TYPE 2 COMPLEXES CONSTRUCTED FROM BROWN–GITLER SPECTRA

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Abstract. In a previous paper, one of us interpreted mod 2 Dyer-Lashof operations as explicit A–module extensions between Brown-Gitler modules, and showed these A–modules can be topologically realized by finite spectra occurring as fibers of maps between 2-local dual Brown–Gitler spectra.

Starting from these constructions, in this paper, we show that infinite families of these finite spectra are of chromatic type 2, with mod 2 cohomology that is free over $A(1)$. Applications include classifying the dual Brown–Gitler spectra after localization with respect to K–theory.

1. Introduction

Localized at a prime p, finite spectra are beautifully organized by chromatic type: X is of type n if $K(n)^*(X) \neq 0$ while $K(n+1)^*(X) = 0$, where $K(n)$ is the nth Morava K–theory at p. Closed manifolds are all of type 0 or 1, but general theory [HS98] tells us that finite spectra of all types abound: any finite type n spectrum X admits a v_n-self map $v : \Sigma^d X \to X$ with cofiber of type $(n+1)$.

In spite of this, it is a bit hard to find explicit examples of spectra of higher height. In particular, explicit v_n–self maps are notoriously difficult to construct, with proofs involving deep dives into either the classical Adams spectral sequence or the Adams-Novikov spectral sequence[1]. Some higher height examples have also been constructed as retracts of manifolds by using representation theory to construct splitting idempotents[2], but such examples are usually quite large.

In this paper we mine a new source of type 2 complexes. We find infinite families of type 2 complexes at the prime 2 among the family of spectra $X(n,m)$ recently constructed by the third author [K23] as the fibers of maps $f(n,m) : T(n) \to T(m)$, where $T(n)$ is the n-dual of the nth Brown-Gitler

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[1]See [BE20] for a recent example.

[2]Steve Mitchell [M85] used the representation theory of the finite general linear groups to construct the first spectra of all heights, and a more flexible method using the symmetric groups was introduced by Jeff Smith (see [Rav92, Appendix C] or [KL24a]).

spectrum. Thus our type 2 complexes are built out of two finite spectra which have been much studied for rather different purposes.

2. Main Results

2.1. **A little bit of background.** To state our results, we need to recall a bit of background material; see §3 and §4 for more detail and references.

We work at the prime 2, and A denotes the mod 2 Steenrod algebra. We let $V^\vee$ denote the dual of a vector space over $\mathbb{Z}/2$. Given $n \in \mathbb{N}$, $\alpha_2(n)$ denotes the number of ones in the binary expansion of n, and $\nu_2(n)$ denotes the greatest i such that 2^i divides n.

We recall a few facts about computing $K(1)^*(X) = KU^*(X;\mathbb{Z}/p)$. The Atiyah–Hirzebruch spectral sequence converging to $K(1)^*(X)$ has $E_2 = E_3 = H^*(X;\mathbb{Z}/2[v^{\pm 1}])$ with $|v| = -2$. The formula for the next differential is given by

$$d_3(x) = Q_1(x)v,$$

where $Q_1 = Sq^2Sq^1 + Sq^1Sq^2$ is a primitive in A satisfying $Q_1^2 = 0$. If M is an A–module, let

$$H(M;Q_1) = \ker\{Q_1 : M \to M\}/\operatorname{im}\{Q_1 : M \to M\}$$

denote the Q_1-Margolis homology of M. Letting $H^*(X)$ denote mod 2 cohomology, it follows that X will be $K(1)^*$–acyclic if $H(H^*(X);Q_1) = 0$. Furthermore, J. Palmieri [P96, Cor.A.6] showed that if M is a finite dimensional A–module, then $H(M;Q_1) = 0$ if and only if M is free over $A(1)$, the 8 dimensional subalgebra of A generated by Sq^1 and Sq^2.

We recall a few facts about (dual) Brown–Gitler spectra and their cohomology. Let $J(n) = H^*(T(n))$. Then $J(n)$ is an unstable A–module, where we recall that an A–module M is *unstable* if $Sq^i x = 0$ for $i > n$ if $x \in M^n$. We let $\mathcal{U}$ denote the category of such modules; this is a full subcategory of the abelian category of all left A–modules. Note that $H^*(X) \in \mathcal{U}$ if X is any *spacelike* spectrum, i.e. a retract of a suspension spectrum.

The modules $J(n)$ are, in fact, $\mathcal{U}$–injectives: $J(n)$ has a unique nonzero top degree class of degree n, and the natural map

$$\operatorname{Hom}_A(M, J(n)) \simeq M^{n\vee} \tag{2.1}$$

sending $f : M \to J(n)$ to $f^n : M^n \to J(n)^n \simeq \mathbb{Z}/2$ is an isomorphism for $M \in \mathcal{U}$. Computation shows that $J(n)$ has a unique nonzero bottom degree class in degree $\alpha_2(n)$.

The algebraic properties of $J(n)$ are reflected in two properties of $T(n)$:

- (Brown-Gitler property) $[T(n), X] \to H_n(X)$ is onto if X is spacelike.
- $T(n)$ is spacelike.

We note that $T(2n+1) = \Sigma T(2n)$, and the first few even examples are $T(0) = S$, $T(2) = \Sigma^\infty \mathbb{R}P^2$, and $T(4) = \Sigma^\infty \mathbb{R}P^4$.

Finally we recall a bit about the results in [K23]. There, a certain finite spectrum $X(n,m)$ is defined as the fiber of well-chosen map $f(n,m) : T(m) \to$

$T(n)$, so that there is a fibration sequence of spectra

$$\Sigma^{-1}T(n) \to X(n,m) \to T(m) \xrightarrow{f(n,m)} T(n). \tag{2.2}$$

By construction, the map $f(n,m)$ induces zero in mod 2 cohomology, and thus (2.2) induces a short exact sequence of A–modules

$$0 \to J(m) \to Q(n,m) \to \Sigma^{-1}J(n) \to 0, \tag{2.3}$$

where $Q(n,m) = H^*(X(n,m))$.

Note that $Q(n,m)$ can be viewed as an element in $\mathrm{Ext}_A^{1,1}(J(n), J(m))$. The main result in [K23] is that these induce a known action of the Dyer-Lashof algebra on the bigraded vector space $\mathrm{Ext}_A^{\star,\star}(M, J(*))$ for all A–modules M, by letting

$$Q^r : \mathrm{Ext}_A^{s,s}(M, J(n)) \to \mathrm{Ext}_A^{s+1,s+1}(M, J(n+r))$$

be the linear map induced by Yoneda splice with $Q(n, n+r)$. We will need a consequence of this theorem: the short exact sequence (2.3) is split if $m < 2n - 1$.

2.2. **The main theorem and consequences.** The goal of this paper is to compute exactly when $Q(n,m)$ is Q_1–acyclic, and thus $X(n,m)$ is a type 2 complex.

For starters, $m \geq 2n$ will be seen to be necessary using [K23]. Our main theorem then goes as follows.

Theorem 2.1. *(a) Let m and n be even, $m \geq 2n$, and write m in the form $m = l + 2^k$ with $0 \leq l < 2^k$. Then $Q(n,m)$ is Q_1–acyclic if and only if*

1. *$\nu_2(n) = \nu_2(l)$.*
2. *$\alpha_2(n) = \alpha_2(l)$.*
3. *If $l = 2^{i_1} + \cdots + 2^{i_d}$ with $i_1 < \cdots < i_d$, and $n = 2^{j_1} + \cdots + 2^{j_d}$ with $j_1 < \cdots < j_d$, then $i_1 = j_1 < i_2 \leq j_2 < i_3 \leq \cdots < i_d \leq j_d$.*

(b) If m and n are both even, then $Q(n+1, m+1) = \Sigma Q(n,m)$, and so is Q_1–acyclic if and only if $Q(n,m)$ is.
(c) If m and n are of different parities, then $Q(n,m)$ is not Q_1–acyclic.

Note that condition (3) of part (a) implies that $l \leq n$. The special case when $n = l$ was first proved by Brian Thomas [T19] and reads as follows.

Corollary 2.2. *If n is even and $2^k > n$, then $Q(n, n+2^k)$ is Q_1–acyclic.*

Examples 2.3. Pairs satisfying the conditions of Theorem 2.1(a) include

- $(n,m) = (2,6)$ with $l = 2$ and $k = 2$. In this case, $Q(2,6)$ is isomorphic to $A(1)$ as an $A(1)$–module.
- $(n,m) = (10,22)$ with $l = 6$ and $k = 4$.

From Theorem 2.1, one immediately deduces the following consequences.

Theorem 2.4. *If (n,m) satisfies the conditions in Theorem 2.1(a), then*

(a) *$Q(n,m)$ is a free $A(1)$–module.*

(b) *If M is any A–module of the form $M = A \otimes_{A(1)} N$, with N an $A(1)$–module, then Yoneda splice with $Q(n,m)$,*

$$Q(n,m)\circ : \mathrm{Ext}_A^{s,t}(M, J(n)) \to \mathrm{Ext}_A^{s+1,t+1}(M, J(m)),$$

will be an isomorphism for $s \geq 1$ and an epimorphism for $s = 0$.

(c) *$X(n,m)$ is a type 2 complex.*

(d) *$f(n,m) : T(m) \to T(n)$ becomes an equivalence after L_1–localization.*

Here L_1 is localization with respect to complex K–theory, localized at 2. For statement (c), we need to know that $X(n,m)$ is not $K(2)^*$–acyclic, or, equivalently, $f(n,m)$ does not induce an isomorphism on $K(2)^*$, proved as Corollary 4.10.

Examples 2.5. Important A–modules M as in Theorem 2.4(b) include $H^*(ku) = A \otimes_{A(1)} (A(1) \otimes_{E(1)} \mathbb{Z}/2)$ and $H^*(ko) = A \otimes_{A(1)} \mathbb{Z}/2$. Here $E(1)$ is the subalgebra of A generated by Sq^1 and Q_1.

Theorem 2.4(d) will let us classify the spectra $T(n)$ after L_1–localization.

Theorem 2.6. *(a) If n is even and $i = \nu_2(n)$ then there is an L_1–equivalence $f : T(n) \to T(2^i)$ of Adams filtration at least $\alpha_2(n) - 1$.*
(b) There are maps $\alpha : T(2^{i-1}) \to T(2^i)$ and $\beta : T(2^i) \to \Sigma^2 T(2)$ such that

$$L_1T(2^{i-1}) \xrightarrow{L_1\alpha} L_1T(2^i) \xrightarrow{L_1\beta} \Sigma^2 L_1T(2)$$

is a cofibration sequence of L_1–local spectra.
(c) $K^0(T(2^i))_{(2)} \simeq \mathbb{Z}/2^i$, $K^1(T(2^i))_{(2)} \simeq 0$, and, on K^0, the cofibration sequence of part (b) induces the short exact sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2^i \to \mathbb{Z}/2^{i-1} \to 0.$$

2.3. **First steps in the proof of Theorem 2.1.** We need to say a bit more about the construction of $f(n,m) : T(m) \to T(n)$ from [K23]. Let $P_1 = \Sigma^\infty \mathbb{R}P^\infty$, so that $H^*(P_1) = (t) \subset \mathbb{Z}/2[t] = H^*(\mathbb{R}P^\infty)$. Then let $\tilde{P}_{-1}$ denote the fiber of the Kahn-Priddy map $t : P_1 \to S$, so there is a fibration sequence of spectra

$$S^{-1} \to \tilde{P}_{-1} \to P_1 \xrightarrow{t} S. \tag{2.4}$$

Smashing this sequence with $T(n)$ yields a fibration sequence of spectra

$$\Sigma^{-1}T(n) \to T(n) \wedge \tilde{P}_{-1} \to T(n) \wedge P_1 \xrightarrow{1\wedge t} T(n) \tag{2.5}$$

inducing a short exact sequence of A–modules

$$0 \to J(n) \otimes (t) \to J(n) \otimes H^*(\tilde{P}_{-1}) \to \Sigma^{-1}J(n) \to 0. \tag{2.6}$$

A well chosen A–module map $q(n,m)^* : J(n) \otimes (t) \to J(m)$ is realized by a map $q(n,m) : T(m) \to T(n) \wedge P_1$, and $f(n,m)$ is defined as the composite

$T(m) \xrightarrow{q(n,m)} T(n) \wedge P_1 \xrightarrow{1\wedge t} T(n)$. It follows that there is a map of fibration sequences

$$\begin{tikzcd}
\Sigma^{-1}T(n) \arrow[r] & T(n)\wedge \widetilde{P}_{-1} \arrow[r] & T(n)\wedge P_1 \arrow[r, "1\wedge t"] & T(n) \\
\Sigma^{-1}T(n) \arrow[r] \arrow[u, equal] & X(n,r) \arrow[r] \arrow[u] & T(m) \arrow[r, "f(n,m)"] \arrow[u, "q(n,m)"'] & T(n) \arrow[u, equal]
\end{tikzcd}$$

inducing a map of A–module extensions

$$\begin{tikzcd}
0 \arrow[r] & J(n)\otimes(t) \arrow[r] \arrow[d, "q(n,m)^*"] & J(n)\otimes H^*(\tilde{P}_{-1}) \arrow[r] \arrow[d] & \Sigma^{-1}J(n) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J(m) \arrow[r] & Q(n,m) \arrow[r] & \Sigma^{-1}J(n) \arrow[r] & 0.
\end{tikzcd} \tag{2.7}$$

It is easy to check – see Example 4.1 – that $H^*(\tilde{P}_{-1})$ is Q_1–acyclic and then that diagram chasing yields the next lemma.

Lemma 2.7. *The A-module $Q(n,m)$ is Q_1–acyclic if and only if $q(n,m)^*$ is a Q_1–isomorphism.*

Thus the proof of Theorem 2.1 amounts to analyzing exactly when

$$q(n,m)^* : H(J(n)\otimes(t);Q_1) \to H(J(m);Q_1)$$

is an isomorphism.

The good news here is that the domain and range are always 2-dimensional, with classes represented by explicit elements in $J(n)\otimes(t)$ and $J(m)$.

Less good, and the cause of most of the work in this paper, is that the A–module map $q(n,m)^* : J(n)\otimes(t) \to J(m)$ is defined implicitly using (2.1), and thus is only explicitly described in degree m.

2.4. **Organization of the rest of the paper.** In §3 we recall basic properties of the dual Brown-Gitler modules; in particular, the elegant description of $\bigoplus_{n=0}^{\infty} J(n)$, viewed as an algebra in $\mathcal{U}$. This allows us to precisely define our key A–module maps

$$q(n,m)^* : J(n)\otimes(t) \to J(m),$$

as well as another family of A–module maps

$$p(n,l)^* : J(n) \to J(l).$$

Like $q(n,m)^*$, the map $p(n,l)^*$ can also be geometrically realized by a map $p(n,l) : T(l) \to T(n)$, and we let $Y(n,l)$ denote the fiber.

Then in §4 we use the description of $\bigoplus_{n=0}^{\infty} J(n)$ to recover implicitly known calculations of $K(m)^*(T(n))$ for all n and m, calculating the Margolis homology groups $H(J(n);Q_m)$ enroute. Also proved here will be Lemma 2.7.

We prove Theorem 2.1 in the three sections that follow this.

In §5, we first recall results from [K23] allowing us to conclude that if $q(n,m)^*$ is a Q_1–isomorphism, then necessarily $m \geq 2n$. When this condition holds, we show that $q(n,m)^*$ is a Q_1–isomorphism if and only if $p(n,l)^*$ is a Q_1–isomorphism, where $m = l + 2^k$ with $0 \leq l < 2^k$. As $p(n,n)^* : J(n) \to J(n)$ is the identity, this is already sufficient to deduce Corollary 2.2.

In §6, we focus on pairs (n,l) when n and l are even, and first show that $H(J(n);Q_1) \simeq H(J(l);Q_1)$ as graded vector spaces if and only if $\alpha_2(n) = \alpha_2(l)$ and $\nu_2(n) = \nu_2(l)$. Under these conditions, we then study the much more delicate question of when $p(n,l)^*$ is a Q_1–isomorphism, ultimately leading to condition (3) listed in Theorem 2.1(a).

In §7, we use similar methods to prove Theorem 2.1(c): $p(n,l)^*$ is never a Q_1–isomorphism when n and l are of different parities.

In the final section §8 we first prove Theorem 2.6 and then discuss how our work fits with the thesis of Brian Thomas [T19].

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Our work builds on the 2019 University of Virginia Ph.D. thesis work of Brian Thomas [T19], working under the supervision of the third author. In particular, knowing Corollary 2.2 was true was useful, even though our proof of this is ultimately different.

3. The mod 2 cohomology of $T(n)$

3.1. **The A–module $H^*(\mathbb{R}P^\infty)$ and related modules.** Recall that, as an unstable A–algebra $H^*(\mathbb{R}P^\infty) \simeq \mathbb{Z}/2[t]$, with $Sq^i t^j = \binom{j}{i} t^{j+i}$.

For $d \in \mathbb{Z}$, we let P_d be the Thom spectrum of d copies of the canonical line bundle over $\mathbb{R}P^\infty$. Then $P_0 = \Sigma^\infty \mathbb{R}P^\infty_+$ with cohomology $\mathbb{Z}/2[t]$ as just described and $P_1 = \Sigma^\infty \mathbb{R}P^\infty$ with cohomology equal to the ideal (t). For general d, $H^*(P_d)$ has basis given by t^j for $j \geq d$ with the formula $Sq^i t^j = \binom{j}{i} t^{j+i}$ still applying[3]. In particular $Sq^i t^{-1} = t^{i-1}$ for all i.

For $d \leq 0$, let $\tilde{P}_d$ be the fiber of the composite $P_d \to P_0 \xrightarrow{\pi} S$, where π is the standard retraction. Then $\tilde{P}_0 \simeq P_1$, and the cofibration sequence

$$S^{-1} \to P_1 \to P_0 \xrightarrow{tr} S,$$

with tr the C_2–transfer, induces the cofibration sequence

$$S^{-1} \to \tilde{P}_{-1} \to P_1 \xrightarrow{t} S$$

of the introduction. The first terms of this induce a short exact sequence of A–modules

$$0 \to (t) \to H^*(\tilde{P}_{-1}) \to \Sigma^{-1}\mathbb{Z}/2 \to 0. \tag{3.1}$$

[3]For j possibly negative, $\binom{j}{i}$ is defined by the identity $(x+1)^j = \sum_{i=0}^\infty \binom{j}{i} x^i$

3.2. **Projectives and injectives in $\mathcal{U}$.** A good general reference for much of the algebra here is [S94, Chapter 2].

Let $F(m) \in \mathcal{U}$ be the free unstable module on a class of degree m. There is a natural isomorphism

$$\mathrm{Hom}_A(F(m), N) \simeq N^m \tag{3.2}$$

for all unstable modules N. Particularly important for us is the explicit structure of $F(1)$: it is the sub-A–module of $\mathbb{Z}/2[t] = H^*(\mathbb{R}P^\infty)$ spanned by the elements t^{2^k} with $k \geq 0$. One has $Sq^{2^k} t^{2^k} = t^{2^{k+1}}$ with all other nonidentity Steenrod operations acting as 0.

Dually, the dual Brown-Gitler module $J(n)$ is the unstable module such that there is a natural isomorphisms

$$\mathrm{Hom}_A(M, J(n)) \simeq M^{n\vee} \tag{3.3}$$

for all unstable modules M.

Combining (3.2) and (3.3) shows that $J(n)^m \simeq F(m)^{n\vee}$. From this, one sees that $J(n)^n \simeq \mathbb{Z}/2$ and $J(n)^m = 0$ for $m \geq n$. One also sees that $J(n)^1 \simeq \mathbb{Z}/2$ if n is a power of 2 and is 0 otherwise.

Notation 3.1. For $k \geq 0$, let x_k be the nonzero element of $J(2^k)^1$ and let $p_k : F(1) \to J(2^k)$ be the unique nonzero A–module map. Note that $p_k(t) = x_k$.

Using (3.3), the natural transformations $Sq^i : M^n \to M^{n+i}$ induce corresponding maps of left A–modules $\cdot Sq^i : J(n+i) \to J(n)$.

Lemma 3.2. *$\cdot Sq^{2^k} : J(2^{k+1}) \to J(2^k)$ maps x_{k+1} to x_k.*

Proof. Unwinding the definitions, one sees that the formula $\cdot Sq^{2^k}(x_{k+1}) = x_k$ is dual to $Sq^{2^k}(t^{2^k}) = t^{2^{k+1}}$. □

3.3. **The bigraded algebra $J(\star)^*$.** We recall the computation of the bigraded object $J(\star)^* = \bigoplus_{n=0}^{\infty} J(n)$. This is a graded commutative algebra in the category $\mathcal{U}$ with multiplication $\mu : J(m) \otimes J(n) \to J(m+n)$ corresponding to the unique A–module map that is nonzero in degree $m+n$.

Theorem 3.3. [S94, Thm.2.4.7] *There is an isomorphism*

$$J(\star)^* \simeq \mathbb{Z}/2[x_0, x_1, \dots].$$

The A–module structure is determined by the instability condition, the Cartan formula, and the formulae $Sq^1(x_{k+1}) = x_k^2$ and $Sq^1 x_0 = 0$.

In particular, each $J(n)$ has a basis given by some of the monomials in the x_k's. The top degree nonzero class is $x_0^n \in J(n)^n$ and there is a unique bottom degree nonzero class in degree $\alpha_2(n)$ as follows.

Notation 3.4. Let $x(n) \in J(n)^{\alpha_2(n)}$ be the class $x(n) = x_{i_1} \cdots x_{i_d}$ where $n = 2^{i_1} + \cdots + 2^{i_d}$ with $i_1 < \cdots < i_d$ (so $d = \alpha_2(n)$).

The following calculation of the map $p_k : F(1) \to J(2^k)$ will be used by us, and is easily proved by induction on i.

Lemma 3.5. *For $0 \le i \le k$, $p_k(t^{2^i}) = x_{k-i}^{2^i}$.*

3.4. **The Mahowald sequences.** There are isomorphisms of A–modules

$$\Sigma J(2n) \xrightarrow{\sim} J(2n+1)$$

and also short exact sequences [S94, Prop.2.2.3]

$$0 \to \Sigma J(2n-1) \to J(2n) \xrightarrow{\cdot Sq^n} J(n) \to 0. \tag{3.4}$$

These are easy to describe on basis elements. Firstly, $\Sigma J(n) \to J(n+1)$ sends σx to $x_0 x$. Secondly, it is not hard to check that

$$\cdot Sq^\star : J(2\star) \to J(\star)$$

is a ring homomorphism, and identifies with the ring homomorphism

$$\mathbb{Z}/2[x_0^2, x_1, x_2, \dots] \to \mathbb{Z}/2[x_0, x_1, x_2, \dots]$$

sending x_0^2 to zero and x_{k+1} to x_k.

3.5. **Two families of A–module maps and a family of A–modules.** As $J(n)$ has a standard basis, so does $J(n) \otimes (t)$ in the obvious way. We define two families of A–module maps using (3.3).

Definitions 3.6. (a) Let $p(n,l)^* : J(n) \to J(l)$ be the unique A–module map which is nonzero on each standard basis element of $J(n)^l$. (In particular, $p(n,n)^*$ is the identity.)
(b) Let $q(n,m)^* : J(n) \otimes (t) \to J(m)$ be the unique A–module map which is nonzero on each standard basis element of $(J(n) \otimes (t))^m$.

Tensoring (3.1) with $J(n)$ gives a short exact sequence of A–modules

$$0 \to J(n) \otimes (t) \to J(n) \otimes H^*(\tilde{P}_{-1}) \to \Sigma^{-1} J(n) \to 0.$$

Definition 3.7. Define the A–module $Q(n,m)$ to be the pushout of the diagram $J(m) \xleftarrow{q(n,m)^*} J(n) \otimes (t) \to J(n) \otimes H^*(\tilde{P}_{-1})$.

Thus there is a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & J(n) \otimes (t) \arrow[r] \arrow[d, "q(n,m)^*"] & J(n) \otimes H^*(\tilde{P}_{-1}) \arrow[r] \arrow[d] & \Sigma^{-1} J(n) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & J(m) \arrow[r] & Q(n,m) \arrow[r] & \Sigma^{-1} J(n) \arrow[r] & 0. \end{tikzcd} \tag{3.5}$$

3.6. **Geometric realization.** Brown and Gitler [BG73] showed that there are remarkable finite spectra $T(n)$[4] (the n-duals of what were later termed Brown-Gitler spectra) satisfying the following properties:

(1) $H^*(T(n)) \simeq J(n)$.
(2) The natural map $[T(n), X] \to H_n(X)$, sending f to $f_*(\iota_n)$, is onto whenever X is spacelike (i.e. a retract of a suspension spectrum).

Later Goerss [G85] and Lannes [L88] proved a third property:

(3) $T(n)$ is spacelike.

A nice proof of all of this is given in [GLM93], and the equivalence of the second and third properties, assuming the first, is shown in [HK00].

It follows that there exist maps $T(i+j) \to T(i) \wedge T(j)$ inducing $\mu : J(i) \otimes J(j) \to J(i+j)$ in mod 2 cohomology.

Similarly, there exist maps $p(n,l) : T(l) \to T(n)$ and $q(n,m) : T(m) \to T(n) \wedge P_1$ inducing $p(n,l)^*$ and $q(n,m)^*$. We then define $Y(n,l)$ to be the fiber of $p(n,l)$ and $X(n,m)$ to be the fiber of $f(n,m)$, where $f(n,m) : T(m) \to T(n)$ is the composite $(1 \wedge t) \circ q(n,m)$.

From these constructions it follows that $Q(n,m) \simeq H^*(X(n,m))$. Indeed, applying cohomology to the left part of the diagram of fibration sequences

$$\begin{tikzcd}
\Sigma^{-1}T(n) \arrow[r] & T(n) \wedge \widetilde{P}_{-1} \arrow[r] & T(n) \wedge P_1 \arrow[r, "1\wedge t"] & T(n) \\
\Sigma^{-1}T(n) \arrow[r] \arrow[u, equal] & X(n,r) \arrow[r] \arrow[u] & T(m) \arrow[r, "f(n,m)"] \arrow[u, "q(n,m)"'] & T(n) \arrow[u, equal]
\end{tikzcd}$$

yields diagram (3.5).

Remark 3.8. Note that $p(n,l)$ and $q(n,m)$ are only well defined up to maps of positive Adams filtration, and thus $f(n,m)$ can be varied by certain maps of Adams filtration greater than 1. We do not know how this affects the homotopy types of the spectra $Y(n,l)$ and $X(n,m)$.

Finally, we note that the Mahowald short exact sequence can be realized geometrically by a cofibration sequence: there exists a cofibration sequence

$$T(n) \to T(2n) \to \Sigma T(2n-1)$$

inducing (3.4) in cohomology. The Brown-Gitler property makes it clear that two maps exist that realize the algebraic ones, but it is not formal that they can be chosen so that the composite is null, showing that one has a cofibration sequence. The proof that this *can* be done was given in [CMM78].

[4]The notation $T(n)$ was used by them, and in subsequent papers by Lannes, Goerss, the third author, and others, and should not be confused with telescopic $T(n)$, used by many, including some of us.

4. The Morava K—theory of $T(n)$

4.1. **The Morava K–theory A.H.S.S..** Let $m \geq 1$. Recall that the mth Morava K–theory (at the prime 2) is a complex oriented ring theory having coefficient ring $K(m)^* = \mathbb{Z}/2[v_m^{\pm 1}]$, with $|v_m| = 2 - 2^{m+1}$. (This is cohomological grading.) This is a graded field, and it follows that $K(n)^*(X)$ is a graded $K(m)^*$–vector space and, for finite X and all Y, the natural map $K(m)^*(X) \otimes_{K(m)^*} K^*(Y) \to K(m)^*(X \wedge Y)$ is an isomorphism.

The Atiyah–Hirzeburch spectral sequence $\{E_r^*(X)\} \Rightarrow K(m)^*(X)$ has $E_2^*(X) = H^*(X; \mathbb{Z}/2[v_m^{\pm 1}])$. By sparseness, the first possible nonzero differential is $d_{2^{m+1}-1}$, and, indeed, it is not hard to deduce[5] the precise formula:

$$d_{2^{m+1}-2}(x) = v_m Q_m x.$$

Here, $Q_m \in A^{2^{m+1}-1}$ for $m \geq 0$ are primitives recursively defined by $Q_0 = Sq^1$ and $Q_m = [Q_{m-1}, Sq^{2^m}]$. These satisfy $Q_m^2 = 0$.

It follows that there is a natural isomorphism $E^*_{2^{m+1}}(X) \simeq K(m)^* \otimes H(H^*(X); Q_m)$ where the Q_m–homology of an A–module M is defined to be

$$H(M; Q_m) = \frac{\ker Q_m}{\operatorname{im} Q_m}.$$

Note that a short exact sequence of A–modules, $0 \to L \to M \to N \to 0$ will induce a long exact sequence

$$\cdots \to H(L; Q_m) \to H(M; Q_m) \to H(N; Q_m) \xrightarrow{d} H(L; Q_m) \to \cdots$$

where d raises degree by $|Q_m| = 2^{m+1} - 1$.

Example 4.1. It is easy to compute that, in $H^*(\tilde{P}_{-1})$, we have $Q_1 t^j = t^{j+3}$ for all odd j, and thus $H^*(\tilde{P}_{-1})$ is Q_1–acyclic[6]. Since Q_1 is a derivation, Q_1–homology satisfies a Künneth theorem, and so any module of the form $M \otimes H^*(\tilde{P}_{-1})$ will also be Q_1–acyclic. This, together with the 5-lemma applied to the Q_1–homology long exact sequences coming from the diagram

$$\begin{tikzcd}
0 \arrow[r] & J(n) \otimes (t) \arrow[r] \arrow[d, "q(n,m)^*"] & J(n) \otimes H^*(\tilde{P}_{-1}) \arrow[r] \arrow[d] & \Sigma^{-1} J(n) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J(m) \arrow[r] & Q(n,m) \arrow[r] & \Sigma^{-1} J(n) \arrow[r] & 0,
\end{tikzcd}$$

proves Lemma 2.7: $Q(n,m)$ is Q_1–acyclic if and only if $q(n,m)^*$ is a Q_1–isomorphism.

[5]The proof of the odd prime version of this formula in [Y80] works without change at the prime 2, and we sketch the idea. As the differential is natural, it must have the form $d_{2^{m+1}-2}(x) = v_m a x$ for some $a \in A^{2^{m+1}-1}$. Furthermore a must be primitive, as the differential must be a derivation when X is a space, and must be nonzero to be consistent with $K(m)^*(\mathbb{R}P^\infty) \simeq K(m)^*[y]/(y^{2^m})$, calculated using a Gysin sequence. Q_m is the only nonzero primitive in degree $2^{m+1} - 1$.

[6]This illustrates that the hypothesis of finite dimensionality is needed in [P96, Cor.A.6], as $H^*(\tilde{P}_{-1})$ is *not* free over $A(1)$.

Remark 4.2. When $m = 0$, one can replace the Atiyah–Hirzebruch spectral sequence in most of the above with the Bockstein spectral sequence, and these can be identified with localized Adams spectral sequences for connective Morava K–theories $k(n)^*$, where $K(0) = H\mathbb{Q}$ and $k(0) = H\mathbb{Z}$. (See [KL24b, §3.1] for more detail.)

4.2. **The Q_m–homology groups of $J(n)$.** In this subsection, we calculate $H(J(\star)^*; Q_m)$.

Recall that $J(\star)^* = \mathbb{Z}/2[x_0, x_1, \ldots]$, with $x_k \in J(2^k)^1$ and with A–module structure determined by the Cartan formula, the instability condition, $Sq^1(x_0) = 0$, and $Sq^1(x_k) = x_{k-1}^2$ for $k \geq 1$. The action of the derivation Q_m on $J(\star)^*$ is determined by the following lemma, which easily proved by induction on m.

Lemma 4.3. $Q_m x_k = \begin{cases} 0 & \textit{for } k \leq m, \\ x_{k-m-1}^{2^{m+1}} & \textit{for } k > m. \end{cases}$

Proposition 4.4.

$$H(J(\star)^*; Q_m) = \mathbb{Z}/2[x_0, \ldots, x_m, x_{m+1}^2, x_{m+2}^2, \ldots]/(x_k^{2^{m+1}} \mid k \geq 0).$$

Proof. It is convenient to let $R = \mathbb{Z}/2[x_0, \ldots, x_m, x_{m+1}^2, x_{m+2}^2, \ldots]$. Then $(J(\star)^*; Q_m)$ is a chain complex of R modules that decomposes as an infinite tensor product of chain complexes of R–modules

$$J(\star)^* \simeq C(0) \otimes_R C(1) \otimes_R C(2) \otimes \cdots$$

where $C(k) = R \oplus Rx_{m+1+k}$ with differential $d(x_{m+k+1}) = x_k^{2^{m+1}}$.

Using that $x_0^{2^{m+1}}, x_1^{2^{m+1}}, x_2^{2^{m+1}}, \ldots$ is a regular sequence in R, we see first that $H(C(k)) = R/(x_k^{2^{m+1}})$ and then that a Künneth isomorphism holds:

$$H(J(\star)^*; Q_m) \simeq H(C(0); Q_m) \otimes_R H(C(1); Q_m) \otimes_R H(C(2); Q_m) \otimes \cdots$$

Thus $H(J(\star)^*; Q_m) = R/(x_k^{2^{m+1}} \mid k \geq 0)$ as an R module, and the proposition follows. □

4.3. **Calculation of $K(m)^*(T(\star))$.** We now calculate $K(m)^*(T(n))$ using the Atiyah-Hirzebruch spectral sequence by assembling these for all n at once. The existence of maps $T(i + j) \to T(i) \wedge T(j)$ inducing our multiplication $J(i) \otimes J(j) \to J(i + j)$ implies that the Atiyah–Hirzebruch spectral sequence $\{E_r^*(T(\star))\} \Rightarrow K(m)^*(T(\star))$ will be a spectral sequence of differential graded algebras.

By Proposition 4.4 we have the calculation

$$E_{2^{m+1}}^*(T(\star)) = \mathbb{Z}/2[v_m^{\pm 1}, x_0, \ldots, x_m, x_{m+1}^2, x_{m+2}^2, \ldots]/(x_k^{2^{m+1}} \mid k \geq 0).$$

Proposition 4.5. *The spectral sequence collapses after this:* $E_{2^{m+1}}^*(T(\star)) = E_\infty^*(T(\star))$.

Proof. We show that all the algebra generators of $E_{2^{m+1}}^{*,\star}(T(\star))$ are permanent cycles.

The generator $x_k \in E_{2^{m+1}}^1(T(2^k))$ will be a permanent cycle for $k \leq m$ for degree reasons.

Let $y \in H^*(\mathbb{C}P^\infty; \mathbb{Z}/2)$ be the nonzero algebra generator. Then $x_k^2 \in E_{2^{m+1}}^2(T(2^{k+1}))$ will be a permanent cycle as it will be in the image of $y \in E_{2^{m+1}}^2(\mathbb{C}P^\infty)$ under any map $T(2^{k+1}) \to \Sigma^\infty \mathbb{C}P^\infty$ inducing a map that is nonzero on mod 2 cohomology in degree 2^{m+1}. $\square$

Remark 4.6. We confess that what we have done here appears (in dual form) in the literature. Yamaguchi [Y88] computes $K(m)_*(\Omega^2 S^{2r+1})$ which, up to a shift of degrees of generators, is equivalent to calculating $K(m)^*(J(\star))$. The link here is that $\Sigma^\infty \Omega^2 S^{2r+1}$ decomposes as an infinite wedge of finite spectra dual to the $T(n)$ after appropriate suspension. (It is a significant theorem that these wedge summands really are dual to Brown and Gitler's $T(n)$'s [BG78, HK00].) Yamaguchi's argument is similar to ours; indeed our Proposition 4.4 and Proposition 4.5 are $p = 2$ versions of his Lemma (2.1) and Lemma (2.2), though our argument showing that x_k^2 is an A.H.S.S. permanent cycle is quite different than his: he uses that his spectral sequence is a spectral sequence of Hopf algebra, and we use the Brown-Gitler property of the $T(2^k)$'s. Ravenel revisits Yamaguchi's argument in [Rav93], and both of them conclude that (formulated our way) $K(m)^*(T(\star))$ is isomorphic as a $K(m)^*$–algebra to $K(m)^*[\tilde{x}_0, \ldots, \tilde{x}_m, c_{m+2}, c_{m+3}, \ldots]/(\tilde{x}_k^{2^{m+1}}, c_l^{2^m})$, with $\tilde{x}_k \in K(m)^1(T(2^k))$ and $c_l \in K(m)^2(T(2^l))$ respectively represented by x_k and x_{l-1}^2 in the A.H.S.S.

4.4. **Specialization to $m \leq 2$.** We remind the reader that $T(0) = S^0$ and $T(1) = S^1$, and also that $x(n) \in J(n)^{\alpha_2(n)}$ denotes the class $x_{i_1} \cdots x_{i_d}$ where $n = 2^{i_1} + \cdots + 2^{i_d}$ with $i_1 < \cdots < i_d$ (so $d = \alpha_2(n)$).

It is convenient to let $k_{m,n}$ denote the dimension of $H(J(n); Q_m)$, which also equals the rank of $K(m)^*(T(n))$ as a $K(m)^*$–module. So, for example, $k_{m,0} = k_{m,1} = 1$ for all m. Then let $k_m(t) = \displaystyle\sum_{n=0}^{\infty} k_{m,n} t^n$.

We specialize Proposition 4.4 to the case $m = 0$.

Corollary 4.7. *$J(n)$ is Sq^1–acyclic for all $n \geq 2$, and thus $T(n)$ is rationally acyclic for all $n \geq 2$.*

Proof. $H^*(J(\star); Q_0) = \mathbb{Z}/2[x_0]/(x_0^2)$ and $k_0(t) = 1 + t$. $\square$

We specialize Proposition 4.4 to the case $m = 1$.

Corollary 4.8. *$k_{1,n} = 2$ for all $n \geq 2$. For $n \geq 1$ and e equal to 0 or 1, cycle representatives for $H(J(2n+e); Q_1)$ are given by $x_0^e x(n)^2$ and $x_0^e x_1 x(n-1)^2$.*

Proof. From the computation

$$H(J(\star);Q_1) = \bigotimes_{k=0}^{1} \mathbb{Z}/2[x_k]/(x_k^4) \otimes \bigotimes_{k=2}^{\infty} \mathbb{Z}/2[x_k^2]/(x_k^4),$$

we compute that

$$\begin{aligned} k_1(t) &= \frac{(1-t^4)}{(1-t)}\frac{(1-t^8)}{(1-t^2)} \cdot \prod_{k=2}^{\infty} \frac{(1-t^{2^{k+2}})}{(1-t^{2^{k+1}})} \\ &= \frac{(1-t^4)}{(1-t)}\frac{(1-t^8)}{(1-t^2)} \cdot \frac{1}{(1-t^8)} \\ &= \frac{(1+t^2)}{(1-t)} = (1+t^2) \cdot \sum_{n=0}^{\infty} t^n = 1 + t + \sum_{n=2}^{\infty} 2t^n. \end{aligned}$$

Thus $k_1(n) = 2$ for all $n \geq 2$. The given elements *are* Q_1–cycles in $J(n)$ of different degrees. □

We specialize Proposition 4.4 to the case $m = 2$.

Corollary 4.9. *$k_{2,2n+e} = 2n$ for all $n \geq 1$ and e equal to 0 or 1.*

Proof. We first show that $k_2(t) = \dfrac{(1+t^4)}{(1-t)(1-t^2)}$ in a manner similar to our last proof. We have shown that

$$H(J(\star);Q_2) = \bigotimes_{k=0}^{2} \mathbb{Z}/2[x_k]/(x_k^8) \otimes \bigotimes_{k=3}^{\infty} \mathbb{Z}/2[x_k^2]/(x_k^8),$$

and so

$$\begin{aligned} k_2(t) &= \prod_{k=0}^{2} \frac{(1-t^{2^{k+3}})}{(1-t^{2^k})} \cdot \prod_{k=3}^{\infty} \frac{(1-t^{2^{k+3}})}{(1-t^{2^{k+1}})} \\ &= \frac{(1-t^8)}{(1-t)}\frac{(1-t^{16})}{(1-t^2)}\frac{(1-t^{32})}{(1-t^4)} \cdot \frac{1}{(1-t^{16})}\frac{1}{(1-t^{32})} \\ &= \frac{(1+t^4)}{(1-t)(1-t^2)}. \end{aligned}$$

Now we show that $\dfrac{(1+t^4)}{(1-t)(1-t^2)} = (1+t)\left(1 + \displaystyle\sum_{n=1}^{\infty} 2nt^{2n}\right)$, as the right hand side has coefficients as in the statement of the corollary.

Recall that $\dfrac{1}{(1-u)} = \displaystyle\sum_{i=0}^{\infty} u^i$, and then that $\dfrac{1}{(1-u)^2} = \displaystyle\sum_{i=0}^{\infty}(i+1)u^i$. Multiplying by $2u$ and letting $n = i+1$, we see that $\dfrac{2u}{(1-u)^2} = \displaystyle\sum_{n=1}^{\infty} 2nu^n$.

Letting $u = t^2$, we learn that $\dfrac{2t^2}{(1-t^2)^2} = \sum_{n=1}^{\infty} 2nt^{2n}$. Thus

$$\begin{aligned}(1+t)\left(1+\sum_{n=1}^{\infty} 2nt^{2n}\right) &= (1+t)\left(1+\frac{2t^2}{(1-t^2)^2}\right)\\ &= (1+t)\frac{1+t^4}{(1-t^2)^2} = \frac{(1+t^4)}{(1-t)(1-t^2)}.\end{aligned}$$

□

Corollary 4.9 has an obvious consequence.

Corollary 4.10. *If m and n are even and $m > n$, then $f(n,m) : T(m) \to T(n)$ is not a $K(2)^*$–isomorphism and thus $X(n,m)$ is not $K(2)^*$-acyclic.*

5. Proof of Theorem 2.1: part 1

In this and the next two sections, we determine for which pairs (n,m), the map

$$q(n,m)^* : J(n) \otimes (t) \to J(m)$$

induces an isomorphism

$$q(n,m)^* : H(J(n) \otimes (t); Q_1) \to H(J(m); Q_1).$$

Equivalently, we determine when the A–module $Q(n,m)$ is Q_1–acyclic.

First of all, $q(n,m)^*$ is not a Q_1-isomorphism if $0 \leq n \leq 1$, since then $H(J(n) \otimes (t); Q_1)$ is one-dimensional and concentrated in degree 2 or 3, and this is not isomorphic to $H(J(m); Q_1)$ for any m. Thus we assume hereafter that $n \geq 2$.

From [K23], we learn that the short exact sequence

$$0 \to J(m) \to Q(n,m) \to \Sigma^{-1} J(n) \to 0$$

splits when $m < 2n-1$, so that $Q(n,m) \simeq J(m) \oplus \Sigma^{-1} J(n)$ and is not Q_1–acyclic. Furthermore, when $m = 2n-1$, the sequence identifies with one desuspension of a Mahowald sequence, and we learn that $Q(n, 2n-1) \simeq \Sigma^{-1} J(2n)$ and so also not Q_1–acyclic. Summarizing:

Proposition 5.1. *If $q(n,m)^*$ is a Q_1–isomorphism, then $m \geq 2n$.*

Now comes our first use of a key trick.

Since $H(F(1); Q_1) = \langle t^2, t^8, t^{16}, \dots \rangle$ while $H((t); Q_1) = \langle t^2 \rangle$, the inclusion $F(1) \hookrightarrow (t)$ clearly induces a surjection on Q_1–homology. The next lemma follows.

Lemma 5.2. *Let $\bar{q}(n,m) : J(n) \otimes F(1) \to J(m)$ be the composite*

$$J(n) \otimes F(1) \hookrightarrow J(n) \otimes (t) \xrightarrow{q(n,m)^*} J(m).$$

Then $q(n,m)^*$ will be a Q_1–isomorphism if and only if $\bar{q}(n,m)$ induces a surjection on Q_1–homology.

Under the condition that $m \geq 2n$, the map $\bar{q}(n,m)$ turns out to be much easier to describe than $q(n,m)^*$.

Lemma 5.3. *Let $m \geq 2n$. If we write m as $m = l + 2^k$ with $l < 2^k$, then $\bar{q}(n,m)$ agrees with the composite*

$$J(n) \otimes F(1) \xrightarrow{p(n,l)^* \otimes p_k} J(l) \otimes J(2^k) \xrightarrow{\mu} J(m). \tag{5.1}$$

Proof. $\bar{q}(n,m)$ is the unique A–module map that is nonzero on every basis element of $(J(n) \otimes F(1))^m$, so we need to show that our composite has this same property. Elements of the form $x \otimes t^{2^k}$, with x a standard basis element of $J(n)^l$, certainly map nonzero:

$$\bar{q}(n,m)(x \otimes t^{2^k}) = \mu(p(n,l)^*(x) \otimes p_k(t^{2^k})) = \mu(x_0^l \otimes x_0^{2^k}) = x_0^m.$$

We now observe that our condition that $m \geq 2n$ implies that every basis element of $[J(n) \otimes F(1)]^m$ has this form: if $x \otimes t^{2^i} \in [J(n) \otimes F(1)]^m$ then $i = k$. If $i > k$, then $|x| = l + (2^k - 2^i) \leq l - 2^k < 0$, which can't happen. If $i < k$, then $2n \geq 2|x| = 2(l + 2^k - 2^i) \geq 2(l + 2^{k-1}) = 2l + 2^k > l + 2^k = m$, which also can't happen. □

Proposition 5.4. *Let $m \geq 2n$. If we write m as $m = l + 2^k$ with $l < 2^k$, then $q(n,m)^*$ will be a Q_1–isomorphism if and only if $p(n,l)^*$ is a Q_1–isomorphism.*

Proof. We need to check that the composite (5.1) is a Q_1–surjection if and only if $p(n,l)^*$ is a Q_1–isomorphism.

The first observation is that the image of $p(n,l)^* \otimes p_k$ in Q_1-homology will be precisely $(\operatorname{im} p(n,l)^*) \otimes \langle x_{k-1}^2 \rangle$. If $0 \leq l \leq 1$, then neither $p(n,l)^*$ nor (5.1) is a Q_1-surjection since $\dim(H(J(l); Q_1)) = 1$. If $l \geq 2$, then $\mu : J(l) \otimes J(2^k) \to J(m)$ induces an isomorphism $\mu_* : H(J(l); Q_1) \otimes \langle x_{k-1}^2 \rangle \simeq H(J(m); Q_1)$, and the proposition follows. □

6. Proof of Theorem 2.1: part 2

Now we focus on the case when our pair are both even, so we are determining for which pairs (n,l) the map

$$p(2n, 2l)^* : H(J(2n); Q_1) \to H(J(2l); Q_1)$$

is an isomorphism of graded vector spaces of total dimension 2.

For starters, one clearly needs that these two graded vector spaces are isomorphic. Call (n,l) a *good pair* if this is the case.

Proposition 6.1. *(n,l) is a good pair if and only if $\alpha_2(n) = \alpha_2(l)$ and $\nu_2(n) = \nu_2(l)$.*

Proof. $H(J(2n); Q_1)$ is represented by the classes $x(n)^2$ and $x_1 x(n-1)^2$ in degrees $2\alpha_2(n)$ and $2\alpha_2(n-1)+1$, and similarly $H(J(2l) \otimes (t); Q_1)$ is represented by the classes $x(l)^2$ and $x_1 x(l-1)^2$ in degrees $2\alpha_2(l)$ and $2\alpha_2(l-1)+1$.

Matching degrees and simplifying, we find that (n, l) will be good if and only if

$$\alpha_2(n) = \alpha_2(l) \quad \text{and} \quad \alpha_2(n-1) = \alpha_2(l-1),$$

or, equivalently, if

$$\alpha_2(n) = \alpha_2(l) \quad \text{and} \quad \alpha_2(n-1) - \alpha_2(n) = \alpha_2(l-1) - \alpha_2(l).$$

The next observation finishes the proof. □

Lemma 6.2. *For all $n \geq 1$, $\alpha_2(n-1) - \alpha_2(n) = \nu_2(n) - 1$.*

Now suppose that $\alpha_2(n) = \alpha_2(l)$ and $\nu_2(n) = \nu_2(l)$. If we let $d = \alpha_2(n) - 1$ and $i = \nu_2(n)$, then these two conditions say that n and l have the form

$$l = 2^i + 2^{j_1} + \cdots + 2^{j_d} \text{ with } i < j_1 < \cdots < j_d$$

and

$$n = 2^i + 2^{k_1} + \cdots + 2^{k_d} \text{ with } i < k_1 < \cdots < k_d.$$

Theorem 6.3. *In this situation, $p(2n, 2l)^* : H(J(2n); Q_1) \to H(J(2l); Q_1)$ is an isomorphism if*

$$(\clubsuit) \qquad j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_d \leq k_d$$

and is zero otherwise.

Note that this theorem combines with Proposition 5.4 to prove Theorem 2.1(a).

The theorem follows immediately from the next two propositions. To state these, we need some definitions and notation.

Let $J = \{j_1, \ldots, j_d\}$ and $K = \{k_1, \ldots k_d\}$. Let S_d denote the dth symmetric group, and then let $S(J, K) \subset S_d$ be the set

$$S(J, K) = \{\sigma \in S_d \mid k_c \geq j_{\sigma(c)} \text{ for all } c\},$$

with cardinality $|S(J, K)|$.

Proposition 6.4. *$p(2n, 2l)^* : H(J(2n); Q_1) \to H(J(2l); Q_1)$ is 'multiplication by $|S(J, K)|$'. More precisely,*

$$p(2n, 2l)^*(x(n)^2) = |S(J, K)| x(l)^2 \text{ and}$$
$$p(2n, 2l)^*(x_1 x(n-1)^2) = |S(J, K)| x_1 x(l-1)^2.$$

Proposition 6.5. *$S(J, K)$ has the single element 'identity' if ($\clubsuit$) holds, and has an even number of elements otherwise.*

Proof of Proposition 6.4. For notational simplicity, in this proof we let $p = p(2n, 2l)^* : J(2n) \to J(2l)$. Our proof makes heavy use of Lemma 3.5 which said that $p_k(t^{2^j}) = x_{k-j}^{2^j}$ for $j \leq k$, where $p_k : F(1) \to J(2^k)$ is the nonzero A–module map.

Let $g : J(2^{i+1}) \otimes F(1)^{\otimes d} \to J(2n)$ be the composite

$$J(2^{i+1}) \otimes F(1)^{\otimes d} \to J(2^{i+1}) \otimes J(2^{k_1+1}) \otimes \cdots \otimes J(2^{k_d+1}) \xrightarrow{\mu} J(2n),$$

where the first map is $\mathrm{id} \otimes p_{k_1+1} \otimes \cdots \otimes p_{k_d+1}$ and the second is multiplication.

The map g induces an epimorphism in Q_1–homology; more precisely,

$$g(x_i^2 t_1^2 t_2^2 \cdots t_d^2) = x_i^2 x_{k_1}^2 \cdots x_{k_d}^2 = x(n)^2 \quad \text{and}$$

$$g(x_1 x_0^2 \cdots x_{i-1}^2 t_1^2 t_2^2 \cdots t_d^2) = x_1 x_0^2 \cdots x_{i-1}^2 x_{k_1}^2 \cdots x_{k_d}^2 = x_1 x(n-1)^2.$$

Here t_c is the one-dimensional class in the cth copy of $F(1)$ in the d–fold tensor product.

Thus it suffices to show that

$$p(g(x_i^2 t_1^2 t_2^2 \cdots t_d^2)) = |S(J,K)| x(l)^2 \tag{6.1}$$

and

$$p(g(x_1 x_0^2 \cdots x_{i-1}^2 t_1^2 t_2^2 \cdots t_d^2)) = |S(J,K)| x_1 x(l-1)^2. \tag{6.2}$$

To do this, we identify $p \circ g : J(2^{i+1}) \otimes F(1)^{\otimes d} \to J(2l)$ in a way that makes this easy to see.

Recall that $p \circ g$ is determined by its values on $(J(2^{i+1}) \otimes F(1)^{\otimes d})^{2l}$. Note that this has basis $\mathcal{B} = \{b_\sigma \mid \sigma \in S_d\}$ where $b_\sigma = x_0^{2^{i+1}} t_1^{2^{1+j_{\sigma(1)}}} \cdots t_d^{2^{1+j_{\sigma(d)}}}$.

For all c,

$$p_{k_c+1}(t^{2^{1+j_{\sigma(c)}}}) = \begin{cases} x_{k_c - j_{\sigma(c)}}^{2^{1+j_{\sigma(c)}}} & \text{if } k_c \geq j_{\sigma(c)} \\ 0 & \text{otherwise,} \end{cases}$$

and it follows that $g(b_\sigma)$ is a standard basis element in $J(2n)^{2l}$ for $\sigma \in S(J,K)$ and 0 otherwise. Thus

$$p(g(b_\sigma)) = \begin{cases} x_0^{2l} & \text{if } \sigma \in S(J,K) \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

Now we identify the basis of $\mathrm{Hom}_A(J(2^{i+1}) \otimes F(1)^{\otimes d}, J(2l))$ dual to $\mathcal{B}$.

Similar to our definition of g, we let $h : J(2^{i+1}) \otimes F(1)^{\otimes d} \to J(2l)$ be the composite

$$J(2^{i+1}) \otimes F(1)^{\otimes d} \to J(2^{i+1}) \otimes J(2^{j_1+1}) \otimes \cdots \otimes J(2^{j_d+1}) \xrightarrow{\mu} J(2l).$$

Similar to, but simpler than, our computation of $g(b_\sigma)$, one computes that

$$h(b_\sigma) = \begin{cases} x_0^{2l} & \text{if } \sigma = \mathrm{id} \\ 0 & \text{otherwise.} \end{cases}$$

It follows that if we let h_τ be the composite

$$J(2^{i+1}) \otimes F(1)^{\otimes d} \xrightarrow{\mathrm{id} \otimes \tau^{-1}} J(2^{i+1}) \otimes F(1)^{\otimes d} \xrightarrow{h} J(2l),$$

for $\tau \in S_d$, then

$$h_\tau(b_\sigma) = \begin{cases} x_0^{2l} & \text{if } \sigma = \tau \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

Comparing (6.3) with (6.4), we conclude that $p \circ g = \sum_{\sigma \in S(J,K)} h_\sigma$.

Now one checks that for all $\sigma \in S_d$, one has

$$h_\sigma(x_i^2 t_1^2 t_2^2 \cdots t_d^2) = x(l)^2 \tag{6.5}$$

and

$$h_\sigma(x_1 x_0^2 \cdots x_{i-1}^2 t_1^2 t_2^2 \cdots t_d^2) = x_1 x(l-1)^2. \tag{6.6}$$

Comparing these to (6.1) and (6.2), the proof of Proposition 6.4 is complete. □

Proof of Proposition 6.5. Recall that $j_1 < \cdots < j_d$, $k_1 < \cdots < k_d$, and we wish to show that the set

$$S(J,K) = \{\sigma \in S_d \mid j_{\sigma(c)} \leq k_c\}$$

will consist of only the identity element if

$$j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_d \leq k_d, \tag{♣}$$

and will have an even number of elements otherwise.

For $1 \leq c \leq d$, define a function s by $s(c) = |\{b \mid j_b \leq k_c\}|$. Then $0 \leq s(1) \leq \cdots \leq s(d) \leq d$, and if $s(c) > 0$ then $s(c)$ is the maximal b such that $j_b \leq k_c$.

From this it follows that $S(J,K) = S(s)$ where

$$S(s) = \{\sigma \in S_d \mid \sigma(c) \leq s(c) \text{ for all } c\}.$$

If $s(c) < c$ for any c, e.g. if $s(c) = 0$, $S(s)$ will be empty. The condition $s(c) \geq c$ for all c corresponds to the condition $j_c \leq k_c$ for all c.

Here is an interpretation of $S(s)$: it is set of the ways of fitting blocks of size $1, 2, \ldots, d$ under the graph of s.

If any two of the $s(c)$'s equal each other, then one clearly gets an even number of ways of fitting the blocks in. (More generally our set is acted on freely by an easily defined symmetry group which will have even order if any two values of the $s(c)$'s are equal.)

If all the $s(c)$'s are positive and distinct then $s(c) = c$ for all c, and this holds if and only if (♣) holds. In this case, one easily sees that the identity element is the only element in $S(s)$. □

7. Proof of Theorem 2.1: part 3

Theorem 2.1(c) asserted that $Q(n,m)$ can only be a Q_1-acyclic if m and n are of the same parity. Thanks to Proposition 5.4, to prove this it suffices to prove the next two propositions.

Proposition 7.1. *$p(2n, 2l+1)^* : J(2n) \to J(2l+1)$ does not induce an isomorphism in Q_1–homology, since $p(2n, 2l+1)^*(x(n)^2) = 0$.*

Proposition 7.2. *$p(2n+1, 2l)^* : J(2n+1) \to J(2l)$ does not induce an isomorphism in Q_1–homology, since $p(2n+1, 2l)^*(x_0 x(n)^2) = 0$.*

Proof of Proposition 7.1. For notational simplicity, in this proof we let $p = p(2n, 2l+1)^* : J(2n) \to J(2l+1)$. As we are assuming $2n \geq 2$, if $l = 0$ then p is certainly not a Q_1-isomorphism because the dimensions of the source and target differ. Thus we may assume $l \geq 1$ for the remainder of the proof.

As in Section 6, we begin by identifying the *good pairs*, i.e. the pairs (n, l) for which the graded vector spaces $H(J(2n); Q_1)$ and $H(J(2l+1); Q_1)$ are isomorphic. The former is represented by the classes $x(n)^2$ and $x_1 x(n-1)^2$ in degrees $2\alpha_2(n)$ and $2\alpha_2(n-1)+1$, and the latter by the classes $x_0 x(l)^2$ and $x_0 x_1 x(l-1)^2$ in degrees $2\alpha_2(l)+1$ and $2\alpha_2(l-1)+2$.

Matching degrees and simplifying, we find that (n, l) is a good pair if and only if

$$\alpha_2(n) = \alpha_2(l-1) + 1 \quad \text{and} \quad \alpha_2(n-1) = \alpha_2(l),$$

or, making use of Lemma 6.2, if and only if

$$\alpha_2(n) = \alpha_2(l-1) + 1 \quad \text{and} \quad \nu_2(n) + \nu_2(l) = 1.$$

Thus there are two cases to consider.

Case #1: $\nu_2(n) = 0$ and $\nu_2(l) = 1$.

These conditions, together with the condition $\alpha_2(n) = \alpha_2(l-1)+1$, imply that n and l have the form

$$l = 2 + 2^{j_1} + \cdots + 2^{j_d} \text{ with } 1 < j_1 < \cdots < j_d$$

and

$$n = 1 + 2^{k_1} + \cdots + 2^{k_{d+1}} \text{ with } 0 < k_1 < \cdots < k_{d+1}$$

where $d := \alpha_2(l) - 1 = \alpha_2(n) - 2$.

Let $g : J(2) \otimes F(1)^{\otimes d+1} \to J(2n)$ be the composite

$$J(2) \otimes J(2^{k_1+1}) \otimes \cdots J(2^{k_{d+1}+1}) \xrightarrow{\mu} J(2n),$$

where the first map is $\mathrm{id} \otimes p_{k_1+1} \otimes \cdots \otimes p_{k_{d+1}+1}$ and the second is multiplication. Since

$$g(x_0^2 \otimes t_1^2 \otimes t_2^2 \otimes \cdots \otimes t_{d+1}^2) = x_0^2 x_{k_1}^2 \cdots x_{k_{d+1}}^2 = x(n)^2,$$

(where t_c denotes the one-dimensional class in the cth copy of $F(1)$ in the $(d+1)$-fold tensor product), it suffices to show that

$$p(g(x_0^2 \otimes t_1^2 \otimes \cdots \otimes t_{d+1}^2)) = 0.$$

To do this, we will produce a basis for $\mathrm{Hom}_A(J(2) \otimes F(1)^{\otimes d+1}, J(2l+1))$ and show that each element of this basis maps $x_0^2 \otimes t_1^2 \otimes \cdots \otimes t_{d+1}^2$ to zero. A map of this form is determined by its values on $(J(2) \otimes F(1)^{\otimes d+1})^{2l+1}$, which has basis $\mathcal{B} = \{b_\sigma \mid \sigma \in S_{\{0,\ldots,d\}}\}$, where

$$b_\sigma = x_1 \otimes t^{2^{j_{\sigma(0)}+1}} \otimes t^{2^{j_{\sigma(1)}+1}} \otimes \cdots \otimes t^{2^{j_{\sigma(d)}+1}}$$

and $j_0 := 1$. Now we identify the basis of $\mathrm{Hom}_A(J(2) \otimes F(1)^{\otimes d+1}, J(2l+1))$ dual to $\mathcal{B}$. Let $h : J(2) \otimes F(1)^{\otimes d+1} \to J(2l+1)$ be the composite

$$J(2) \otimes F(1)^{\otimes d+1} \to J(1) \otimes J(4) \otimes J(2^{j_1+1}) \otimes \cdots \otimes J(2^{j_d+1}) \xrightarrow{\mu} J(2l+1),$$

where the first map is $p(2,1)^* \otimes p_2 \otimes p_{j_1+2} \otimes \cdots \otimes p_{j_d+2}$, and then let h_τ be the composite

$$J(2) \otimes F(1)^{\otimes d+1} \xrightarrow{\mathrm{id}\otimes\tau^{-1}} J(2) \otimes F(1)^{\otimes d+1} \xrightarrow{h} J(2l+1)$$

for $\tau \in S_{d+1}$. One checks that

$$h_\tau(b_\sigma) = \begin{cases} x_0^{2l+1} & \text{if } \sigma = \tau, \\ 0 & \text{otherwise,} \end{cases}$$

verifying that $\{h_\tau \,|\, \tau \in S_{d+1}\}$ is the basis dual to $\mathcal{B}$. But then

$$h_\tau(x_0^2 \otimes t_1^2 \otimes \cdots \otimes t_{d+1}^2) = h(x_0^2 \otimes t_1^2 \otimes \cdots \otimes t_{d+1}^2) = 0$$

for all $\tau \in S_{d+1}$, for the simple reason that $p(2,1)^*(x_0^2) = 0$. From this it follows that any map $J(2) \otimes F(1)^{\otimes d+1} \to J(2l+1)$, in particular the map $p \circ g$, annihilates $x_0^2 \otimes t_1^2 \otimes \cdots \otimes t_{d+1}^2$.

Case #2: $\nu_2(n) = 1$ and $\nu_2(l) = 0$.

In this case, n and l must have the form

$$l = 1 + 2^{j_1} + \cdots + 2^{j_d} \text{ with } 0 < j_1 < \cdots < j_d$$

and

$$n = 2 + 2^{k_1} + \cdots + 2^{k_d} \text{ with } 1 < k_1 < \cdots < k_d$$

where $d := \alpha_2(l) - 1 = \alpha_2(n) - 1$. Let g be the composite

$$J(4) \otimes F(1)^{\otimes d} \to J(4) \otimes J(2^{k_1+1}) \otimes \cdots \otimes J(2^{k_d+1}) \xrightarrow{\mu} J(2n),$$

where the first map is $\mathrm{id} \otimes p_{k_1+1} \otimes \cdots \otimes p_{k_d+1}$. Since

$$g(x_1^2 \otimes t_1^2 \otimes \cdots \otimes t_d^2) = x_1^2 x_{k_1}^2 \cdots x_{k_d}^2 = x(n)^2,$$

it suffices to show that $p(g(x_1^2 \otimes t_1^2 \otimes \cdots \otimes t_d^2)) = 0$.

A basis for $(J(4) \otimes F(1)^{\otimes d})^{2l+1}$ is given by $\{b_\sigma\}_{\sigma \in S_d}$, where

$$b_\sigma = x_0^2 x_1 \otimes t^{2^{k_{\sigma(1)}+1}} \otimes \cdots \otimes t^{2^{k_{\sigma(d)}+1}},$$

and the corresponding basis of $\mathrm{Hom}_A(J(4) \otimes F(1)^{\otimes d}, J(2l+1))$ is $\{h_\tau\}_{\tau \in S_d}$, where h_τ is the map

$$J(4) \otimes F(1)^{\otimes d} \xrightarrow{1\otimes\tau^{-1}} J(4) \otimes F(1)^{\otimes d} \xrightarrow{h} J(2l+1)$$

and $h = \mu \circ (p(4,3)^* \otimes p_{j_1+1} \otimes \cdots \otimes p_{j_d+1})$. But then

$$h_\tau(x_1^2 \otimes t_1^2 \otimes \cdots \otimes t_d^2) = h(x_1^2 \otimes t_1^2 \otimes \cdots \otimes t_d^2) = 0$$

because

$$p(4,3)^*(x_1^2) = p(4,3)^*(Sq^1(x_2)) = Sq^1(p(4,3)^*(x_2)) = Sq^1(0) = 0.$$

It follows that any map $J(4) \otimes F(1)^d \to J(2l+1)$, in particular $p \circ g$, annihilates $x_1^2 \otimes t_1^2 \otimes \cdots \otimes t_d^2$. $\square$

Proof of Proposition 7.2. As with Proposition 7.1, we can assume $l \geq 1$. By reversing the roles of n and l in the proof of Proposition 7.1, we find that (n, l) is a *good pair*, i.e. $H(J(2n+1); Q_1)$ and $H(J(2l); Q_1)$ are isomorphic, if and only if

$$\alpha_2(n) = \alpha_2(l-1) \quad \text{and} \quad \nu_2(n) + \nu_2(l) = 1.$$

So we consider two cases.

Case #1: $\nu_2(n) = 0$ and $\nu_2(l) = 1$.

In this case, n and l have the form

$$l = 2 + 2^{j_1} + \cdots + 2^{j_d} \text{ with } 1 < j_1 < \cdots < j_d$$

and

$$n = 1 + 2^{k_1} + \cdots + 2^{k_d} \text{ with } 0 < k_1 < \cdots < k_d,$$

where $d = \alpha(n) - 1 = \alpha(l) - 1$. Let g be the composite

$$J(3) \otimes F(1)^{\otimes d} \to J(3) \otimes J(2^{k_1+1}) \otimes \cdots \otimes J(2^{k_d+1}) \xrightarrow{\mu} J(2n+1),$$

where the first map is $\mathrm{id} \otimes p_{k_1+1} \otimes \cdots \otimes p_{k_d+1}$. Since

$$g(x_0^3 \otimes t_1^2 \otimes \cdots \otimes t_d^2) = x_0^3 x_{k_1}^2 \cdots x_{k_d}^2 = x_0 x(n)^2,$$

it suffices to show that $p(2n+1, 2l)^*(g(x_0^3 \otimes t_1^2 \otimes \cdots \otimes t_d^2)) = 0$.

Now $p(2n+1, 2l)^* \circ g$ is a map $J(3) \otimes F(1)^{\otimes d} \to J(2l)$, but the only such map is the zero map since $(J(3) \otimes F(1)^{\otimes d})^{2l} = 0$; to see this, observe that $\alpha(2l - i) = d + 1$ for $2 \leq i \leq 3$ (the range of degrees in which $J(3)$ is nontrivial).

Case #2: $\nu_2(n) = 1$ and $\nu_2(l) = 0$.

In this case, n and l have the form

$$l = 1 + 2^{j_1} + \cdots + 2^{j_{d+1}} \text{ with } 0 < j_1 < \cdots < j_{d+1}$$

and

$$n = 2 + 2^{k_1} + \cdots + 2^{k_d} \text{ with } 1 < k_1 < \cdots < k_d,$$

where $d = \alpha(n) - 1 = \alpha(l) - 2$. Let g be the composite

$$J(5) \otimes F(1)^{\otimes d} \to J(5) \otimes J(2^{k_1+1}) \otimes \cdots \otimes J(2^{k_d+1}) \xrightarrow{\mu} J(2n+1),$$

where the first map is $\mathrm{id} \otimes p_{k_1+1} \otimes \cdots \otimes p_{k_d+1}$. Since

$$g(x_0 x_1^2 \otimes t_1^2 \otimes \cdots \otimes t_d^2) = x_0 x_1^2 x_{k_1}^2 \cdots x_{k_d}^2 = x_0 x(n)^2,$$

it suffices to show that $p(2n+1, 2l)^*(g(x_0 x_1^2 \otimes t_1^2 \otimes \cdots \otimes t_d^2)) = 0$.

Now $p(2n+1, 2l)^* \circ g$ is a map $J(5) \otimes F(1)^{\otimes d} \to J(2l)$, but the only such map is the zero map since $(J(5) \otimes F(1)^{\otimes d})^{2l} = 0$; to see this, observe that $\alpha(2l - i) \geq d + 1$ for $2 \leq i \leq 5$ (the range of degrees in which $J(5)$ is nontrivial). $\square$

8. Applications

8.1. **Proof of Theorem 2.6.** We prove the various statements in Theorem 2.6.

Proof of Theorem 2.6(a). Let n be even, $\nu_2(n) = i$ and suppose $n = 2^i + 2^{k_1} + \cdots + 2^{k_d}$ with $i < k_1 < \cdots < k_d$, so $d = \alpha_2(n) - 1$. It is convenient to let $n_c = 2^i + \sum_{j=1}^{c} 2^{k_j}$ for $c = 0, 1, \ldots d$, so $2^i = n_0 < \cdots < n_d = n$.

If we let $f : T(n) \to T(2^i)$ be the composite of the maps

$$T(n) \xrightarrow{f(n_{d-1},n_d)} T(n_{d-1}) \xrightarrow{f(n_{d-2},n_{d-1})} \cdots \xrightarrow{f(n_1,n_2)} T(n_1) \xrightarrow{f(n_0,n_1)} T(n_0),$$

then f is the composite of L_1–equivalences, so is an L_1–equivalence. As the composition of d maps, each inducing 0 in mod 2 cohomology, f has Adams filtration at least d. □

Proof of Theorem 2.6(b). As discussed at the end of §3, there is a Mahowald cofibration sequence

$$T(2^{i-1}) \xrightarrow{\alpha} T(2^i) \xrightarrow{\beta'} \Sigma T(2i-1).$$

Now $\Sigma T(2i-1) \simeq \Sigma^2 T(2^i - 2)$, and by part (a), there is an L_1–equivalence $f : T(2^i - 2) \to T(2)$. Thus if we let β be the composite $\Sigma^2 f \circ \beta'$, the sequence

$$T(2^{i-1}) \xrightarrow{\alpha} T(2^i) \xrightarrow{\beta} \Sigma^2 T(2) \tag{8.1}$$

will be a cofibration sequence after L_1–localization. □

Proof of Theorem 2.6(c). For $i \geq 1$, we prove that $K^0(T(2^i))_{(2)} \simeq \mathbb{Z}/2^i$, and $K^1(T(2^i))_{(2)} \simeq 0$ by induction on i. Recalling that $T(2) = \Sigma^\infty \mathbb{R}P^2$, one sees that this is true for $i = 1$.

For the inductive step, assume the calculation for $i - 1$. Because the 2–local K–theory of both $T(2^{i-1})$ and $T(2)$ is concentrated in degree 0, the sequence (8.1) will induce a short exact sequence

$$0 \to K^0(T(2))_{(2)} \to K^0(T(2^i))_{(2)} \to K^0(T(2^{i-1}))_{(2)} \to 0, \tag{8.2}$$

and we also learn that $K^1(T(2^i))_{(2)} \simeq 0$.

In particular, we learn that $K^0(T(2^i))_{(2)}$ is a finite abelian group of order 2^i. This group must be cyclic, since $K(1)^*(T(2^i))$ can be calculated from this group by the universal coefficient theorem, and we know independently that $K(1)^*(T(2^i))$ is 2–dimensional.

Thus $K^0(T(2^i))_{(2)} \simeq \mathbb{Z}/2^i$ and (8.2) must be equivalent to

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2^i \to \mathbb{Z}/2^{i-1} \to 0.$$ □

8.2. **Connections to the thesis of Brian Thomas.** Mod 2 homology applied to the Goodwillie tower of the functor $X \rightsquigarrow \Sigma^\infty\Omega^\infty X$ yields a spectral sequence converging to $H_*(\Omega^\infty X)$ for all 0–connected spectra X. The Hopf algebra $H_*(\Omega^\infty X)$ is a module over the Dyer–Lashof algebra $\mathcal{R}$, and, as studied in [KMcC13], this is reflected in the spectral sequence: $E^\infty_{*,*}(X)$ is always a primitively generated Hopf algebra equipped with Dyer–Lashof operations compatible with those acting on $H_*(\Omega^\infty X)$..

Let $E(X) = \bigoplus_{s=0}^{\infty} \operatorname{Ext}_A^{s,s}(H^*(X), J(\star))$. [KMcC13, Cor. 1.14] implies that, if a certain 'geometric condition' holds and $E(X)$ is concentrated in even degrees and is generated by $\operatorname{Hom}_A(H^*(X), J(\star))$ as an $\mathcal{R}$–module, then

$$E^\infty_{*,*}(X) = S^*(E(X))/(Q_0x - x^2),$$

where we let $Q_0 x = Q^{|x|}x$.

In Thomas' thesis [T19], his goal is to verify that this applies when $X = \Sigma^2 ku$, so that $\Omega^\infty X = BU$. One knows that $H^*(ku) = A \otimes_{E(1)} \mathbb{Z}/2$, and Thomas carefully computes $E(\Sigma^2 ku) = \bigoplus_{s=0}^{\infty} \operatorname{Ext}^{*,*}_{E(1)}(\Sigma^2\mathbb{Z}/2, J(\star))$ with some of its $\mathcal{R}$–module structure.

The punchline is that $E(\Sigma^2 ku)$ has a basis of elements

$$\bar{c}_n(k) \in \operatorname{Ext}_A^{\alpha_2(n)-1+k,\alpha_2(n)-1+k}(H^*(\Sigma^2 ku), J(2^{k+1}n))$$

with $n \geq 1$ and $k \geq 0$, and one has the following formula: if $n = 2^i + 2^{j_1} + \cdots + 2^{j_d}$ with $i < j_1 < \cdots < j_d$, then

$$\bar{c}_n(k) = Q_0^k Q^{2^{j_d}} \cdots Q^{2^{j_1}} \bar{c}_{2^i}(0).$$

(The behavior of Q_0 here is a bit more than is proved in [T19].)

One concludes that $E^\infty_{*,*}(\Sigma^2 ku) = \mathbb{Z}/2[\bar{c}_1(0), \bar{c}_2(0), \ldots]$. This recovers the known calculation of $H_*(BU)$, usually computed using very different means!

A key to the calculations is to use Theorem 2.4(b) to prove algebraic analogues of statements (a) and (b) of Theorem 2.6.

Proposition 8.1. *If $n = 2^i + 2^{j_1} + \cdots + 2^{j_d}$ with $1 \leq i < j_1 < \cdots < j_d$, then*

$$Q^{2^{j_d}} \cdots Q^{2^{j_1}} : \operatorname{Ext}^{s,s}_{E(1)}(\Sigma^2\mathbb{Z}/2, J(2^i)) \to \operatorname{Ext}^{s-1+\alpha(n),s-1+\alpha(n)}_{E(1)}(\Sigma^2\mathbb{Z}/2, J(n)).$$

is an isomorphism.

Proposition 8.2. *The Mahowald sequences induce short exact sequences*

$$0 \to \operatorname{Ext}^{s,s}_{E(1)}(\mathbb{Z}/2, J(2)) \to \operatorname{Ext}^{s+i,s+i}_{E(1)}(\Sigma^2\mathbb{Z}/2, J(2^{i+1})) \to \operatorname{Ext}^{s+i,s+i}_{E(1)}(\Sigma^2\mathbb{Z}/2, J(2^i)) \to 0.$$

More details about these calculations will appear elsewhere.

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