}$$

The endomorphism ring of $\widetilde{\Gamma}$ has the explicit description

$$\mathrm{End}(\widetilde{\Gamma}) \cong \mathbb{W}(\mathbb{F}_{p^n})\langle \mathrm{T} \rangle / (\mathrm{T}a - a^\phi \mathrm{T}, \mathrm{T}^n - up), \tag{3.3}$$

where T is a uniformizer of $\mathrm{End}(\widetilde{\Gamma})$ and $u \in \mathbb{W}(\mathbb{F}_{p^n})^\times$ is a unit depending on $\widetilde{\Gamma}$ and T.

From the identification (3.3), one can check that the center of $\mathbb{D}(\widetilde{\Gamma})$ is equal to $\mathbb{Q}_p$, so $\mathbb{D}(\widetilde{\Gamma})$ is a central division algebra over $\mathbb{Q}_p$ and represents an element of the Brauer group $\mathrm{Br}(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}$. Because $\nu(\mathrm{T}) = 1/n$ we see that $\mathbb{D}(\widetilde{\Gamma})$ has Hasse invariant

$$\frac{1}{n} \in \mathbb{Q}/\mathbb{Z} \cong \mathrm{Br}(\mathbb{Q}_p).$$

A theorem of Serre ([Ser67, page 138]) says that a central division algebra over $\mathbb{Q}_p$ of Hasse invariant $\frac{a}{b}$ with $(a,b)=1$ contains a field extension L of $\mathbb{Q}_p$ if and only if the degree of L over $\mathbb{Q}_p$ divides b. If ζ is a pth root of unity then $\mathbb{Q}_p(\zeta)$ is a field extension of degree $p-1$. Thus if $p-1$ divides n then by Serre's theorem there is a map $\mathbb{Q}_p(\zeta)\to\mathbb{D}(\widetilde{\Gamma})$. Because the equation for ζ is integral, ζ lies in $\mathcal{O}_{\mathbb{D}(\widetilde{\Gamma})}$. In this case we see that

$$\{1,\zeta,\ldots,\zeta^{p-1}\}\subset\mathcal{O}^\times_{\mathbb{D}(\widetilde{\Gamma})}\cong\mathrm{Aut}(\widetilde{\Gamma})$$

is a cyclic subgroup of $\mathrm{Aut}(\widetilde{\Gamma})$ of order p. By the Skolem-Noether theorem, the subgroup C_p is unique up to conjugation. If $\mathbb{F}$ does not contain $\mathbb{F}_{p^n}$ then $\mathrm{Aut}(\Gamma)$ need not contain a subgroup of order p. If $\mathrm{Aut}(\Gamma)$ does contain a subgroup of order p, it need not be unique up to conjugation.

Remark 3.4. Using Lubin-Tate theory [LT65], it is easy to produce formal groups defined over $\mathbb{F}_p$ with C_{p^d}-actions, but it is difficult in general to find nice coordinates on the formal groups produced in this way. If we are interested in C_p-actions we can find a formal group with a nice coordinate: the formal group law defined over $\mathbb{F}_p$ with $[-p](x)=x^{p^{k(p-1)}}$ admits a C_p-action.

Notation 3.5. We will use the following notations and conventions for the remainder of the paper.

- Fix a prime p and an integer k. Let $n=k(p-1)$.
- Fix a perfect field $\mathbb{F}$ of characteristic p and a formal group Γ of height n over $\mathbb{F}$. Let E_Γ denote the associated Lubin-Tate theory. By Lubin-Tate theory
$$\pi_*\mathrm{E}_\Gamma\cong\mathbb{W}(\mathbb{F})[\![u_1,\ldots,u_{n-1}]\!][u^{\pm}],$$
where $\mathbb{W}(\mathbb{F})$ is the ring of Witt vectors over $\mathbb{F}$. The elements $u_1,\ldots,u_{n-1}$ are elements of $\pi_0\mathrm{E}_\Gamma$ and $u\in\pi_{-2}\mathrm{E}_\Gamma$.
- Let $\mathfrak{m}$ denote the maximal ideal $(p,u_1,\ldots,u_{n-1})$ of $\pi_*\mathrm{E}_\Gamma$ and let K_Γ denote the corresponding height n Morava K-theory so that
$$\pi_*\mathrm{K}_\Gamma\simeq\pi_*(\mathrm{E}_\Gamma)/\mathfrak{m}.$$
- Fix an faithful action $\iota\colon \mathrm{C}_p\to\mathrm{Aut}(\Gamma)$ of C_p on Γ and let
$$\mathrm{EO}_\Gamma\coloneqq\mathrm{E}_\Gamma^{\mathrm{hC}_p}.$$
- Abbreviate E_Γ by E, K_Γ by K, and EO_Γ by EO, leaving the dependence on Γ and ι implicit.
- Let $\mathrm{E}^\wedge_*(-)\coloneqq\pi_*(\mathrm{L_K}(\mathrm{E}\wedge-))$, $\mathrm{E}^{\mathrm{EO}}_*(-)\coloneqq\pi_*(\mathrm{E}\wedge_{\mathrm{EO}}-)$, and $\mathrm{K}^{\mathrm{EO}}_*(-)=\pi_*(\mathrm{K}\wedge_{\mathrm{KO}}-)$.
- Fix a p-typical coordinate on Γ and let $\pi_{\mathrm{E}}\colon\mathrm{BP}\to\mathrm{E}$ be the associated map. Let $\pi_{\mathrm{K}}\colon\mathrm{BP}\to\mathrm{K}$ denote the composite of π_{E} with the reduction map from $\mathrm{E}\to\mathrm{K}$.
- Let $\pi_{\mathbb{F}_p}\colon\mathrm{BP}\to\mathrm{H}\mathbb{F}_p$ be the standard reduction map.

Remark 3.6. The Bousfield class of E_Γ only depends on the prime p and the height of Γ. The same is true for K_Γ.

The map $\pi_{\mathrm{E}}\colon \mathrm{BP} \to \mathrm{E}$ induces a map $\mathrm{BP}\wedge\mathrm{BP} \to \mathrm{E}\wedge\mathrm{E} \to \mathrm{L}_{\mathrm{K}}(\mathrm{E}\wedge\mathrm{E})$. There is a homotopy coequalizer map $\mathrm{L}_{\mathrm{K}}(\mathrm{E}\wedge\mathrm{E}) \to \mathrm{E}\wedge_{\mathrm{EO}}\mathrm{E}$. These maps fit into a diagram

$$\begin{tikzcd} \mathrm{BP}_*\mathrm{BP} \arrow[r, "\rho_{\mathrm{E}}"] & \mathrm{E}^{\wedge}_*\mathrm{E} \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{E}^{\mathrm{EO}}_*\mathrm{E} \arrow[d, phantom, "\cong"] \\ & \mathrm{Maps}^{\mathrm{cts}}(\mathrm{Aut}(\Gamma),\mathrm{E}_*) \arrow[r, "\mathrm{Res}^{\mathrm{C}_p}"] & \mathrm{Map}(\mathrm{C}_p,\mathrm{E}_*). \end{tikzcd} \tag{3.7}$$

The vertical isomorphisms are by Galois theory [Rog08, Theorem 5.4.4 and Definition 4.1.3]. The map $\mathrm{Res}^{\mathrm{C}_p}$ is restriction along the inclusion map $\mathrm{C}_p \to \mathrm{Aut}(\Gamma)$. To understand the action of C_p on $\mathrm{E}_*(X)$ we need to understand the map $\mathrm{Res}^{\mathrm{C}_p}\circ\rho_{\mathrm{E}}$.

Notation 3.8. Given $\theta \in \mathrm{BP}_*\mathrm{BP}$ and $g \in \mathrm{Aut}(\Gamma)$, the map

$$\rho_{\mathrm{E}}\colon \mathrm{BP}_*\mathrm{BP} \to \mathrm{E}^{\wedge}_*\mathrm{E} = \mathrm{Map}^c(\mathrm{Aut}(\Gamma),\mathrm{E}_*)$$

allows us to interpret the element $\theta(g) \coloneqq \rho_{\mathrm{E}}(\theta)(g) \in \mathrm{E}_*$. Let $\overline{\theta}(g)$ denote the image of $\theta(g)$ under the quotient map

$$\mathrm{E}_* \twoheadrightarrow \mathrm{E}_*/\mathfrak{m} \cong \mathrm{K}_*.$$

There is an isomorphism $\mathrm{BP}_*\mathrm{BP} \cong \mathrm{BP}_*[t_1, t_2, \ldots]$.

Lemma 3.9. *Let $\zeta \in \mathrm{Aut}(\Gamma)$ be an element of order p. Then $\overline{t}_i(\zeta) = 0$ for $i < k$ and $\overline{t}_k(\zeta)$ is a unit.*

Proof. Let $\widetilde{\mathbb{F}}$ is the compositum $\mathbb{F}\cdot\mathbb{F}_{p^n}$ and let $\widetilde{\Gamma}$ be the base change of Γ along the field extension $\mathbb{F} \to \widetilde{\mathbb{F}}$. Let $\widetilde{\mathrm{K}} = \mathrm{K}_{\widetilde{\Gamma}}$, the Morava K-theory associated to $\widetilde{\Gamma}$. Let

$$f_1\colon \mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\widetilde{\Gamma})$$
$$f_2\colon \mathrm{K}_* \to \widetilde{\mathrm{K}}_*$$

be the maps induced by $\mathbb{F} \to \widetilde{\mathbb{F}}$. Since f_2 is an injection, $\overline{t}_i(\zeta) = 0$ if and only if $f_2(\overline{t}_i(\zeta)) = 0$. Since $\overline{t}_i(f_1(\zeta)) = f_2(\overline{t}_i(\zeta))$, it suffices to check if $\zeta \in \mathrm{Aut}(\widetilde{\Gamma})$ has order p then $\overline{t}_i(\zeta) = 0$ for $i < k$ and $\overline{t}_k(\zeta)$ is a unit.

We use the identification (3.3) of $\mathrm{End}(\widetilde{\Gamma})$. Let $\phi\colon \mathbb{F}_{p^n} \to \mathbb{W}(\mathbb{F}_{p^n})$ denote the Teichmuller lift. If $\zeta = 1 + \sum_{i>0}\phi(a_i)\mathrm{T}^i$ then

$$t_i(\zeta) \equiv a_i u^{1-p^i} \mod \mathfrak{m} \tag{3.10}$$

where $\mathfrak{m}$ is the maximal ideal $(p, u_1, \ldots, u_{n-1}) \subset \mathrm{E}_*$. Because $\mathbb{Q}_p(\zeta)$ is a totally ramified extension of $\mathbb{Q}_p$ of degree $p-1$ with $\zeta - 1$ as a uniformizer, we see that

$$\nu(\zeta - 1) = \frac{1}{p-1} = \nu(\mathrm{T}^k).$$

We conclude that $a_i = 0$ if $i < k$ in (3.10) and that a_k is a unit. □

For any spectrum X there is an $\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}$-coaction on $\mathrm{E}_*(X)$ given by the composition

$$\Psi\colon \mathrm{E}_*(X) \xrightarrow{\Psi_{\mathrm{E}}} \mathrm{E}^{\wedge}_*\mathrm{E}\otimes_{\mathrm{E}_*}\mathrm{E}_*(X) \xrightarrow{\mathrm{Res}^{\mathrm{C}_p}\otimes\mathrm{id}} \mathrm{E}^{\mathrm{EO}}_*\mathrm{E}\otimes_{\mathrm{E}_*}\mathrm{E}_*(X). \tag{3.11}$$

The isomorphism $\mathrm{E}_*^{\mathrm{EO}}\mathrm{E} \cong \mathrm{Map}(\mathrm{C}_p, \mathrm{E}_*)$ leads to a contragradient C_p-action on $\mathrm{E}_*(X)$: the action of $g \in \mathrm{C}_p$ is given by the map

$$\mathrm{E}_*(X) \xrightarrow{\Psi} \mathrm{E}_*^{\mathrm{EO}}\mathrm{E} \otimes_{\mathrm{E}_*} \mathrm{E}_*(X) \xrightarrow{\mathrm{ev}_g \otimes 1} \mathrm{E}_*(X)$$

Note that $\mathrm{E}_*^{\mathrm{EO}}\mathrm{E} \cong \mathrm{Map}(\mathrm{C}_p, \mathrm{E}_*)$ is a quotient Hopf algebra of $\mathrm{E}_*^{\wedge}\mathrm{E}$ and dual to the Hopf algebra $\pi_*(\mathrm{EO}\text{–}\mathrm{Mod}(\mathrm{E}, \mathrm{E})) = \mathrm{E}_*[\mathrm{C}_p]^\sigma$ (as defined in Remark 3.12).

Remark 3.12. There is a nontrivial action of C_p on E_*, so as a ring

$$\pi_*(\mathrm{EO}\text{–}\mathrm{Mod}(\mathrm{E}, \mathrm{E}))$$

is not isomorphic to the group ring $\mathrm{E}_*[\mathrm{C}_p]$. The ring $\pi_*(\mathrm{EO}\text{–}\mathrm{Mod}(\mathrm{E}, \mathrm{E}))$ is a *twisted* group ring

$$\mathrm{E}_*[\mathrm{C}_p]^\sigma := \mathrm{E}_*\langle g\rangle/(g^p = 1, ga = \zeta(a)g).$$

The action of C_p on K_* is trivial so $\mathrm{E}_*[\mathrm{C}_p]^\sigma/\mathfrak{m}$ is isomorphic to the untwisted group ring $\mathrm{K}_*[\mathrm{C}_p]$.

Remark 3.13. The Steenrod algebra is $\pi_*(\mathrm{Spectra}(\mathrm{H}\mathbb{F}_p, \mathrm{H}\mathbb{F}_p))$ and the dual Steenrod algebra is $\pi_*(\mathrm{H}\mathbb{F}_p \wedge_{\mathbb{S}} \mathrm{H}\mathbb{F}_p)$. We think of $\pi_*(\mathrm{EO}\text{–}\mathrm{Mod}(\mathrm{E}, \mathrm{E}))$ as "the relative Steenrod algebra for the map $\mathrm{EO} \to \mathrm{E}$" and of $\mathrm{E}_*^{\mathrm{EO}}\mathrm{E}$ as "the relative dual Steenrod algebra for the map $\mathrm{EO} \to \mathrm{E}$." In Section 4 we describe the relative Adams spectral sequence for the map $\mathrm{EO} \to \mathrm{E}$. The E_2-page of the relative Adams spectral sequence is calculated by Ext over $\mathrm{E}_*^{\mathrm{EO}}\mathrm{E}$ just like the way that the E_2-page of the classical Adams spectral sequence is calculated by Ext over the classical dual Steenrod algebra.

Definition 3.14. A spectrum X is even if X is bounded below, $\mathrm{H}_{2i+1}(X) = 0$ and $\mathrm{H}_{2i}(X)$ is a finitely generated $\mathbb{Z}$-module for all integers i.

If X is an even spectrum, there is an isomorphism

$$\mathrm{K}_*X \cong \mathrm{E}_*X/\mathfrak{m}$$

which makes K_*X into a $\mathrm{K}_*[\mathrm{C}_p]$-module. Note that $\mathbb{CP}^\infty$ is an even space.

If R is a complex orientable cohomology theory and X is an even spectrum, the AHSS for R_*X collapses on the E_2-page. The associated Atiyah-Hirzebruch filtration on R_*X is given by

$$\mathrm{Fil}_d\,\mathrm{R}_*X = \mathrm{R}_*X^{(d)} \subseteq \mathrm{R}_*X,$$

where $X^{(d)}$ is the d-skeleton of X. Let $\mathrm{gr}_*\mathrm{R}_*X$ be the Atiyah-Hirzebruch associated graded

$$\mathrm{gr}_*\mathrm{R}_*X := \bigoplus \frac{\mathrm{Fil}_d\,\mathrm{R}_*X}{\mathrm{Fil}_{d-1}\,\mathrm{R}_*X}.$$

The associated graded $\mathrm{gr}_*\mathrm{R}_*X$ is bigraded, where one grading comes from the grading of R_*X and the other grading is induced by the Atiyah-Hirzebruch filtration. The AHSS collapse induces an isomorphism

$$\mathrm{gr}_*\mathrm{R}_*X \cong \mathrm{R}_* \otimes_{\mathbb{Z}} \mathrm{H}_*X$$

and we let

$$\iota_{\mathrm{R}} \colon \mathrm{H}_*X \to \mathrm{gr}_*\mathrm{R}_*X \tag{3.15}$$

be the map $x \mapsto 1 \otimes x$.

Remark 3.16. The Atiyah-Hirzebruch spectral sequence has signature $\mathrm{gr}_a \mathrm{R}_b X \Rightarrow \mathrm{R}_{a+b} X$, which means that if

$$\overline{x} \in \mathrm{gr}_a \mathrm{R}_b X,$$

then $\overline{x}$ detects a subset of $\mathrm{R}_{a+b} X$.

Lemma 3.17. *Let X be an even spectrum and let $\chi = \zeta - 1 \in \mathrm{K}_*[\mathrm{C}_p]$. Then*

$$\chi_* \operatorname{Fil}_d \mathrm{K}_* X \subseteq \operatorname{Fil}_{d-2p^k+2} \mathrm{K}_* X.$$

Proof. We required that X is even so the map $\mathrm{BP}_*(X) \to \mathrm{K}_*(X)$ is surjective and we can check our claim on the image.

Pick $x^{\mathrm{BP}} \in \operatorname{Fil}_d \mathrm{BP}_* X$. Write $x^{\mathrm{K}} = \pi_{\mathrm{K}*}(x^{\mathrm{BP}})$ and $x^{\mathbb{F}_p} = \pi_{\mathbb{F}_p *}(x^{\mathrm{BP}})$. Write the BP-coaction on x^{BP} as

$$\psi(x^{\mathrm{BP}}) = 1 \otimes x^{\mathrm{BP}} + \sum \theta_{(1)} \otimes x^{\mathrm{BP}}_{(2)}.$$

The counit axiom says that

$$(\epsilon \otimes 1)(\psi(x^{\mathrm{BP}})) = x^{\mathrm{BP}} = (\epsilon \otimes 1)(1 \otimes x^{\mathrm{BP}})$$

so we may assume that $\theta_{(1)} \in \ker \epsilon = (t_1, t_2, \ldots)$. By definition

$$(\zeta - 1)_*(x^{\mathrm{K}}) = \sum \overline{\theta}_{(1)}(\zeta) \cdot x^{\mathrm{K}}_{(2)}$$

where $\overline{\theta}_{(1)}(\zeta)$ is as in Notation 3.8 and where $x^{\mathrm{K}}_{(2)} = \pi_{\mathrm{K}*}(x^{\mathrm{BP}}_{(2)})$. The coaction is graded, so $|x^{\mathrm{K}}_{(2)}| = d - |\theta_{(1)}|$. By Lemma 3.9, if θ is in the ideal

$$(t_1, \ldots, t_{k-1}) \subseteq \mathrm{BP}_*\mathrm{BP}$$

then $\overline{\theta}(\zeta) = 0$. Thus, we need only consider the $\mathrm{BP}_*\mathrm{BP}$-coaction modulo $(t_1, \ldots, t_{k-1})$. The lowest degree element of $\ker(\epsilon)/(t_1, \ldots, t_{k-1})$ is t_k. If $\theta \in \mathrm{BP}_*\mathrm{BP}$ has $|\theta| = i$ then $\overline{\theta}(\zeta) \in \mathrm{K}_i$ so by Remark 3.16 the filtration of

$$\overline{t}_k(\theta) \cdot x_{(2)}$$

is less than or equal to $d - |t_k| = d - 2p^k + 2$. □

It is clear from the proof of Lemma 3.17 that the leading term of $\chi_*(x^{\mathrm{K}})$ is determined by the t_k-coaction on x^{BP}. The image of t_k under the map

$$\mathrm{BP}_*\mathrm{BP} \to \mathcal{A}_*$$

is nonzero so can identify the leading term using the Steenrod coaction on $\mathrm{H}_*(X)$.

First we introduce some relevant subquotient Hopf algebras of the Steenrod algebra. Let $\mathcal{P}$ be the quotient Hopf algebra of the Steenrod algebra

$$\mathcal{P} = \mathcal{A} /\!\!/ (\mathrm{Q}_0, \mathrm{Q}_1, \ldots)$$

where Q_i are the Milnor primitives. The dual of $\mathcal{P}$ is the sub Hopf algebra of $\mathcal{A}_*$ generated by the even degree generators:

$$\mathcal{P}_* \cong \begin{cases} \mathbb{F}_p[\xi_1, \xi_2, \ldots] \subset \mathcal{A}_* & \text{if } p \text{ is odd} \\ \mathbb{F}_p[\xi_1^2, \xi_2^2, \ldots] \subset \mathcal{A}_* & \text{if } p = 2. \end{cases}$$

Let

$$\eta_k = \begin{cases} c(\xi_k) & \text{if } p \text{ is odd} \\ c(\xi_k^2) & \text{if } p = 2, \end{cases}$$

where c denotes the Steenrod antipode map. Let P_k be the element of $\mathcal{P}$ dual to $c(\eta_k) \in \mathcal{P}_*$. There is a formula for P_k in terms of commutator brackets:

$$\mathrm{P}_k = \begin{cases} [[\cdots[[\mathrm{P}^1, \mathrm{P}^p], \mathrm{P}^{p^2}]\cdots], \mathrm{P}^{p^k}] & \text{if } p \text{ is odd} \\ [[\cdots[[\mathrm{Sq}^2, \mathrm{Sq}^4], \mathrm{Sq}^8]\cdots], \mathrm{Sq}^{2^{k+1}}] & \text{if } p = 2. \end{cases} \tag{3.18}$$

Definition 3.19. Let $\mathcal{B}(k) \subset \mathcal{P}$ denote the sub Hopf algebra generated by P_k. As a Hopf algebra

$$\mathcal{B}(k) \cong \mathbb{F}_p[\mathrm{P}_k]/(\mathrm{P}_k^p)$$

where P_k is primitive. Let $\beta_k = |\mathrm{P}_k^{p-1}| = (p-1)(p^k-1)$.

The dual Hopf algebra $\mathcal{B}(k)_*$ is the quotient of $\mathcal{P}_*$

$$\mathcal{B}(k)_* = \mathbb{F}_p[\eta_k]/(\eta_k)^p.$$

The action of $\mathrm{gr}\,\mathrm{K}_*[\mathrm{C}_p]$ on $\mathrm{gr}\,\mathrm{K}_*X$ is closely related to the contragradient action of $\mathcal{B}(k)$ on $\mathrm{H}_*(X;\mathbb{F}_p)$.

Lemma 3.20. *Let X be an even spectrum. Suppose $x \in \mathrm{H}_dX$ and let $y = \mathrm{P}_k(x)$. Then*

$$\chi_*(\iota_{\mathrm{K}}(x)) = -\overline{t}_k(\zeta)\iota_{\mathrm{K}}(y) \mod \mathrm{Fil}_{d-2p^k}\,\mathrm{K}_*X.$$

Proof. Write $x^{\mathbb{F}_p} = x$ and $y^{\mathbb{F}_p} = y = \mathrm{P}_k(x)$. Write $\mathcal{I}_n^{\mathbb{F}_p} = \mathcal{P}_* \otimes \mathrm{Fil}_{d-2p^k+2}\,\mathrm{H}_*(X)$ and $\mathcal{I}_n^{\mathrm{BP}} = \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathrm{Fil}_n\,\mathrm{BP}_*(X)$. The $\mathcal{A}_*$ coaction on $x^{\mathbb{F}_p}$ is

$$\Psi(x^{\mathbb{F}_p}) = 1 \otimes x^{\mathbb{F}_p} + \eta_k \otimes y^{\mathbb{F}_p} \mod (\xi_1, \ldots, \xi_{k-1}) + \mathcal{I}_{d-2p^k}^{\mathbb{F}_p}.$$

Let x^{BP} be a lift of x to $\mathrm{BP}_d(X)$. It follows that the BP coaction on x^{BP} is

$$\Psi(x^{\mathrm{BP}}) = 1 \otimes x^{\mathrm{BP}} + t_k \otimes y^{\mathrm{BP}} \mod (p, \ldots, v_{n-1}, t_1, \ldots, t_{k-1}) + \mathcal{I}_{d-2p^k}^{\mathrm{BP}},$$

where y^{BP} is any lift of $y^{\mathbb{F}_p}$ (the difference between two such lifts is contained in $\mathcal{I}_{d-2p^k}^{\mathrm{BP}}$). Since $\overline{\theta}(\zeta) = 0$ for all $\theta \in (t_1, \ldots, t_{k-1})$, we see that

$$\zeta_*(x^{\mathrm{K}}) = x^{\mathrm{K}} + \overline{t}_k(\zeta)y^{\mathrm{K}} \mod \mathrm{Fil}_{d-2p^k}\,\mathrm{K}_*(X).$$ □

If we put χ in filtration $2p^k-2$ then Lemma 3.17 shows that the action of $\mathrm{K}_*[\mathrm{C}_p]$ on $\mathrm{K}_*(X)$ is filtered when X is an even spectrum. It follows that there is an associated graded action of $\mathrm{gr}\,\mathrm{K}_*[\mathrm{C}_p]$ on $\mathrm{gr}\,\mathrm{K}_*(X)$. Let

$$\overline{\chi} \in \mathrm{gr}\,\mathrm{K}_*[\mathrm{C}_p]$$

be the image of χ under the map $\mathrm{Fil}_{2p^k-2}\,\mathrm{K}_*[\mathrm{C}_p] \to \mathrm{gr}_{2p^k-2}\,\mathrm{K}_*[\mathrm{C}_p]$.

Lemma 3.21. *The bigraded Hopf algebra $\mathrm{gr}\,\mathrm{K}_*[\mathrm{C}_p]$ is isomorphic to*

$$\mathrm{K}_*[\overline{\chi}]/(\overline{\chi}^p)$$

where the elements in K_{2i} are scalars in bidegree $(2i, 0)$ and $\overline{\chi}$ is a primitive in bidegree $(0, 2-2p^k)$.

Proof. Since $\Delta(\zeta) = \zeta \otimes \zeta$, using $\chi = \zeta - 1$ we see $\Delta(\chi) = \chi \otimes 1 + 1 \otimes \chi + \chi \otimes \chi$. In the associated graded,

$$\Delta(\overline{\chi}) = \overline{\chi} \otimes 1 + 1 \otimes \overline{\chi}.$$ □

Construction 3.22. Let F_k be the map of Hopf algebras

$$\begin{aligned} \mathrm{F}_k\colon \mathcal{B}(k) &\longrightarrow \operatorname{gr}\mathrm{K}_*[\mathrm{C}_p] \\ \mathrm{P}_k &\longmapsto -\bar{t}_k(\zeta)\cdot\overline{\chi} \end{aligned}$$

and let $\mathcal{F}_k$ be the functor

$$\begin{aligned} \mathcal{F}_k\colon \mathcal{B}(k)\text{-mod} &\longrightarrow \operatorname{gr}\mathrm{K}_*[\mathrm{C}_p]\text{-mod} \\ M &\longmapsto \operatorname{gr}\mathrm{K}_*[\mathrm{C}_p]\otimes_{\mathcal{B}(k)} M \end{aligned}$$

where the tensor product in the definition of $\mathcal{F}_k$ is along F_k. The map F_k extends to an isomorphism

$$\overline{\mathrm{F}}_k\colon \mathrm{K}_*\otimes_{\mathbb{F}_p}\mathcal{B}(k)\to \operatorname{gr}\mathrm{K}_*[\mathrm{C}_p],$$

so there is an isomorphism of K_*-modules

$$\operatorname{gr}\mathrm{K}_*[\mathrm{C}_p]\otimes_{\mathcal{B}(k)} M\cong(\mathrm{K}_*\otimes\mathcal{B}(k))\otimes_{\mathcal{B}(k)} M\cong \mathrm{K}_*\otimes_{\mathbb{F}_p} M.$$

Combining Lemma 3.17 and Lemma 3.20 we deduce:

Proposition 3.23. *For any even spectrum X there is an isomorphism*

$$\operatorname{gr}\mathrm{K}_*(X)\cong\mathcal{F}_k(\mathrm{H}_*(X;\mathbb{F}_p))$$

of $\operatorname{gr}\mathrm{K}_*[\mathrm{C}_p]$*-modules.*

Notation 3.24. Let x be the generator of $\mathrm{H}^*\mathbb{CP}^\infty_+\cong\mathbb{F}_p[\![x]\!]$ and let $b_i\in\mathrm{H}_{2i}\mathbb{CP}^\infty_+$ be the linear dual of x^i. By the $\mathrm{H}\mathbb{F}_p$-Thom isomorphism for $(\gamma_1)^{\oplus c}$ we see that $\mathrm{H}^*\mathbb{CP}^\infty_c$ is a free module over $\mathrm{H}^*\mathbb{CP}^\infty_+$. Fix an $\mathrm{H}\mathbb{F}_p$-Thom class u_c for $(\gamma_1)^{\oplus c}$. We write

$$b_i\in\mathrm{H}_{2i}\mathbb{CP}^\infty_c$$

for the element that is the linear dual to $x^{i-n}\cdot u_c$. We also write b_i for the generator $\mathrm{H}_{2i}\mathbb{CP}^{a+c}_c$ which maps to $b_i\in\mathrm{H}_{2i}\mathbb{CP}^\infty_c$ under the skeletal inclusion.

If R is any complex oriented theory, we will also write b_i for $\iota_{\mathrm{R}}(b_i)\in\mathrm{R}_{2i}\mathbb{CP}^\infty$ where ι_{R} is the map in (3.15).

Proposition 3.25. *Let c be an integer. The action of $\overline{\chi}$ on $\operatorname{gr}\mathrm{K}_*(\mathbb{CP}^\infty_c)\cong\mathrm{K}_*\{b_c,b_{c-1},\ldots\}$ is given by*

$$\overline{\chi}_*(b_i)=\begin{cases}(i-p^k+1)b_{i-p^k+1} & \text{if } i\geq p^k-1+c\\ 0 & \text{otherwise.}\end{cases}$$

Moreover, there is a $\operatorname{gr}\mathrm{K}_*[\mathrm{C}_p]$*-module isomorphism*

$$\operatorname{gr}\mathrm{K}_*\mathbb{CP}^\infty_c\cong\operatorname{gr}\mathrm{F}_c\oplus\operatorname{gr}\mathrm{M}_c$$

where $\operatorname{gr}\mathrm{F}_c$ *is a free* $\operatorname{gr}\mathrm{K}_*[\mathrm{C}_p]$*-module of infinite rank and* $\operatorname{gr}\mathrm{M}_c$ *is a finite dimensional* K_**-module.*

Remark 3.26. The Hopf algebras $\mathrm{K}_*[\mathrm{C}_p]$ and $\operatorname{gr}\mathrm{K}_*[\mathrm{C}_p]$ are self injective. In other words, if M is a $\mathrm{K}_*[\mathrm{C}_p]$-module and

$$i\colon\mathrm{K}_*[\mathrm{C}_p]\{\iota\}\to M$$

is an injective map then i admits a splitting (and similarly for $\operatorname{gr}\mathrm{K}_*[\mathrm{C}_p]$-modules).

Proof. By Proposition 3.23, it suffices to compute the $\mathcal{B}(k)$-coaction on $\mathrm{H}_*\mathbb{CP}_c^\infty$. The generator $\eta_k \in \mathcal{B}(k)$ is dual to $\mathrm{P}_k \in \mathcal{P}$. By Remark 3.27, the element P_k is primitive in $\mathcal{P}$.

Because P_k is primitive, the action of P_k on $\mathrm{H}^*\mathbb{CP}^\infty \cong \mathbb{F}_p[\![x]\!]$ satisfies the Leibniz rule

$$\mathrm{P}_k(x^i) = ix^{i-1}\mathrm{P}_k(x).$$

The action of P_k on $\mathrm{H}^*\mathbb{CP}^\infty \cong \mathbb{F}_p[\![x]\!] \cdot u_c$ can be calculated using the formula

$$\mathrm{P}_k(x^i \cdot u_c) = \mathrm{P}_k(x^i) \cdot u_c + x^i \cdot \mathrm{P}_k(u_c).$$

By Formula 3.18, $\mathrm{P}_k(x) = x^{p^k}$ and $\mathrm{P}_k(u_c) = cx^{p^k-1} \cdot u_c$. Therefore,

$$\mathrm{P}_k(x^i \cdot u_c) = (i+c)x^{i+p^k-1} \cdot u_c.$$

Dually, the map P_{k*} is given by

$$\mathrm{P}_{k*} \colon \mathrm{H}_*\mathbb{CP}_c^\infty \longrightarrow \mathrm{H}_{*-2\upbeta_k}\mathbb{CP}_c^\infty$$

$$b_i \longmapsto (i-p^k+1)b_{i-p^k+1}$$

where it is understood that $\mathrm{P}_{k*}(b_i) = 0$ if $i - p^k + 1 < c$.

We see that $\operatorname{gr}\mathrm{K}_*\mathbb{CP}_c^\infty$ contains an infinite rank free $\operatorname{gr}\mathrm{K}_*[\mathrm{C}_p]$-submodule generated by the set

$$\{b_{pi} \colon i > (\upbeta_k - c)/p\}.$$

By Remark 3.26,

$$\operatorname{gr}\mathrm{K}_*[\mathrm{C}_p]\,\{b_{pi} \colon i > (\upbeta_k - c)/p\}$$

is a summand of $\operatorname{gr}\mathrm{K}_*\mathbb{CP}_c^\infty$. $\square$

Remark 3.27. If $p = 2$, then $\Delta(\mathrm{P}_k) = \mathrm{P}_k \otimes 1 + \mathrm{Q}_k \otimes \mathrm{Q}_k + 1 \otimes \mathrm{P}_k$ in $\mathcal{A}$ but the quotient map $\mathcal{A} \to \mathcal{P}$ sends $\mathrm{Q}_k \mapsto 0$ and so we see that P_k is primitive as an element of $\mathcal{P}$ even though it is not a primitive element of $\mathcal{A}$. If p is odd, P_k is primitive even in $\mathcal{A}$.

Lemma 3.28. *Suppose that* M *is a filtered* $\mathrm{K}_*[\mathrm{C}_p]$*-module and there is a* $\operatorname{gr}\mathrm{K}_*[\mathrm{C}_p]$*-module injection*

$$\overline{\psi} \colon \overline{\mathrm{F}} \to \operatorname{gr}\mathrm{M}$$

where $\overline{\mathrm{F}}$ *is a free* $\operatorname{gr}\mathrm{K}_*[\mathrm{C}_p]$*-module. Then there is a lift of* $\overline{\psi}$ *to an injection*

$$\psi \colon \mathrm{F} \to \mathrm{M}$$

where F *is a free* $\mathrm{K}_*[\mathrm{C}_p]$*-module. Furthermore* ψ *is the inclusion of a summand.*

Proof. It suffices to consider the case when $\overline{\mathrm{F}} = \operatorname{gr}\mathrm{K}_*[\mathrm{C}_p]\{\overline{x}\}$ has rank one. Let $\overline{y} = \overline{\psi}(\overline{x})$. Let $y \in \mathrm{F}$ be an arbitrary lift of $\overline{x}$. Define

$$\psi \colon \mathrm{K}_*[\mathrm{C}_p]\{\iota\} \to \mathrm{M}$$

by setting $\psi(x) = y$. Suppose $y \in \mathrm{Fil}_d(\mathrm{M})$. The image of

$$\chi^{p-1}(y) \in \mathrm{Fil}_{d-2\upbeta_k}(\mathrm{M})$$

in $\operatorname{gr}_{d-2\upbeta_k}(\mathrm{M})$ is $\overline{\chi}^{p-1}(y) = \overline{\psi}(\chi^{p-1}(x))$. Because $\overline{\psi}$ is injective, we see that $\chi^{p-1}(y) \neq 0$. We see that each element of

$$S = \{y, \chi(y), \ldots, \chi^{p-1}(y)\}$$

is nonzero and detected in distinct filtration. Thus S is an independent set and so ψ is injective. By Remark 3.26, ψ admits a splitting. ☐

Proposition 3.25 and Lemma 3.28 imply:

Corollary 3.29. *Let c be an integer. As a $\mathrm{K}_*[\mathrm{C}_p]$-module $\mathrm{K}_*\mathbb{CP}^\infty_c \cong \mathrm{F}_c \oplus \mathrm{M}_c$, where F_c is a free $\mathrm{K}_*[\mathrm{C}_p]$-module and M_c is a finitely generated $\mathrm{K}_*[\mathrm{C}_p]$-module.*

4. A splitting of $\mathrm{EO} \wedge \mathbb{CP}^\infty$

The primary goal of this section is to use the algebraic information about $\mathrm{K}_*\mathbb{CP}^\infty_c$ computed in Section 3 to deduce information about the homotopy type of $\mathrm{EO} \wedge \mathbb{CP}^\infty$. We will lift the splitting given in Corollary 3.29 to an EO-module splitting of $\mathrm{EO} \wedge \mathbb{CP}^\infty_c$ (Lemma 4.12) which leads to the proof of Main Theorem 1.4. Lemma 4.12 also helps us understand the homotopy type of the S^1-Tate spectrum of EO_Γ and prove Main Theorem 1.7.

Our main tool is the relative Adams spectral sequence for the map $\mathrm{EO} \to \mathrm{E}$. The relative Adams spectral sequence is extremely well behaved because the map $\mathrm{EO} \to \mathrm{E}$ is Galois in the sense of [Rog08].

Definition 4.1. An EO-module $\mathcal{M}$ is *relatively free* if $\mathrm{E}^{\mathrm{EO}}_*\mathcal{M}$ is a projective E_*-module.

For example, E is a relatively free EO-module.

Theorem 4.2 ([Dev05, Corollary 3.4])**.** *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are EO-modules such that $\mathcal{M}$ is finite and relatively free. The Adams spectral sequence relative to the map $\mathrm{EO} \to \mathrm{E}$*

$$\mathrm{E}_2^{s,t} := \mathrm{Ext}^{s,t}_{\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}}(\mathrm{E}^{\mathrm{EO}}_*\mathcal{M}, \mathrm{E}^{\mathrm{EO}}_*\mathcal{N}) \Rightarrow \pi_{t-s}\,\mathrm{EO}\text{–}\mathrm{Mod}(\mathcal{M},\mathcal{N}), \tag{4.3}$$

is strongly convergent.

The relative Adams spectral sequence was developed by Baker and Lazarev [BL01]. Devinatz [Dev05] defined the homotopy fixed points E^{hG} for G an arbitrary closed subgroup of $\mathrm{Aut}(\Gamma)$. He identified the E_2-page of the relative Adams spectral sequence for $\mathrm{E}^{\mathrm{hG}} \to \mathrm{E}$ with the group cohomology of G and showed that for every E^{hG}-module the relative Adams spectral sequence is strongly convergent and that there is a uniform horizontal vanishing line independent of the E^{hG}-module. Rognes [Rog08] developed the theory of Galois extensions of ring spectra and reinterpreted the results of Devinatz as saying that the map $\mathrm{EO} \to \mathrm{E}$ is Galois in [Rog08, Theorem 5.4.4].

Definition 4.4. A *cell structure* on an EO-module $\mathcal{M}$ is a filtration

$$\mathcal{M}^{(0)} \to \mathcal{M}^{(1)} \to \cdots \to \mathcal{M}$$

such that

(1) $\mathcal{M}^{(0)}$ is contractible,

(2) $\mathrm{colim}_d\,\mathcal{M}^{(d)} \simeq \mathcal{M}$, and

(3) there are cofiber sequences $\bigvee_{i\in I_d} \Sigma^{s_i}\mathrm{EO} \to \mathcal{M}^{(d)} \to \mathcal{M}^{(d+1)}$.

An EO-module $\mathcal{M}$ is *cellular* if there exists a cell structure on $\mathcal{M}$.

If X is a connective spectrum, then $\mathrm{EO} \wedge X$ has a cell structure induced by the Atiyah-Hirzebruch filtration on X. If $\mathcal{M}$ is an EO-module with a cell structure then any summand of $\mathcal{M}$ has an induced cell structure. For these two reasons, every EO-module we consider is cellular.

As a consequence of the convergence of the relative Adams spectral sequence, we deduce that $\mathrm{K}^{\mathrm{EO}}_*(-)$ detects equivalences of EO-modules:

Corollary 4.5. *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are* EO*-modules such that $\mathcal{M}$ is cellular and relative free. A map $f\colon \mathcal{M} \to \mathcal{N}$ is an equivalence if and only if it induces an isomorphism $f_*\colon \mathrm{K}^{\mathrm{EO}}_*\mathcal{M} \to \mathrm{K}^{\mathrm{EO}}_*\mathcal{N}$.*

Proof. By the Nakayama lemma, if $f_*\colon \mathrm{K}^{\mathrm{EO}}_*\mathcal{M} \to \mathrm{K}^{\mathrm{EO}}_*\mathcal{N}$ is an isomorphism, then so is $f_*\colon \mathrm{E}^{\mathrm{EO}}_*\mathcal{M} \to \mathrm{E}^{\mathrm{EO}}_*\mathcal{N}$. Thus f induces an isomorphism of relative Adams E_2-pages. It follows that f induces an isomorphism on π_*. $\square$

Let

$$\mathrm{D}_{\mathrm{EO}}(-) \coloneqq \mathrm{EO}\text{–}\mathrm{Mod}(-, \mathrm{EO})\colon \mathrm{EO}\text{-module} \to \mathrm{EO}\text{-module}$$

denote the relative Spanier-Whitehead dual functor. Because the map $\mathrm{EO} \to \mathrm{E}$ is Galois, [Rog08, Proposition 6.4.7] implies that E is a self-dual EO-module:

Proposition 4.6. *There is an equivalence $\mathrm{D}_{\mathrm{EO}}(\mathrm{E}) \simeq \mathrm{E}$.*

Lemma 4.7. *The edge homomorphism for the relative Adams spectral sequence* (4.3) *is the map*

$$\pi_*\, \mathrm{EO}\text{–}\mathrm{Mod}(\mathcal{M}, \mathcal{N}) \to \mathrm{Hom}_{\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}}(\mathrm{E}^{\mathrm{EO}}_*\mathcal{M}, \mathrm{E}^{\mathrm{EO}}_*\mathcal{N})$$

given by $f \mapsto \mathrm{E}^{\mathrm{EO}}_*(f)$. The edge homomorphism is an isomorphism

(I) *when $\mathcal{M} = \mathrm{E}$ and $\mathcal{N}$ is an arbitrary* EO*-module, and also*

(II) *when $\mathcal{N} = \mathrm{E}$ and $\mathcal{M}$ is a finite relatively-free* EO*-module.*

Proof. When $\mathcal{M}$ is finite relatively-free EO-module and $\mathcal{N}$ is an arbitrary EO-module there is a relative Adams spectral sequence

$$\mathrm{E}_2^{s,t}\colon\ \mathrm{Ext}^{s,t}_{\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}}(\mathrm{E}^{\mathrm{EO}}_*\mathcal{M}, \mathrm{E}^{\mathrm{EO}}_*\mathcal{N}) \Rightarrow \pi_{t-s}(\mathrm{EO}\text{–}\mathrm{Mod}(\mathcal{M}, \mathcal{N})) \tag{4.8}$$

Consider the case (I), when $\mathcal{M} = \mathrm{E}$. Because E is a finite and self-dual EO-module, there is an equivalence

$$\mathrm{E} \wedge_{\mathrm{EO}} \mathcal{N} \simeq \mathrm{D}_{\mathrm{EO}}(\mathrm{E}) \wedge_{\mathrm{EO}} \mathcal{N} \simeq \mathrm{EO}\text{–}\mathrm{Mod}(\mathrm{E}, \mathcal{N}).$$

Since $\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}$ is E_*-free, there is a Kunneth isomorphism

$$\mathrm{E}^{\mathrm{EO}}_*(\mathrm{E} \wedge_{\mathrm{EO}} \mathcal{N}) \cong \mathrm{E}^{\mathrm{EO}}_*(\mathrm{E}) \otimes_{\mathrm{E}_*} \mathrm{E}^{\mathrm{EO}}_*\mathcal{N}.$$

The spectral sequence (4.8) is concentrated on the zero line because $\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}$ is cofree as an $\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}$-comodule and

$$\pi_*\, \mathrm{EO}\text{–}\mathrm{Mod}(\mathrm{E}, \mathcal{N}) \to \mathrm{Hom}_{\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}}(\mathrm{E}_*, \mathrm{E}^{\mathrm{EO}}_*\mathrm{E} \otimes_{\mathrm{E}_*} \mathrm{E}^{\mathrm{EO}}_*\mathcal{N})$$

is an equivalence. Because $\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}$ is a self-dual $\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}$-comodule,

$$\mathrm{Hom}_{\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}}(\mathrm{E}_*, \mathrm{E}^{\mathrm{EO}}_*\mathrm{E} \otimes_{\mathrm{E}_*} \mathrm{E}^{\mathrm{EO}}_*\mathcal{M}) \cong \mathrm{Hom}_{\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}}(\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}, \mathrm{E}^{\mathrm{EO}}_*\mathcal{M})$$

and so the edge map is an isomorphism.

Now consider the case (II), when $\mathcal{N} = \mathrm{E}$ and $\mathcal{M}$ is finite and relatively free. Because $\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}$ is a cofree $\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}$-comodule, the E_2-page of the relative Adams spectral sequence (4.8) is concentrated the zero line and the edge map is an isomorphism. □

Before applying Lemma 4.7, we need to lift the splitting of (3.29) to an $\mathrm{E}^{\mathrm{EO}}_*\mathrm{EO}$-module splitting. If $\mathcal{M}$ is a relatively-free EO-module then $\mathrm{E}^{\mathrm{EO}}_*\mathcal{M}$ is E_*-free. If f is an injection of $\mathrm{K}_*[\mathrm{C}_p]$-modules $f\colon \mathrm{K}_*[\mathrm{C}_p]\{x\} \to \mathrm{K}^{\mathrm{EO}}_*\mathcal{M}$ then an arbitrary lift of $f(x)$ to an element of $\mathrm{E}^{\mathrm{EO}}_*\mathcal{M}$ leads to an injection of C_p-modules $\widetilde{f}\colon \mathrm{E}_*[\mathrm{C}_p]^\sigma \to \mathrm{E}^{\mathrm{EO}}_*\mathcal{M}$. Then we find a splitting of $\widetilde{f}$ using the fact that the pair $(\mathrm{E}_*, \mathrm{E}^{\mathrm{EO}}_*\mathrm{E})$ is relatively injective:

Definition 4.9. Suppose that S is a commutative ring and that R is a subring of S. The pair (R, S) is *relatively injective* if every map $\mathrm{M} \to \mathrm{N}$ of S-modules that splits in the category of R-modules also splits in the category of S-modules.

Proposition 4.10. *The pair* $(\mathrm{E}_*, \mathrm{E}_*[\mathrm{C}_p]^\sigma)$ *is relatively injective.*

In particular, this implies that if F is a free $\mathrm{E}_*[\mathrm{C}_p]^\sigma$-module and M is an $\mathrm{E}_*[\mathrm{C}_p]^\sigma$-module which is free over E_* then any $\mathrm{E}_*[\mathrm{C}_p]^\sigma$-module injective map $\mathrm{F} \to \mathrm{M}$ splits in the category of $\mathrm{E}_*[\mathrm{C}_p]^\sigma$-modules.

Proof. Hochschild [Hoc56, Lemma 1] showed that when S is a subring of R and A is an S-module, then the pair $(\mathrm{S}, \mathrm{Hom}_\mathrm{S}(\mathrm{R}, \mathrm{A}))$ is relatively injective. When $(\mathrm{S}, \mathrm{R}) = (\mathrm{E}_*, \mathrm{E}_*[\mathrm{C}_p]^\sigma)$ and $\mathrm{A} = \mathrm{E}_*$, there is a $\mathrm{E}_*[\mathrm{C}_p]^\sigma$-module isomorphism

$$\mathrm{Hom}_{\mathrm{E}_*}(\mathrm{E}_*[\mathrm{C}_p]^\sigma, \mathrm{E}_*) \cong \mathrm{E}_*[\mathrm{C}_p]^\sigma$$

because $\mathrm{EO} \to \mathrm{E}$ is Galois. Therefore, $(\mathrm{E}_*, \mathrm{E}_*[\mathrm{C}_p]^\sigma)$ is relatively injective. □

Corollary 4.11. *Suppose that* $\mathcal{M}$ *is a relatively-free cellular* EO*-module such that*

$$\mathrm{E}^{\mathrm{EO}}_*(\mathcal{M})/\mathfrak{m} \cong \mathrm{F} \oplus \mathrm{N}$$

where $\mathrm{F} \cong \mathrm{K}_*[\mathrm{C}_p]\{\mathrm{A}\}$ *is a free* $\mathrm{K}_*[\mathrm{C}_p]$*-module on a set* A *and* N *is a complement of* F*. Then there exists an* EO*-module splitting*

$$\mathcal{M} \simeq \mathcal{F} \vee \mathcal{N}$$

where $\mathcal{F} \simeq \bigvee_{a\in A} \mathrm{E}$ *and there is a* $\mathrm{K}_*[\mathrm{C}_p]$*-module isomorphism* $\mathrm{E}^{\mathrm{EO}}_*(\mathcal{N})/\mathfrak{m} \cong \mathrm{N}$.

Proof. It suffices to consider the case where A contains only one element. In this case F is free of rank one. Let

$$b \in \mathrm{F} \subseteq \mathrm{E}^{\mathrm{EO}}_*(\mathcal{M})/\mathfrak{m}$$

be a $\mathrm{K}_*[\mathrm{C}_p]$-module generator of F and let $\widetilde{b} \in \mathrm{E}^{\mathrm{EO}}_*\mathcal{M}$ be an arbitrary lift of b. Then there is an $\mathrm{E}_*[\mathrm{C}_p]^\sigma$-module map

$$\mathrm{E}_*[\mathrm{C}_p]^\sigma\{x\} \to \mathrm{E}^{\mathrm{EO}}_*\mathcal{M}$$

sending $x \mapsto \widetilde{b}$. Because $\mathcal{M}$ is relatively free, this map splits as an E_*-module map. By Proposition 4.10 the splitting map can be chosen to be an $\mathrm{E}_*[\mathrm{C}_p]^\sigma$-module map.

By Lemma 4.7 (I), $\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}$-comodule map

$$\mathrm{E}^{\mathrm{EO}}_*\mathrm{E} \hookrightarrow \mathrm{E}^{\mathrm{EO}}_*\mathcal{M}$$

can be realized by an EO-module map $f\colon \mathrm{E} \to \mathcal{M}$.

We obtain the splitting map as follows. Fix a cell structure on $\mathcal{M}$. The ith skeleton $\mathcal{M}^{(i)}$ is relatively free and finite so Lemma 4.7 (II) says there is an equivalence

$$\pi_0(\mathrm{EO}\text{–}\mathrm{Mod}(\mathcal{M}^{(i)}, \mathrm{E})) \cong \mathrm{Hom}_{\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}}(\pi_*(\mathcal{M}^{(i)}), \mathrm{E}^{\mathrm{EO}}_*\mathrm{E})$$

so the composite

$$\mathrm{E}^{\mathrm{EO}}_*(\mathcal{M}^{(i)}) \to \mathrm{E}^{\mathrm{EO}}_*(\mathcal{M}) \to \mathrm{E}^{\mathrm{EO}}_*\mathrm{E}$$

is realizable by a map $s^{(i)}\colon \mathcal{M}^{(i)} \to \mathrm{E}$. The isomorphism

$$\pi_0(\mathrm{EO}\text{–}\mathrm{Mod}(\mathcal{M}^{(i)}, \mathrm{E})) \cong \mathrm{Hom}_{\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}}(\pi_*(\mathcal{M}^{(i)}), \mathrm{E}^{\mathrm{EO}}_*\mathrm{E})$$

is natural in $\mathcal{M}^{(i)}$, so the maps $s^{(i)}$ assemble into a colimit diagram. Passing to the colimit gives a map $s\colon \mathcal{M} = \mathrm{colim}_i\, \mathcal{M}^{(i)} \to \mathrm{E}$ splitting f as desired. □

For any space X there is an isomorphism of $\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}$-comodules

$$\mathrm{E}^{\mathrm{EO}}_*(\mathrm{EO} \wedge X) \cong \mathrm{E}_*X,$$

where the $\mathrm{E}^{\mathrm{EO}}_*\mathrm{E}$-coaction on $\mathrm{E}_*(X)$ is as indicated in (3.11). Combining Corollary 4.11 and Corollary 3.29 we deduce a splitting of $\mathrm{EO} \wedge \mathbb{CP}^\infty_c$:

Lemma 4.12. *There exists an* EO*-module splitting*

$$\mathrm{EO} \wedge \mathbb{CP}^\infty_c \simeq \mathcal{F}_c \vee \mathcal{M}_c,$$

where $\mathcal{F}_c$ is an infinite rank free E*-module and $\mathcal{M}_c$ is some finite* EO*-module.*

We will now prove that $\mathcal{M}_c \simeq \mathcal{M}_{c'}$ for certain pairs c and c', which will imply that $\Theta(\mathrm{EO}, \gamma_1)$ is finite (see Proposition 4.17).

Lemma 4.13. *The Thom isomorphism $\mathrm{H}_*\mathbb{CP}^\infty_+ \to \mathrm{H}_{*+pc}\mathbb{CP}^\infty_{pc}$ is a map of $\mathcal{B}(k)$-modules.*

Proof. The Thom class in $\mathrm{H}^*\mathbb{CP}^\infty_{pc}$ is x^{pc}. Because P_k is primitive,

$$\mathrm{P}_k(x^{pc}) = 0$$

and the result follows. □

By Lemma 4.13 and Proposition 3.23 we deduce that there is an isomorphism

$$\mathrm{gr}\, \mathrm{K}_*\mathbb{CP}^\infty_{pc} \cong \mathrm{gr}\, \mathrm{K}_*\mathbb{CP}^\infty_+$$

as $\mathrm{gr}\, \mathrm{K}[\mathrm{C}_p]$-modules.

Lemma 4.14. *There exist a positive integer N and a corresponding integer r, such that for all $j \in \mathbb{Z}$, the pair (r, N) satisfies the following conditions:*

(i) there is an equivalence of EO*-modules $\Sigma^{2j\mathsf{r}}\mathcal{M}_0 \xrightarrow{\simeq} \mathcal{M}_{j\mathsf{r}}$, and*

(ii) the composite map $\Sigma^{2j\mathsf{r}}\mathcal{M}_0 \to \mathcal{M}_{j\mathsf{r}} \to \mathrm{EO} \wedge \mathbb{CP}^\infty_{j\mathsf{r}}$ admits a factorization

$$\begin{tikzcd} & \mathrm{EO} \wedge \mathbb{CP}^{\mathsf{N}+j\mathsf{r}}_{j\mathsf{r}} \arrow[d, hook] \\ \Sigma^{2j\mathsf{r}}\mathcal{M}_0 \arrow[ur] \arrow[r] & \mathrm{EO} \wedge \mathbb{CP}^\infty_{j\mathsf{r}}. \end{tikzcd} \tag{4.15}$$

Proof. First consider the case that $j = 0$. In this case, (i) is tautological. For (ii), pick an N so that there is a factorization

$$\begin{tikzcd} & \mathrm{EO} \wedge \mathbb{CP}^{\mathsf{N}}_{+} \arrow[d, hook] \\ \mathcal{M}_0 \arrow[ur, dashed] \arrow[r] & \mathrm{EO} \wedge \mathbb{CP}^{\infty}_{+}. \end{tikzcd}$$

Such an N exists because Lemma 4.12 implies that $\mathcal{M}_0$ is a compact EO-module.

Let $\mathsf{r} = \Theta(\mathrm{EO}, \mathbb{CP}^{\mathsf{N}}_{+})$. Note that r divides $\Theta(\mathbb{S}_{(p)}, \mathbb{CP}^{\mathsf{N}}_{+})$ so it is finite. For any integer j, there is an EO-Thom isomorphism

$$\mathrm{EO} \wedge \mathbb{CP}^{\mathsf{N}+j\mathsf{r}}_{j\mathsf{r}} \simeq \mathrm{EO} \wedge \Sigma^{2j\mathsf{r}} \mathbb{CP}^{\mathsf{N}}_{+}.$$

We use this Thom isomorphism to construct an EO-module map

$$\tau\colon \Sigma^{2j\mathsf{r}} \mathcal{M}_0 \to \mathrm{EO} \wedge \Sigma^{2j\mathsf{r}} \mathbb{CP}^{\mathsf{N}}_{+} \xrightarrow{\simeq} \mathrm{EO} \wedge \mathbb{CP}^{\mathsf{N}+j\mathsf{r}}_{j\mathsf{r}} \to \mathrm{EO} \wedge \mathbb{CP}^{\infty}_{j\mathsf{r}} \to \mathcal{M}_{j\mathsf{r}}.$$

Notice that on applying the functor $\mathrm{gr}\,\mathrm{K}^{\mathrm{EO}}_{*}(-)$ to the Thom isomorphism, we get a $\mathrm{gr}\,\mathrm{K}_*[\mathrm{C}_p]$-isomorphism. Therefore, by Lemma 4.13, Proposition 3.23 and Proposition 3.25, τ induces a $\mathrm{gr}\,\mathrm{K}_*[\mathrm{C}_p]$-isomorphism

$$\Sigma^{2j\mathsf{r}} \mathrm{M}_0 \cong \mathrm{gr}\,\mathrm{K}^{\mathrm{EO}}_{*}(\Sigma^{2j\mathsf{r}} \mathcal{M}_0) \cong \mathrm{gr}\,\mathrm{K}^{\mathrm{EO}}_{*} \mathcal{M}_{j\mathsf{r}} \cong \mathrm{M}_{j\mathsf{r}}.$$

Thus τ induces an isomorphism of $\mathrm{K}_*[\mathrm{C}_p]$-module. By Corollary 4.5, we conclude that τ is an equivalence of EO-modules, which proves (i) as well as (ii). $\square$

Notation 4.16. Let N be the smallest positive integer satisfying Lemma 4.14, and set $\mathsf{r} = \Theta(\mathrm{EO}, \mathbb{CP}^{\mathsf{N}}_{+})$.

Proposition 4.17. $\Theta(\mathrm{EO}, \gamma_1) = \mathsf{r}$

Proof. By Lemma 4.14, there is a map $\Sigma^{2\mathsf{r}} \mathcal{M}_0 \to \mathrm{EO} \wedge \mathbb{CP}^{\infty}_{\mathsf{r}}$. Let

$$\mathcal{F}'_{\mathsf{r}} := \mathrm{Cofiber}(\Sigma^{2\mathsf{r}} \mathcal{M}_0 \to \mathrm{EO} \wedge \mathbb{CP}^{\infty}_{\mathsf{r}}). \tag{4.18}$$

Clearly $\mathrm{gr}\,\mathrm{K}^{\mathrm{EO}}_{*} \mathcal{F}'_{\mathsf{r}}$ is a free $\mathrm{gr}\,\mathrm{K}_*[\mathrm{C}_p]$-module. Corollary 4.11 gives an EO-module equivalence

$$\mathrm{EO} \wedge \mathbb{CP}^{\infty}_{\mathsf{r}} \simeq \Sigma^{2\mathsf{r}} \mathcal{M}_0 \vee \mathcal{F}'_{\mathsf{r}}.$$

There is a commutative diagram of $\mathrm{gr}\,\mathrm{K}_*[\mathrm{C}_p]$-modules:

$$\begin{tikzcd} & \mathrm{gr}\,\mathrm{K}^{\mathrm{EO}}_{*}(\Sigma^{2\mathsf{r}} \mathcal{M}_0) \arrow[dl, "(1)"'] \arrow[dr, "(2)"] & \\ \mathrm{gr}\,\mathrm{K}_* \mathbb{CP}^{\mathsf{N}+\mathsf{r}}_{\mathsf{r}} \arrow[rr, "\simeq"] \arrow[d, "(3)"'] & & \mathrm{gr}\,\mathrm{K}_*(\Sigma^{2\mathsf{r}} \mathbb{CP}^{\mathsf{N}}_{+}) \arrow[d, "(4)"] \\ \mathrm{gr}\,\mathrm{K}_* \mathbb{CP}^{\infty}_{\mathsf{r}} \arrow[rr, "\simeq"] & & \mathrm{gr}\,\mathrm{K}_*(\Sigma^{2\mathsf{r}} \mathbb{CP}^{\infty}_{+}) \end{tikzcd} \tag{4.19}$$

where the horizontal maps are induced by K_*-Thom isomorphisms, the map (2) is induced by the map

$$\mathcal{M}_0 \to \mathrm{EO} \wedge \mathbb{CP}^{\mathsf{N}}_{+} \tag{4.20}$$

and the maps (3) and (4) are the skeletal inclusion maps. The Thom isomorphisms are $\mathrm{gr}\,\mathrm{K}_*[\mathrm{C}_p]$-module maps by Proposition 3.23 and Lemma 4.13.

Since the map in (4.20) is the inclusion of a split summand with cofiber $\mathcal{F}_0$, the map $(3)\circ(1)$ is an injection whose cokernel is the free $\operatorname{gr}\mathrm{K}_*[\mathrm{C}_p]$-module isomorphic to $\operatorname{gr}\mathrm{K}_*^{\mathrm{EO}}\mathcal{F}_0$. Diagram (4.19) implies that the same is true for the cokernel of map $(4)\circ(2)$. The map

$$\sigma\colon \mathcal{F}_0 \to \mathrm{EO}\wedge\mathbb{CP}^\infty_+ \to \mathrm{EO}\wedge\mathbb{CP}^\infty_{\mathsf{r}} \to \mathcal{F}_{\mathsf{r}}$$

thus induces a $\operatorname{gr}\mathrm{K}_*[\mathrm{C}_p]$-module isomorphism between $\operatorname{coker}((3)\circ(1))$ and $\operatorname{coker}((4)\circ(2))$. Therefore, σ induces a $\mathrm{K}_*[\mathrm{C}_p]$-module isomorphism

$$\mathrm{K}_*^{\mathrm{EO}}\mathcal{F}'_{\mathsf{r}} \simeq \mathrm{K}_*^{\mathrm{EO}}(\Sigma^{2\mathsf{r}}\mathcal{F}_0)$$

By Lemma 4.7, σ is an EO-module equivalence

$$\mathcal{F}'_{\mathsf{r}} \simeq \Sigma^{2\mathsf{r}}\mathcal{F}_0$$

Hence, there is an EO-Thom isomorphism

$$\mathrm{EO}\wedge\mathbb{CP}^\infty_{\mathsf{r}} \simeq \Sigma^{2\mathsf{r}}\mathcal{M}_0\vee\mathcal{F}'_{\mathsf{r}} \simeq \Sigma^{2\mathsf{r}}(\mathcal{M}_0\vee\mathcal{F}_0) \simeq \mathrm{EO}\wedge\Sigma^{2\mathsf{r}}\mathbb{CP}^\infty_+$$

as desired. □

Let $\hat{\beta}_k = p^{k+1}-p^k$ be the smallest multiple of p which is greater than β_k. By Formula 2.19,

$$\Theta(\mathbb{S}_{(p)},\gamma_1^{\hat{\beta}_k-1}) = p^{p^k-1}.$$

We will finish the proof of Main Theorem 1.4 by showing that $\mathsf{N}\leq\hat{\beta}_k-1$.

Let f and s respectively denote the projection map and the section map for the splitting of $\mathrm{EO}\wedge\mathbb{CP}^\infty$

$$\mathcal{F}_0 \xrightarrow{\mathsf{s}} \mathrm{EO}\wedge\mathbb{CP}^\infty_+ \xrightarrow{\mathsf{f}} \mathcal{F}_0$$

so that $\mathsf{f}\circ\mathsf{s}$ is homotopic to the identity. Similarly, let $\bar{\mathsf{s}}$ and $\bar{\mathsf{f}}$ respectively denote the projection and the section map for the complement of $\mathcal{F}_0$

$$\mathcal{M}_0 \xrightarrow{\mathsf{s}'} \mathrm{EO}\wedge\mathbb{CP}^\infty_+ \xrightarrow{\mathsf{f}'} \mathcal{M}_0.$$

The free summand F_0 of the $\mathcal{B}(k)$-module $\mathrm{H}_*\mathbb{CP}^\infty$ surjects onto $\mathrm{H}_*\mathbb{CP}^\infty_{\hat{\beta}_k}$ under the coskeletal map. It follows that the composite

$$\mathcal{F}_0 \to \mathrm{EO}\wedge\mathbb{CP}^\infty_+ \to \mathrm{EO}\wedge\mathbb{CP}^\infty_{\hat{\beta}_k}$$

induces a surjection on $\mathrm{K}_*^{\mathrm{EO}}$. We wish to show that the composite

$$\mathcal{M}_0 \to \mathrm{EO}\wedge\mathbb{CP}^\infty_+ \to \mathrm{EO}\wedge\mathbb{CP}^\infty_{\hat{\beta}_k}$$

is null. We will prove that it is null after potentially modifying the inclusion map $\bar{\mathsf{s}}\colon\mathcal{M}_0\to\mathrm{EO}\wedge\mathbb{CP}^\infty_+$. This implies that $\mathsf{N}\leq\hat{\beta}_k-1$.

Proof of Main Theorem 1.4. Consider the composite

$$\mathrm{EO}\wedge\mathbb{CP}^\infty_+ \xrightarrow{\mathsf{f}} \mathcal{F}_0 \xrightarrow{\mathsf{s}} \mathrm{EO}\wedge\mathbb{CP}^\infty_+ \xrightarrow{\mathsf{cosk}} \mathrm{EO}\wedge\mathbb{CP}^\infty_{\hat{\beta}_k}$$

Let $\mathcal{R}$ be the fiber of $\mathsf{cosk}\circ\mathsf{s}\circ\mathsf{f}$. Since $\mathsf{cosk}\circ\mathsf{s}\circ\mathsf{f}$ induces a surjection on $\mathrm{K}_*^{\mathrm{EO}}$, the inclusion map $\mathcal{R}\to\mathrm{EO}\wedge\mathbb{CP}^\infty_+$ induces an injection on $\mathrm{K}_*^{\mathrm{EO}}$. Because $\mathrm{EO}\wedge\mathbb{CP}_+^{\hat{\beta}_k-1} = \mathrm{Fiber}(\mathsf{cosk})$, there is a map

$$\alpha\colon\mathcal{R}\to\mathrm{EO}\wedge\mathbb{CP}_+^{\hat{\beta}_k-1}.$$

The two maps $\mathsf{cosk}\circ\mathsf{s}\circ\mathsf{f}$ and cosk induce the same map on $\mathrm{K}_*^{\mathrm{EO}}$ so the inclusion map $\mathrm{K}_*^{\mathrm{EO}}(\mathcal{R}) \to \mathrm{K}_*(\mathbb{CP}_+^\infty)$ has the same image as the map $\mathrm{K}_*(\mathbb{CP}_+^{\hat{\beta}_k-1}) \to \mathrm{K}_*(\mathbb{CP}_+^\infty)$. Thus α must induce an injection on $\mathrm{K}_*^{\mathrm{EO}}$ and since $\mathrm{K}_*^{\mathrm{EO}}(\alpha)$ is a map of finite dimensional K_*-modules, it follows that $\mathrm{K}_*^{\mathrm{EO}}(\alpha)$ is an isomorphism. By Corollary 4.5, α is a weak equivalence of EO-modules. Because $\mathsf{f}\circ\bar{\mathsf{s}}$ is null, the composite $(\mathsf{cosk}\circ\mathsf{s}\circ\mathsf{f})\circ\bar{\mathsf{s}}$ is also null and there is a map $\mathcal{M}_0 \to \mathcal{R}$. Now let $\bar{\mathsf{s}}'$ be

$$\bar{\mathsf{s}}'\colon \mathcal{M}_0 \longrightarrow \mathcal{R} \xrightarrow{\simeq} \mathrm{EO}\wedge\mathbb{CP}_+^{\hat{\beta}_k-1}.$$

Note that $\mathrm{K}_*^{\mathrm{EO}}(\mathrm{Cofiber}(\bar{\mathsf{s}}'))$ is a free $\mathrm{gr}\,\mathrm{K}_*[\mathrm{C}_p]$-module so by Corollary 4.11, $\bar{\mathsf{s}}$ is the inclusion of a summand. Main Theorem 1.4 follows from Lemma 4.14 and Proposition 4.17. $\square$

Now we shift our focus to identifying the S^1-Tate spectrum of EO. If a group G acts on a spectrum X, the Tate spectrum X^{tG} of X is the cofiber of the norm map from the homotopy orbits spectrum X_{hG} of X to the homotopy fixed points spectrum X^{hG} of X

$$X^{\mathrm{tG}} \coloneqq \mathrm{Cofiber}(\mathrm{Nm}\colon X_{\mathrm{hG}} \to X^{\mathrm{hG}}).$$

If $\mathrm{G} = \mathrm{S}^1$ and the action on X is trivial, [GM95] gives an alternate description of the Tate fixed points:

$$X^{\mathrm{tS}^1} \simeq \lim_{c\to\infty} \Sigma^2\mathrm{R}\wedge\mathbb{CP}_{-c}^\infty. \tag{4.21}$$

There is a technical difficulty which needs to be resolved in order to use (4.21). Inverse limits do not commute with $\pi_*(-)$. Instead, there is a short exact sequence

$$0 \longrightarrow \lim_c^1(\mathrm{R}_*\mathbb{CP}_{-c}^\infty) \longrightarrow \pi_*\mathrm{R}^{\mathrm{tS}^1} \longrightarrow \lim_c(\mathrm{R}_*\mathbb{CP}_{-c}^\infty) \longrightarrow 0$$

called the “Milnor sequence”. To compute $\mathrm{EO}^{\mathrm{tS}^1}$, we need to prove that the $\lim^1$ vanishes. Before we prove the $\lim^1$ vanishing, we make one observation.

Lemma 4.22. *Let $\mathcal{F}_{j\mathsf{r}}$ be the free summand of $\mathrm{EO}\wedge\mathbb{CP}_{j\mathsf{r}}^\infty$ as in Lemma 4.12. Then the composition of the splitting map, coskeletal collapse map and the projection map*

$$\mathcal{F}_{j\mathsf{r}} \longrightarrow \mathrm{EO}\wedge\mathbb{CP}_{j\mathsf{r}}^\infty \xrightarrow{\mathsf{cosk}} \mathrm{EO}\wedge\mathbb{CP}_{(j+1)\mathsf{r}}^\infty \longrightarrow \mathcal{F}_{(j+1)\mathsf{r}}$$

is a surjection in homotopy.

Proof. By Proposition 3.25 we get a surjection after applying the functor $\mathrm{gr}\,\mathrm{K}_*^{\mathrm{EO}}(-)$. The result follows from the collapse of the relative Adams spectral sequence. $\square$

We now give the proof of Main Theorem 1.7 that there is an equivalence

$$\mathrm{EO}_\Gamma^{\mathrm{tS}^1} \simeq \prod_{-\infty<k<\infty} \mathrm{E}_\Gamma.$$

Proof of Main Theorem 1.7. Let $\mathsf{r}' = p^i\cdot\mathsf{r}$ where i is a positive integer large enough that $\mathsf{r}' > \mathsf{N}$. The subdiagram $\lim_j \mathrm{EO}\wedge\mathbb{CP}_{j\mathsf{r}'}^\infty$ is homotopy cofinal so we restrict our attention to it. The map

$$\Sigma^{j\mathsf{r}}\mathcal{M}_0 \longrightarrow \mathrm{EO}\wedge\mathbb{CP}_{j\mathsf{r}'}^\infty \xrightarrow{\mathsf{cosk}} \mathrm{EO}\wedge\mathbb{CP}_{(j+1)\mathsf{r}'}^\infty$$

is null homotopic. As a result there is a homotopy commutative diagram
(4.23)

$$\begin{tikzcd}
\cdots \arrow[r] & \Sigma^{2(i-1)\mathsf{r}'}\mathcal{M}_0 \arrow[r, "0"] \arrow[d] & \Sigma^{2i\mathsf{r}'}\mathcal{M}_0 \arrow[r, "0"] \arrow[d] & \Sigma^{2(i+1)\mathsf{r}'}\mathcal{M}_0 \arrow[r, "0"] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{EO} \wedge \mathbb{CP}^{\infty}_{(i-1)\mathsf{r}'} \arrow[r, "\mathrm{sk}_{\mathsf{r}'-1}"] \arrow[d] & \mathrm{EO} \wedge \mathbb{CP}^{\infty}_{i\mathsf{r}'} \arrow[r, "\mathrm{sk}_{\mathsf{r}'}"] \arrow[d] & \mathrm{EO} \wedge \mathbb{CP}^{\infty}_{(i+1)\mathsf{r}'} \arrow[r, "\mathrm{sk}_{\mathsf{r}'+1}"] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathcal{F}_{(i-1)\mathsf{r}'} \arrow[r] & \mathcal{F}_{i\mathsf{r}'} \arrow[r] & \mathcal{F}_{(i+1)\mathsf{r}'} \arrow[r] & \cdots
\end{tikzcd}$$

such that the horizontal maps in the top row are null maps, the horizontal maps in the middle row are coskeletal collapse maps, and the horizontal maps in the bottom row induce surjections on π_* by Lemma 4.22. Consequently, we get a six-term exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \lim_i \pi_* \Sigma^{-2i\mathsf{r}'}\mathcal{M}_0 \arrow[r] & \lim_i \mathrm{EO}_*\mathbb{CP}^{\infty}_{-i\mathsf{r}'} \arrow[r] & \lim_i \pi_*\mathcal{F}_{-i\mathsf{r}'} \arrow[dll] \\
 & \lim^1_i \pi_* \Sigma^{-2i\mathsf{r}'}\mathcal{M}_0 \arrow[r] & \lim^1_i \mathrm{EO}_*\mathbb{CP}^{\infty}_{-i\mathsf{r}'} \arrow[r] & \lim^1_i \pi_*\mathcal{F}_{-i\mathsf{r}'} \arrow[r] & 0.
\end{tikzcd}$$

From Diagram (4.23),

$$\lim_i \pi_* \Sigma^{-2i\mathsf{r}}\mathcal{M}_0 = 0$$
$$\lim^1_i \pi_* \Sigma^{-2i\mathsf{r}}\mathcal{M}_0 = 0.$$

and from Lemma 4.22 we get

$$\lim_i \pi_*\mathcal{F}_{-i\mathsf{r}} = 0.$$

Lemma 4.22 also shows that

$$\lim_i \mathcal{F}_{-i\mathsf{r}} \simeq \lim_i \prod_{-i<k<\infty} \mathrm{E}.$$

This finishes the proof of Main Theorem 1.7. □

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University of Virginia

E-mail address: pb9wh@virginia.edu

Massachusetts Institute of Technology

E-mail address: hood@mit.edu