Ambidexterity in $K(n)$-Local Stable Homotopy Theory

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Introduction

Let G be a finite group, and let M be a spectrum equipped with an action of G. We let M^{hG} denote the homotopy fixed point spectrum for the action of G on M, and M_{hG} the homotopy orbit spectrum for the action of G on X. These spectra are related by a canonical *norm map* $\mathrm{Nm} : M_{hG} \to M^{hG}$. Our starting point is the following result of Hovey and Sadofsky (see [9]):

Theorem 0.0.1. *Let $K(n)$ be a Morava K-theory, let M be a spectrum which is $K(n)$-local, and let G be a finite group acting on M. Then the norm map*

$$\mathrm{Nm} : M_{hG} \to M^{hG}$$

exhibits M^{hG} as a $K(n)$-localization of M_{hG}. In other words, the map Nm *induces an isomorphism of $K(n)$-homology groups $K(n)_* M_{hG} \to K(n)_* M^{hG}$.*

Our goal in this paper is to place Theorem 0.0.1 into a larger context. The collection of all $K(n)$-local spectra can be organized into an ∞-category, which we will denote by $\mathrm{Sp}_{K(n)}$. If M is a $K(n)$-local spectrum with an action of a finite group G, then M determines a diagram $\rho : BG \to \mathrm{Sp}_{K(n)}$. The homotopy fixed point spectrum M^{hG} can be identified with a limit of the diagram ρ, and the localized homotopy orbit spectrum $L_{K(n)} M_{hG}$ can be identified with a colimit of the diagram ρ. The main result of this paper is the following variant of Theorem 0.0.1:

Theorem 0.0.2. *Let X be a Kan complex. Assume that, for every vertex $x \in X$, the sets $\pi_n(X, x)$ are finite for every integer n, and trivial for $n \gg 0$. Let $\rho : X \to \mathrm{Sp}_{K(n)}$ be a diagram of $K(n)$-local spectra, indexed by X. Then there is a canonical homotopy equivalence*

$$\mathrm{Nm}_X : \varinjlim(\rho) \xrightarrow{\sim} \varprojlim(\rho).$$

Remark 0.0.3. In the special case where X is the classifying space of a finite group G, and $\rho : X \to \mathrm{Sp}_{K(n)}$ classifies an action of G on a $K(n)$-local spectrum M, the map $\mathrm{Nm}_X : \varinjlim(\rho) \to \varprojlim(\rho)$ we will construct in our proof of Theorem 0.0.2 agrees with the $K(n)$-localization of the usual norm map $M_{hG} \to M^{hG}$. Consequently, Theorem 0.0.2 can be regarded as a generalization of Theorem 0.0.1.

Example 0.0.4. The simplest instance of Theorem 0.0.2 occurs when the Kan complex X is discrete. In this case, Theorem 0.0.2 asserts that for any finite collection of objects $M_1, \ldots, M_k \in \mathrm{Sp}_{K(n)}$, the product $\prod_{1 \leq i \leq k} M_i$ and the coproduct $\coprod_{1 \leq i \leq k} M_i$ are canonically equivalent.

Let us briefly outline our approach to Theorem 0.0.2. Our assumptions on X guarantee that there exists an integer $n \geq 0$ such that the homotopy groups $\pi_m(X, x)$ vanish for $m > n$. We proceed by induction on n. The case $n = 0$ reduces to Example 0.0.4. The inductive step can be broken into two parts:

(a) The construction of the norm map $\mathrm{Nm}_X : \varinjlim(\rho) \to \varprojlim(\rho)$.

(b) The proof that $\mathrm{Nm}_X(\rho)$ is an equivalence.

To carry out (a), we note that a map from $\varinjlim(\rho)$ to $\varprojlim(\rho)$ can be identified with a collection of maps $\phi_{x,y} : \rho(x) \to \rho(y)$, depending functorially on the pair $(x, y) \in X \times X$. Note that every point e of the path space $P_{x,y} = \{x\} \times_X X^{\Delta^1} \times_X \{y\}$ determines an equivalence $\rho(e) : \rho(x) \to \rho(y)$ in the ∞-category $\mathcal{C}$. We will choose Nm_X to correspond to the family of maps $\phi_{x,y}$ given heuristically by the formula

$$\phi_{x,y} = \int_{e \in P_{x,y}} \rho(e) d\mu,$$

where the integral is taken with respect to a "measure" μ which is defined using the inverse of the norm map $\mathrm{Nm}_{P_{x,y}}$ (which exists by virtue of our inductive hypothesis). Making this idea precise will require some rather intricate categorical constructions, which we explain in detail in §4.

The core of the proof is in the verification of (b). Using formal properties of the norm maps Nm_X, we can reduce to proving that the map $\mathrm{Nm}_X : \varinjlim(\rho) \to \varprojlim(\rho)$ is an equivalence in the special case where X is an Eilenberg-MacLane space $K(\mathbf{Z}/p\mathbf{Z}, m)$, and ρ is the constant functor taking the value $K(n) \in \mathrm{Sp}_{K(n)}$. After passing to homotopy groups, Nm_X induces a map of graded abelian groups $\iota : K(n)_* X \to K(n)^{-*} X$, and we wish to show that ι is an isomorphism.

Here we proceed by explicit calculation. The groups $K(n)_* K(\mathbf{Z}/p\mathbf{Z}, m)$ and $K(n)^* K(\mathbf{Z}/p\mathbf{Z}, m)$ were computed by Ravenel and Wilson in [18]. Their results are most neatly summarized using the language of Dieudonne theory. Let $\kappa = \pi_0 K(n)$, let $\mathbf{G}$ denote the formal group over κ given by $\mathrm{Spf}\, K(n)^0\, \mathbf{CP}^\infty$, and let M be the covariant Dieudonne module of $\mathbf{G}$ (so that M is a module over the Dieudonne ring $W(\kappa)[F, V]$, which is free of n as a module over $W(\kappa)$). For each $d \geq 1$, the exterior power $\wedge^d_{W(\kappa)} M$ inherits an action of $W(\kappa)[F, V]$, which is the covariant Dieudonne module of a smooth formal group $\mathbf{G}^{(d)}$ of height $\binom{n}{d}$ and dimension $\binom{n-1}{d-1}$. We have a canonical isomorphism

$$K(n)^* K(\mathbf{Z}/p\mathbf{Z}, m) \simeq \pi_* K(n) \otimes_\kappa A,$$

where A is the ring of functions on p-torsion subgroup of $\mathbf{G}^{(m)}$. We will give a proof of this result in §2 which is somewhat different from the proof given in [18]: it relies on multiplicative aspects of the theory of Dieudonne modules, which we review in §1.

We can summarize the preceding discussion by saying that the group scheme $\mathrm{Spec}\, K(n)^0 K(\mathbf{Z}/p\mathbf{Z}, m)$ behaves, in some sense, like an mth exterior power of the group scheme $\mathrm{Spec}\, K(n)^0 K(\mathbf{Z}/p\mathbf{Z}, 1)$. In §4, we will make this idea more precise by introducing, for every integer $m > 0$ and every finite flat commutative group scheme G over a commutative ring R, another group scheme $\mathrm{Alt}^{(m)}_G$. We will see that the calculation of Ravenel and Wilson yields an isomorphism of commutative group schemes

$$\mathrm{Spec}\, K(n)_0 K(\mathbf{Z}/p\mathbf{Z}, m) \simeq \mathrm{Alt}^{(m)}_{\mathbf{G}[p]},$$

where $\mathbf{G}[p]$ denotes the p-torsion subgroup of the formal group $\mathbf{G}$ (Theorem 2.4.10). Moreover, this isomorphism can be lifted to characteristic zero: if E denotes the Lubin-Tate spectrum associated to the formal group $\mathbf{G}$, and $\overline{\mathbf{G}}$ denotes the universal deformation of $\mathbf{G}$ over the Lubin-Tate ring $R = \pi_0 E$, then we have a canonical isomorphism

$$\mathrm{Spec}\, E_0^\wedge K(\mathbf{Z}/p\mathbf{Z}, m) \simeq \mathrm{Alt}^{(m)}_{\overline{\mathbf{G}}[p]}$$

of finite flat group schemes over R (Theorem 3.4.1). We will use this isomorphism to identify the bilinear form β with (the reduction of) a certain multiple of the trace pairing on the algebra $E_0^\wedge K(\mathbf{Z}/p\mathbf{Z}, m)$. In §5, we will use this identification to prove the nondegeneracy of β, and thereby obtain a proof of Theorem 0.0.2.

Remark 0.0.5. Most of the results proven in the first three sections of this paper have appeared elsewhere in print, though with a somewhat different exposition. In particular, the material of §1 was inspired by [4] (see also [2]). The calculations of §2 were originally carried out by in [18] (at least for odd primes; an extension to the prime 2 is indicated in [12]). The algebraic results of §3 concerning alternating powers of finite flat group schemes can be found in [8] (at least for odd primes), and the relationship with the generalized cohomology of Eilenberg-MacLane spaces is described in [2] (for Morava K-theory) and [17] (for Morava E-theory).

Notation and Terminology

In the later sections of this paper, we will freely use the language of ∞-categories. We refer the reader to [13] for the foundations of this theory, and to [14] for an exposition of stable homotopy theory from the ∞-categorical point of view. We will generally adopt the notation of [13] and [14]. This leads to a few nonstandard conventions:

- If $\mathcal{C}$ is a monoidal ∞-category, we will generally let $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ indicate the tensor product functor on $\mathcal{C}$. In particular, we will employ this notation when discussing the smash product of spectra (and the $K(n)$-localized smash product of $K(n)$-local spectra). That is, if X and Y are spectra, we denote their smash product by $X \otimes Y$ rather than $X \wedge Y$.

- Let A be an associative ring spectrum (that is, an associative algebra object of the ∞-category Sp). We let LMod_A and RMod_A denote the ∞-categories of left A-module spectra and right A-module spectra, respectively. If A is a commutative algebra object of Sp (that is, an $\mathbb{E}_\infty$-ring), then we will simply right Mod_A in place of LMod_A and RMod_A.

- If A is an commutative ring, we let $\mathbf{Mod}_A$ denote the abelian category of A-modules. We will generally abuse notation by identifying A with the corresponding (discrete) ring spectrum. In this case, the notation Mod_A will always indicate the ∞-category of A-module *spectra*. This ∞-category is related to (but larger than) $\mathbf{Mod}_A$: more precisely, the homotopy category of Mod_A can be identified with the unbounded derived category of $\mathbf{Mod}_A$.

- More generally, we will use boldface notations such as **CAlg**, **CoAlg**, **Hopf** in cases where we consider classical algebraic objects (commutative algebras, commutative coalgebras, and Hopf algebras, respectively), which we will be our emphasis throughout §1. Later in this paper we will consider algebras, coalgebras, and Hopf algebras over ring spectra. These are organized into ∞-categories which will be denoted by CAlg, CoAlg, and Hopf (with additional subscripts indicating the ground ring).

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1 Multiplicative Aspects of Dieudonne Theory

Let κ be a field. In this section, we will study commutative and cocommutative Hopf algebras over κ, which we shall henceforth refer to simply as *Hopf algebras*. We let $\mathbf{Hopf}_\kappa$ denote the category of Hopf algebras over κ. In this section, we will study the structure of the category $\mathbf{Hopf}_\kappa$.

If H is a Hopf algebra over κ, then the counit and comultiplication on H are given by maps

$$\epsilon : H \to \kappa \qquad \Delta : H \to H \otimes_\kappa H.$$

We let $\mathrm{Prim}(H) = \{x \in H : \Delta(x) = 1 \otimes x + x \otimes 1\}$ denote the collection of primitive elements of H, and $\mathrm{GLike}(H) = \{x \in H : \Delta(x) = x \otimes x, \epsilon(x) = 1\}$ denote the collection of grouplike elements of H. Note that $\mathrm{Prim}(H)$ is a vector space over κ, and that the multiplication on H makes $\mathrm{GLike}(H)$ into an abelian group.

Definition 1.0.6. Let H be a Hopf algebra over a field κ. We will say that H is *multiplicative* if $\mathrm{Prim}(H) \simeq 0$. We let $\mathbf{Hopf}^m_\kappa$ denote the full subcategory of $\mathbf{Hopf}_\kappa$ spanned by the multiplicative Hopf algebras over κ.

Let $\overline{\kappa}$ be an algebraic closure of κ, and let $H_{\overline{\kappa}} = \overline{\kappa} \otimes_\kappa H$ be the induced Hopf algebra over $\overline{\kappa}$. We will say that H is *connected* if the group $\mathrm{GLike}(H_{\overline{\kappa}})$ is trivial. We let $\mathbf{Hopf}^c_\kappa$ denote the full subcategory of $\mathbf{Hopf}_\kappa$ spanned by the connected Hopf algebras over κ.

Remark 1.0.7. The construction $H \mapsto \operatorname{Spec} H$ determines a contravariant equivalence from the category $\mathbf{Hopf}_\kappa$ to the category of commutative affine group schemes over κ. A Hopf algebra H is multiplicative if the group scheme $\operatorname{Spec} H$ is pro-reductive, and connected if the group scheme $\operatorname{Spec} H$ is pro-unipotent.

If the field κ is perfect, then every Hopf algebra H over κ admits an essentially unique factorization $H \simeq H^c \otimes_\kappa H^m$, where H^c is connected and H^m is multiplicative. This induces an equivalence of categories $\mathbf{Hopf}_\kappa \simeq \mathbf{Hopf}^c_\kappa \times \mathbf{Hopf}^m_\kappa$, where $\mathbf{Hopf}^c_\kappa$ and $\mathbf{Hopf}^m_\kappa$ denote the full subcategories of $\mathbf{Hopf}_\kappa$ spanned by those Hopf algebras which are connective and multiplicative, respectively.

Let H be an arbitrary Hopf algebra over κ, and let $M = \mathrm{GLike}(H_{\overline{\kappa}})$ be the collection of group-like elements of $H_{\overline{\kappa}}$. Then M is an abelian group equipped with a continuous action of the Galois group $\mathrm{Gal}(\overline{\kappa}/\kappa)$. Let

$\overline{\kappa}[M]$ denote the group algebra of M over $\overline{\kappa}$. The inclusion $M = \mathrm{GLike}(H_{\overline{\kappa}}) \subseteq H_{\overline{\kappa}}$ extends uniquely to a map $\overline{\kappa}[M] \to H_{\overline{\kappa}}$ of Hopf algebras over $\overline{\kappa}$, which is $\mathrm{Gal}(\overline{\kappa}/\kappa)$-equivariant. Restricting to fixed points, we obtain a map $\mu : H_0 \to H$ of Hopf algebras over κ, where H_0 denotes the algebra of $\mathrm{Gal}(\overline{\kappa}/\kappa)$-fixed points on $\overline{\kappa}[M]$. One can show that μ is always injective, and is an isomorphism if and only if H is multiplicative. Consequently, the construction

$$H \mapsto \mathrm{GLike}(H_{\overline{\kappa}})$$

determines an equivalence from the category $\mathbf{Hopf}_{\kappa}^{m}$ to the category of abelian groups equipped with a continuous action of the Galois group $\mathrm{Gal}(\overline{\kappa}/\kappa)$.

One can attempt to carry out a similar analysis in the setting of connected Hopf algebras, using primitive elements rather than group-like elements. Let H be an arbitrary Hopf algebra over κ, and let $V = \mathrm{Prim}(H)$ be the κ-vector space of primitive elements of H. The inclusion $V \hookrightarrow H$ extends uniquely to a map of Hopf algebras $\nu : \mathrm{Sym}^* V \to H$ (where we regard $\mathrm{Sym}^* V$ as endowed with the unique Hopf algebra structure compatible with its multiplication, having the property that the elements of V are primitive). If κ is a field of characteristic zero, then the map ν is always an injection, and is an isomorphism if and only if the Hopf algebra H is connected. In this case, the construction $H \mapsto \mathrm{Prim}(H)$ determines an equivalence from the category $\mathbf{Hopf}_{\kappa}^{c}$ to the category Vect_{κ} of vector spaces over κ.

If κ is a perfect field of characteristic $p > 0$, the situation is much more complicated. The map $\nu : \mathrm{Sym}^* \mathrm{Prim}(H) \to H$ is generally neither injective nor surjective, even if we assume that H is connected. To understand the structure of H, it is necessary to replace the vector space $\mathrm{Prim}(H)$ by a more sophisticated invariant, called the *Dieudonne module* of H. Let $W(\kappa)$ denote the ring of (p-typical) Witt vectors of κ, let $\varphi : W(\kappa) \to W(\kappa)$ denote the automorphism induced by the Frobenius map from κ to itself, and let $\mathrm{D}_{\kappa} = W(\kappa)[F, V]$ denote the non-commutative ring obtained by adjoining to $W(\kappa)$ a pair of elements F and V satisfying the identities

$$VF = FV = p \qquad F\lambda = \varphi(\lambda)F \qquad V\varphi(\lambda) = \lambda V$$

where λ ranges over $W(\kappa)$. The following is a foundational result of Dieudonne theory:

Theorem 1.0.8. *Let κ be a perfect field of characteristic $p > 0$. Then there is a fully faithful embedding* DM *from the category $\mathbf{Hopf}_{\kappa}^{c}$ of connected Hopf algebras over κ to the category of left D_{κ}-modules. The essential image of this functor is the collection of those left D_{κ}-modules M having the property that each element $x \in M$ is annihilated by V^n, for some $n \gg 0$.*

If H is a connected Hopf algebra over κ, we will refer to $\mathrm{DM}(H)$ as the *Dieudonne module* of H. It can be regarded as an enlargement of the set $\mathrm{Prim}(H)$ of primitive elements of H, in the sense that there is a canonical isomorphism $\mathrm{Prim}(H) \simeq \{x \in \mathrm{DM}(H) : Vx = 0\}$.

As indicated in Remark 1.0.7, the category $\mathbf{Hopf}_{\kappa}$ of Hopf algebras over κ can be identified with (the opposite of) the category of commutative affine group schemes over κ. However, it has another algebro-geometric interpretation which will play an important role throughout this section. Let $\mathbf{CAlg}_{\kappa}^{\mathrm{fd}}$ denote the category of finite dimensional commutative algebras over κ, and let $\mathcal{A}b$ denote the category of abelian groups. A *commutative formal group* over κ is a functor $G : \mathbf{CAlg}_{\kappa}^{\mathrm{fd}} \to \mathcal{A}b$ which preserves finite limits. If H is a Hopf algebra over κ, then the construction $R \mapsto \mathrm{GLike}(H_R)$ is a formal group over κ (here H_R denotes the Hopf algebra $R \otimes_{\kappa} H$ over R), which we will denote by $\mathrm{Spf}\, H^{\vee}$. One can show that the construction $H \mapsto \mathrm{Spf}\, H^{\vee}$ determines an equivalence from the category of Hopf algebras over κ to the category of commutative formal groups over κ.

Suppose that G, G', and G'' are commutative formal groups over κ. We can then consider the notion of a *bilinear map* $G \times G' \to G''$: that is, a natural transformation of functors $G \times G' \to G''$ which induces a bilinear map

$$G(R) \times G'(R) \to G''(R)$$

for every $R \in \mathbf{CAlg}_{\kappa}^{\mathrm{fd}}$. If G and G' are fixed, then there is a universal example of a commutative formal group G'' equipped with a bilinear map $G \times G' \to G''$, which we will denote by $G \otimes G'$. Writing $G = \mathrm{Spf}\, H^{\vee}$

and $G' = \operatorname{Spf} H'^{\vee}$, we can write $G'' = \operatorname{Spf} H''^{\vee}$ for some Hopf algebra H''. We will indicate the dependence of H'' on H and H' by writing $H'' = H \boxtimes H'$. The operation $\boxtimes$ determines a symmetric monoidal structure on the category $\mathbf{Hopf}_{\kappa}$. If the Hopf algebras H and H' are connected, we will see that $H \boxtimes H'$ is also connected. Consequently, $H \boxtimes H'$ is determined by its Dieudonne module $\mathrm{DM}(H \boxtimes H')$. Our main goal in this section is to prove the following result, which gives a linear-algebraic description of $\mathrm{DM}(H \boxtimes H')$ in terms of $\mathrm{DM}(H)$ and $\mathrm{DM}(H')$:

Theorem 1.0.9 (Goerss, Buchstaber-Lazarev)**.** *Let H and H' be connected Hopf algebras over κ. Then the Diuedonne module $\mathrm{DM}(H \boxtimes H')$ is characterized by the following universal property: for any left D_{κ}-module M, there is a bijective correspondence between D_{κ}-module maps $\mathrm{DM}(H \boxtimes H') \to M$ and $W(\kappa)$-bilinear maps $\lambda : \mathrm{DM}(H) \times \mathrm{DM}(H') \to M$ satisfying the following identities*

$$V\lambda(x, y) = \lambda(Vx, Vy) \qquad F\lambda(Vx, y) = \lambda(x, Fy) \qquad F\lambda(x, Vy) = \lambda(Fx, y).$$

Let us now outline the contents of this section. We will begin in §1.1 with some generalities on bialgebras and Hopf algebras, and give a construction of the tensor product $\boxtimes : \mathbf{Hopf}_{\kappa} \times \mathbf{Hopf}_{\kappa} \to \mathbf{Hopf}_{\kappa}$. In §1.3, we will recall the definition of the Dieudonne module functor DM, and give a proof of Theorem 1.0.9. Both the definition and the proof will require some general facts about Witt vectors, which we review in §1.2. Finally, in §1.4, we describe some extensions of Theorem 1.0.9 to the case of Hopf algebras which are not necessarily connected.

1.1 Tensor Products of Hopf Algebras

Let k be a commutative ring, which we regard as fixed throughout this section, and let $\mathbf{Hopf}_k$ denote the category of (commutative and cocommutative) Hopf algebras over k. In this section, we will introduce a symmetric monoidal structure on the category $\mathbf{Hopf}_k$, which will play an important role throughout this paper. When k is a field, the tensor product functor $\boxtimes : \mathbf{Hopf}_k \times \mathbf{Hopf}_k \to \mathbf{Hopf}_k$ can be described by the following universal property: given Hopf algebras $H, H', H'' \in \mathbf{Hopf}_k$, there is a bijective correspondence between Hopf algebra homomorphisms $H \boxtimes H' \to H''$ and bilinear maps of formal k-schemes

$$\operatorname{Spf} H^{\vee} \times \operatorname{Spf} H'^{\vee} \to \operatorname{Spf} H''^{\vee}.$$

We begin with some general remarks. We let $\mathbf{Mod}_k$ denote the category of (discrete) k-modules. We regard $\mathbf{Mod}_k$ as symmetric monoidal category by means of the usual tensor product $(M, N) \mapsto M \otimes_k N$.

Definition 1.1.1. A *k-coalgebra* is an object $C \in \mathbf{CoAlg}_k$ which is equipped with a comultiplication $\Delta_C : C \to C \otimes_k C$ which is commutative, associative, and admits a counit $\epsilon_C : C \to k$.

Warning 1.1.2. The notion of k-coalgebra introduced in Definition 1.1.1 is more often referred to as a cocommutative, coassociative, counital coalgebra. We will generally omit these adjectives: in this paper, we will never consider coalgebras which are not cocommutative and coassociative.

If C and D are k-coalgebras, a *coalgebra homomorphism* from C to D is a map $f : C \to D$ for which the diagram

$$\begin{tikzcd}
k \arrow[d, "\mathrm{id}"] & C \arrow[l, "\epsilon_C"'] \arrow[r, "\Delta_C"] \arrow[d, "f"] & C \otimes_k C \arrow[d, "f \otimes f"] \\
k & D \arrow[l, "\epsilon_D"'] \arrow[r, "\Delta_D"] & D \otimes_k D
\end{tikzcd}$$

commutes. The collection of k-coalgebras and coalgebra homomorphisms determines a category, which we will denote by $\mathbf{CoAlg}_k$.

We will use the following category-theoretic fact:

Proposition 1.1.3. *The category $\mathbf{CoAlg}_k$ is locally presentable: that is, it admits small colimits and is generated by a set of τ-compact objects, for some regular cardinal τ.*

Proof. The existence of small colimits in $\mathbf{CoAlg}_k$ follows immediately from the existence of small colimits in $\mathbf{Mod}_k$ (note also that the forgetful functor $\mathbf{CoAlg}_k \to \mathbf{Mod}_k$ preserves small colimits). The accessibility of $\mathbf{CoAlg}_k$ follows from the observation that $\mathbf{CoAlg}$ can be identified with a lax limit of accessible categories (see, for example, [15]). $\square$

Corollary 1.1.4. *The category $\mathbf{CoAlg}_k$ admits small limits and colimits.*

Example 1.1.5. Given a pair of k-coalgebras C and D, the tensor product $C \otimes_k D$ is a product of C and D in the category $\mathbf{CoAlg}_k$.

Remark 1.1.6. Suppose that k is a field. We let $\mathbf{CoAlg}_k^{\mathrm{fd}}$ denote the full subcategory of $\mathbf{CoAlg}_k$ spanned by those coalgebras which are finite-dimensional when regarded as vector spaces over k. The objects of $\mathbf{CoAlg}_k^{\mathrm{fd}}$ are compact when regarded as objects of $\mathbf{CoAlg}_k$, so that the inclusion $\mathbf{CoAlg}_k^{\mathrm{fd}} \hookrightarrow \mathbf{CoAlg}_k$ extends to a fully faithful embedding $\theta : \mathrm{Ind}(\mathbf{CoAlg}_k^{\mathrm{fd}}) \to \mathbf{CoAlg}_k$. The functor θ is an equivalence of categories: essentially surjectivity follows from the fact that every k-coalgebra can be written as a union of its finite-dimensional subcoalgebras. Consequently, we obtain (in the case where k is a field) a stronger version of Proposition 1.1.3: the category $\mathbf{CoAlg}_k$ is compactly generated.

Notation 1.1.7. For every k-coalgebra C, we let $h_C : \mathbf{CoAlg}_k^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ denote the functor represented by C, so that h_C is described by the formula

$$h_C(D) = \mathrm{Hom}_{\mathbf{CoAlg}_k}(D, C).$$

According to Yoneda's Lemma, the construction $C \mapsto h_C$ determines a fully faithful embedding

$$\mathbf{CoAlg}_k \to \mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathcal{S}\mathrm{et}).$$

Remark 1.1.8. Suppose that k is a field. Using Remark 1.1.6, we see that the Yoneda embedding $C \mapsto h_C$ induces a fully faithful embedding

$$\mathbf{CoAlg}_k \to \mathrm{Fun}((\mathbf{CoAlg}_k^{\mathrm{fd}})^{\mathrm{op}}, \mathcal{S}\mathrm{et}).$$

Let $\mathbf{CAlg}_k^{\mathrm{fd}}$ denote the category of (discrete) commutative algebras over k which are finite-dimensional when regarded as vector spaces over k, so that vector space duality induces an equivalence of categories

$$(\mathbf{CoAlg}_k^{\mathrm{fd}})^{\mathrm{op}} \simeq \mathbf{CAlg}_k^{\mathrm{fd}}.$$

We then obtain a fully faithful embedding

$$\mathbf{CoAlg}_k \to \mathrm{Fun}(\mathbf{CAlg}_k^{\mathrm{fd}}, \mathcal{S}\mathrm{et}),$$

whose essential image consists of those functors $\mathbf{CAlg}_k^{\mathrm{fd}} \to \mathcal{S}\mathrm{et}$ which preserve finite limits. We will sometimes denote this latter embedding by $C \mapsto \mathrm{Spf}\, C^\vee$. Here we can regard $C^\vee$ as a topological ring, and $\mathrm{Spf}\, C^\vee$ carries an object $A \in \mathbf{CAlg}_k^{\mathrm{fd}}$ to the set of continuous k-algebra homomorphisms $C^\vee \to A$.

Proposition 1.1.9. *The Yoneda embedding $h : \mathbf{CoAlg}_k \to \mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ admits a left adjoint $L : \mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathbf{CAlg}_k$. Moreover, the functor L commutes with finite products.*

Proof. The existence of L is a formal consequence of Proposition 1.1.3. Let us review a proof, since we will need it to show that L commutes with finite products. We first note that the essential image of h is the full subcategory $\mathcal{E}_0 \subseteq \mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ spanned by those functors which carry small colimits in $\mathbf{CoAlg}_k$ to limits in $\mathcal{S}\mathrm{et}$. Choose a regular cardinal τ such that $\mathbf{CoAlg}_k^{\mathrm{op}}$ is τ-compactly generated, let $\mathcal{C}$ be the category of τ-compact objects of $\mathbf{CoAlg}_k$, and let $\mathcal{E}_1$ be the full subcategory of $\mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ spanned by those functors which carry κ-filtered colimits to limits in $\mathcal{S}\mathrm{et}$. Since $\mathbf{CoAlg}_k^{\mathrm{op}}$ is τ-compactly generated, a functor $F \in \mathcal{E}_1$ is determined by its restriction $F|\,\mathcal{C}$. More precisely, $\mathcal{E}_1$ is the full subcategory of $\mathrm{Fun}(\mathbf{CoAlg}_k^{op}, \mathcal{S}\mathrm{et})$ spanned by those functors F which are right Kan extensions of $F|\,\mathcal{C}^{op}$. It follows that the inclusion $\mathcal{E}_1 \hookrightarrow \mathbf{CAlg}_k$ admits a left adjoint, which is equivalent to the restriction functor

$$\mathrm{Fun}(\mathbf{CoAlg}_k^{op}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et})$$

(and therefore commutes with all small limits). To complete the proof, it will suffice to show that the inclusion $\mathcal{E}_0 \hookrightarrow \mathcal{E}_1$ admits a left adjoint which commutes with finite products.

In what follows, let us identify $\mathcal{E}_1$ with the presheaf category $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et})$. Under this identification, the inclusion $i : \mathcal{E}_0 \hookrightarrow \mathcal{E}_1$ is given by the restricted Yoneda embedding $C \mapsto h_C|\,\mathcal{C}^{op}$. This functor obviously preserves small limits and τ-filtered colimits. Using the adjoint functor theorem, we deduce that i admits a left adjoint L_0.

We will complete the proof by showing that L_0 commutes with finite products. Fix a pair of functors $F, F' : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$; we wish to show that the canonical map $\theta_{F,F'} : L_0(F \times F') \to L_0(F) \otimes_k L_0(F')$ is an isomorphism. Note that if F' is fixed, the constructions $F \mapsto L_0(F \times F')$ and $F \mapsto L_0(F) \otimes_k L_0(F')$ carry colimits in $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et})$ to colimits in $\mathbf{CoAlg}_k$. We may therefore reduce to the case where F is the functor represented by a coalgebra C. Similarly, we may suppose that F' is represented by a coalgebra C'. In this case, $\theta_{F,F'}$ is induced by the identity map from $C \otimes_k C'$ to itself. $\square$

Definition 1.1.10. Let $\mathcal{C}$ be a category which admits finite products. A *commutative monoid object* of $\mathcal{C}$ is an object $M \in \mathcal{C}$ equipped with a multiplication map $m : M \times M \to M$ which is commutative, associative, and unital. We let $\mathrm{CMon}(\mathcal{C})$ denote the category of commutative monoid objects in $\mathcal{C}$.

Example 1.1.11. Let $\mathcal{C}$ be the category of sets. In this case, $\mathrm{CMon}(\mathcal{C}) = \mathrm{CMon}(\mathcal{S}\mathrm{et})$ is the category of commutative monoids. We will denote this category simply by CMon.

Example 1.1.12. Let $\mathcal{C} = \mathbf{CoAlg}_k$ be the category of k-coalgebras. In this case, we will denote $\mathrm{CMon}(\mathcal{C})$ by $\mathbf{BiAlg}_k$. We will refer to $\mathbf{BiAlg}_k$ as the *category of k-bialgebras*. By definition, an object of $\mathbf{BiAlg}_k$ is a k-module H which is equipped with a comultiplication $\Delta : H \to H \otimes_k H$ and a multiplication $H \otimes_k H \to H$ which is a map of k-coalgebras. Here we always require the multiplication and comultiplication on H to be commutative, associative, and unital.

Let $h : \mathbf{CoAlg}_k \to \mathrm{Fun}(\mathbf{CoAlg}_k^{op}, \mathcal{S}\mathrm{et})$ be the Yoneda embedding, and let L denote the left adjoint to h supplied by Proposition 1.1.9. Since h and L commute with finite products, they determine an adjunction

$$\mathrm{CMon}(\mathrm{Fun}(\mathbf{CoAlg}_k^{op}, \mathcal{S}\mathrm{et})) \underset{\mathrm{CMon}(h)}{\overset{\mathrm{CMon}(L)}{\rightleftarrows}} \mathrm{CMon}(\mathbf{CoAlg}_k),$$

which we will denote simply by

$$\mathrm{Fun}(\mathbf{CoAlg}_k^{op}, \mathrm{CMon}) \underset{h}{\overset{L}{\rightleftarrows}} \mathbf{BiAlg}_k\,.$$

More concretely, we can summarize the situation as follows:

- Let H be a k-bialgebra. Then for every k-coalgebra C, the multiplication on H determines a commutative monoid structure on the set $\mathrm{Hom}_{\mathbf{CoAlg}_k}(C, H)$. Consequently, we can view h_H as a contravariant functor from $\mathbf{CoAlg}_k$ to the category of commutative monoids.

- The functor $h : \mathbf{BiAlg}_k \to \mathrm{Fun}(\mathbf{CoAlg}_k^{op}, \mathrm{CMon})$ admits a left adjoint, given on the level of coalgebras by the construction $F \mapsto L(F)$ where L is defined as in Proposition 1.1.9.

Remark 1.1.13. Suppose that k is a field. Using Remark 1.1.8, we see that the Yoneda embedding determines a fully faithful functor $\mathbf{BiAlg}_k \to \mathrm{Fun}(\mathbf{CAlg}_k^{fd}, \mathrm{CMon}(\mathcal{S}\mathrm{et}))$, whose essential image is spanned by those functors $X : \mathbf{CAlg}_k^{fd} \to \mathrm{CMon}(\mathcal{S}\mathrm{et})$ which preserve finite limits.

Construction 1.1.14. Let M, M', and M'' be commutative monoids (whose monoid structure will be denoted additively). We will say that a map $\lambda : M \times M' \to M''$ is *bilinear* if it satisfies the identities

$$\lambda(x + x', y) = \lambda(x, y) + \lambda(x', y) \qquad \lambda(0, y) = 0 = \lambda(x, 0) \qquad \lambda(x, y + y') = \lambda(x, y) + \lambda(x, y').$$

Given commutative monoids M and M', there exists another commutative monoid $M \otimes M'$ and a bilinear map $\lambda_u : M \times M' \to M \otimes M'$ which is *universal* in the following sense: for every commutative monoid M'', composition with λ_u induces a bijection from the set $\mathrm{Hom}_{\mathrm{CMon}}(M \otimes M', M'')$ to the set of bilinear maps $M \times M' \to M''$. We will refer to $M \otimes M'$ as the *tensor product* of M with M'. The tensor product of commutative monoids is commutative and associative up to coherent isomorphism, and the formation of tensor products endows CMon with the structure of a symmetric monoidal category.

Remark 1.1.15. If M and M' are abelian groups, then the tensor product $M \otimes M'$ in the category CMon agrees with their tensor product in the category of abelian groups. However, the inclusion from the category of abelian groups to the category CMon is not symmetric monoidal, because it does not preserve unit objects: the unit object of CMon is not the free abelian group $\mathbf{Z}$ on one generator, but rather the free commutative monoid $\mathbf{Z}_{\geq 0}$ on one generator.

Proposition 1.1.16. *Let $\mathcal{C}$ denote the functor category $\mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathrm{CMon})$, and regard $\mathcal{C}$ as a symmetric monoidal category using Construction 1.1.14 objectwise. $L : \mathcal{C} \to \mathbf{BiAlg}_k$ denote a left adjoint to the Yoneda embedding. Then L is compatible with the symmetric monoidal structure on $\mathcal{C}$. That is, if $\alpha : F \to F'$ is a morphism in $\mathcal{C}$ which induces an isomorphism of bialgebras $L(F) \to L(F')$, and G is an arbitrary object of $\mathcal{C}$, then the induced map $\beta : L(F \otimes G) \to L(F' \otimes G)$ is also an isomorphism of bialgebras.*

Proof. Let H be a bialgebra over k; we wish to show that composition with β induces a bijection

$$\theta : \mathrm{Hom}_{\mathbf{BiAlg}_k}(L(F' \otimes G), H) \to \mathrm{Hom}_{\mathbf{BiAlg}_k}(L(F \otimes G), H).$$

Unwinding the definitions, we can identify θ with a map

$$\mathrm{Hom}_{\mathcal{C}}(F' \otimes G, h_H) \to \mathrm{Hom}_{\mathcal{C}}(F \otimes G, h_H).$$

We can identify the left hand side with the set of bilinear maps $F' \times G \to h_H$, and the right hand side with the set of bilinear maps $F \times G \to h_H$. Using the fact that the functor L commutes with finite products, we can identify both sides with the same subset of the mapping set

$$\mathrm{Hom}_{\mathbf{CoAlg}_k}(L(F') \otimes_k L(G), H) \simeq \mathrm{Hom}_{\mathbf{CoAlg}_k}(L(F) \otimes_k L(G), H).$$

$\square$

Corollary 1.1.17. *The category $\mathbf{BiAlg}_k$ inherits a symmetric monoidal structure from the symmetric monoidal structure on $\mathcal{C} = \mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathrm{CMon})$. That is, there is a symmetric monoidal structure on $\mathbf{BiAlg}_k$ (which is unique up to canonical isomorphism) for which the localization functor $L : \mathcal{C} \to \mathbf{BiAlg}_k$ is symmetric monoidal.*

Notation 1.1.18. We will indicate the symmetric monoidal structure of Corollary 1.1.17 by

$$\boxtimes : \mathbf{BiAlg}_k \times \mathbf{BiAlg}_k \to \mathbf{BiAlg}_k .$$

Note that $H \boxtimes H'$ is very different from the k-linear tensor product $H \otimes_k H'$. Unwinding the definitions, we see that giving a bialgebra map $H \boxtimes H' \to H''$ is equivalent to giving a coalgebra map $\lambda : H \otimes_k H' \to H''$ satisfying the identities

$$\lambda(1 \otimes y) = \epsilon_{H'}(y) \qquad \lambda(x \otimes 1) = \epsilon_H(x)$$
$$\lambda(xx' \otimes y) = \sum c_\alpha \lambda(x, z_\alpha) \lambda(x', z'_\alpha) \text{ if } \Delta_{H'}(y) = \sum c_\alpha z_\alpha \otimes z'_\alpha$$
$$\lambda(x \otimes yy') = \sum c_\alpha \lambda(z_\alpha \otimes y) \lambda(z'_\alpha \otimes y') \text{ if } \Delta_{H'}(x) = \sum c_\alpha z_\alpha \otimes z'_\alpha.$$

Concretely, we can describe $H \boxtimes H'$ as the quotient of the symmetric algebra $\mathrm{Sym}^*(H \otimes_k H')$ by the ideal which enforces these relations.

Remark 1.1.19. We have a diagram of categories and functors

$$\begin{tikzcd} \mathbf{BiAlg}_k \arrow[r, "h"] \arrow[d] & \mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathrm{CMon}) \arrow[d] \\ \mathbf{CoAlg}_k \arrow[r] & \mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \end{tikzcd}$$

which commutes up to canonical isomorphism; here the vertical maps are given by the forgetful functors. Each of these functors admits a left adjoint. The left adjoint to the forgetful functor $\mathbf{BiAlg}_k \to \mathbf{CoAlg}_k$ is given by the formation of symmetric algebras $C \mapsto \mathrm{Sym}^*(C)$, while the left adjoint to the right vertical map is given by pointwise composition with the free commutative monoid functor $\mathcal{S}\mathrm{et} \to \mathrm{CMon}$ given by $S \mapsto \mathbf{Z}_{\geq 0}[S]$. We therefore obtain a diagram of left adjoints

$$\begin{tikzcd} \mathbf{BiAlg}_k & \mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathrm{CMon}) \arrow[l, "L"'] \\ \mathbf{CoAlg}_k \arrow[u, "\mathrm{Sym}^*"] & \mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \arrow[l, "L"'] \arrow[u] \end{tikzcd}$$

which commutes up to canonical isomorphism. The free commutative monoid functor $\mathcal{S}\mathrm{et} \to \mathrm{CMon}$ is symmetric monoidal: that is, it carries products of sets to tensor products of commutative monoids. It follows that the functor $\mathrm{Sym}^* : \mathbf{CoAlg}_k \to \mathbf{BiAlg}_k$ is also symmetric monoidal. In particular, if C and D are k-coalgebras, we have a canonical isomorphism of k-bialgebras $\mathrm{Sym}^*(C \otimes_k D) \simeq (\mathrm{Sym}^* C) \boxtimes (\mathrm{Sym}^* D)$.

Example 1.1.20. Let us regard k as a coalgebra over itself. Then k is the unit object of $\mathbf{CoAlg}_k$ (with respect to the Cartesian product on $\mathbf{CoAlg}_k$, given by tensor product over k). It follows that $\mathrm{Sym}^* k \simeq k[x]$ is the unit object of $\mathbf{BiAlg}_k$ with respect to the tensor product $\boxtimes$. Here the polynomial ring $k[x]$ is equipped with its usual multiplication, and its coalgebra structure is determined by the relation $\Delta(x) = x \otimes x$.

Notation 1.1.21. Let $\mathbf{Z}$ be the group of integers, which we regard as an object of the category CMon of commutative monoids. We have an evident inclusion $\mathbf{Z}_{\geq 0} \hookrightarrow \mathbf{Z}$, which induces a map of commutative monoids

$$\mathbf{Z} \simeq \mathbf{Z}_{\geq 0} \otimes \mathbf{Z} \to \mathbf{Z} \otimes \mathbf{Z}.$$

A simple calculation shows that this map is an isomorphism: that is, we can regard $\mathbf{Z}$ as an idempotent object in the symmetric monoidal category CMon. If follows that the category $\mathrm{Mod}_{\mathbf{Z}}(\mathrm{CMon})$ of $\mathbf{Z}$-module objects of CMon can be identified with a full subcategory of CMon. Unwinding the definitions, we see that a commutative monoid $M \in \mathrm{CMon}$ admits a $\mathbf{Z}$-module structure if and only if M is an abelian group. Let $\mathcal{A}b$ denote the category of abelian groups, which we identify with a full subcategory of CMon. It follows that $\mathcal{A}b$ inherits a symmetric monoidal structure from the symmetric monoidal structure on CMon, with the same tensor product (but a different unit object).

Let H be a k-bialgebra. We will say that H is a *Hopf algebra* if the functor $h_H : (\mathbf{CoAlg}_k)^{\mathrm{op}} \to \mathrm{CMon}$ factors through the full subcategory $\mathcal{A}b \subseteq \mathrm{CMon}$. Let $\mathbf{Hopf}_k$ denote the full subcategory of $\mathbf{BiAlg}_k$ spanned by the Hopf algebras over k.

Let $\underline{\mathbf{Z}}$ denote the constant functor $\mathbf{CoAlg}_k^{\mathrm{op}} \to \mathrm{CMon}$ taking the value $\mathbf{Z}$, and let

$$L : \mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathrm{CMon}) \to \mathbf{BiAlg}_k$$

be a left adjoint to the Yoneda embedding. Unwinding the definitions, we can identify $L(\underline{Z})$ with the ring of Laurent polynomials $k[\mathbf{Z}] = k[t^{\pm 1}]$, with comultiplication given by $\Delta(t) = t \otimes t$. Since the functor L is symmetric monoidal, we conclude that $k[t^{\pm 1}]$ is an idempotent object of $\mathbf{BiAlg}_k$. Note that $k[t^{\pm 1}]$ is a Hopf algebra over k.

For any bialgebra H, we have $h_{H\boxtimes k[t^{\pm 1}]} \simeq h_H \otimes h_{k[t^{\pm 1}]} \in \mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathcal{A}b)$ so that $H \boxtimes k[t^{\pm 1}]$ is a Hopf algebra. Conversely, if H is a Hopf algebra, then we have

$$H \simeq L(h_H) \simeq L(h_H \otimes \underline{\mathbf{Z}}) \simeq L(h_H) \boxtimes k[t^{\pm 1}] \simeq H \boxtimes k[t^{\pm 1}].$$

It follows that we can identify $\mathbf{Hopf}_k$ with the category of modules over the idempotent object $k[t^{\pm 1}]$ in $\mathbf{BiAlg}_k$. In particular, the category $\mathbf{Hopf}_k$ of Hopf algebras over k inherits a symmetric monoidal structure from that of $\mathbf{BiAlg}_k$, with tensor product given by $(H, H') \mapsto H \boxtimes H'$ and unit object given by $k[t^{\pm 1}]$.

We close this section with a few observations which will be helpful when computing with the tensor product $\boxtimes$ on $\mathbf{BiAlg}_k$ and $\mathbf{Hopf}_k$.

Definition 1.1.22. Let C be a k-coalgebra. A *coaugmentation* on C is a k-coalgebra morphism $\lambda : k \to C$. We let $\mathbf{CoAlg}_k^{\mathrm{aug}}$ denote the category $(\mathbf{CoAlg}_k)_{k/}$ whose objects are k-coalgebras equipped with a coaugmentation.

Remark 1.1.23. A coaugmentation $\lambda : k \to C$ is uniquely determined by the element $\lambda(1) \in C$. Conversely, an arbitrary element $x \in C$ determines a coaugmentation on C if and only if it is *grouplike*: that is, if and only if it satisfies the equations

$$\Delta_C(x) = x \otimes x \qquad \epsilon_C(x) = 1.$$

Remark 1.1.24. Let C be a coaugmented k-coalgebra. Then, as a k-module, C splits as a direct sum $k \oplus C_0$, where C_0 denotes the kernel of the counit map $\epsilon_C : C \to k$.

Remark 1.1.25. Let $\mathcal{S}\mathrm{et}_*$ denote the category of pointed sets. The adjunction

$$\mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \underset{h}{\overset{L}{\rightleftarrows}} \mathbf{CoAlg}_k$$

determines another adjunction

$$\mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathcal{S}\mathrm{et}_*) \underset{h}{\overset{L}{\rightleftarrows}} \mathbf{CoAlg}_k^{\mathrm{aug}}.$$

We will regard $\mathcal{S}\mathrm{et}_*$ as a symmetric monoidal category via the *smash product* of pointed sets. If X is a set, we let $X_* = X \cup \{*\}$ be the set obtained from X by adjoining a disjoint base point. We then have canonical isomorphisms $X_* \wedge Y_* \simeq (X \times Y)_*$, which exhibit the construction $X \mapsto X_*$ as a symmetric monoidal functor from $(\mathcal{S}\mathrm{et}, \times)$ to $(\mathcal{S}\mathrm{et}_*, \wedge)$. The smash product induces a symmetric monoidal structure on $\mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathcal{S}\mathrm{et}_*)$, which is compatible with the localization functor L (as in the proof of Proposition 1.1.16). It follows that the category $\mathbf{CoAlg}_k^{\mathrm{aug}}$ inherits a symmetric monoidal structure, which we will denote by $\wedge : \mathbf{CoAlg}_k^{\mathrm{aug}} \times \mathbf{CoAlg}_k^{\mathrm{aug}} \to \mathbf{CoAlg}_k^{\mathrm{aug}}$. More concretely, if C and D are coaugmented coalgebras, then we can describe $C \wedge D$ as the quotient of the product $C \otimes_k D$ obtained by identifying the maps

$$C \simeq C \otimes_k k \to C \otimes_k D \leftarrow k \otimes_k D \simeq D$$

with those given by the coaugmentation on $C \wedge D$. Writing $C \simeq k \oplus C_0$, $D \simeq k \oplus D_0$, and $C \wedge D = k \oplus (C \wedge D)_0$ as in Remark 1.1.24, we obtain an isomorphism of k-modules $(C \wedge D)_0 \simeq C_0 \otimes_k D_0$.

Remark 1.1.26. If H is a k-bialgebra, then we can regard H as a k-coalgebra with a coaugmentation given by the unit map $k \to H$. This construction determines a forgetful functor $\mathbf{BiAlg}_k \to \mathrm{CoAlg}_k^{\heartsuit,\mathrm{aug}}$, which fits into a diagram

$$\begin{tikzcd}
\mathbf{BiAlg}_k \arrow[r, "h"] \arrow[d] & \mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathrm{CMon}) \arrow[d] \\
\mathbf{CoAlg}_k^{\mathrm{aug}} \arrow[r] & \mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathcal{S}\mathrm{et}_*)
\end{tikzcd}$$

which commutes up to canonical isomorphism. Each of these functors admits a left adjoint. The left adjoint to the forgetful functor $\mathbf{BiAlg}_k \to \mathbf{CoAlg}_k^{\mathrm{aug}}$ is given by the reduced symmetric algebra construction

$$\mathrm{Sym}^*_{\mathrm{red}} : \mathbf{CoAlg}_k^{\mathrm{aug}} \to \mathbf{BiAlg}_k,$$

which carries a coalgebra C with distinguished grouplike element $x \in C$ to the quotient $\mathrm{Sym}^*(C)/(x-1)$. Writing $C = k \oplus C_0$ as in Remark 1.1.24, we have an isomorphism of k-algebras $\mathrm{Sym}^*_{\mathrm{red}}(C) \simeq \mathrm{Sym}^*(C_0)$. The left adjoint to the right vertical map is induced by pointwise composition with the reduced free commutative monoid functor $F : \mathcal{S}\mathrm{et}_* \to \mathrm{CMon}$, given on objects by the formula $F(S_*) = \mathbf{Z}_{\geq 0}[S]$. We therefore obtain a diagram of left adjoints

$$\begin{tikzcd}
\mathbf{BiAlg}_k & \mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathrm{CMon}) \arrow[l, "L"'] \\
\mathbf{CoAlg}_k \arrow[u, "\mathrm{Sym}^*_{\mathrm{red}}"] & \mathrm{Fun}(\mathbf{CoAlg}_k^{\mathrm{op}}, \mathcal{S}\mathrm{et}_*) \arrow[l, "L"'] \arrow[u]
\end{tikzcd}$$

which commutes up to canonical isomorphism. Since the functor F is symmetric monoidal, it follows that the functor $\mathrm{Sym}^*_{\mathrm{red}}$ is also symmetric monoidal. More concretely, if we are given coaugmented coalgebras $C \simeq k \oplus C_0$, $D \simeq k \oplus D_0$, then there is a canonical bialgebra isomorphism

$$\mathrm{Sym}^*_{\mathrm{red}}(C) \boxtimes \mathrm{Sym}^*_{\mathrm{red}}(D) \simeq \mathrm{Sym}^*_{\mathrm{red}}(C \wedge D),$$

which we can write more informally as $\mathrm{Sym}^*(C_0) \boxtimes \mathrm{Sym}^*(D_0) \simeq \mathrm{Sym}^*(C_0 \otimes D_0)$.

Example 1.1.27. Let us regard $k[x]$ as a Hopf algebra over k, with comultiplication given by $\Delta(x) = 1 \otimes x + x \otimes 1$. Then $k[x] \simeq \mathrm{Sym}^*_{\mathrm{red}}(C)$, where C is the subcoalgebra of $k[x]$ generated by 1 and x. Note that there is a canonical isomorphism $C \wedge C \simeq C$, given by $x \otimes x \mapsto x$. Applying Remark 1.1.26, we obtain an isomorphism $k[x] \boxtimes k[x] \simeq k[x]$, given by $x \boxtimes x \mapsto x$.

1.2 Witt Vectors

In this section, we will review some aspects of the theory of Witt vectors which are needed in this paper. For a more comprehensive discussion, we refer the reader to [7].

Notation 1.2.1. For every commutative ring R, we let $W_{\mathrm{Big}}(R)$ denote the subset of $R[[t]]$ consisting of power series of the form $1 + c_1 t + c_2 t^2 + \cdots$ (that is, power series with constant term 1). The set $W_{\mathrm{Big}}(R)$ has the structure of an abelian group, given by multiplication of power series. We will refer to $W_{\mathrm{Big}}(R)$ as the *group of big Witt vectors of* R (in fact, $W_{\mathrm{Big}}(R)$ has the structure of a commutative ring, but the multiplication on $W_{\mathrm{Big}}(R)$ will not concern us in this section).

Let $\mathrm{Wt}_{\mathrm{Big}}$ denote the polynomial ring $\mathbf{Z}[c_1, c_2, \ldots]$ on infinitely many variables. For any commutative ring R, we have a canonical bijection. $W_{\mathrm{Big}}(R) \simeq \mathrm{Hom}_{\mathrm{Ring}}(\mathrm{Wt}_{\mathrm{Big}}, R)$. Since the functor $R \mapsto W_{\mathrm{Big}}(R)$ takes values in the category of abelian groups, we can regard $\mathrm{Wt}_{\mathrm{Big}}$ as a Hopf algebra over the ring of integers $\mathbf{Z}$.

Unwinding the definitions, we see that the comultiplication on $\mathrm{Wt}_{\mathrm{Big}}$ is given by

$$c_n \mapsto \sum_{i+j=n} c_i \otimes c_j,$$

where by convention we set $c_0 = 1$.

Remark 1.2.2. We can identify the commutative ring $\mathrm{Wt}_{\mathrm{Big}}$ with the cohomology ring $\mathrm{H}^*(\mathrm{BU}; \mathbf{Z})$. Here the identification carries each $c_n \in \mathrm{Wt}_{\mathrm{Big}}$ to the nth Chern class of the tautological (virtual) bundle on BU.

Remark 1.2.3. Every formal power series $1 + c_1 t + c_2 t^2 + \cdots \in R[[t]]$ can be written uniquely in the form

$$\prod_{n>0} (1 - a_n t^n)$$

for some $a_n \in R$. Here we can write each a_n as a polynomial (with integer coefficients) in variables $\{c_m\}_{m>0}$, and each c_m as a polynomial (with integer coefficients) in the variables $\{a_n\}_{n>0}$. Applying this to the universal case $R = \mathrm{Wt}_{\mathrm{Big}}$, we obtain elements $\{a_n \in \mathrm{Wt}_{\mathrm{Big}}\}_{n>0}$ which determine an isomorphism $\mathbf{Z}[a_1, a_2, \ldots] \simeq \mathrm{Wt}_{\mathrm{Big}}$. The element $a_n \in \mathrm{Wt}_{\mathrm{Big}}$ is called the *nth Witt component*.

Notation 1.2.4. Let $f(t) \in \mathrm{Wt}_{\mathrm{Big}}[[t]]$ denote the tautological element of $W(\mathrm{Wt}_{\mathrm{Big}})$, given by the formal power series $1 + c_1 t + c_2 t^2 + \cdots$. Write

$$t \operatorname{dlog}(f(t)) = \frac{tf'(t)}{f(t)} = w_1 t + w_2 t^2 + w_3 t^3 + \cdots$$

for some elements $w_1, w_2, \ldots \in \mathrm{Wt}_{\mathrm{Big}}$. We will refer to w_n as the *nth ghost component*. Note that each w_n is a primitive element of $\mathrm{Wt}_{\mathrm{Big}}$: that is, we have $\Delta(w_n) = w_n \otimes 1 + 1 \otimes w_n$.

Let $\mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}} = \mathbf{Q}[c_1, c_2, \ldots]$ denote the tensor product of $\mathrm{Wt}_{\mathrm{Big}}$ with the rational numbers. In the power series ring $\mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}}[[t]]$, we have the identity

$$\log(f(t)) = \sum_{n>0} \frac{w_n}{n} t^n,$$

so that $f(t) = \exp(\sum \frac{w_n}{n} t^n)$. We therefore have a canonical isomorphism $\mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}} \simeq \mathbf{Q}[w_1, w_2, \ldots]$: in particular, each c_n can be written as a polynomial in the ghost components $\{w_n\}$ with rational coefficients.

Remark 1.2.5. Writing $f(t) = \prod_{n>0}(1 - a_n t^n)$, we obtain the formula

$$\sum_{n>0} \frac{w_n}{n} t^n = \log f(t) = \sum_{m>0} \log(1 - a_m t^m) = \sum_{m>0} \sum_{d>0} \frac{a_m^d t^{md}}{d} = \sum_{n>0} \sum_{d|n} \frac{a_{n/d}^d t^n}{d}.$$

Extracting coefficients, we obtain for each $n > 0$

$$w_n = \sum_{d|n} \frac{n}{d} a_{n/d}^d.$$

In particular, we see that each ghost component w_n can be written as a polynomial in the Witt components $\{a_m\}_{m|n}$ with integer coefficients. Conversely, each Witt component a_n can be written as polynomial in the ghost components $\{w_m\}_{m|n}$ with rational coefficients.

Remark 1.2.6. Let S be a set of positive integers which is closed under divisibility: that is, if $n \in S$ and $d|n$, then $d|S$. Let Wt_S denote the subalgebra of $\mathrm{Wt}_{\mathrm{Big}}$ generated by the Witt components a_n for $n \in S$, and $\mathrm{Wt}_S^{\mathbf{Q}}$ the tensor product $\mathrm{Wt}_S \otimes \mathbf{Q}$. We make the following observations concerning Wt_S:

- An element of $\mathrm{Wt}_{\mathrm{Big}}$ belongs to Wt_S if and only if, when written as a polynomial in the Witt components $\{a_n\}_{n>0}$, the only Witt components which appear (with nonzero coefficients) are those a_n for which $n \in S$. In particular, we have $\mathrm{Wt}_S = \mathrm{Wt}_S^{\mathbf{Q}} \cap \mathrm{Wt}_{\mathrm{Big}}$ (where the intersection is formed in the larger ring $\mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}} \simeq \mathbf{Q}[a_1, a_2, \ldots]$).
- For each $n \in S$, the ghost component w_n is contained in Wt_S. Moreover, $\mathrm{Wt}_S^{\mathbf{Q}}$ is a polynomial algebra (over $\mathbf{Q}$) on the ghost components $\{w_n\}_{n \in S}$ (see Remark 1.2.5).
- Since the antipode of $\mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}}$ carries each w_n to $-w_n$, it preserves the subalgebra $\mathrm{Wt}_S^{\mathbf{Q}}$ and therefore also the subalgebra $\mathrm{Wt}_S = \mathrm{Wt}_S^{\mathbf{Q}} \cap \mathrm{Wt}_{\mathrm{Big}}$.
- If $n \in S$, then $\Delta(w_n) = 1 \otimes w_n + w_n \otimes 1$ belongs to $\mathrm{Wt}_S \otimes \mathrm{Wt}_S \subseteq \mathrm{Wt}_{\mathrm{Big}} \otimes \mathrm{Wt}_{\mathrm{Big}}$. It follows that the comultiplication $\Delta : \mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}} \to \mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}}$ carries $\mathrm{Wt}_S^{\mathbf{Q}}$ into $\mathrm{Wt}_S^{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathrm{Wt}_S^{\mathbf{Q}}$. Since $\mathrm{Wt}_S \otimes \mathrm{Wt}_S \simeq (\mathrm{Wt}_S^{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathrm{Wt}_S^{\mathbf{Q}}) \cap (\mathrm{Wt}_{\mathrm{Big}} \otimes \mathrm{Wt}_{\mathrm{Big}})$, we conclude that the comultiplication on $\mathrm{Wt}_{\mathrm{Big}}$ carries Wt_S into $\mathrm{Wt}_S \otimes \mathrm{Wt}_S$: that is, Wt_S inherits the structure of a Hopf algebra from $\mathrm{Wt}_{\mathrm{Big}}$.

Example 1.2.7. Let p be a prime number. We let Wt_{p^∞} denote the subalgebra $\mathbf{Z}[a_1, a_p, a_{p^2}, \ldots] \subseteq \mathrm{Wt}_{\mathrm{Big}}$. We refer to Wt_{p^∞} as the *Hopf algebra of p-typical Witt vectors.*

We now study the relationship between Witt vectors and the Hopf algebra tensor product introduced in §1.1.

Lemma 1.2.8. *Let $H_1, H_2, \ldots, H_n$ be a finite collection of bialgebras over $\mathbf{Z}$ which are free when regarded as $\mathbf{Z}$-modules. If each H_i is finitely generated as a commutative ring, then $H_1 \boxtimes \cdots \boxtimes H_n$ is finitely generated as a commutative ring.*

Proof. Let $H_i^{\mathbf{Q}}$ denote the tensor product of H_i with the rational numbers. Then $H_i^{\mathbf{Q}}$ is a coalgebra over $\mathbf{Q}$, and can therefore be written as a filtered colimit $\varinjlim H_{i,\alpha}^{\mathbf{Q}}$ of finite-dimensional subcoalgebras $H_{i,\alpha}^{\mathbf{Q}} \subseteq H_i^{\mathbf{Q}}$. For each index α, let $H_{i,\alpha} = H_\alpha^{\mathbf{Q}} \cap H_i$ (where we identify $H_{i,\alpha}^{\mathbf{Q}}$ and H_i with subsets of $H_i^{\mathbf{Q}}$). Since H_i is free as a $\mathbf{Z}$-module, $H_{i,\alpha}$ is finitely generated as a $\mathbf{Z}$-module. Note that $H_{i,\alpha}$ is a subcomodule of H_i. Since H_i is finitely generated as a commutative ring, we may choose α so that H_i is generated (as a commutative ring) by $H_{i,\alpha}$. Set $C_i = H_{i,\alpha}$, so that we have a surjective bialgebra map $\mathrm{Sym}^*(C_i) \to H_i$. We then obtain a surjective bialgebra map $\mathrm{Sym}^*(C_1) \boxtimes \cdots \boxtimes \mathrm{Sym}^*(C_n) \to H_1 \boxtimes \cdots \boxtimes H_n$. Using the isomorphism $\mathrm{Sym}^*(C_1) \boxtimes \cdots \boxtimes \mathrm{Sym}^*(C_n) \simeq \mathrm{Sym}^*(C_1 \otimes \cdots \otimes C_n)$, we deduce that $H_1 \boxtimes \cdots \boxtimes H_n$ is finitely generated as a commutative ring. □

Notation 1.2.9. Let H be a bialgebra over a commutative ring k. We let I_H denote the augmentation ideal of H: that is, the kernel of the counit map $\epsilon : H \to k$. We let $Q(H)$ denote the quotient I_H/I_H^2. Then $Q(H)$ is a k-module, which we will refer as the *k-module of indecomposables of H*. Note that if H is finitely generated as a k-algebra, then $Q(H)$ is finitely generated as a k-module.

Suppose we are given a bialgebra map $\phi : H \to H'$. We let $\mathrm{coker}(\phi)$ denote the quotient of H' by the ideal generated by $\phi(I_H)$. Then $\mathrm{coker}(\phi)$ inherits the structure of a bialgebra: it is the cokernel of the map ϕ in the (pointed) category of k-bialgebras. In the language of affine schemes, we can describe $\mathrm{Spec}\,\mathrm{coker}(\phi)$ as the kernel of the map of commutative monoid schemes $\mathrm{Spec}\, H' \to \mathrm{Spec}\, H$ determined by ϕ. Note that we have an exact sequence of k-modules

$$Q(H) \to Q(H') \to Q(\mathrm{coker}(\phi)) \to 0.$$

Proposition 1.2.10. *Let H be a Hopf algebra which is finitely generated over $\mathbf{Z}$. Assume that for each prime number p, the affine scheme $\mathrm{Spec}\, H/pH$ is connected. The following conditions are equivalent:*

(1) *The Hopf algebra H is smooth as an algebra over $\mathbf{Z}$.*

(2) *The module of indecomposables $Q(H)$ is free.*

(3) *For every prime number p, we have*

$$\dim_{\mathbf{Q}}(Q(H) \otimes_{\mathbf{Z}} \mathbf{Q}) \geq \dim_{\mathbf{F}_p}(Q(H) \otimes_{\mathbf{Z}} \mathbf{F}_p).$$

Proof. Let $\Omega_{H/\mathbf{Z}}$ denote the module of Kähler differentials of H over $\mathbf{Z}$. Then $Q(H) \simeq \Omega_{H/\mathbf{Z}} \otimes_H \mathbf{Z}$, where the tensor product is taken along the counit map $H \to \mathbf{Z}$. If H is smooth over $\mathbf{Z}$, then $\Omega_{H/\mathbf{Z}}$ is a projective H-module of finite rank, so that $Q(H)$ is a projective $\mathbf{Z}$-module of finite rank, and therefore free. This proves (1) ⇒ (2). The implication (2) ⇒ (3) is obvious.

Let us now suppose that (3) is satisfied, and prove (1). We begin by showing that each fiber of the map $\mathrm{Spec}\, H \to \mathrm{Spec}\, \mathbf{Z}$ is smooth. For the generic fiber, this is clear (any algebraic group over a field of characteristic zero is smooth). For the fiber over a prime number p, let Ω denote the sheaf of relative Kähler differentials of the map $G = \mathrm{Spec}\, H/pH \to \mathrm{Spec}\, \mathbf{F}_p$. Then Ω is equivariant with respect to the translation action of G on itself, and therefore a locally free sheaf of rank $r = \dim_{\mathbf{F}_p}(Q(H) \otimes_{\mathbf{Z}} \mathbf{F}_p)$. To prove that G is smooth, it will suffice to show r is equal to the Krull dimension d of G. Let d' denote the Krull dimension of the generic fiber $\mathrm{Spec}(H \otimes_{\mathbf{Z}} \mathbf{Q})$, so that we have inequalities $d' \leq d \leq r$. Condition (3) (and the smoothness of $H \otimes_{\mathbf{Z}} \mathbf{Q}$ over $\mathbf{Q}$) imply that equality holds throughout, so that G is smooth over $\mathbf{F}_p$.

To complete the proof that H is smooth over $\mathbf{Z}$, it will suffice to show that it is flat over $\mathbf{Z}$: that is, that H is torsion-free as an abelian group. Let $I \subseteq H$ denote the torsion submodule of H. Then I is an ideal of H, and therefore finitely generated as an H-module (since H is Noetherian). It follows that there exists an integer $n > 0$ such that $nI = 0$. Choose n as small as possible. We wish to prove that $n = 1$ (so that $I = 0$ and therefore H is torsion-free). Assume otherwise; then we can write $n = pn'$ for some prime number p. The minimality of n implies that $n'I \neq 0 = nI = n'(pI)$, so that the quotient I/pI is nontrivial. Since H/I is torsion-free, we have an exact sequence

$$0 \to I/pI \to H/pH \to H/I \otimes_{\mathbf{Z}} \mathbf{F}_p \to 0.$$

Since $G = \mathrm{Spec}\, H/pH$ is a connected, smooth $\mathbf{F}_p$-scheme of dimension d, any proper closed subscheme of G has dimension $< d$. It follows that $\mathrm{Spec}(H/I \otimes_{\mathbf{Z}} \mathbf{F}_p)$ has dimension $< d$. This is a contradiction, since the generic fiber of $\mathrm{Spec}\, H/I$ coincides with the generic fiber of $\mathrm{Spec}\, H$, which has dimension d. $\square$

Notation 1.2.11. Let R be a commutative ring, and let $n \geq 1$ be an integer. There is a canonical group homomorphism $W_{\mathrm{Big}}(R) \to W_{\mathrm{Big}}(R)$, given on power series by $f(t) \mapsto f(t^n)$. This homomorphism depends functorially on R, and is therefore induced by a Hopf algebra homomorphism

$$V_n : \mathrm{Wt}_{\mathrm{Big}} \to \mathrm{Wt}_{\mathrm{Big}}.$$

We will refer to V_n as the *nth Verschiebung map*. Concretely, it is given by

$$V_n(c_m) = \begin{cases} c_{m/n} & \text{if } n|m \\ 0 & \text{otherwise} \end{cases}$$

or equivalently by

$$V_n(a_m) = \begin{cases} a_{m/n} & \text{if } n|m \\ 0 & \text{otherwise.} \end{cases}$$

On ghost components, the Verschiebung map is given by $V_n(w_m) = \begin{cases} nw_{m/n} & \text{if } n|m \\ 0 & \text{otherwise.} \end{cases}$

Let S and T be subsets of $\mathbf{Z}_{>0}$ which are closed under divisibility. Suppose that for every integer $d > 0$ such that $nd \in S$, we have $d \in T$. Then the Verschiebung map V_n restricts to a Hopf algebra map $\mathrm{Wt}_S \to \mathrm{Wt}_T$, which we will also denote by V_n.

Remark 1.2.12. Let S be a set of integers which is closed under divisibility, and let $n \in S$ be an element which does not divide any other element of S, so that $S' = S - \{n\}$ is also closed under divisibility. Then the Verschiebung map $V_n : \mathrm{Wt}_S \to \mathrm{Wt}_{\{1\}} \simeq \mathbf{Z}[c_1]$ exhibits $\mathrm{Wt}_{\{1\}}$ as a cokernel of the inclusion $\mathrm{Wt}_{S'} \to \mathrm{Wt}_S$, in the category of Hopf algebras over $\mathbf{Z}$.

Proposition 1.2.13. *Let S and T be finite subsets of $\mathbf{Z}_{>0}$ which are closed under divisibility. Then the Hopf algebra $\mathrm{Wt}_S \boxtimes \mathrm{Wt}_T$ is smooth over $\mathbf{Z}$.*

Proof. Let s denote the cardinality of the set S, and t denote the cardinality of the set T. We may assume without loss of generality that $s, t > 0$ (otherwise the result is vacuous). It follows from Lemma 1.2.8 that $H = \mathrm{Wt}_S \boxtimes \mathrm{Wt}_T$ is finitely generated as a $\mathbf{Z}$-algebra. Let us first describe the rationalization $H_{\mathbf{Q}} = H \otimes_{\mathbf{Z}} \mathbf{Q}$. Let C denote the $\mathbf{Q}$-submodule $\mathrm{Wt}_S^{\mathbf{Q}}$ generated by the unit element 1 together with the ghost components $\{w_m\}_{m \in S}$, and define $C' \subseteq \mathrm{Wt}_T^{\mathbf{Q}}$ similarly. Since each ghost component is a primitive element of $\mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}}$, C and C' are coaugmented coalgebras over $\mathbf{Q}$, with coaugmentation given by the unit element. It follows from Remark 1.2.6 that the inclusions

$$C \hookrightarrow \mathrm{Wt}_S^{\mathbf{Q}} \qquad C' \hookrightarrow \mathrm{Wt}_T^{\mathbf{Q}}$$

induce Hopf algebra isomorphisms $\mathrm{Sym}^*_{\mathrm{red}}(C) \simeq \mathrm{Wt}^{\mathbf{Q}}_S$ and $\mathrm{Sym}^*_{\mathrm{red}}(C') \simeq \mathrm{Wt}^{\mathbf{Q}}_T$, so that

$$H_{\mathbf{Q}} \simeq \mathrm{Wt}^{\mathbf{Q}}_S \boxtimes_{\mathbf{Q}} \mathrm{Wt}^{\mathbf{Q}}_T \simeq \mathrm{Sym}^*_{\mathrm{red}}(C) \boxtimes_{\mathbf{Q}} \mathrm{Sym}^*_{\mathrm{red}}(C') \simeq \mathrm{Sym}^*_{\mathrm{red}}(C \wedge C')$$

is a polynomial algebra on generators $w_m \boxtimes w_n$, where $m \in S$ and $n \in T$. In particular, we see that $H_{\mathbf{Q}}$ is a smooth $\mathbf{Q}$-algebra of dimension st.

To complete the proof, it will suffice (by Proposition 1.2.10) to show that for each prime number p, the affine scheme $\mathrm{Spec}\, H/pH$ is connected and has dimension st. We will proceed by induction on the product st. If $st = 1$, then $S = T = \{1\}$ and the desired result follows from Example 1.1.27. Let us therefore assume that $st > 1$. We may assume without loss of generality that $s > 1$. Let n be the largest element of S and let $S' = S - \{n\}$, so that we have a cofiber sequence

$$\mathrm{Wt}_{S'} \to \mathrm{Wt}_S \xrightarrow{V_n} \mathrm{Wt}_1$$

of Hopf algebras over $\mathbf{Z}$. Set $H' = \mathrm{Wt}_{S'} \boxtimes \mathrm{Wt}_T$ and $H'' = \mathrm{Wt}_1 \boxtimes \mathrm{Wt}_T$. We then have a cofiber sequence of Hopf algebras over $\mathbf{F}_p$

$$H'/pH' \to H/pH \to H''/pH''.$$

Set $G = \mathrm{Spec}\, H/pH$, $G' = \mathrm{Spec}\, H'/pH'$, and $G'' = \mathrm{Spec}\, H''/pH''$, so that we have an exact sequence of commutative group schemes over $\mathbf{F}_p$

$$0 \to G'' \to G \xrightarrow{u} G'.$$

The inductive hypothesis implies that G'' and G' are connected smooth group schemes over $\mathbf{F}_p$, having dimensions t and $(s-1)t$, respectively. Since the generic fiber of the map $\mathrm{Spec}\, H \to \mathrm{Spec}\, \mathbf{Z}$ has dimension st, it follows that the dimension of G is at least st. It follows that the image of u is a closed subgroup $G'_0 \subseteq G'$ of dimension at least $\dim(G) - \dim(G'') \geq (s-1)t$. It follows that $G'_0 = G'$: that is, the map u is a flat surjection. Since G'' is smooth, the map u is smooth, so that G is smooth of dimension $\dim(G') + \dim(G'') = st$. The connectedness of G' and G'' now imply the connectedness of G. $\square$

Remark 1.2.14. The proof of Proposition 1.2.13 shows more generally that for any collection $S_1, \ldots, S_k \subseteq \mathbf{Z}_{>0}$ of finite sets which are closed under divisibility, the iterated Hopf algebra tensor product $\mathrm{Wt}_{S_1} \boxtimes \cdots \boxtimes \mathrm{Wt}_{S_k}$ is smooth over $\mathbf{Z}$.

Scholium 1.2.15. Let $S' \subseteq S \subseteq \mathbf{Z}_{>0}$ and $T' \subseteq T \subseteq \mathbf{Z}_{>0}$ be subsets which are closed under divisibility. Then the inclusion maps

$$\mathrm{Wt}_{S'} \hookrightarrow \mathrm{Wt}_S \qquad \mathrm{Wt}_{T'} \hookrightarrow \mathrm{Wt}_T$$

induce a faithfully flat map

$$\phi : \mathrm{Wt}_{S'} \boxtimes \mathrm{Wt}_{T'} \to \mathrm{Wt}_S \boxtimes \mathrm{Wt}_T .$$

Proof. Using a direct limit argument, we can reduce to the case where the sets S and T are finite. Note that the map ϕ factors as a composition

$$\mathrm{Wt}_{S'} \boxtimes \mathrm{Wt}_{T'} \to \mathrm{Wt}_{S'} \boxtimes \mathrm{Wt}_T \to \mathrm{Wt}_S \boxtimes \mathrm{Wt}_T .$$

We may therefore reduce to the case where either $S = S'$ or $T = T'$. Let us assume $T = T'$ (the proof in the other case is the same). Working by induction on the number of elements in $S - S'$, we may reduce to the case where $S = S' \cup \{n\}$, so that we have a cofiber sequence of Hopf algebras

$$\mathrm{Wt}_{S'} \to \mathrm{Wt}_S \xrightarrow{V_n} \mathrm{Wt}_1 .$$

Set $H = \mathrm{Wt}_S \boxtimes \mathrm{Wt}_T$, $H' = \mathrm{Wt}_{S'} \boxtimes \mathrm{Wt}_T$, and $H'' = \mathrm{Wt}_1 \boxtimes \mathrm{Wt}_T$, so that we have a cofiber sequence

$$H' \xrightarrow{\phi} H \to H''.$$

Note that $H \otimes_{\mathbf{Z}} \mathbf{Q}$ is a polynomial ring over $\mathbf{Q}$ with generators given by $\{w_a \boxtimes w_b\}_{a \in S, b \in T}$, and that ϕ identifies $H' \otimes_{\mathbf{Z}} \mathbf{Q}$ with the algebra generated by $\{w_a \boxtimes w_b\}_{a \in S', b \in T}$. It follows that ϕ is flat after tensoring with $\mathbf{Q}$. Using the fiber-by-fiber flatness criterion (Corollary 11.3.11 of [6]), it will suffice to show that ϕ induces a flat map $\phi_p : H'/pH' \to H/pH$ for every prime number p. This was established in the proof of Proposition 1.2.13. □

Corollary 1.2.16. *Let S be a set of positive integers which is closed under divisibility. Then the canonical map $\mathrm{Wt}_S \boxtimes \mathrm{Wt}_S \to \mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$ is injective. Moreover, $\mathrm{Wt}_S \boxtimes \mathrm{Wt}_S$ can be identified with the intersection of $\mathrm{Wt}_S^{\mathbf{Q}} \boxtimes_{\mathbf{Q}} \mathrm{Wt}_S^{\mathbf{Q}}$ with $\mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$ in $\mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}} \boxtimes_{\mathbf{Q}} \mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}}$.*

Proof. Using Scholium 1.2.15, we are reduced to verifying the following general assertion: if $\phi : A \to B$ is a faithfully flat map between torsion free commutative rings, then we can identify A with the intersection of B with $A_{\mathbf{Q}} = A \otimes_{\mathbf{Z}} \mathbf{Q}$ inside $B_{\mathbf{Q}} = B \otimes_{\mathbf{Z}} \mathbf{Q}$. To prove this, suppose we are given an element $x \in A_{\mathbf{Q}}$ whose image in $B_{\mathbf{Q}}$ belongs to B. Choose $n > 0$ so that $y = nx \in A$. Then $\phi(y) \in nB$, so that the image of y vanishes under the map $A/nA \to B/nB$. Since ϕ is faithfully flat, we conclude that $y \equiv 0 \mod n$, so that $x \in A$. □

Remark 1.2.17. Let C denote the $\mathbf{Z}$-submodule of $\mathrm{Wt}_{\mathrm{Big}}$ generated by the elements $\{c_n\}_{n \geq 0}$. Using the formulas

$$\Delta(c_n) = \sum_{i+j=n} c_i \otimes c_j \qquad \epsilon(c_n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

we see that C is a subcoalgebra over $\mathrm{Wt}_{\mathrm{Big}}$, equipped with a coaugmentation given by the grouplike element $1 = c_0 \in C$. The inclusion $C \hookrightarrow \mathrm{Wt}_{\mathrm{Big}}$ extends uniquely to a map $\mathrm{Sym}^*_{\mathrm{red}}(C) \to \mathrm{Wt}_{\mathrm{Big}}$, which is easily seen to be an isomorphism. In other words, we can identify $\mathrm{Wt}_{\mathrm{Big}}$ with the free bialgebra generated by the coaugmented coalgebra C.

Notation 1.2.18. Let H and H' be bialgebras over $\mathbf{Z}$, and let $\nu : H \otimes H' \to H \boxtimes H'$ be the canonical coalgebra map. For every pair of elements $x \in H$, $y \in H'$, we let $x \boxtimes y$ denote the element $\nu(x \otimes y) \in H \boxtimes H'$.

Remark 1.2.19. Combining Remarks 1.2.17 and 1.1.26, we see that the Hopf algebra $\mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$ is freely generated (as a commutative ring) by the images of the elements $\{c_i \boxtimes c_j \in \mathrm{Wt}_{\mathrm{Big}} \otimes \mathrm{Wt}_{\mathrm{Big}}\}_{i,j>0}$. In particular, $\mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$ is a polynomial algebra over $\mathbf{Z}$.

Proposition 1.2.20. *There exists a unique Hopf algebra map $\iota : \mathrm{Wt}_{\mathrm{Big}} \to \mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$ with the property that $\iota(w_n) = \frac{w_n \boxtimes w_n}{n}$.*

Proof. Let us first prove the analogous result working over the field $\mathbf{Q}$, rather than over the integers. Let $\boxtimes_{\mathbf{Q}}$ denote the tensor product operation on bialgebras over $\mathbf{Q}$. Since $\mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}}$ is a polynomial ring generated by the ghost components $\{w_n\}_{n \geq 1}$, there is a unique ring homomorphism $\iota_{\mathbf{Q}} : \mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}} \to \mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}} \boxtimes_{\mathbf{Q}} \mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}}$ satisfying $\iota_{\mathbf{Q}}(w_n) = \frac{w_n \boxtimes w_n}{n}$. We claim that $\iota_{\mathbf{Q}}$ is a bialgebra homomorphism. To prove this, it will suffice to show that $\iota_{\mathbf{Q}}$ carries each w_n to a primitive element of $\mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}} \boxtimes_{\mathbf{Q}} \mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}}$. Equivalently, we must show that $w_n \boxtimes w_n$ is primitive for $n \geq 1$. We now compute

$$\begin{aligned} \Delta(w_n \boxtimes w_n) &= (w_n \otimes 1 + 1 \otimes w_n) \boxtimes (w_n \otimes 1 + 1 \otimes w_n) \\ &= (w_n \boxtimes w_n) \otimes (1 \boxtimes 1) + (1 \boxtimes 1)(w_n \boxtimes w_n) + (1 \boxtimes w_n) \otimes (w_n \boxtimes 1) + (w_n \boxtimes 1) \otimes (1 \boxtimes w_n). \end{aligned}$$

We now conclude by observing that $1 \boxtimes w_n = \epsilon(w_n) = 0$ and $1 \boxtimes 1 = \epsilon(1) = 1$.

Let us now work over the ring $\mathbf{Z}$ of integers. Since $\mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$ is a polynomial ring over $\mathbf{Z}$ (Remark 1.2.19), it is torsion free. We may therefore identify $\mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$ with its image in the $\mathbf{Q}$-bialgebra $(\mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}) \otimes \mathbf{Q} \simeq \mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}} \boxtimes_{\mathbf{Q}} \mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}}$. The first part of proof shows that there is a unique algebra homomorphism $\iota : \mathrm{Wt}_{\mathrm{Big}} \to \mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}} \boxtimes_{\mathbf{Q}} \mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}}$ satisfying $\iota(w_n) = \frac{w_n \boxtimes w_n}{n}$. We will complete the proof by showing that ι factors through $\mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$.

If R is a commutative ring and $G(x,y) \in R[[x,y]]$ is a power series in two variables given by $G(x,y) = \sum_{i,j \geq 0} \lambda_{i,j} x^i y^j$, we let $G^\delta(t) = \sum_{i \geq 0} \lambda_{i,i} t^i$ denote the "diagonal part" of G. Write $f(t) = 1 + c_1 t + c_2 t^2 + \cdots \in \mathrm{Wt}_{\mathrm{Big}}[[t]]$, so that $\log f(t) \in \mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}}[[t]]$ is given by $\sum_{n \geq 1} \frac{w_n}{n} t^n$. It follows that

$$\iota_{\mathbf{Q}}(\log f(t)) = \sum_{n \geq 1} \iota(\frac{w_n}{n}) t^n = \sum_{n \geq 1} (\frac{w_n}{n} \boxtimes \frac{w_n}{n}) t^n = F^\delta(t),$$

where $F^\delta(x,y) = \log f(x) \boxtimes \log f(y)$. Since $\iota_{\mathbf{Q}}$ is a ring homomorphism, we obtain

$$\iota(f(t)) = \iota_{\mathbf{Q}}(f(t)) = \exp(\iota_{\mathbf{Q}} \log f(t)) = \exp(F^\delta(t)).$$

We wish to show that each coefficient of this power series belongs to $\mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$.

Let I denote the augmentation ideal of $\mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}}[[y]]$. Since $\log f(x)$ is a primitive element of $\mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}}[[x]]$, the construction $g(y) \mapsto \log f(x) \boxtimes g(y)$ annihilates the ideal I^2. Since $f(y) \in 1 + I$, we have $\log f(y) \equiv f(y) - 1 \mod I^2$. It follows that $\log f(x) \boxtimes \log f(y) = \log f(x) \boxtimes (f(y) - 1) = \log f(x) \boxtimes f(y)$. Since $f(y)$ is a grouplike element of $\mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}}[[y]]$, the construction $g(x) \mapsto g(x) \boxtimes f(y)$ is a ring homomorphism. It follows that

$$(\log f(x)) \boxtimes (\log f(y)) = (\log f(x)) \boxtimes f(y) = \log(f(x) \boxtimes f(y)).$$

Note that the coefficients of the power series $f(x) \boxtimes f(y)$ belong to $\mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$. To complete the proof, it will suffice to verify the following:

- $(*)$ Let R be a torsion-free ring and let $R_{\mathbf{Q}} = R \otimes \mathbf{Q}$. Let $H(x,y) \in R[[x,y]]$ be a power series with constant term 1, let $G(x,y) = \log H(x,y) \in R_{\mathbf{Q}}[[x,y]]$, and let $G^\delta(t) \in R_{\mathbf{Q}}[[t]]$ be defined as above. Then $\exp G^\delta(t) \in R[[t]]$.

To prove $(*)$, we can write $H(x,y)$ formally as a product $\prod_{i,j}(1 + \lambda_{i,j} x^i y^j$, where $\lambda_{i,j} \in R$ and the product is taken over all pairs $(i,j) \in \mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$ such that $(i,j) \neq (0,0)$. Then $G(x,y) = \sum_{i,j} \log(1 + \lambda_{i,j} x^i y^j)$, so that $G^\delta(t) = \sum_{i > 0} \log(1 + \lambda_i t^i)$. It follows that $\exp(G^\delta(t)) = \prod_{i > 0}(1 + \lambda_i t^i)$ has coefficients in R, as desired. ◻

Corollary 1.2.21. *Let S be a subset of $\mathbf{Z}_{>0}$ which is closed under divisibility. Then there exists a unique Hopf algebra map $\iota_S : \mathrm{Wt}_S \to \mathrm{Wt}_S \boxtimes \mathrm{Wt}_S$ with the property that $\iota_S(w_n) = \frac{w_n \boxtimes w_n}{n}$ for $n \in S$.*

Proof. Since $\mathrm{Wt}_S \boxtimes \mathrm{Wt}_S$ is flat over $\mathbf{Z}$ (Scholium 1.2.15), the uniqueness can be checked after tensoring with $\mathbf{Q}$, where it follows from the observation that $\mathrm{Wt}_S^{\mathbf{Q}}$ is a polynomial ring on generators $\{w_n\}_{n \in S}$. Using Scholium 1.2.15, we can identify $\mathrm{Wt}_S \boxtimes \mathrm{Wt}_S$ with its image in $\mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$. Let $\iota : \mathrm{Wt}_{\mathrm{Big}} \to \mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$ be as in Proposition 1.2.20. To prove the existence of ι_S, it will suffice to show that ι carries $\mathrm{Wt}_S \subseteq \mathrm{Wt}_{\mathrm{Big}}$ into $\mathrm{Wt}_S \boxtimes \mathrm{Wt}_S \subseteq \mathrm{Wt}_{\mathrm{Big}} \boxtimes \mathrm{Wt}_{\mathrm{Big}}$. Using Corollary 1.2.16, we are reduced to proving that the image of Wt_S in $\mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}} \boxtimes_{\mathbf{Q}} \mathrm{Wt}_{\mathrm{Big}}^{\mathbf{Q}}$ is contained in $\mathrm{Wt}_S^{\mathbf{Q}} \boxtimes_S \mathrm{Wt}_S^{\mathbf{Q}}$, which follows immediately from the formula $\iota(w_n) = \frac{w_n \boxtimes w_n}{n}$. ◻

Remark 1.2.22. Let S be a subset of $\mathbf{Z}_{>0}$ which is closed under divisibility. Then the diagram

$$\begin{tikzcd}
\mathrm{Wt}_S \arrow[r, "\iota_S"] \arrow[d, "\iota_S"] & \mathrm{Wt}_S \boxtimes \mathrm{Wt}_S \arrow[d, "\mathrm{id} \boxtimes \iota_S"] \\
\mathrm{Wt}_S \boxtimes \mathrm{Wt}_S \arrow[r, "\iota_S \boxtimes \mathrm{id}"] & \mathrm{Wt}_S \boxtimes \mathrm{Wt}_S \boxtimes \mathrm{Wt}_S
\end{tikzcd}$$

commutes. By virtue of Remark 1.2.14, it suffices to check this after tensoring with $\mathbf{Q}$, where it follows from the observation that both maps are given by

$$w_n \mapsto \frac{w_n \boxtimes w_n \boxtimes w_n}{n^2}.$$

Remark 1.2.23. Let $S, T \subseteq \mathbf{Z}_{>0}$ be subsets which are closed under divisibility, let n be a positive integer, and suppose that whenever $nm \in S$, we have $m \in T$. Then the diagram of Hopf algebras

$$\begin{tikzcd} \mathrm{Wt}_S \arrow[r, "\iota_S"] \arrow[d, "V_n"] & \mathrm{Wt}_S \boxtimes \mathrm{Wt}_S \arrow[d, "\iota_S \boxtimes \iota_S"] \\ \mathrm{Wt}_T \arrow[r, "\iota_T"] & \mathrm{Wt}_T \boxtimes \mathrm{Wt}_T \end{tikzcd}$$

commutes. To prove this, we may work rationally; it now suffices to observe that both compositions are given by

$$w_m \mapsto \begin{cases} \frac{n^2}{m}(w_{m/n} \boxtimes w_{m/n}) & \text{if } n|m \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.2.24. Let $S = \{1\}$. Then the map $\iota_S : \mathrm{Wt}_S \to \mathrm{Wt}_S \boxtimes \mathrm{Wt}_S$ can be identified with the isomorphism $\mathbf{Z}[t] \simeq \mathbf{Z}[t] \boxtimes \mathbf{Z}[t]$ of Example 1.1.27.

1.3 Dieudonne Modules

Throughout this section, we fix a field κ of characteristic $p > 0$. Recall that a Hopf algebra H over κ is said to be *connected* if the Hopf algebra $\overline{\kappa} \otimes_\kappa H$ does not contain any nontrivial grouplike elements, where $\overline{\kappa}$ is an algebraic closure of κ (Definition 1.0.6). We let $\mathbf{Hopf}_\kappa^c$ denote the full subcategory of $\mathbf{Hopf}_\kappa$ spanned by the connected Hopf algebras over κ. In this section, we will review the theory of Dieudonne modules, which provides a fully faithful embedding

$$\mathrm{DM} : \mathbf{Hopf}_\kappa^c \hookrightarrow \mathbf{LMod}_{\mathrm{D}_\kappa},$$

where $\mathbf{LMod}_{\mathrm{D}_\kappa}$ denotes the category of left modules over a certain noncommutative ring $\mathrm{D}_\kappa \simeq W(\kappa)[F, V]$. Our main goal is to prove a result of Goerss, which asserts that DM is a (nonunital) symmetric monoidal functor (see Theorem 1.3.28).

Notation 1.3.1. For each integer $n \geq 0$, let Wt_n^κ denote the Hopf algebra over κ given by

$$\kappa \otimes_{\mathbf{Z}} \mathrm{Wt}_{\{1,p,p^2,\ldots,p^{n-1}\}},$$

where $\mathrm{Wt}_{\{1,\ldots,p^{n-1}\}}$ is defined as in Remark 1.2.6. We will refer to Wt_n^κ as the *Hopf algebra of n-truncated p-typical Witt vectors*. We can write $\mathrm{Wt}_n^\kappa = \kappa[a_1, \ldots, a_{p^{n-1}}]$, where the Witt components a_i are defined as in Remark 1.2.3.

Notation 1.3.2. Let H be a Hopf algebra over κ. For every integer n, we let $[n] : H \to H$ be the Hopf algebra homomorphism which classifies the map of group schemes $\operatorname{Spec} H \to \operatorname{Spec} H$ given by multiplication by n. If $n \geq 0$, this map is given by the composition

$$H \to H^{\otimes n} \to H,$$

where the first map is given by iterated comultiplication and the second by iterated multiplication.

We let $H^{(p)} \in \mathbf{Hopf}_\kappa$ denote the base change of H along the Frobenius isomorphism $\varphi : \kappa \to \kappa$. Then we have a canonical isomorphism of commutative rings $H \simeq H^{(p)}$, which we will denote by $x \mapsto x^{(p)}$. The κ-algebra structure on $H^{(p)}$ is then characterized by the formula $\lambda^p x^{(p)} = (\lambda x)^{(p)}$, where $\lambda \in \kappa$ and $x \in H$.

There is a canonical Hopf algebra homomorphism $F : H^{(p)} \to H$, given by $x^{(p)} \mapsto x^p$. We will refer to F as the *Frobenius map*. A dual construction yields a Hopf algebra homomorphism $V : H \to H^{(p)}$, called the *Verschiebung map*. The composite maps

$$H \xrightarrow{V} H^{(p)} \xrightarrow{F} H$$
$$H^{(p)} \xrightarrow{F} H \xrightarrow{V} H^{(p)}$$

are given by the Hopf algebra homomorphisms $[p]$ on H and $H^{(p)}$, respectively.

In the special case where $H = \mathrm{Wt}_n^{\kappa}$, we have a canonical isomorphism $H \simeq H^{(p)}$ (since H is defined over the prime field $\mathbf{F}_p$). Under this isomorphism, the Verscheibung map $V : H \to H$ agrees with the Verscheibung map V_p of Notation 1.2.11. In particular, it is given by the formula

$$V(a_{p^i}) = \begin{cases} 0 & \text{if } i = 0 \\ a_{p^{i-1}} & \text{if } i > 0. \end{cases}$$

Definition 1.3.3. Let H be a connected Hopf algebra over κ. For each integer $n \geq 1$, we let $\mathrm{DM}(H)_n$ denote the set of all elements $x \in H$ which satisfy the following condition: there exists a Hopf algebra homomorphism $f : \mathrm{Wt}_n^{\kappa} \to H$ such that $f(a_{p^{n-1}}) = x$. We let $\mathrm{DM}(H)$ denote the union $\bigcup_{n>0} \mathrm{DM}(H)_n$. We will refer to $\mathrm{DM}(H)$ as the *Dieudonne module* of H.

Remark 1.3.4. Let H be a connected Hopf algebra over κ. By definition, evaluation at $a_{p^{n-1}} \in \mathrm{Wt}_n^{\kappa}$ induces a surjective map

$$\mathrm{Hom}_{\mathbf{Hopf}_{\kappa}}(\mathrm{Wt}_n^{\kappa}, H) \to \mathrm{DM}(H)_n.$$

In fact, this map is bijective: that is, a Hopf algebra homomorphism $f : \mathrm{Wt}_n^{\kappa} \to H$ is determined by the element $f(a_{p^{n-1}}) \in H$. This is clear, since Wt_n^{κ} is generated as an algebra by the elements

$$a_{p^{n-1}}, V a_{p^{n-1}} = a_{p^{n-2}}, V^2 a_{p^{n-1}} = a_{p^{n-3}}, \ldots, V^{n-1} a_{p^{n-1}} = a_1.$$

Remark 1.3.5. Let H be a connected Hopf algebra over κ. Then we have inclusions

$$\mathrm{DM}(H)_1 \subseteq \mathrm{DM}(H)_2 \subseteq \mathrm{DM}(H)_3 \subseteq \cdots.$$

To prove this, we note that if $x \in \mathrm{DM}(H)_n$, then there exists a Hopf algebra map $f : \mathrm{Wt}_n^{\kappa} \to H$ with $f(a_{p^{n-1}}) = x$. Then x is the image of $a_{p^n} \in \mathrm{Wt}_{n+1}^{\kappa}$ under the composite map

$$\mathrm{Wt}_{n+1}^{\kappa} \xrightarrow{V_p} \mathrm{Wt}_n^{\kappa} \xrightarrow{f} H,$$

so that $x \in \mathrm{DM}(H)_{n+1}$.

Remark 1.3.6. Let H be a connected Hopf algebra over κ. For each $n \geq 1$, the identification $\mathrm{DM}(H)_n \simeq \mathrm{Hom}_{\mathbf{Hopf}_{\kappa}}(\mathrm{Wt}_n^{\kappa}, H)$ determines an abelian group structure on $\mathrm{DM}(H)_n$. Moreover, the inclusions

$$\mathrm{DM}(H)_1 \subseteq \mathrm{DM}(H)_2 \subseteq \ldots$$

are group homomorphisms, so that the union $\mathrm{DM}(H)$ inherits the structure of an abelian group.

Example 1.3.7. Let H be a connected Hopf algebra over κ. Then $\mathrm{DM}(H)_1$ is the subset $\mathrm{Prim}(H) \subseteq H$ consisting of primitive elements. Consequently, we can identify $\mathrm{Prim}(H)$ with a subset of the Dieudonne module $\mathrm{DM}(H)$. Moreover, this identification is additive: that is, the addition on $\mathrm{DM}(H)_1$ described in Remark 1.3.6 agrees with the usual addition in H.

Remark 1.3.8. Let H be a connected Hopf algebra over κ. If $f : \mathrm{Wt}_n^{\kappa} \to H$ is a Hopf algebra homomorphism, then f induces another Hopf algebra homomorphism

$$f^{(p)} : \mathrm{Wt}_n^{\kappa} \simeq (\mathrm{Wt}_n^{\kappa})^{(p)} \to H^{(p)},$$

satisfying $f^{(p)}(a_{p^{n-1}}) = f(a_{p^{n-1}})^{(p)}$. It follows that the construction $x \mapsto x^{(p)}$ determines a bijection $\mathrm{DM}(H) \to \mathrm{DM}(H^{(p)})$.

We now investigate the structure of the Dieudonne module $\mathrm{DM}(H)$.

Construction 1.3.9. The construction $H \mapsto \mathrm{DM}(H)$ is functorial in H: that is, if $f : H \to H'$ is a map of Hopf algebras over κ, then f induces a group homomorphism from $\mathrm{DM}(H)$ to $\mathrm{DM}(H')$. In particular, the Frobenius and Verschiebung maps

$$F : H^{(p)} \to H \qquad V : H \to H^{(p)}$$

induce maps

$$\mathrm{DM}(H) \simeq \mathrm{DM}(H^{(p)}) \to \mathrm{DM}(H) \qquad \mathrm{DM}(H) \to \mathrm{DM}(H^{(p)}) \simeq \mathrm{DM}(H),$$

which we will also denote by F and V, respectively.

More concretely, for $x \in \mathrm{DM}(H) \subseteq H$, we have $Fx = x^p \in \mathrm{DM}(H) \subseteq H$. If $x = f(a_{p^{n-1}})$ for some Hopf algebra homomorphism $f : \mathrm{Wt}_n^\kappa \to H$, then $Vx = f(Va_{p^{n-1}}) = f(a_{p^{n-2}})$.

Remark 1.3.10. Let H be a connected Hopf algebra over κ, and let $x \in \mathrm{DM}(H)$ be represented by a Hopf algebra homomorphism $f : \mathrm{Wt}_n^\kappa \to H$. For $m \leq n$, we have an exact sequence of Hopf algebras over κ

$$\mathrm{Wt}_{n-m}^\kappa \hookrightarrow \mathrm{Wt}_n^\kappa \overset{V^{n-m}}{\to} \mathrm{Wt}_m^\kappa .$$

Note that x belongs to $\mathrm{DM}(H)$ if and only if f factors through the map $V^{n-m} : \mathrm{Wt}_\kappa^n \to \mathrm{Wt}_m^\kappa$, which is equivalent to the requirement that the restriction $f|\,\mathrm{Wt}_{n-m}^\kappa$ is trivial. In other words, $x \in \mathrm{DM}(H)$ if and only if $f(a_{p^{n-m-1}}) = 0$. Since $f(a_{p^{n-m-1}}) = V^m f(a_{p^{n-1}}) = V^m x$, we conclude that $\mathrm{DM}(H)_m$ is the kernel of the map $V^m : \mathrm{DM}(H) \to \mathrm{DM}(H)$. In particular, we can identify $\mathrm{Prim}(H) = \mathrm{DM}(H)_1$ with the kernel of the map $V : \mathrm{DM}(H) \to \mathrm{DM}(H)$.

Notation 1.3.11. Let $W(\kappa)$ denote the ring of p-typical Witt vectors of the field κ. For each $x \in \kappa$, we let $\tau(x) \in W(\kappa)$ denote its Teichmüller representative. Let $\varphi : W(\kappa) \to W(\kappa)$ denote the Frobenius map: that is, the unique ring homomorphism from $W(\kappa)$ to itself satisfying $\varphi(\tau(x)) = \tau(x^p)$.

Construction 1.3.12. For each integer $n \geq 1$, the affine scheme $\operatorname{Spec} \mathrm{Wt}_n^\kappa$ is a ring-scheme, whose ring of κ-points is given by the quotient $W(\kappa)/p^n$. In particular, the commutative ring $W(\kappa)/p^n$ acts on the Hopf algebra Wt_n^κ (as an object of the abelian category $\mathbf{Hopf}_\kappa$). We will regard Wt_n^κ as a module over the ring $W(\kappa)$ via the composite map

$$W(\kappa) \overset{\phi^{n-1}}{\to} W(\kappa) \to W(\kappa)/p^n W(\kappa).$$

For each $a \in W(\kappa)$, we let $[a] : \mathrm{Wt}_n^\kappa \to \mathrm{Wt}_n^\kappa$ denote the corresponding endomorphism of Wt_n^κ (note that when λ is an integer, this agrees with Notation 1.3.2). The action of $W(\kappa)$ on Wt_n^κ is uniquely characterized by the formula

$$[\tau(\lambda)](a_{p^i}) = \lambda^{p^{1+i-n}} a_{p^i}$$

for $\lambda \in \kappa$.

For every connected Hopf algebra H, the action of $W(\kappa)$ on Wt_n^κ determines an action of $W(\kappa)$ on the abelian group

$$\mathrm{DM}(H)_n \simeq \mathrm{Hom}_{\mathbf{Hopf}_\kappa}(\mathrm{Wt}_n^\kappa, H).$$

This action is normalized so that for each $\lambda \in \kappa$, the map $\tau(\lambda) : \mathrm{DM}(H)_n \to \mathrm{DM}(H)_n$ is given by multiplication by λ. Note that the inclusions $\mathrm{DM}(H)_n \hookrightarrow \mathrm{DM}(H)_{n+1}$ are $W(\kappa)$-linear, so that $\mathrm{DM}(H)$ inherits the structure of a $W(\kappa)$-module.

Remark 1.3.13. The action of $W(\kappa)/p^n$ on Wt_n^κ determines, by composition with the Teichmüller map $\tau : \kappa \to W(\kappa)/p^n$, an action of the multiplicative group $\kappa^\times$ on Wt_n^κ. Unwinding the definitions, we see that this action is determined by the grading of Wt_n^κ, where we regard a_{p^i} as a homogeneous element of degree p^i.

Notation 1.3.14. We let D_κ denote the associative ring generated by $W(\kappa)$ together with symbols F and V, subject to the relations

$$F\lambda = \varphi(\lambda)F \qquad V\varphi(\lambda) = \lambda V \qquad FV = VF = p,$$

where λ ranges over $W(\kappa)$. We will refer to D_κ as the *Dieudonne ring.* We let $\mathbf{LMod}_{\mathrm{D}_\kappa}$ denote the category of (discrete) left modules over D_κ.

We will say that a left D_κ-module M is *V-nilpotent* if, for every element $x \in M$, we have $V^n x = 0$ for $n \gg 0$. Let $\mathbf{LMod}^{\mathrm{Nil}}_{\mathrm{D}_\kappa}$ denote the full subcategory of $\mathbf{LMod}_{\mathrm{D}_\kappa}$ spanned by the V-nilpotent D_κ-modules.

Proposition 1.3.15. *Let H be a connected Hopf algebra over κ. Then the maps $F, V : \mathrm{DM}(H) \to \mathrm{DM}(H)$ and the action of $W(\kappa)$ on $\mathrm{DM}(H)$ exhibit $\mathrm{DM}(H)$ as a V-nilpotent left module over the Dieudonne ring D_κ of Notation 1.3.14.*

Proof. If $x \in \mathrm{DM}(H)$, the equalities $VFx = px = FVx$ follow from Notation 1.3.2. It follows from Remark 1.3.10 that if $x \in \mathrm{DM}(H)_n$, then x is annihilated by V^n. Let $\lambda \in W(\kappa)$; we wish to show that

$$F(\lambda x) = \varphi(\lambda)Fx \qquad V\varphi(\lambda)x = \lambda Vx.$$

We will prove the first equality; the proof of the second is similar. Assume that $x \in \mathrm{DM}(H)_n$, so that x is annihilated by V^n and therefore also by p^n. To verify the equality $F(\lambda x) = \varphi(\lambda)Fx$, we may replace λ by any element of $\lambda + p^n W(\kappa)$. We may therefore assume without loss of generality that λ has a finite Teichmüller expansion

$$\lambda = \sum_{0 \leq i < n} p^i \tau(a_i)$$

for some $a_i \in \kappa$. We may therefore reduce to the case where $\lambda = \tau(a)$ for some $a \in \kappa$. In this case, we have $\varphi(\lambda) = \tau(a^p)$, and the desired equality reduces to the formula $(ax)^p = a^p x^p$. $\square$

We refer the reader to [3] for a proof of the following result:

Theorem 1.3.16. *The construction $H \mapsto \mathrm{DM}(H)$ determines an equivalence of categories $\mathrm{DM} : \mathbf{Hopf}^c_\kappa \to \mathbf{LMod}^{\mathrm{Nil}}_{\mathrm{D}_\kappa}$.*

Remark 1.3.17. For each $n \geq 0$, the functor DM carries Wt^κ_n to the module $\mathrm{D}_\kappa / \mathrm{D}_\kappa V^n$. Since the category of V-nilpotent left D_κ-modules is generated under small colimits by the objects $\mathrm{D}_\kappa / \mathrm{D}_\kappa V^n$, Theorem 1.3.16 implies that $\mathbf{Hopf}^c(\kappa)$ is generated under small colimits by the Hopf algebras Wt^κ_n.

Corollary 1.3.18. *Let H be a connected Hopf algebra over κ. Then H is generated as an algebra by the subset $\mathrm{DM}(H) \subseteq H$.*

Proof. Let H' denote the subalgebra of H generated by $\mathrm{DM}(H)$. We first claim that the comultiplication of H restricts to a comultiplication on H'. To prove this, we let $A \subseteq H$ denote the inverse image of $H' \otimes H'$ under the comultiplication map $\Delta : H \to H \otimes H$. Then A is a subalgebra of H. Consequently, to prove that $H' \subseteq A$, it will suffice to show that $\mathrm{DM}(H) \subseteq A$. Let $x \in \mathrm{DM}(H)_n$, so that x is the image of $a_{p^{n-1}} \in \mathrm{Wt}^\kappa_n$ under some Hopf algebra homomorphism $\phi : \mathrm{Wt}^\kappa_n \to H$. Then $\Delta(x) = (\phi \otimes \phi)\Delta(a_{p^{n-1}})$. Consequently, to show that $\Delta(x) \in H' \otimes H'$, it will suffice to show that ϕ carries Wt^κ_n into H'. Since Wt^κ_n is generated by the elements a_{p^i} for $0 \leq i < n$, we are reduced to proving that $\phi(a_{p^i}) \in H'$ for $0 \leq i < n$. This is clear, since $\phi(a_{p^i}) = V^{n-1-i}x \in \mathrm{DM}(H)$.

Since $\mathrm{DM}(H)$ is closed under the antipodal map from H to itself, the subalgebra H' is also invariant under the antipodal map, and is therefore a sub-Hopf algebra of H. By construction, we have $\mathrm{DM}(H') = \mathrm{DM}(H)$. It follows from Theorem 1.3.16 that the inclusion $H' \hookrightarrow H$ is an isomorphism, so that H is generated by $\mathrm{DM}(H)$. $\square$

Remark 1.3.19. Let H be a Hopf algebra over κ. The inclusion $\rho_H : \mathrm{DM}(H) \hookrightarrow H$ is generally not additive. Given classes $x, x' \in \mathrm{DM}(H)$, represented by Hopf algebra homomorphisms $\phi, \phi' : \mathrm{Wt}_n^\kappa \to H$, we can identify $\rho_H(x + x')$ with the image of $a_{p^{n-1}}$ under the composite map

$$\psi : \mathrm{Wt}_n^\kappa \xrightarrow{\Delta} \mathrm{Wt}_n^\kappa \otimes_\kappa \mathrm{Wt}_n^\kappa \xrightarrow{\phi \otimes \phi'} H \otimes_\kappa H \xrightarrow{m} H.$$

Let $\mathfrak{m}_H$ denote the augmentation ideal of H, and let $\mathfrak{n}$ denote the augmentation ideal of $\mathrm{DM}(H)$. Then $\Delta(a_{p^{n-1}}) - a_{p^{n-1}} \otimes 1 - 1 \otimes a_{p^{n-1}} \in \mathfrak{n} \otimes \mathfrak{n}$, we have $\rho_H(x+y) - \rho_H(x) - \rho_H(y) \in \mathfrak{m}_H^2$. Consequently, ρ_H induces an additive map $\mathrm{DM}(H) \to \mathfrak{m}_H/\mathfrak{m}_H^2$. Note that if $x \in \mathrm{DM}(H)$, then $\rho_H(Fx) = \rho_H(x)^p \in \mathfrak{m}_H^p \subseteq \mathfrak{m}_H^2$. Consequently, ρ_H descends to a map

$$\overline{\rho}_H : \mathrm{DM}(H)/F\,\mathrm{DM}(H) \to \mathfrak{m}_H/\mathfrak{m}_H^2.$$

Using the description of the action of $\tau(\kappa) \subseteq W(\kappa)$ on $\mathrm{DM}(H)$ supplied by Construction 1.3.12, we see that $\overline{\rho}$ is κ-linear.

Proposition 1.3.20. *Let H be a connected Hopf algebra over κ. Then the map $\overline{\rho}_H : \mathrm{DM}(H)/F\,\mathrm{DM}(H) \to \mathfrak{m}_H/\mathfrak{m}_H^2$ of Remark 1.3.19 is an isomorphism of vector spaces over κ.*

Proof. Let $\mathcal{C}$ denote the full subcategory of $\mathbf{LMod}_{\mathrm{D}_\kappa}^{\mathrm{Nil}}$ spanned by those Dieudonne modules M for which there exists a connected Hopf algebra H with $\mathrm{DM}(H) \simeq M$ and $\overline{\rho}_H : \mathrm{DM}(H)/F\,\mathrm{DM}(H) \to \mathfrak{m}_H/\mathfrak{m}_H^2$ an isomorphism. Using Theorem 1.3.16, we see that $\mathcal{C}$ is closed under the formation of colimits. Consequently, it will suffice to show that $\mathcal{C}$ contains every Dieudonne module of the form $\mathrm{D}_\kappa / V^n\,\mathrm{D}_\kappa$. In other words, it suffices to treat the case where $H = \mathrm{Wt}_n^\kappa$. In this case, $\mathrm{DM}(H)$ is generated by an element x (corresponding to identity map in $\mathrm{Hom}_{\mathbf{Hopf}_\kappa}(\mathrm{Wt}_n^\kappa, H))$ satisfying $V^n x = 0$, and $\mathrm{DM}(H)/F\,\mathrm{DM}(H)$ has a basis (as a κ-vector space) given by the images of the elements $x, Vx, V^2x, \ldots, V^{n-1}x$. Unwinding the definitions, we ahve $\rho_H(V^i x) = a_{p^{n-1-i}} \in H$, from which we immediately deduce that $\overline{\rho}_H$ carries the images of $x, Vx, \ldots, V^{n-1}x$ to a basis for $\mathfrak{m}_H/\mathfrak{m}_H^2$. ☐

We now investigate the relationship between the theory of Dieudonne modules and the Hopf algebra tensor product introduced in §1.1.

Proposition 1.3.21. *Let H and H' be connected Hopf algebras over κ. Then $H \boxtimes H'$ is connected.*

Proof. Without loss of generality, we may assume that κ is algebraically closed. In this case, every Hopf algebra K over κ splits as a tensor product $K^c \otimes_\kappa K^d$, where K^c is connected and $K^d \simeq \kappa[M]$ is the group algebra of some abelian group M. Consequently, if $H \boxtimes H'$ is not connected, then there exists an abelian group M and a nontrivial Hopf algebra homomorphism $H \boxtimes H' \to \kappa[M]$. In particular, there must be a nontrivial coalgebra map $H \otimes_\kappa H' \to \kappa[M]$. It follows that the formal scheme $\mathrm{Spf}(H \otimes_\kappa H')^\vee \simeq \mathrm{Spf}\, H^\vee \times \mathrm{Spf}\, H'^\vee$ is disconnected, contracting our assumption that H and H' are connected. ☐

Notation 1.3.22. Let $\mathbf{Hopf}_\kappa^c$ denote the full subcategory of $\mathbf{Hopf}_\kappa$ spanned by the connected Hopf algebras over κ. It follows from Proposition 1.3.21 that $\mathbf{Hopf}_\kappa^c$ has the structure of a nonunital symmetric monoidal category, with tensor product given by $\boxtimes$.

Warning 1.3.23. The unit object for the operation $\boxtimes$ on $\mathbf{Hopf}_\kappa$ is the Hopf algebra of Laurent polynomials $\kappa[t^{\pm 1}]$, which is *not* connected.

Definition 1.3.24. Let M, M', and M'' be left D_κ-modules. A *pairing* of M and M' into M'' is a $W(\kappa)$-bilinear map

$$\mu : M \times M' \to M''$$

satisfying the identities

$$V\mu(x,y) = \mu(Vx, Vy) \qquad F\mu(x, Vy) = \mu(Fx, y) \qquad F\mu(Vx, y) = \mu(x, Fy).$$

Remark 1.3.25. Let M and M' be left D_κ-modules. Then there exists a left D_κ-module $M \widetilde{\otimes} M'$ and a pairing $\mu_0 : M \times M' \to M \widetilde{\otimes} M'$ with the following universal property: for every left D_κ-module M'', composition with μ_0 induces a bijection from the set $\mathrm{Hom}_{\mathbf{LMod}_{\mathrm{D}_\kappa}}(M \widetilde{\otimes} M', M'')$ to the set of pairings $M \times M' \to M''$.

Note that the module $M \widetilde{\otimes} M'$ depends functorially on M and M'. Moreover, the construction $(M, M') \mapsto M \widetilde{\otimes} M'$ preserves small colimits in each variable (in particular, it is right exact in each variable).

Construction 1.3.26. For each $n \geq 0$, let $\iota_n : \mathrm{Wt}_n^\kappa \to \mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa$ be the map given by Corollary 1.2.21. Given a pair of objects $H, H' \in \mathbf{Hopf}_\kappa^c$, we let

$$\mu_n : \mathrm{Hom}_{\mathbf{Hopf}_\kappa}(\mathrm{Wt}_n^\kappa, H) \times \mathrm{Hom}_{\mathbf{Hopf}_\kappa}(\mathrm{Wt}_n^\kappa, H') \to \mathrm{Hom}_{\mathbf{Hopf}_\kappa}(\mathrm{Wt}_n^\kappa, H \boxtimes H')$$

denote the composition of the evident map

$$\mathrm{Hom}_{\mathbf{Hopf}_\kappa}(\mathrm{Wt}_n^\kappa, H) \times \mathrm{Hom}_{\mathbf{Hopf}_\kappa}(\mathrm{Wt}_n^\kappa, H') \to \mathrm{Hom}_{\mathbf{Hopf}_\kappa}(\mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa\, H \boxtimes H')$$

with the map $\mathrm{Hom}_{\mathbf{Hopf}_\kappa}(\mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa\, H \boxtimes H') \to \mathrm{Hom}_{\mathbf{Hopf}_\kappa}(\mathrm{Wt}_n^\kappa, H \boxtimes H')$ given by composition with ι_n. Let $V : \mathrm{Wt}_{n+1}^\kappa \to \mathrm{Wt}_n^\kappa$ be the Verschiebung map (see Notation 1.2.11). Using Remark 1.2.23, we see that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Hopf}_\kappa}(\mathrm{Wt}_n^\kappa, H) \times \mathrm{Hom}_{\mathbf{Hopf}_\kappa}(\mathrm{Wt}_n^\kappa, H') \arrow[r, "\mu_n"] \arrow[d, "\circ V \times V"] & \mathrm{Hom}_{\mathbf{Hopf}_\kappa}(\mathrm{Wt}_n^\kappa, H \boxtimes H') \arrow[d, "\circ V"] \\
\mathrm{Hom}_{\mathbf{Hopf}_\kappa}(\mathrm{Wt}_{n+1}^\kappa, H) \times \mathrm{Hom}_{\mathbf{Hopf}_\kappa}(\mathrm{Wt}_{n+1}^\kappa, H') \arrow[r, "\mu_{n+1}"] & \mathrm{Hom}_{\mathbf{Hopf}_\kappa}(\mathrm{Wt}_{n+1}^\kappa, H \boxtimes H')
\end{tikzcd}$$

commutes. Consequently, the maps $\{\mu_n\}$ determine a map $\mu : \mathrm{DM}(H) \times \mathrm{DM}(H') \to \mathrm{DM}(H \boxtimes H')$.

Proposition 1.3.27. *Let H and H' be Hopf algebras over κ. Then the map $\mu : \mathrm{DM}(H) \times \mathrm{DM}(H') \to \mathrm{DM}(H \boxtimes H')$ of Construction 1.3.26 is a pairing, in the sense of Definition 1.3.24.*

Proof. We first show that μ is $W(\kappa)$-bilinear. Choose $x \in \mathrm{DM}(H)$, $y \in \mathrm{DM}(H')$, and $\lambda \in W(\kappa)$; we wish to show that $\mu(\lambda x, y) = \lambda \mu(x, y) = \mu(x, \lambda y)$. Write $\lambda = \sum_{i \geq 0} \tau(\lambda_i) p^i$ for $\lambda_i \in \kappa$. Choose an integer n such that $V^n x = 0 = V^n y$. Then x, y, and $\mu(x, y)$ are annihilated by p^n. We may therefore replace λ by the finite sum $\tau(\lambda_0) + \cdots + p^{n-1}\tau(\lambda_{n-1})$. Since μ is $\mathbf{Z}$-bilinear, we are reduced to proving the identity

$$\mu(\tau(z)x, y) = \tau(z)\mu(x, y) = \mu(x, \tau(z)y)$$

for $z \in \kappa$. Since $V^n x = 0$, x and y determine a Hopf algebra homomorphisms $\mathrm{Wt}_n^\kappa \to H$. We may therefore reduce to the universal case where $H = H' = \mathrm{Wt}_n^\kappa$, in which case $\mu(x, y)$ is given by the map $\iota_n : \mathrm{Wt}_n^\kappa \to \mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa$. Unwinding the definitions, we must show that the diagram

$$\begin{tikzcd}
\mathrm{Wt}_n^\kappa \arrow[d, "\iota"] & \mathrm{Wt}_n^\kappa \arrow[l, "{[z^{p^{1-n}}]}"'] \arrow[r, "{[z^{p^{1-n}}]}"] \arrow[d, "\iota_n"] & \mathrm{Wt}_n^\kappa \arrow[d, "\iota_n"] \\
\mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa & \mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa \arrow[l, "{[z^{p^{1-n}}] \boxtimes \mathrm{id}}"'] \arrow[r, "{\mathrm{id} \boxtimes [z^{p^{1-n}}]}"] & \mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa
\end{tikzcd}$$

commutes. This follows from the observation that the map $\iota_n : \mathrm{Wt}_n^\kappa \to \mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa$ carries homogeneous elements of degree m to homogeneous elements of bidegree (m, m) (see Remark 1.3.13).

We next prove that $V\mu(x, y) = \mu(Vx, Vy)$ for $x \in \mathrm{DM}(H)$ and $y \in \mathrm{DM}(H')$. As above, we may assume that $V^n x = 0 = V^n y$ for some n, and then reduce to the universal case $H = H' = \mathrm{Wt}_n^\kappa$. Unwinding the definitions, we must show that the diagram

$$\begin{tikzcd}
\mathrm{Wt}_n^\kappa \arrow[r, "V"] \arrow[d, "\iota_n"] & \mathrm{Wt}_n^\kappa \arrow[d, "\iota_n"] \\
\mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa \arrow[r, "V \boxtimes V"] & \mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa
\end{tikzcd}$$

commutes. To prove this, we may assume without loss of generality that $\kappa = \mathbf{F}_p$. It will then suffice to show that the bottom horizontal map $V \boxtimes V : \mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa \to \mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa$ coincides with the (absolute) Verschiebung map V' of $\mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa$. Let $\xi : \mathrm{Wt}_n^\kappa \otimes \mathrm{Wt}_n^\kappa \to \mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa$ be the canonical coalgebra map. Since the image of ξ generates $\mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa$ as an algebra, it will suffice to show that the composite maps

$$\mathrm{Wt}_n^\kappa \otimes \mathrm{Wt}_n^\kappa \xrightarrow{\xi} \mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa \xrightarrow{V \boxtimes V} \mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa$$

$$\mathrm{Wt}_n^\kappa \otimes \mathrm{Wt}_n^\kappa \xrightarrow{\xi} \mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa \xrightarrow{V'} \mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa .$$

coincide. This is clear, since both maps are obtained by precomposing ξ with the Verschiebung map on the coalgebra $\mathrm{Wt}_n^\kappa \otimes \mathrm{Wt}_n^\kappa$.

We now complete the proof by showing that $F\mu(Vx, y) = \mu(x, Fy)$ and $F\mu(x, Vy) = \mu(Fx, y)$ for $x \in \mathrm{DM}(H)$ and $y \in \mathrm{DM}(H')$. By symmetry, it will suffice to prove the first of these identities. As above, we may suppose that $V^n x = V^n y = 0$, and reduce to the universal case where $H = H' = \mathrm{Wt}_n^\kappa$. Unwinding the definitions, we must prove that the diagram

$$\begin{tikzcd}
\mathrm{Wt}_n^\kappa \arrow[r, "F"] \arrow[d, "\iota_n"] & \mathrm{Wt}_n^\kappa \arrow[d, "(V \boxtimes \mathrm{id}) \circ \iota_n"] \\
\mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa \arrow[r, "\mathrm{id} \boxtimes F"] & \mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa
\end{tikzcd}$$

commutes. As before, we may reduce to the case $\kappa = \mathbf{F}_p$, so that F and V coincide with the absolute Frobenius and Verschiebung maps on Wt_n^κ, respectively. Let $F^{(2)}$ be the Frobenius map from $\mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa$ to itself, so that $F^{(2)} \circ \iota_n = \iota_n \circ F$. Consequently, we are reduced to proving the identity

$$(V \boxtimes \mathrm{id}) \circ F^{(2)} \circ \iota_n = (\mathrm{id} \boxtimes F) \circ \iota_n.$$

In fact, we claim that $(V \boxtimes \mathrm{id}) \circ F^{(2)}$ and $\mathrm{id} \boxtimes F$ coincide as Hopf algebra homomorphisms from $\mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa$ to itself. To prove this, it will suffice to show that both homomorphisms agree on $u \boxtimes v$, for every pair of elements $u, v \in \mathrm{Wt}_n^\kappa$. In other words, we must verify the identity $(V(u) \boxtimes v)^p = u \boxtimes v^p$, which follows from the identities given in Notation 1.1.18. $\square$

We are now ready to state the main result of this section.

Theorem 1.3.28 (Goerss, Buchstaber-Lazarev)**.** *Let H and H' be connected Hopf algebras over κ. Then the pairing $\mu : \mathrm{DM}(H) \times \mathrm{DM}(H') \to \mathrm{DM}(H \boxtimes H')$ of Proposition 1.3.27 induces an isomorphism of left D_κ-modules*

$$\theta_{H,H'} : \mathrm{DM}(H) \widetilde{\otimes} \mathrm{DM}(H') \to \mathrm{DM}(H \boxtimes H').$$

Proof. As functors of H and H', both the domain and codomain of θ preserve small colimits in each variable. Using Remark 1.3.17, we can reduce to the case where $H = \mathrm{Wt}_m^\kappa$ and $H' = \mathrm{Wt}_n^\kappa$ for some integers m and n. We now proceed by induction on m. The case $m = 0$ is trivial. If $m > 0$, we have an exact sequence of Hopf algebras

$$\kappa \to \mathrm{Wt}_1^\kappa \to \mathrm{Wt}_m^\kappa \xrightarrow{T} \mathrm{Wt}_{m-1}^\kappa \to \kappa,$$

giving rise to a commutative diagram of exact sequences

$$\begin{tikzcd}
\mathrm{DM}(\mathrm{Wt}_1^\kappa) \widetilde{\otimes} \mathrm{DM}(\mathrm{Wt}_n^\kappa) \arrow[r] \arrow[d, "\theta'"] & \mathrm{DM}(\mathrm{Wt}_m^\kappa) \widetilde{\otimes} \mathrm{DM}(\mathrm{Wt}_n^\kappa) \arrow[r] \arrow[d, "\theta"] & \mathrm{DM}(\mathrm{Wt}_{m-1}^\kappa) \widetilde{\otimes} \mathrm{DM}(\mathrm{Wt}_n^\kappa) \arrow[r] \arrow[d, "\theta''"] & 0 \\
\mathrm{DM}(\mathrm{Wt}_1^\kappa \boxtimes \mathrm{Wt}_n^\kappa) \arrow[r, "\phi"] & \mathrm{DM}(\mathrm{Wt}_m^\kappa \boxtimes \mathrm{Wt}_n^\kappa) \arrow[r] & \mathrm{DM}(\mathrm{Wt}_{m-1}^\kappa \boxtimes \mathrm{Wt}_n^\kappa) \arrow[r] & 0.
\end{tikzcd}$$

It follows from Theorem 1.3.16 and Scholium 1.2.15 that the map ϕ is injective, and from the inductive hypothesis that θ'' is an isomorphism. Consequently, to prove that θ is an isomorphism, it will suffice to show that θ' is an isomorphism. That is, we may reduce to the case where $m = 1$. Similarly, we may reduce to the case $n = 1$. In this case, $\mathrm{DM}(H)$ and $\mathrm{DM}(H')$ can be identified with the free D_κ-module M generated by a single element e satisfying $Ve = 0$. The map $\iota_1 : \mathrm{Wt}_1^\kappa \to \mathrm{Wt}_1^\kappa \boxtimes \mathrm{Wt}_1^\kappa$ is an isomorphism (Example 1.2.24). We may therefore identify θ with a map $M \widetilde{\otimes} M \to M$, induced by a bilinear map $\mu : M \times M \to M$ satisfying $\mu(e,e) = e$. Let $\mu_0 : M \times M \to M \widetilde{\otimes} M$ be the universal pairing. Then $V\mu_0(e,e) = \mu_0(Ve, Ve) = 0$, so there is a unique map of left D_κ-modules $\psi : M \to M \widetilde{\otimes} M$ satisfying $\psi(e) = \mu_0(e,e)$. It is clear that $\theta \circ \psi = \mathrm{id}_M$. To complete the proof that θ is an isomorphism, it will suffice to show that ψ is surjective. Since the image of ψ is a left D_κ-submodule of $M \widetilde{\otimes} M$, it will suffice to show that the image of ψ contains $\mu_0(x,y)$ for all $x, y \in M$. As a κ-vector space, M has a basis given by $F^i e$ for $i \geq 0$. It will therefore suffice to show that $\mu_0(F^i x, F^j y)$ belongs to the image of ψ for $i, j \geq 0$. For $i = j = 0$, this is clear from the construction. If $i > 0$, we have

$$\mu_0(F^i x, F^j y) = F\mu_0(F^{i-1}x, VF^j y) = F\mu_0(F^{i-1}x, 0) = 0.$$

Similarly, if $j > 0$, we have $\mu_0(F^i x, F^j y) = 0$. $\square$

Remark 1.3.29. It is not hard to see that the operation $\widetilde{\otimes}$ endows $\mathbf{LMod}_{\mathrm{D}_\kappa}$ with the structure of a symmetric monoidal category. The unit object of $\mathbf{LMod}_{\mathrm{D}_\kappa}$ is given by $W(\kappa)$, where the action of F and V are given by the formulas

$$F(x) = p\varphi(x) \qquad V(x) = \varphi^{-1}(x).$$

Corollary 1.3.30. *The functor* $\mathrm{DM} : \mathbf{Hopf}_\kappa^c \to \mathbf{LMod}_{\mathrm{D}_\kappa}$ *is a (nonunital) symmetric monoidal functor.*

Proof. For every pair of connected Hopf algebras H and H' over κ, Theorem 1.3.28 provides a canonical isomorphism $\theta_{H,H'} : \mathrm{DM}(H) \widetilde{\otimes} \mathrm{DM}(H') \to \mathrm{DM}(H \boxtimes H')$. To show that this data endows DM with the structure of a (nonunital) symmetric monoidal functor, it will suffice to show that the diagrams

$$\begin{tikzcd}
\mathrm{DM}(H)\widetilde{\otimes}\mathrm{DM}(H') \arrow[r, "\sim"] \arrow[d, "\theta_{H,H'}"] & \mathrm{DM}(H')\widetilde{\otimes}\mathrm{DM}(H) \arrow[d, "\theta_{H',H}"] \\
\mathrm{DM}(H \boxtimes H') \arrow[r, "\sim"] & \mathrm{DM}(H' \boxtimes H)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{DM}(H)\widetilde{\otimes}(\mathrm{DM}(H')\widetilde{\otimes}\mathrm{DM}(H'')) \arrow[r, "\sim"] \arrow[d, "\theta_{H',H''}"] & (\mathrm{DM}(H)\widetilde{\otimes}\mathrm{DM}(H'))\widetilde{\otimes}\mathrm{DM}(H'') \arrow[d, "\theta_{H,H'}"] \\
\mathrm{DM}(H)\widetilde{\otimes}\mathrm{DM}(H' \boxtimes H'') \arrow[d, "\theta_{H,H'}"] & \mathrm{DM}(H \boxtimes H')\widetilde{\otimes}\mathrm{DM}(H'') \arrow[d, "\theta_{H\boxtimes H',H''}"] \\
\mathrm{DM}(H \boxtimes (H' \boxtimes H'')) \arrow[r, "\sim"] & \mathrm{DM}((H \boxtimes H') \boxtimes H'')
\end{tikzcd}$$

commute. In the first case this is obvious, and in the second it follows from Remark 1.2.22. $\square$

Notation 1.3.31. Let M, M', and M'' be left D_κ-modules, and suppose we are given a pairing $\lambda : M \times M' \to M''$. Using the relation $V\lambda(x,y) = \lambda(Vx, Vy)$, we deduce that λ carries $\{x \in M : Vx = 0\} \times M'$ into $\{z \in M'' : Vz = 0\}$. Moreover, if $Vx = 0$, then

$$\lambda(x, Fy) = F\lambda(Vx, y) = F\lambda(0, y) = 0,$$

so that λ induces a map $\overline{\lambda} : \{x \in M : Vx = 0\} \times M'/FM' \to \{z \in M'' : Vz = 0\}$.

Remark 1.3.32. Let H, H', and H'' be connected Hopf algebras over κ and let $\mu : H \otimes_\kappa H' \to H''$ be a bilinear map of Hopf algebras. If $x \in H$ is primitive and $y \in \mathfrak{m}_{H'}$, then $\mu(x \otimes y) \in H''$ is primitive. Moreover, for $y, y' \in \mathfrak{m}_{H'}$, we have

$$\mu(x \otimes yy') = \mu(x \otimes y)\mu(1 \otimes y') + \mu(1 \otimes y)\mu(x \otimes y') = 0.$$

It follows that μ induces a map

$$\overline{\mu} : \mathrm{Prim}(H) \otimes_\kappa \mathfrak{m}_{H'}/\mathfrak{m}_{H'}^2 \to \mathrm{Prim}(H'').$$

We will need the following compatibility between the constructions described in Notation 1.3.31 and Remark 1.3.32:

Proposition 1.3.33. *Let H, H', and H'' be Hopf algebras over κ, let $\mu : H \otimes H' \to H''$ be a bilinear map, and let $\lambda : \mathrm{DM}(H) \times \mathrm{DM}(H') \to \mathrm{DM}(H'')$ be the induced pairing. Let $\rho_H : \mathrm{DM}(H) \to H$ be the inclusion, and let $\overline{\rho}_{H'} : \mathrm{DM}(H')/F\,\mathrm{DM}(H') \to \mathfrak{m}_{H'}/\mathfrak{m}_{H'}^2$ be the isomorphism of Proposition 1.3.20. Then the diagram*

$$\begin{tikzcd}
\{x \in \mathrm{DM}(H) : Vx = 0\} \times \mathrm{DM}(H')/F\,\mathrm{DM}(H') \arrow[r, "\overline{\lambda}"] \arrow[d, "\rho_H \times \overline{\rho}_{H'}"] & \{z \in \mathrm{DM}(H'') : Vz = 0\} \arrow[d] \\
\mathrm{Prim}(H) \otimes_\kappa \mathfrak{m}_{H'}/\mathfrak{m}_{H'}^2 \arrow[r, "\overline{\mu}"] & \mathrm{Prim}(H'')
\end{tikzcd}$$

commutes, where $\overline{\lambda}$ and $\overline{\mu}$ are defined as in Notation 1.3.31 and Remark 1.3.32, respectively.

The proof of Proposition 1.3.33 will require the following preliminary result:

Lemma 1.3.34. *For each $n \geq 0$, the elements $\{(a_1 \circ a_{p^i})^{p^j}\}_{0 \leq i < n, 0 \leq j}$ form a basis for the vector space $\mathrm{Prim}(\mathrm{Wt}_1^\kappa \boxtimes \mathrm{Wt}_n^\kappa)$.*

Proof. We proceed by induction on n, the case $n = 0$ being trivial. Assume that $n > 0$. We have an exact sequence of Hopf algebras

$$\kappa \to \mathrm{Wt}_{n-1}^\kappa \hookrightarrow \mathrm{Wt}_n^\kappa \xrightarrow{V^{n-1}} \mathrm{Wt}_1^\kappa \to \kappa.$$

Using Scholium 1.2.15, we deduce that the induced sequence

$$\kappa \to \mathrm{Wt}_1^\kappa \boxtimes \mathrm{Wt}_{n-1}^\kappa \to \mathrm{Wt}_1^\kappa \boxtimes \mathrm{Wt}_n^\kappa \to \mathrm{Wt}_1^\kappa \boxtimes \mathrm{Wt}_1^\kappa \to \kappa$$

is also exact, so that we have an exact sequence of κ-vector spaces

$$0 \to \mathrm{Prim}(\mathrm{Wt}_1^\kappa \boxtimes \mathrm{Wt}_{n-1}^\kappa) \to \mathrm{Prim}(\mathrm{Wt}_1^\kappa \boxtimes \mathrm{Wt}_n^\kappa) \xrightarrow{q} \mathrm{Prim}(\mathrm{Wt}_1^\kappa \boxtimes \mathrm{Wt}_1^\kappa).$$

The inductive hypothesis implies that the collection of elements $\{(a_1 \circ a_{p^i})^{p^j}\}_{0 \leq i < n-1}$ forms a basis for $\mathrm{Prim}(\mathrm{Wt}_1^\kappa \boxtimes \mathrm{Wt}_{n-1}^\kappa)$. To complete the proof, it will suffice to show that that the elements $\{q((a_1 \boxtimes a_{p^{n-1}})^{p^j})\}_{j \geq 0}$ form a basis for $\mathrm{Wt}_1^\kappa \boxtimes \mathrm{Wt}_1^\kappa$. We now observe that $\mathrm{Wt}_1^\kappa \boxtimes \mathrm{Wt}_1^\kappa$ is isomorphic to a polynomial ring $\kappa[x]$, with comultiplication given by $\Delta(x) = x \otimes 1 + 1 \otimes x$; the collection of primitive elements in $\kappa[x]$ has a basis given by the monomials $\{x^{p^j}\}_{j \geq 0}$. $\square$

Proof of Proposition 1.3.33. Fix $n \geq 1$, and consider the composite map

$$\phi : \mathrm{Wt}_n^\kappa \xrightarrow{\iota_n} \mathrm{Wt}_n^\kappa \boxtimes \mathrm{Wt}_n^\kappa \xrightarrow{T^{n-1} \boxtimes \mathrm{id}} \mathrm{Wt}_1^\kappa \boxtimes \mathrm{Wt}_n^\kappa.$$

It follows from Notation 1.3.31 that $\phi(a_{p^{n-1}})$ is a primitive element of $\mathrm{Wt}_1^\kappa \boxtimes \mathrm{Wt}_n^\kappa$. Using Lemma 1.3.34, we can write $\phi(a_{p^{n-1}})$ as a linear combination of elements $(a_1 \boxtimes a_{p^i})^{p^j}$. Let us regard $\mathrm{Wt}_1 \boxtimes \mathrm{Wt}_n$ as bigraded (where $a_i \boxtimes a_j$ has bidegree (i, j)), so that ϕ carries elements of degree d to elements of bidegree $(p^{1-n}d, d)$. In particular, $\phi(a_{p^{n-1}})$ has bidegree $(1, p^{n-1})$. It follows that $\phi(a_{p^{n-1}}) = c_n(a_1 \boxtimes a_{p^{n-1}})$ for some constant $c_n \in \kappa$. It then follows that for every V-torsion element $x \in \mathrm{DM}(H)$ and every V^n-torsion element $y \in \mathrm{DM}(H')$, we have $\rho_{H''}\overline{\lambda}(x, y) = c_n \overline{\mu}(\rho_H(x), \overline{\rho}_{H'}(y))$. Since this equation holds for every bilinear map $\mu : H \otimes H' \to H''$, we conclude that c_n is independent of n. To complete the proof, it will suffice to show that $c_n = 1$ for all n. Since c_n does not depend on n, it suffices to show that $c_1 = 1$, which is clear (note that $\iota_1(a_1) = a_1 \boxtimes a_1$). $\square$

1.4 Disconnected Formal Groups

Let κ be a perfect field of characteristic $p > 0$, which we regard as fixed throughout this section. The Dieudonne module functor $H \mapsto \mathrm{DM}(H)$ determines a fully faithful embedding from the category $\mathbf{Hopf}_{\kappa}^{c}$ of connected Hopf algebras over κ to the category $\mathbf{LMod}_{\mathrm{D}_{\kappa}}$ of Dieudonne modules. For many applications, it is useful to extend this construction to a somewhat larger class of Hopf algebras: namely, those Hopf algebras H for which multiplication by p is locally nilpotent on the formal group $\operatorname{Spf} H^{\vee}$. In this section, we will prove an analogue of Theorem 1.3.28 in this more general context.

We begin with some general remarks about adjoining units to symmetric monoidal categories.

Notation 1.4.1. Let $\mathcal{A}b$ denote the category of abelian groups, and let $\mathcal{A}$ be an abelian category which admits small colimits. Then there exists a unique functor $\otimes : \mathcal{A}b \times \mathcal{A} \to \mathcal{A}$ which preserves small colimits separately in each variable, having the property that the functor $C \mapsto \mathbf{Z} \otimes C$ is the identity functor from $\mathcal{A}$ to itself. This functor exhibits $\mathcal{A}$ as tensored over the category of abelian groups.

We will say that $\mathcal{A}$ is $\mathbf{Z}/p^n\mathbf{Z}$-*linear* if, for every pair of objects $C, D \in \mathcal{A}$, the abelian group $\operatorname{Hom}_{\mathcal{A}}(C, D)$ is annihilated by p^n. In this case, for every abelian group M and every object $C \in \mathcal{A}$, the canonical epimorphism

$$M \otimes C \to (M/p^n M) \otimes C$$

is an isomorphism. It follows that the functor $\otimes : \mathcal{A}b \times \mathcal{A} \to \mathcal{A}$ factors as a composition

$$\mathcal{A}b \times \mathcal{A} \to \mathbf{Mod}_{\mathbf{Z}/p^n\mathbf{Z}} \times \mathcal{A} \xrightarrow{\otimes_{\mathbf{Z}/p^n\mathbf{Z}}} \mathcal{A}.$$

The functor $\otimes_{\mathbf{Z}/p^n\mathbf{Z}}$ exhibits $\mathcal{A}$ as tensored over the symmetric monoidal category $\mathbf{Mod}_{\mathbf{Z}/p^n\mathbf{Z}}$ of (discrete) $\mathbf{Z}/p^n\mathbf{Z}$-modules.

Construction 1.4.2. Suppose that $\mathcal{A}$ is an abelian category which admits small colimits and is $\mathbf{Z}/p^n\mathbf{Z}$-linear. Suppose further that $\mathcal{A}$ is equipped with a nonunital symmetric monoidal structure for which the tensor product functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ preserves colimits separately in each variable. We let $\mathcal{A}_{+}$ denote the product category $\mathbf{Mod}_{\mathbf{Z}/p^n\mathbf{Z}} \times \mathcal{A}$. We let $\otimes_{+} : \mathcal{A}_{+} \times \mathcal{A}_{+} \to \mathcal{A}_{+}$ denote the functor given by

$$(M, C) \otimes_{+} (M', C') = (M \otimes_{\mathbf{Z}/p^n\mathbf{Z}} M', (M \otimes_{\mathbf{Z}/p^n\mathbf{Z}} C') \oplus (M' \otimes_{\mathbf{Z}/p^n\mathbf{Z}} C) \oplus (C \otimes C')).$$

It is not difficult to show that the tensor product functor $\otimes_{+}$ exhibits $\mathcal{A}_{+}$ as a symmetric monoidal category, with unit object given by $(\mathbf{Z}/p^n\mathbf{Z}, 0)$.

Remark 1.4.3. In the situation of Construction 1.4.2, the symmetric monoidal category $\mathcal{A}_{+}$ has the following universal property: for any $\mathbf{Z}/p^n\mathbf{Z}$-linear abelian category $\mathcal{B}$ equipped with a symmetric monoidal structure for which the tensor product preserves small colimits in each variable, composition with the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{A}_{+}$ induces an equivalence from the category of colimit-preserving symmetric monoidal functors from $\mathcal{A}_{+}$ to $\mathcal{B}$ to the category of colimit-preserving nonunital symmetric monoidal functors from $\mathcal{A}$ to $\mathcal{B}$.

Notation 1.4.4. Let $n \geq 0$ be an integer. We will say that a Hopf algebra H over κ is p^n-*torsion* if it is annihilated by p^n, when regarded as an object of the abelian category $\mathbf{Hopf}_{\kappa}$. We let $\mathbf{Hopf}_{\kappa,n}$ denote the the full subcategory of $\mathbf{Hopf}_{\kappa}$ spanned by the p^n-torsion Hopf algebras, and $\mathbf{Hopf}_{\kappa,n}^{c} = \mathbf{Hopf}_{\kappa}^{c} \cap \mathbf{Hopf}_{\kappa,n}$ the full subcategory of $\mathbf{Hopf}_{\kappa}$ spanned by the connected p^n-torsion Hopf algebras.

Remark 1.4.5. The inclusion functor $\mathbf{Hopf}_{\kappa,n} \hookrightarrow \mathbf{Hopf}_{\kappa}$ admits a left adjoint, which carries each Hopf algebra H to the cokernel (in the abelian category $\mathbf{Hopf}_{\kappa}$ of the map $[p^n] : H \to H$ representing multiplication by p^n. Consequently, we may view $\mathbf{Hopf}_{\kappa,n}$ as a localization of the category $\mathbf{Hopf}_{\kappa}$.

Remark 1.4.6. Let H and H' be Hopf algebras over κ. If either H or H' is p^n-torsion, then the Hopf algebra $H \boxtimes H'$ is p^n-torsion. It follows that the full subcategories $\mathbf{Hopf}_{\kappa,n}, \mathbf{Hopf}_{\kappa,n}^{c} \subseteq \mathbf{Hopf}_{\kappa}$ are closed under the functor $\boxtimes$, and therefore inherit the structure of nonunital symmetric monoidal categories. In fact, $\mathbf{Hopf}_{\kappa,n}$ is even symmetric monoidal: it has a unit object, given by the group algebra $\kappa[\mathbf{Z}/p^n\mathbf{Z}] = \kappa[x]/(x^{p^n} - 1)$.

Let $n \geq 0$. It follows from Remark 1.4.3 that the inclusion $\mathbf{Hopf}^c_{\kappa,n} \hookrightarrow \mathbf{Hopf}_{\kappa,n}$ admits an essentially unique extension to a symmetric monoidal functor $\theta : (\mathbf{Hopf}^c_{\kappa,n})_+ \to \mathbf{Hopf}_{\kappa,n}$.

Proposition 1.4.7. *Suppose that the field κ is algebraically closed. Then for each $n \geq 0$, the functor $\theta : (\mathbf{Hopf}^c_{\kappa,n})_+ \to \mathbf{Hopf}_{\kappa,n}$ is an equivalence of categories.*

Proof. Since κ is algebraically closed, every Hopf algebra H over κ can be written canonically as a tensor product $H^c \otimes_\kappa \kappa[M]$, where H^c is a connected Hopf algebra over κ, and M is an abelian group (which we can identify with the collection of group-like elements of H). This observation determines an equivalence of categories $\mathbf{Hopf}_\kappa \simeq \mathcal{A}b \times \mathbf{Hopf}^c_\kappa$, which restricts to an equivalence $\mathbf{Hopf}_{\kappa,n} \simeq \mathbf{Mod}_{\mathbf{Z}/p^n\mathbf{Z}} \times \mathbf{Hopf}^c_{\kappa,n}$. Under this equivalence, θ corresponds to the identity functor from $\mathbf{Mod}_{\mathbf{Z}/p^n\mathbf{Z}} \times \mathbf{Hopf}^c_{\kappa,n}$ to itself. □

For $n \geq 0$, we let D_κ/p^n denote the quotient of D_κ by the (two-sided) ideal generated by p^n. We will identify $\mathbf{LMod}_{\mathrm{D}_\kappa/p^n}$ with the full subcategory of $\mathbf{LMod}_{\mathrm{D}_\kappa}$ spanned by those left D_κ-modules which are annihilated by p^n, and we let $\mathrm{LMod}^{\mathrm{Nil}}_{\mathrm{D}_\kappa/p^n}$ denote the full subcategory spanned by those left D_κ-modules which are annihilated by p^n on which the action of V is locally nilpotent. Note that $\mathbf{LMod}_{\mathrm{D}_\kappa/p^n}$ and $\mathbf{LMod}^{\mathrm{Nil}}_{\mathrm{D}_\kappa/p^n}$ are closed under the tensor product $\widetilde{\otimes}$, and therefore inherit the structure of nonunital symmetric monoidal ∞-categories. Moreover, $\mathbf{LMod}_{\mathrm{D}_\kappa/p^n}$ has a unit object, given by the quotient $W(\kappa)/p^n W(\kappa)$ (where the actions of F and V are given by $F(x) = p\varphi(x)$ and $V(x) = \varphi^{-1}(x)$, as in Remark 1.3.29). Proposition 1.4.7 implies that the formation of Dieudonne modules defines a nonunital symmetric monoidal functor

$$\mathrm{DM} : \mathbf{Hopf}^c_{\kappa,n} \simeq \mathbf{LMod}^{\mathrm{Nil}}_{\mathrm{D}_\kappa/p^n} \hookrightarrow \mathbf{LMod}_{\mathrm{D}_\kappa/p^n}.$$

This extends uniquely to a symmetric monoidal functor $\theta' : (\mathbf{Hopf}^c_{\kappa,n})_+ \to \mathbf{LMod}_{\mathrm{D}_\kappa/p^n}$.

Proposition 1.4.8. *Let $n \geq 0$. Then the functor $\theta' : (\mathbf{Hopf}^c_{\kappa,n})_+ \to \mathbf{LMod}_{\mathrm{D}_\kappa/p^n}$ is fully faithful.*

Proof. We can identify objects of $(\mathbf{Hopf}^c_{\kappa,n})_+$ with pairs (M, H), where M is an abelian group which is annihilated by p^n, and H is a connected p^n-torsion Hopf algebra over κ. Unwinding the definitions, we see that the functor θ^+ is given by

$$\theta^+(M, H) = (W(\kappa) \otimes_{\mathbf{Z}} M) \oplus \mathrm{DM}(H),$$

where the actions of F and V on the first factor are given by

$$F(\lambda x) = p\varphi(\lambda)x \qquad V(\lambda x) = \varphi^{-1}(\lambda)x$$

for $\lambda \in W(\kappa)$, $x \in M$. Since the functor DM is fully faithful, the assertion that θ' is fully faithful is equivalent to the following:

(1) For every abelian group M which is annihilated by p^n and every connected Hopf algebra H which is annihilated by p^n, there are no nonzero left D_κ-module homomorphisms from $W(\kappa) \otimes_{\mathbf{Z}} M$ to $\mathrm{DM}(H)$.

(2) For every abelian group M which is annihilated by p^n and every connected Hopf algebra H which is annihilated by p^n, there are no nonzero left D_κ-module homomorphisms from $\mathrm{DM}(H)$ to $W(\kappa) \otimes_{\mathbf{Z}} M$.

(3) For every pair of abelian groups M and N which are annihilated by p^n, the canonical map

$$\mathrm{Hom}_{\mathcal{A}b}(M, N) \to \mathrm{Hom}_{\mathrm{D}_\kappa}(W(\kappa) \otimes_{\mathbf{Z}} M, W(\kappa) \otimes_{\mathbf{Z}}, N)$$

is bijective.

To prove (1), suppose we are given a map of left D_κ-modules $\lambda : W(\kappa) \otimes_{\mathbf{Z}} M \to \mathrm{DM}(H)$. For each $x \in M$, we have $V(1 \otimes x) = 1 \otimes x$ in $W(\kappa) \otimes_{\mathbf{Z}} M$, so that

$$V^m \lambda(1 \otimes x) = \lambda(V^n(1 \otimes x)) = \lambda(1 \otimes x)$$

for all m. Since the action of V on $\mathrm{DM}(H)$ is locally nilpotent, we conclude that $\lambda(1 \otimes x) = 0$ for all $x \in M$. Since λ is $W(\kappa)$-linear, we conclude that $\lambda = 0$.

The proof of (2) is similar. Let $\lambda : \mathrm{DM}(H) \to W(\kappa) \otimes_{\mathbf{Z}} M$ be a left D_κ-module homomorphism, and let $x \in \mathrm{DM}(H)$. Then $V^m x = 0$ for $m \gg 0$. It follows that

$$V^m \lambda(x) = \lambda(V^m x) = \lambda(0) = 0.$$

Since φ^{-1} is an automorphism of $W(\kappa)$, the action of V on $W(\kappa) \otimes_{\mathbf{Z}} M$ is invertible, from which we deduce that $\lambda(x) = 0$.

We now prove (3). Let M and N be abelian groups which are annihilated by p^n. Let us identify M and N with their images in $W(\kappa) \otimes_{\mathbf{Z}} M$ and $W(\kappa) \otimes_{\mathbf{Z}} N$, respectively. The map $\mathrm{Hom}_{\mathcal{A}b}(M, N) \to \mathrm{Hom}_{\mathrm{D}_\kappa}(W(\kappa) \otimes_{\mathbf{Z}} M, W(\kappa) \otimes_{\mathbf{Z}} N)$ is evidently injective. To prove the surjectivity, it will suffice to show that every left D_κ-module homomorphism $\lambda : W(\kappa) \otimes_{\mathbf{Z}} M \to W(\kappa) \otimes_{\mathbf{Z}} N$ carries M into N. For this, it will suffice to show that if $y \in W(\kappa) \otimes_{\mathbf{Z}} N$ satisfies $V(y) = y$, then $y \in N$. In other words, we wish to show that the sequence

$$0 \to N \to W(\kappa) \otimes_{\mathbf{Z}} N \xrightarrow{(\varphi^{-1} - \mathrm{id}) \otimes \mathrm{id}} W(\kappa) \otimes_{\mathbf{Z}} N$$

is exact. Writing N as a filtered colimit of finitely generated submodules, we may assume that N is a finitely generated abelian group. Writing N as a direct sum of indecomposable summands, we can reduce to the case where $N = \mathbf{Z}/p^m \mathbf{Z}$ for $m \leq n$. In this case, we must prove the exactness of the sequence

$$0 \to \mathbf{Z}/p^m \mathbf{Z} \to W(\kappa)/p^m W(\kappa) \xrightarrow{\varphi^{-1} - \mathrm{id}} W(\kappa)/p^m W(\kappa).$$

At the level of Witt components, this amounts to the observation that every element $x \in \kappa$ satisfying $x^{p^{-1}} = x$ must belong to the prime field $\mathbf{F}_p$. $\square$

Proposition 1.4.9. *Suppose that κ is algebraically closed, and let $n \geq 0$. Then the essential image of the functor $\theta' : (\mathbf{Hopf}^c_{\kappa,n})_+ \to \mathbf{LMod}_{\mathrm{D}_\kappa/p^n}$ is the full subcategory spanned by those left D_κ/p^n-modules N which satisfy the following condition:*

- $(*)$ *For each element $x \in N$, there is a finite length $W(\kappa)$-submodule of N which contains x and is closed under the action of V.*

Lemma 1.4.10. *Let $\varphi : W(\kappa) \to W(\kappa)$ be the Frobenius map, let N be a $W(\kappa)$-module which is annihilated by p^n for some n, let $V : N \to N$ be a φ^{-1}-semilinear map, and set $M = \{x \in N : Vx = x\}$. Then the canonical map $W(\kappa) \otimes M \to N$ is injective.*

Proof. It will suffice to show that for every finitely generated submodule $M_0 \subseteq M$, the induced map $W(\kappa) \otimes M_0 \to N$ is injective. Write M_0 as a direct sum $\bigoplus_{1 \leq i \leq d} \mathbf{Z}/p^{t_i} \mathbf{Z}$, so that the inclusion of M_0 into M determines a collection of elements $x_1, \ldots, x_d \in N$ satisfying $Vx_i = x_i$ and $p^{t_i} x_i = 0$. Suppose we are given a dependence relation

$$c_1 x_1 + \cdots + c_d x_d = 0$$

in N, where $c_i \in W(\kappa)/p^{t_i} W(\kappa)$. We wish to prove that each c_i vanishes. Suppose otherwise. Without loss of generality, we can choose a counterexample with d as small as possible, so that none of the coefficients c_i vanish. Multiplying each c_i by an invertible element of $W(\kappa)$, we may assume without loss of generality that $c_1 = p^k$ for some integer k.

Let $\varphi : W(\kappa) \to W(\kappa)$ denote the Frobenius morphism. We then have

$$0 = V(c_1 x_1 + \cdots + c_d x_d) = \varphi^{-1}(c_1) x_1 + \cdots + \varphi^{-1}(c_d),$$

so that $\sum_{1 \leq i \leq d} (c_i - \varphi^{-1} c_i) x_i = 0$. Since $c_1 = \varphi^{-1}(c_1)$, the minimality of d guarantees that $c_i = \varphi^{-1} c_i$ for $1 \leq i \leq d$. That is, we can identify each c_i with an element of $\mathbf{Z}/p^{t_i} \mathbf{Z} \subseteq W(\kappa)/p^{t_i} W(\kappa)$. In this case, the sum $c_1 x_1 + \cdots + c_d x_d$ can be identified with an element of M, and the composite map

$$M \to M \otimes W(\kappa) \to N$$

is injective by construction. $\square$

Lemma 1.4.11. *Assume that κ is algebraically closed. Let N be a finite dimensional vector space over κ, and let F be a φ-semilinear automorphism of N. If $N \neq 0$, then N contains a nonzero element which is fixed under the action of F.*

Proof. Choose a nonzero element $v \in N$. Since V is finite-dimensional, the elements $\{v, F(v), F^2(v), \ldots\}$ cannot all be linearly independent. Thus there exists a nonzero dependence relation

$$\sum_{0 \leq i \leq n} \lambda_i F^i(v) = 0.$$

Replacing v by $F^i(v)$ if necessary, we may assume that the coefficient λ_0 is nonzero. Dividing by λ_0, we may assume that $\lambda_0 = 1$: that is, we have

$$v = \sum_{1 \leq i \leq n} -\lambda_i F^i(v).$$

We may assume that n is chosen as small as possible: it follows that the set $\{v, F(v), \ldots, F^{n-1}(v)\}$ is linearly independent, and therefore $\lambda_n \neq 0$. Since $v \neq 0$, we must have $n > 0$.

Note that

$$f(x) = x^{p^n} + \lambda_1^{p^{n-1}} x^{p^{n-1}} + \lambda_2^{p^{n-2}} x^{p^{n-2}} + \cdots + \lambda_n x$$

is a separable polynomial of degree $p^n > 1$, and therefore has p^n distinct roots in the field κ. Consequently, there exists a nonzero element $a \in \kappa$ such that $f(a) = 0$. Let

$$w = av + (a^p + a\lambda_1)F(v) + (a^{p^2} + a^p \lambda_1^p + a\lambda_2)F^2(v) + \cdots + (a^{p^{n-1}} + a^{p^{n-2}} \lambda_1^{p^{n-2}} + \cdots + a\lambda_{n-1})F^{n-1}(v).$$

Since the elements $\{F^i(v)\}_{0 \leq i < n}$ are linearly independent and $a \neq 0$, w is a nonzero element of N. An explicit calculation gives

$$w - F(w) = av + \sum_{0 < i < n} a\lambda_i F^i(v) + (a\lambda_n - f(a))F^n(v) = a(v + \lambda_1 F(v) + \cdots + \lambda_n F^n(v)) = 0,$$

so that w is fixed by F. □

Lemma 1.4.12. *Let $\varphi : W(\kappa) \to W(\kappa)$ be the Frobenius map, let N be a $W(\kappa)$-module of finite length, and let $V : N \to N$ be a φ^{-1}-semilinear map. Suppose that V is injective and that κ is algebraically closed. Then N is generated by $M = \{x \in N : Vx = x\}$ as a module over $W(\kappa)$.*

Proof. Let N' denote the $W(\kappa)$-submodule of N generated by $W(\kappa)$ and set $N'' = N/N'$; we wish to show that $N'' \simeq 0$. We have a diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & N' \arrow[r] \arrow[d, "V-1"] & N \arrow[r] \arrow[d, "V-1"] & N'' \arrow[r] \arrow[d, "V-1"] & 0 \\ 0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0. \end{tikzcd}$$

Since κ is algebraically closed, the Artin-Schreier map $x \mapsto x - x^p$ is a surjection from κ to itself. Composing with φ^{-1}, we deduce that $\varphi^{-1} - \mathrm{id} : \kappa \to \kappa$ is surjective. It follows by induction on t that the map $\varphi^{-1} - \mathrm{id}$ is a surjection from $W(\kappa)/p^t W(\kappa)$ to itself for all t. Consequently, $\varphi^{-1} - \mathrm{id}$ induces a surjection from $W(\kappa) \otimes M$ to itself for every finite abelian p-group M, and therefore for every p^n-torsion abelian group M. Combining this observation with Lemma 1.4.10, we deduce that the map $V - 1 : N' \to N'$ is surjective. We therefore obtain a short exact sequence

$$0 \to \ker(V - 1 : N' \to N') \to \ker(V - 1 : N \to N) \to \ker(V - 1 : N'' \to N'') \to 0.$$

By construction, the first map is an isomorphism, so that $V - 1$ is an injection from N'' to itself. Let N''_0 denote the p-torsion subgroup of N''. Since N is a $W(\kappa)$-module of finite length, the injectivity of V implies that $V : N \to N$ is an isomorphism. Then V induces an isomorphism from N''_0 to itself having no fixed points. Applying Lemma 1.4.11 to the inverse isomorphism, we conclude that $N''_0 = 0$, so that $N'' = 0$ as desired. □

Proof of Proposition 1.4.9. Let us identify objects of $(\mathbf{Hopf}^c_{\kappa,n})_+$ with pairs (M,H) as in the proof of Proposition 1.4.8. We first show that every object belonging to the essential image of θ' satisfies condition $(*)$. We have $\theta'(M,H) \simeq \theta'(M,\kappa) \oplus \mathrm{DM}(H)$. Since the action of V on $\mathrm{DM}(H)$ is locally nilpotent, it automatically satisfies $(*)$. It will therefore suffice to show that $\theta'(M,\kappa) \simeq W(\kappa) \otimes M$ satisfies $(*)$, which is clear.

Conversely, suppose that N is a left D_κ-module which annihilates p^n and satisfies condition $(*)$. We wish to show that N belongs to the essential image of θ'. Let N_0 denote the subset of N consisting of those elements $x \in N$ which are annihilated by some power of V, let $M = \{x \in N : Vx = x\}$, and let N_1 be the $W(\kappa)$-submodule of N which is generated by M. Note that the action of F on M is given by multiplication by p, so we have a surjective map of D_κ-modules $M \otimes W(\kappa) \to N_1$. Since the action of V on N_0 is locally nilpotent, we can write $N_0 \simeq \mathrm{DM}(H)$ for some connected Hopf algebra H over κ. Lemma 1.4.10 implies that $N_1 \simeq W(\kappa) \otimes M$. We therefore have $\theta'(M,H) \simeq N_0 \oplus N_1$. To complete the proof, it will suffice to show that the direct sum $N_0 \oplus N_1$ is isomorphic to N.

Since V acts by an isomorphism on N_1 and the action of V on N_0 is locally nilpotent, we have $N_0 \cap N_1 = \emptyset$. It will therefore suffice to show that every element $x \in N$ can be written in the form $x_0 + x_1$, where $x_0 \in N_0$ and $x_1 \in N_1$. Using condition $(*)$, we can choose a finite length $W(\kappa)$-submodule $N' \subseteq N$ which contains x and is closed under the action of V. It follows that there exists an integer m such that $\ker(V^{m'} : N' \to N')$ is independent of m' for $m' \geq m$. Let $N'_0 = \ker(V^m : N' \to N')$ and let $N'_1 = \mathrm{im}(V^m : N' \to N')$. Note that if $y \in N'_0 \cap N'_1$, then we can write $y = V^m z$ for some $z \in N'$ satisfying $V^{2m} z = V^m y = 0$. Then $z \in \ker(V^{2m}) = \ker(V^m)$, so that $y = V^m z = 0$. It follows that $N'_0 \cap N'_1 = 0$. We have an exact sequence

$$0 \to N'_0 \to N' \stackrel{V^m}{\to} N'_1 \to 0,$$

so that the length of N' over $W(\kappa)$ is the sum of the lengths of N'_0 and N'_1 over $W(\kappa)$. It follows that N' is the direct sum of the submodules N'_0 and N'_1. In particular, we can write $x = x_0 + x_1$, where $x_0 \in N'_0$ and $x_1 \in N'_1$. It is clear that x_0 belongs to N_0, and Lemma 1.4.12 implies that x_1 belongs to N_1.

We now prove (b). It will suffice to show that for every finitely generated submodule $M_0 \subseteq M$, the induced map $W(\kappa) \otimes M_0 \to N$ is injective. Write M_0 as a direct sum $\bigoplus_{1 \leq i \leq d} \mathbf{Z}/p^{t_i}\mathbf{Z}$, so that the inclusion of M_0 into M determines a collection of elements $x_1, \ldots, x_d \in N$ satisfying $Vx_i = x_i$ and $p^{t_i} x_i = 0$. Suppose we are given a dependence relation

$$c_1 x_1 + \cdots + c_d x_d = 0$$

in N, where $c_i \in W(\kappa)/p^{t_i} W(\kappa)$. We wish to prove that each c_i vanishes. Suppose otherwise. Without loss of generality, we can choose a counterexample with d as small as possible, so that none of the coefficients c_i vanish. Multiplying each c_i by an invertible element of $W(\kappa)$, we may assume without loss of generality that $c_1 = p^k$ for some integer k.

Let $\varphi : W(\kappa) \to W(\kappa)$ denote the Frobenius morphism. We then have

$$0 = V(c_1 x_1 + \cdots + c_d x_d) = \varphi^{-1}(c_1) x_1 + \cdots + \varphi^{-1}(c_d),$$

so that $\sum_{1 \leq i \leq d} (c_i - \varphi^{-1} c_i) x_i = 0$. Since $c_1 = \varphi^{-1}(c_1)$, the minimality of d guarantees that $c_i = \varphi^{-1} c_i$ for $1 \leq i \leq d$. That is, we can identify each c_i with an element of $\mathbf{Z}/p^{t_i}\mathbf{Z} \subseteq W(\kappa)/p^{t_i} W(\kappa)$. In this case, the sum $c_1 x_1 + \cdots + c_d x_d$ can be identified with an element of M, and the composite map

$$M \to M \otimes W(\kappa) \to N_1 \subseteq N$$

is injective by construction. $\square$

Remark 1.4.13. Let κ be a perfect field of characteristic $p > 0$, let $\overline{\kappa}$ be an algebraic closure of κ, and let $\mathrm{Gal}(\overline{\kappa}/\kappa)$ denote the Galois group over $\overline{\kappa}$ over κ. Let $\mathrm{D}_{\overline{\kappa}}$ denote the Dieudonne ring of $\overline{\kappa}$, so that the Galois group $\mathrm{Gal}(\overline{\kappa}/\kappa)$ acts on $\mathrm{D}_{\overline{\kappa}}$ and therefore also on the category $\mathbf{LMod}_{\mathrm{D}_{\overline{\kappa}}/p^n}$. Let $(\mathbf{LMod}_{\mathrm{D}_{\overline{\kappa}}/p^n})^{\mathrm{Gal}(\overline{\kappa}/\kappa)}$ denote the category of homotopy fixed points for this action. More concretely, $(\mathbf{LMod}_{\mathrm{D}_{\overline{\kappa}}/p^n})^{\mathrm{Gal}(\overline{\kappa}/\kappa)}$ is the category whose objects are left $\mathrm{D}_{\overline{\kappa}}/p^n$-modules equipped with a compatible action of the Galois group $\mathrm{Gal}(\overline{\kappa}/\kappa)$. The construction $M \mapsto W(\overline{\kappa}) \otimes_{W(\kappa)} M \simeq W(\overline{\kappa})/p^n \otimes_{W(\kappa)/p^n} M$ determines a fully faithful embedding $\mathbf{LMod}_{\mathrm{D}_\kappa/p^n} \to (\mathbf{LMod}_{\mathrm{D}_{\overline{\kappa}}/p^n})^{\mathrm{Gal}(\overline{\kappa}/\kappa)}$. The essential image of this functor is the full subcategory of $(\mathbf{LMod}_{\mathrm{D}_{\overline{\kappa}}/p^n})^{\mathrm{Gal}(\overline{\kappa}/\kappa)}$ spanned by those $\mathrm{D}_{\overline{\kappa}}/p^n$-modules M on which the action of $\mathrm{Gal}(\overline{\kappa}/\kappa)$ is continuous (meaning that every element $x \in M$ is stabilized by an open subgroup of $\mathrm{Gal}(\overline{\kappa}/\kappa)$).

Similarly, the Galois group $\mathrm{Gal}(\overline{\kappa}/\kappa)$ acts on the category $\mathbf{Hopf}_{\overline{\kappa},n}$ of p^n-torsion Hopf algebras over $\overline{\kappa}$, and we have a fully faithful embedding $\mathbf{Hopf}_{\kappa,n} \to \mathbf{Hopf}_{\overline{\kappa},n}^{\mathrm{Gal}(\overline{\kappa}/\kappa)}$.

Propositions 1.4.7 and 1.4.8 determine a fully faithful embedding $\mathbf{Hopf}_{n,\overline{\kappa}} \to \mathbf{LMod}_{\mathrm{D}_{\overline{\kappa}}/p^n}$, which induces a fully faithful embedding

$$\overline{\mathrm{DM}}_+ : \mathbf{Hopf}_{n,\overline{\kappa}}^{\mathrm{Gal}(\overline{\kappa}/\kappa)} \to \mathbf{LMod}_{\mathrm{D}_{\overline{\kappa}}/p^n}^{\mathrm{Gal}(\overline{\kappa}/\kappa)}.$$

Proposition 1.4.14. *For each integer $n \geq 0$, the composite functor*

$$\mathbf{Hopf}_{\kappa,n} \to \mathbf{Hopf}_{\overline{\kappa},n}^{\mathrm{Gal}(\overline{\kappa}/\kappa)} \xrightarrow{\overline{\mathrm{DM}}_+} \mathbf{LMod}_{\mathrm{D}_{\overline{\kappa}}/p^n}^{\mathrm{Gal}(\overline{\kappa}/\kappa)}$$

factors through the essential image of the fully faithful embedding $\mathbf{LMod}_{\mathrm{D}_\kappa/p^n} \to \mathbf{LMod}_{\mathrm{D}_{\overline{\kappa}}/p^n}^{\mathrm{Gal}(\overline{\kappa}/\kappa}$. Consequently, we obtain a fully faithful symmetric monoidal functor

$$\mathrm{DM}_+ : \mathbf{Hopf}_{\kappa,n} \to \mathbf{LMod}_{\mathrm{D}_\kappa/p^n}.$$

Proof. Let H be a p^n-torsion Hopf algebra over κ, let $\overline{H} = \overline{\kappa} \otimes_\kappa H$, and let $M = \mathrm{DM}(\overline{H})$ be the associated $\mathrm{D}_{\overline{\kappa}}$-module. Then M is acted on by the Galois group $\mathrm{Gal}(\overline{\kappa}/\kappa)$; we wish to show that this action is continuous. Since the field κ is perfect, we can write H as a tensor product $H^c \otimes_\kappa H^d$, where H^c is connected and H^d is diagonalizable over $\overline{\kappa}$. It will therefore suffice to prove the result under the assumption that H is either connected or diagonalizable over $\overline{\kappa}$. In the connected case, the continuity of the action of $\mathrm{Gal}(\overline{\kappa}/\kappa)$ on $\mathrm{DM}(\overline{H}) \subseteq \overline{H}$ follows immediately from the continuity of the action of $\mathrm{Gal}(\overline{\kappa}/\kappa)$ on $\overline{H}$.

Now suppose that H is diagonalizable over $\overline{\kappa}$ and let N be the collection of grouplike elements of $\overline{H}$, so that we have a Hopf algebra isomorphism $\overline{\kappa}[N] \simeq \overline{H}$. Since $\mathrm{Gal}(\overline{\kappa}/\kappa)$ acts continuously on $\overline{H}$, it acts continuously on N, and therefore also on the tensor product $W(\overline{\kappa})/p^n \otimes_{\mathbf{Z}/p^n\mathbf{Z}} N \simeq \overline{\mathrm{DM}}_+(\overline{H})$. This completes the proof that DM_+ is well-defined. □

Corollary 1.4.15. *For each $n \geq 0$, the nonunital symmetric monoidal equivalence $\mathrm{DM} : \mathbf{Hopf}^c_{n,\kappa} \to \mathbf{LMod}^{\mathrm{Nil}}_{\mathrm{D}_\kappa/p^n}$ extends to a fully faithful symmetric monoidal functor $\mathrm{DM}_+ : \mathbf{Hopf}_{\kappa,n} \to \mathbf{LMod}_{\mathrm{D}_\kappa/p^n}$. The essential image of this functor is the full subcategory of $\mathbf{LMod}_{\mathrm{D}_\kappa/p^n}$ consisting of those modules which satisfy condition $(*)$ of Proposition 1.4.9.*

Proof. The only nontrivial point is to describe the essential image of DM_+. We first note that for any p^n-torsion Hopf algebra H over κ, the $\mathrm{D}_{\overline{\kappa}}$-module $W(\overline{\kappa}) \otimes_{W(\kappa)} \mathrm{DM}_+(H)$ satisfies $(*)$ by Proposition 1.4.12, so that $\mathrm{DM}_+(H)$ also satisfies $(*)$. Conversely, suppose that M is a left D_κ/p^n-module satisfying $(*)$. Then $\overline{M} = W(\overline{\kappa}) \otimes_{W(\kappa)} M$ is a left $\mathrm{D}_{\overline{\kappa}}/p^n$-module satisfying $(*)$, so that Proposition 1.4.12 implies that we can write $\overline{M} = \overline{\mathrm{DM}}_+(\overline{H})$ for some Hopf algebra $\overline{H}$ over $\overline{\kappa}$ equipped with a semilinear action of $\mathrm{Gal}(\overline{\kappa}/\kappa)$. To complete the proof, we must show that the action of $\mathrm{Gal}(\overline{\kappa}/\kappa)$ on $\overline{H}$ is continuous. As above, it will suffice to prove this in the special cases where $\overline{H}$ is assumed either to be connected or diagonalizable.

If $\overline{H}$ is diagonalizable, we can write $\overline{H} = \overline{\kappa}[N]$ for some $\mathbf{Z}/p^n\mathbf{Z}$-module N, and we are reduced to showing that the action of $\mathrm{Gal}(\overline{\kappa}/\kappa)$ on N is continuous. This is clear, since N can be identified with the set of V-fixed elements of $\overline{\mathrm{DM}}_+(\overline{H})$.

If $\overline{H}$ is connected, then we can identify $\overline{\mathrm{DM}}_+(\overline{H})$ with a subset of $\overline{H}$, and the action of $\mathrm{Gal}(\overline{\kappa}/\kappa)$ on that subset is continuous. Since $\overline{H}$ is generated by $\mathrm{DM}(\overline{H})$ as an algebra over κ (Corollary 1.3.18), we conclude that the action of $\mathrm{Gal}(\overline{\kappa}/\kappa)$ on $\overline{H}$ is continuous. □

Remark 1.4.16. Unwinding the definitions, we see that if H is a Hopf algebra over κ which is diagonalizable over $\overline{\kappa}$, then $\mathrm{DM}_+(H)$ can be identified $(W(\overline{\kappa}) \otimes_{\mathbf{Z}} \mathrm{GLike}(H_{\overline{\kappa}}))^{\mathrm{Gal}(\overline{\kappa}/\kappa)}$. The map $V : \mathrm{DM}_+(H) \to \mathrm{DM}_+(H)$ is induced by the automorphism φ^{-1} of $W(\kappa)$, and the map $F : \mathrm{DM}_+(H) \to \mathrm{DM}_+(H)$ is induced by the map $\lambda \mapsto p\varphi(\lambda)$ from $W(\overline{\kappa})$ to itself.

Let us say that a Hopf algebra H over κ is *p-nilpotent* if it is the union of the subalgebas $\{H[p^n]\}_{n\geq 0}$. Let $\mathbf{Hopf}_\kappa^{p-nil}$ denote the full subcategory of $\mathbf{Hopf}_\kappa$ spanned by the p-nilpotent Hopf algebras over κ. Then the construction $H \mapsto \{H[p^n]\}_{n\geq 0}$ determines an equivalence from $\mathbf{Hopf}_\kappa^{p-nil}$ to the homotopy inverse limit of the tower of categories $\{\mathbf{Hopf}_{\kappa,n}\}_{n\geq 0}$. Passing to the limit over n, we obtain the following version of Corollary 1.4.15:

Corollary 1.4.17. *The nonunital symmetric monoidal equivalence* $\mathrm{DM} : \mathbf{Hopf}_\kappa^c \to \mathbf{LMod}_{\mathrm{D}_\kappa}^{\mathrm{Nil}}$ *extends to a fully faithful nonunital symmetric monoidal functor* $\mathrm{DM}_+ : \mathbf{Hopf}_\kappa^{p-nil} \to \mathbf{LMod}_{\mathrm{D}_\kappa}$. *The essential image of this functor is the full subcategory of* $\mathbf{LMod}_{\mathrm{D}_\kappa}$ *consisting of those modules which satisfy condition* $(*)$ *of Proposition 1.4.9.*

Example 1.4.18 (Cartier Duality)**.** Let κ be a field, let $G = \operatorname{Spec} H$ be a finite flat commutative group scheme over κ, and let $\mathbf{D}(G) = \operatorname{Spec} H^\vee$ be its Cartier dual. We then have a bilinear map $\mu : G \times_{\operatorname{Spec}\kappa} \mathbf{D}(G) \to \mathbf{G}_m$. Suppose that G is annihilated by p^n, so that μ factors through the subscheme $\mu_{p^n} \subseteq \mathbf{G}_m$. Then μ is given by a map of Hopf algebras

$$H^\vee \boxtimes H \to \kappa^{\mathbf{Z}/p^n\mathbf{Z}},$$

and therefore induces a pairing of Dieudonne modules

$$\nu : \mathrm{DM}(H^\vee) \times \mathrm{DM}(H) \to \mathrm{DM}(\kappa^{\mathbf{Z}/p^n\mathbf{Z}}).$$

Note that we can identify $\mathrm{DM}(\kappa^{\mathbf{Z}/p^n\mathbf{Z}})$ with the quotient $p^{-n}W(\kappa)/W(\kappa)$. The bilinear pairing therefore induces a map

$$\theta_H : \mathrm{DM}(H^\vee) \to \operatorname{Hom}_{W(\kappa)}(\mathrm{DM}(H), p^{-n}W(\kappa)/W(\kappa)) \simeq \operatorname{Hom}_{W(\kappa)}(\mathrm{DM}(H), W(\kappa)[p^{-1}]/W(\kappa)).$$

Note that the automorphism $\varphi : W(\kappa) \to W(\kappa)$ induces an automorphism of $W(\kappa)[p^{-1}]/W(\kappa)$, which we will also denote by φ. The action of F on $\mathrm{DM}(\kappa^{\mathbf{Z}/p^n\mathbf{Z}}) \simeq p^{-n}W(\kappa)/W(\kappa)$ is then given by φ, while the action of V is given by $z \mapsto p\varphi^{-1}(z)$. Since ν is a pairing of D_κ-modules, we obtain the identities

$$\varphi(\nu(x, Vy)) = \nu(Fx, y) \qquad \nu(Vx, Vy) = p\varphi^{-1}(\nu(x, y)) \qquad \varphi(\nu(Vx, y)) = \nu(x, Fy).$$

Note that the second identity is superfluous (it follows from either of the other identities). The first and third identities imply that θ_H is a map of D_κ-modules, where we regard $\operatorname{Hom}_{W(\kappa)}(\mathrm{DM}(H), W(\kappa)[p^{-1}]/W(\kappa))$ as a left D_κ-module via the action given by

$$(F\lambda)(y) = \varphi(\lambda(Vy)) \qquad (V\lambda)(y) = \varphi^{-1}(\lambda(Fy)).$$

It follows that the kernel of θ_H is a D_κ-submodule of $\mathrm{DM}(H^\vee)$, which classifies a closed subgroup of G on which the pairing μ vanishes. Such a subgroup is automatically trivial, so that θ_H is injective. Since the domain and codomain of θ_H have the same length as $W(\kappa)$-modules, θ_H is an isomorphism. That is, we have a canonical isomorphism of D_κ-modules

$$\mathrm{DM}(H^\vee) \simeq \operatorname{Hom}_{W(\kappa)}(\mathrm{DM}(H), W(\kappa)[p^{-1}]/W(\kappa)),$$

where the action of F and V on the right hand side are given as above.

2 The Morava K-Theory of Eilenberg-MacLane Spaces

Let κ be a perfect field of characteristic $p > 0$, and let $\mathbf{G}_0$ be a 1-dimensional formal group of height $0 < n < \infty$ over κ. To the pair $(\mathbf{G}_0, \kappa)$ one can associate a cohomology theory $K(n)$, called the *nth Morava K-theory.* The Morava K-groups $K(n)_* K(\mathbf{Z}/p^t\mathbf{Z}, d)$ were computed by Ravenel-Wilson in [18] (in the case $p > 2$) and Johnson-Wilson in [12] (in the case $p = 2$). Their results are conveniently stated in the language of Dieudonne modules:

Theorem 2.0.1. *For each $t \geq 0$ and $d > 0$, the canonical map*

$$(\pi_* K(n)) \otimes_\kappa K(n)_0 K(\mathbf{Z}/p^t\,\mathbf{Z}, d) \to K(n)_* K(\mathbf{Z}/p^t\,\mathbf{Z}, d)$$

is an isomorphism (that is, the $K(n)$-homology groups $K(n)_* K(\mathbf{Z}/p^t\,\mathbf{Z}, d)$ are concentrated in even degrees). The group $K(n)_0 K(\mathbf{Z}/p^t\,\mathbf{Z}, d)$ is a finite-dimensional connected Hopf algebra over κ, which is determined by its Dieudonne module $M(d) = \mathrm{DM}(K(n)_0 K(\mathbf{Z}/p^t\,\mathbf{Z}, d))$ (Definition 1.3.3). Moreover, we have an isomorphism of Dieudonne modules $M(d) \simeq \bigwedge_\kappa^d M(1)$, where the action of F and V on $\bigwedge_\kappa^d M(1)$ are determined by the formulas

$$V(x_1 \wedge \cdots \wedge x_d) = Vx_1 \wedge \cdots \wedge Vx_d$$

$$F(Vx_1 \wedge \cdots \wedge Vx_{i-1} \wedge x_i \wedge Vx_{i+1} \wedge \cdots \wedge x_d) = x_1 \wedge \cdots \wedge x_{i-1} \wedge Fx_i \wedge x_{i+1} \wedge \cdots \wedge x_d.$$

We will give the proof of Theorem 2.0.1 in §2.4. Our strategy is to verify the following three assertions, using induction on d:

- (a) The ring $R(d) = K(n)^0 K(\mathbf{Q}_p/\mathbf{Z}_p, d)$ is isomorphic to a formal power series ring over κ.
- (b) The formal group $\operatorname{Spf} R(d)$ is p-divisible, and its Dieudonne module is given by the dth exterior power of the Dieudonne module of the formal group $\mathbf{G}_0 \simeq \operatorname{Spf} R(1)$.
- (c) The group scheme $\operatorname{Spec} K(n)^0 K(\mathbf{Z}/pt\,\mathbf{Z}, d)$ can be identified with the p^t-torsion subgroup of the formal group $\operatorname{Spf} R(d)$.

To carry out the inductive step, we use the Rothenberg-Steenrod spectral sequence to compute the $K(n)$-cohomology groups of $K(\mathbf{Q}_p/\mathbf{Z}_p, d)$ in terms of the $K(n)$-homology groups of $K(\mathbf{Q}_p/\mathbf{Z}_p, d-1)$. We will review the definition of this spectral sequence in §2.3, since the precise construction plays an important role in our proof. The other main ingredient is a purely algebraic result about the cohomology of p-divisible groups (Theorem 2.2.10), which we prove in §2.2.

For the reader's convenience, we include in §2.1 a brief review of some aspects of chromatic homotopy theory that are relevant to this paper, such as the theory of Lubin-Tate spectra and Morava K-theories, and the associated localizations of stable homotopy theory.

2.1 Lubin-Tate Spectra

In this section, we briefly review some concepts from stable homotopy theory which will play an essential role in this paper: specifically, the theory of Lubin-Tate spectra, their associated Morava K-theories, and the corresponding localizations of the stable homotopy category. Our exposition is rather terse, and for the most part proofs have been omitted.

We begin with some general remarks about localization in the setting of stable homotopy theory.

Proposition 2.1.1. *Let Sp denote the ∞-category of spectra, and let $\mathcal{C} \subseteq \mathrm{Sp}$ be a full subcategory. The following conditions are equivalent:*

1. *The inclusion $\iota : \mathcal{C} \hookrightarrow \mathrm{Sp}$ admits a left adjoint F. Moreover, the composite functor $L = \iota \circ F$ is accessible and exact.*
2. *The full subcategory $\mathcal{C} \subseteq \mathrm{Sp}$ is presentable, stable, closed under small limits, and closed under κ-filtered colimits for some sufficiently large regular cardinal κ.*

Proof. If (1) is satisfied, then the exactness of L implies that $\mathcal{C} \simeq L\,\mathrm{Sp}$ is a stable ∞-category; the remaining conditions are established in §HTT.5.5.4. Conversely, suppose that $\mathcal{C}$ satisfies (2). Using Corollary HTT.5.5.2.9, we deduce that the inclusion $\iota : \mathcal{C} \hookrightarrow \mathrm{Sp}$ admits a left adjoint F. Since $\mathcal{C}$ and Sp are presentable and stable, the functors ι and F are exact and accessible (Proposition HTT.5.4.7.7), so that the composition $L = \iota \circ F$ is also accessible and exact. $\square$

In the situation of Proposition 2.1.1, the localization functor L is automatically compatible with the symmetric monoidal structure on the ∞-category Sp:

Proposition 2.1.2. *Let $L : \mathrm{Sp} \to \mathrm{Sp}$ be as in Proposition 2.1.1. Then L is compatible with the smash product of spectra. That is, if X is a spectrum and $f : Y \to Z$ is an L-equivalence of spectra, then the induced map*

$$X \otimes Y \to X \otimes Z$$

is also an L-equivalence.

Proof. The collection of L-equivalences is closed under small colimits. Since the ∞-category Sp is generated (under small colimits) by the collection of spheres S^n, we may reduce to the case where $X = S^n$. In other words, we are reduced to proving that if $f : X \to Y$ is an L-equivalence, then the induced map $\Sigma^n X \to \Sigma^n Y$ is an L-equivalence. This follows immediately from the exactness of L. □

Corollary 2.1.3. *Let $\mathcal{C} \subseteq \mathrm{Sp}$ be a full subcategory satisfying the requirement of Proposition 2.1.1. Then the smash product of spectra induces a symmetric monoidal structure on the ∞-category $\mathcal{C}$. Moreover, the inclusion $\mathcal{C} \hookrightarrow \mathrm{Sp}$ is lax symmetric monoidal, and its left adjoint $L : \mathrm{Sp} \to \mathcal{C}$ is symmetric monoidal.*

Proof. Combine Propositions 2.1.2 and HA.2.2.1.9. □

Notation 2.1.4. In the situation of Corollary 2.1.3, we will sometimes denote the tensor product on the ∞-category $\mathcal{C}$ by $\hat{\otimes}$ (to avoid confusion with the smash product on the ambient ∞-category of spectra). Concretely, this operation is given by

$$X \hat{\otimes} Y = L(X \otimes Y).$$

Proposition 2.1.5. *Let $\mathcal{C} \subseteq \mathrm{Sp}$ be the essential image of an accessible exact localization functor L. Then the localized smash product functor $\hat{\otimes} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ determines a fully faithful embedding $\alpha : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$. The essential image of this embedding is the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ spanned by those functors which preserve small colimits.*

Proof. Let $\mathrm{Fun}'(\mathcal{C}, \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ spanned by those functors which preserve small colimits. It is clear that α factors through $\mathrm{Fun}'(\mathcal{C}, \mathcal{C})$. Let $S \in \mathrm{Sp}$ denote the sphere spectrum. Then evaluation on $LS \in \mathcal{C}$ induces a functor $\beta : \mathrm{Fun}'(\mathcal{C}, \mathcal{C}) \to \mathcal{C}$, and the composition $\beta \circ \alpha$ is homotopic to the identity functor $\mathrm{id}_{\mathcal{C}}$. To prove that α is an equivalence, it will suffice to show that β is fully faithful. Note that β is given by a composition

$$\mathrm{Fun}'(\mathcal{C}, \mathcal{C}) \xrightarrow{\circ L} \mathrm{Fun}'(\mathrm{Sp}, \mathcal{C}) \xrightarrow{\beta'} \mathcal{C},$$

where $\mathrm{Fun}'(\mathrm{Sp}, \mathcal{C})$ is the full subcategory of $\mathrm{Fun}(\mathrm{Sp}, \mathcal{C})$ spanned by those functors which preserve small colimits, and β' is given by evaluation at the sphere spectrum. The map β' is an equivalence of ∞-categories (Corollary HA.1.4.4.6), and the first map is fully faithful by virtue of our assumption that L is a localization functor. □

Remark 2.1.6. In the situation of Proposition 2.1.5, the equivalence $\mathcal{C} \simeq \mathrm{Fun}'(\mathcal{C}, \mathcal{C})$ is a monoidal functor (where we regard $\mathcal{C}$ as endowed with the symmetric monoidal structure given by Corollary 2.1.3, and $\mathrm{Fun}'(\mathcal{C}, \mathcal{C})$ with the monoidal structure given by composition of functors).

In this paper, we will be most interested in the localization of the stable homotopy category with respect to Morava K-theory spectra.

Notation 2.1.7. We define a category $\mathcal{FG}$ as follows:

(1) The objects of $\mathcal{FG}$ are pairs $(R, \mathbf{G})$, where R is a commutative ring and $\mathbf{G}$ is a 1-dimensional formal group over R.

(2) A morphism from $(R, \mathbf{G})$ to $(R', \mathbf{G})$ is given by a pair (ϕ, α), where $\phi : R \to R'$ is a ring homomorphism and $\alpha : \phi^* \mathbf{G} \simeq \mathbf{G}'$ is an isomorphism of formal groups of R'.

We will refer to $\mathcal{FG}$ as the *category of formal groups.* We let $\mathcal{FG}_{\text{pf}}$ denote the full subcategory of $\mathcal{FG}$ spanned by those pairs $(R, \mathbf{G})$ where R is a perfect field of characteristic $p > 0$ and $\mathbf{G}$ is a formal group of finite height over R.

Definition 2.1.8. Let E be an $\mathbb{E}_\infty$-ring spectrum. We will say that E is a *Lubin-Tate spectrum* if the following conditions are satisfied:

(1) As a ring spectrum, E is even periodic. That is, the homotopy groups $\pi_i E$ vanish when i is odd, and there exists an element $\beta \in \pi_{-2}E$ such that multiplication by β induces isomorphisms $\pi_n E \to \pi_{n-2}E$.

(2) The ring $R = \pi_0 E$ is a complete local Noetherian ring having maximal ideal $\mathfrak{m}$, whose residue field $\kappa(E) = R/\mathfrak{m}$ is perfect of characteristic $p > 0$.

(3) Let $\mathbf{G}$ denote the formal group $\operatorname{Spf} \pi_0 E^{\mathbf{CP}^\infty}$ over the commutative ring R, and let $\mathbf{G}_0$ denote the induced formal group over the residue field $\kappa(E)$. Then $\mathbf{G}_0$ has finite height, and $\mathbf{G}$ is a universal deformation of $\mathbf{G}_0$.

In this case, we define the *height* of E to be the height of the formal group $\mathbf{G}_0$.

Let CAlg denote the ∞-category of $\mathbb{E}_\infty$-rings, and $\mathrm{CAlg}_{\mathrm{LT}}$ the full subcategory of CAlg spanned by the Lubin-Tate spectra.

Theorem 2.1.9 (Goerss-Hopkins-Miller)**.** *The construction $E \mapsto (\kappa(E), \mathbf{G}_0)$ determines an equivalence from the ∞-category $\mathrm{CAlg}_{\mathrm{LT}}$ of Lubin-Tate spectra to the (nerve of the) category $\mathcal{FG}_{\mathrm{pf}}$ of one-dimensional formal groups of finite height over perfect fields.*

Notation 2.1.10. Let κ be a perfect field of characteristic $p > 0$, let $\mathbf{G}_0$ be a smooth 1-dimensional formal group over κ of height $0 < n < \infty$. According to Theorem 2.1.9, there exists an (essentially unique) Lubin-Tate spectrum $E = E(\kappa, \mathbf{G}_0)$ for which the formal group $\operatorname{Spf} \pi_0 E^{\mathbf{CP}^\infty}$ is the universal deformation of $\mathbf{G}_0$. We will refer to E as the *Lubin-Tate spectrum* associated to the pair $(\kappa, \mathbf{G}_0)$. Note that $R = \pi_0 E$ is the Lubin-Tate deformation ring of the formal group $\mathbf{G}_0$, which is non-canonically isomorphic to the formal power series ring $W(\kappa)[[u_1, \ldots, u_{n-1}]]$. Set $u_0 = p$, and for $0 \leq i < n$ let $M(i)$ denote the cofiber of the map of E-module spectra $u_i : E \to E$. We let $K(n)$ denote the E-module $\bigotimes_{0 \leq i < n} M_i$ (where the smash product is formed in the symmetric monoidal ∞-category Mod_E of E-module spectra). We will refer to $K(n)$ as the *Morava K-theory spectrum* associated to the pair $(\kappa, \mathbf{G}_0)$. One can show that the homotopy equivalence class of $K(n)$ is independent of system of generators $(u_1, \ldots, u_{n-1})$ chosen for R. It is an E-module spectrum whose homotopy groups are given by

$$\pi_i K(n) \simeq \begin{cases} \kappa & \text{if } i = 2j \\ 0 & \text{if } i = 2j+1. \end{cases}$$

Remark 2.1.11. Lubin-Tate spectra are often referred to in the literature as *Morava E-theories.*

Warning 2.1.12. Our terminology is somewhat nonstandard. Many authors use the notation $K(n)$ to indicate a summand of the spectrum introduced in Notation 2.1.10, whose associated homology theory is periodic of period $2(p^n - 1)$. In this paper, we work exclusively with 2-periodic versions of Morava K-theory.

Definition 2.1.13. Let κ be a perfect field of characteristic $p > 0$, let $\mathbf{G}_0$ be a smooth 1-dimensional formal group of height $0 < n < \infty$ over κ, and let $K(n)$ denote the associated Morava K-theory. We will say that a spectrum X is *$K(n)$-acyclic* if the $K(n)$-homology groups $K(n)_* X$ vanish. We will say that a spectrum Y is *$K(n)$-local* if the mapping space $\mathrm{Map}_{\mathrm{Sp}}(X, Y)$ is contractible whenever X is $K(n)$-acyclic. We let $\mathrm{Sp}_{K(n)}$ denote the full subcategory of Sp spanned by the $K(n)$-local spectra. We refer to $\mathrm{Sp}_{K(n)}$ as the *∞-category of $K(n)$-local spectra.*

Remark 2.1.14. In the situation of Definition 2.1.13, the spectrum $K(n)$ depends on the perfect field κ and the formal group $\mathbf{G}_0$. However, the full subcategory $\mathrm{Sp}_{K(n)} \subseteq \mathrm{Sp}$ is mostly independent of those choices: it depends only on the characteristic p of the field κ, and the height n of the formal group $\mathbf{G}_0$.

Proposition 2.1.15. *Let E be a Lubin-Tate spectrum of height n. Then $\mathrm{Sp}_{K(n)}$ is the essential image of an accessible exact localization functor $L_{K(n)} : \mathrm{Sp} \to \mathrm{Sp}$. Moreover, $\mathrm{Sp}_{K(n)}$ depends only on the integer $n > 0$ and on the characteristic p of the residue field $\kappa(E)$.*

We now discuss the multiplicative structures on Morava K-theories.

Proposition 2.1.16. *Let E be a Lubin-Tate spectrum and let $K(n)$ be the associated Morava E-theory. Then $K(n)$ admits the structure of an $\mathbb{E}_1$-algebra object of Mod_E.*

Remark 2.1.17. The $\mathbb{E}_1$-algebra structure on $K(n)$ is not unique. In fact, one can show that $K(n)$ admits uncountably many pairwise inequivalent $\mathbb{E}_1$-algebra structures.

Notation 2.1.18. Let E be an $\mathbb{E}_\infty$-ring, and let A be an $\mathbb{E}_1$-algebra over E. We let ${}_A\mathrm{BMod}_A(\mathrm{Mod}_E)$ denote the ∞-category of A-A bimodule objects of Mod_E. Then ${}_A\mathrm{BMod}_A(\mathrm{Mod}_E)$ is a presentable monoidal ∞-category (with monoidal structure given by the relative smash product $\otimes_A$), and the tensor product on ${}_A\mathrm{BMod}_A(\mathrm{Mod}_E)$ preserves small colimits separately in each variable. It follows that there is a unique monoidal functor $\mathcal{S} \to {}_A\mathrm{BMod}_A(\mathrm{Mod}_E)$ which preserves small colimits (here we regard $\mathcal{S}$ as a monoidal ∞-category via the Cartesian product). We will denote this functor by $X \mapsto A[X]$. Note that, as a spectrum, we can identify $A[X]$ with the smash product $A \otimes \Sigma^\infty_+ X$. In particular, the homotopy groups $\pi_* A[X]$ can be identified with the A-homology groups $A_*(X)$.

Remark 2.1.19. In the situation of Notation 2.1.18, the construction $M \mapsto \pi_* M$ determines a lax monoidal functor from the ∞-category of A-A bimodule objects of Mod_E to the (nerve of the) ordinary category of graded $(\pi_* A)$-$\pi_* A$ bimodules in $\mathbf{Mod}_{\pi_* E}$. That is, for every pair of objects $M, N \in {}_A\mathrm{BMod}_A(\mathrm{Mod}_E)$, we have a canonical map

$$\mathrm{Tor}_0^{\pi_* E}(\pi_* M, \pi_* N) \to \pi_*(M \otimes_A N).$$

This map is an isomorphism if $\pi_* M$ is flat as a right $\pi_* A$-module, or if $\pi_* N$ is flat as a left $\pi_* A$-module (see Proposition HA.8.2.1.19).

In particular, suppose that E is the Lubin-Tate spectrum associated to a perfect field κ of characteristic $p > 0$ and a one-dimensional formal group $\mathbf{G}_0$ of finite height over κ, and let $A = K(n)$ be the associated Morava K-theory spectrum. Let us regard $K(n)$ as an $\mathbb{E}_1$-algebra over E (Proposition 2.1.16). Since the map $\pi_* E \to \pi_* K(n)$ is surjective, the category of graded $\pi_* K(n)$-bimodule objects of $\mathbf{Mod}_{\pi_* E}$ is equivalent to the category of graded modules over $\pi_* K(n)$. Combining this observation with Notation 2.1.18, we can regard the construction

$$X \mapsto \pi_* K(n)[X] \simeq K(n)_*(X)$$

as a monoidal functor from the ∞-category $\mathcal{S}$ of spaces to (the nerve of) the ordinary category of graded modules over $\pi_* K(n)$.

Definition 2.1.20. Let E be a Lubin-Tate spectrum, and let $K(n)$ be the associated Morava K-theory spectrum. We will say that an object $M \in {}_{K(n)}\mathrm{BMod}_{K(n)}(()\,\mathrm{Mod}_E)$ is *even* if the homotopy groups $\pi_d M$ vanish when d is odd. We let ${}_{K(n)}\mathrm{BMod}^{ev}_{K(n)}(\mathrm{Mod}_E)$ denote the full subcategory of ${}_{K(n)}\mathrm{BMod}_{K(n)}(\mathrm{Mod}_E)$ spanned by the even objects.

We will say that a space X is $K(n)$-*even* if the bimodule $K(n)[X]$ is even. We let $\mathcal{S}^{\mathrm{e}}$ denote the full subcategory of $\mathcal{S}$ spanned by the $K(n)$-even spaces.

Remark 2.1.21. In the situation of Definition 2.1.20, let $M, N \in {}_{K(n)}\mathrm{BMod}_{K(n)}(\mathrm{Mod}_E)$. From the isomorphism $\pi_*(M \otimes_{K(n)} N) \simeq \mathrm{Tor}_0^{\pi_* K(n)}(\pi_* M, \pi_* N)$, we deduce that if M and N are even, then $M \otimes_{K(n)} N$ is even. It follows that the full subcategory ${}_{K(n)}\mathrm{BMod}^{ev}_{K(n)}(E)$ inherits the structure of a monoidal ∞-category. Since the functor $X \mapsto K(n)[X]$ is monoidal, it follows that $\mathcal{S}^{\mathrm{e}}$ is closed under finite products in $\mathcal{S}$.

Remark 2.1.22. Let $K(n)$ be the Morava K-theory spectrum associated to a formal group of height $n < \infty$ over a perfect field κ, and let $\mathcal{C}$ be the category of graded $\pi_* K(n)$-modules which are concentrated in even degrees. Then there is a monoidal equivalence of categories $\mathcal{C} \to \mathrm{Vect}_\kappa$, given by $M_* \mapsto M_0$ (an inverse equivalence $\mathrm{Vect}_\kappa \to \mathcal{C}$ is given by $V \mapsto (\pi_* K(n)) \otimes_\kappa V)$. Applying Remark 2.1.19, we deduce that the construction

$$X \mapsto K(n)_0(X)$$

is a monoidal functor from the ∞-category $\mathcal{S}^{\mathrm{e}}$ to (the nerve of) the category Vect_κ.

Proposition 2.1.23. *Let $K(n)$ be the Morava K-theory spectrum associated to a formal group $\mathbf{G}_0$ of height $n < \infty$ over a perfect field κ. Then the monoidal functor*

$$X \mapsto K(n)_0 X$$

of Remark 2.1.22 is symmetric monoidal. That is, for every pair of $K(n)$-even spaces $X, Y \in \mathcal{S}^{\mathrm{e}}$, the diagram

$$\begin{tikzcd} K(n)_0(X) \otimes_\kappa K(n)_0(Y) \arrow[r] \arrow[d] & K(n)_0(Y) \otimes_\kappa K(n)_0(X) \arrow[d] \\ K(n)_0(X \times Y) \arrow[r] & K(n)_0(Y \times X) \end{tikzcd}$$

commutes.

The proof of Proposition 2.1.23 will require a few preliminaries.

Notation 2.1.24. Let E be a Lubin-Tate spectrum. For every space X, we let $E^\wedge_*(X)$ denote the homotopy groups of the spectrum $L_{K(n)}(E \otimes \Sigma^\infty_+ X)$.

Lemma 2.1.25. *Let E be a Lubin-Tate spectrum and let $K(n)$ denote the associated Morava K-theory. Suppose that N is an E-module spectrum. Assume that N is $K(n)$-local and the homotopy groups $\pi_d(N \otimes_E K(n))$ vanish when d is odd. Then the homotopy groups $\pi_d N$ vanish when d is odd, and the canonical map*

$$\pi_0 N \to \pi_0(N \otimes_E K(n))$$

is surjective.

Proof. Let $R = \pi_0 E$, write $R = W(\kappa)[[u_1, \ldots, u_{n-1}]]$, and set $u_0 = p$. For $0 \leq i < n$, let $M(i)$ denote the cofiber of the map $u_i : E \to E$. For $0 \leq j \leq n$, let $N(j)$ denote the tensor product $N \otimes \bigotimes_{0 \leq i < j} M(i)$ (formed in the symmetric monoidal ∞-category Mod_E), so that we have a sequence of maps

$$N = N(0) \to N(1) \to \cdots \to N(n) = N \otimes_E K(n).$$

We will prove the following assertions using descending induction on j:

(a_j) The homotopy groups $\pi_d N(j)$ vanish when d is odd.

(b_j) The map $\pi_0 N(j) \to \pi_0 N(n)$ is surjective.

Assume that $j < n$ and that assertions (a_{j+1}) and (b_{j+1}) have been verified. For each integer $m \geq 0$, let $T(m)$ denote the cofiber of the map $u_j^m : N(j) \to N(j)$. We have fiber sequences

$$N(j+1) \to T(m+1) \to T(m).$$

It follows by induction on m that the homotopy groups $\pi_d T(m)$ vanish when d is odd, and that the maps $\pi_* T(m+1) \to \pi_* T(m)$ are surjective. Let $T(\infty) = \varprojlim T(m)$. It follows that the groups $\pi_d T(\infty)$ vanishes when d is odd, and that the map $\pi_0 T(\infty) \to \pi_0 T(1) = \pi_0 N(j+1)$ is surjective. To complete the proofs of (a_j) and (b_j), it will suffice to show that the canonical map $\theta : N(j) \to T(\infty)$ is an equivalence. Note that θ becomes an equivalence after tensoring with $M(j)$, and is therefore an equivalence after $K(n)$-localization. Since both $N(j)$ and $T(\infty)$ are $K(n)$-local, we conclude that θ is an equivalence as desired. $\square$

Example 2.1.26. Let X be a space and set $N = L_{K(n)}E[X]$. Then N is $K(n)$-local, and $N \otimes_E K(n) \simeq L_{K(n)}(K(n)[X]) \simeq K(n)[X]$. Consequently, if X is $K(n)$-even, then N satisfies the hypotheses of Lemma 2.1.25, so that the homology groups $E^\wedge_*(X)$ are concentrated in even degrees, and the canonical map $E^\wedge_0(X) \to K(n)_0(X)$ is a surjection.

Remark 2.1.27. It follows from Lemma 2.1.25 that the monoidal structure on the functor

$$\mathcal{S}^{\mathrm{e}} \to \mathrm{Vect}_\kappa$$

$$X \mapsto K(n)_0 X$$

is independent of our choice of ring structure on $K(n)$. That is, if X and Y are $K(n)$-even spaces, then the isomorphism

$$K(n)_0 X \otimes_\kappa K(n)_0 Y \to K(n)_0(X \times Y)$$

does not depend on the multiplication chosen on $K(n)$. To prove this, it will suffice (by virtue of Example 2.1.26) to show that the composite map

$$\mathrm{Tor}_0^{\pi_0 E}(E^\wedge_0(X), E^\wedge_0(Y)) \to K(n)_0 X \otimes_\kappa K(n)_0 Y \to K(n)_0(X \times Y)$$

does not depend on the multiplication of $K(n)$. But this map can also be written as a composition

$$\mathrm{Tor}_0^{\pi_0 E}(E^\wedge_0(X), E^\wedge_0(Y)) \to E^\wedge_0(X \times Y) \to K(n)_0(X \times Y)$$

of maps which do not depend on the $\mathbb{E}_1$-algebra structure of $K(n)$.

Proof of Proposition 2.1.23. Let E be the Lubin-Tate spectrum associated to $(\kappa, \mathbf{G}_0)$, and let $R = \pi_0 E$. Since E is an $\mathbb{E}_\infty$-ring, the construction $X \mapsto E^\wedge_0(X)$ is a lax symmetric monoidal functor from the ∞-category of spaces to the (nerve of the) ordinary category of discrete R-modules. It follows that the diagram

$$\begin{tikzcd}
\mathrm{Tor}_0^R(E^\wedge_0(X), E^\wedge_0(Y)) \arrow[r] \arrow[d] & \mathrm{Tor}_0^R(E_0(Y), E_0(X)) \arrow[d] \\
E_0(X \times Y) \arrow[r] & E_0(Y \times X)
\end{tikzcd}$$

commutes. From this, we deduce the commutativity of the outer rectangle in the diagram

$$\begin{tikzcd}
\mathrm{Tor}_0^R(E^\wedge_0(X), E^\wedge_0(Y)) \arrow[r] \arrow[d] & \mathrm{Tor}_0^R(E^\wedge_0(Y), E^\wedge_0(X)) \arrow[d] \\
K(n)_0(X) \otimes_\kappa K(n)_0(Y) \arrow[r] \arrow[d] & K(n)_0(Y) \otimes_\kappa K(n)_0(X) \arrow[d] \\
K(n)_0(X \times Y) \arrow[r] & K(n)_0(Y \times X).
\end{tikzcd}$$

The desired result now follows from Example 2.1.26. $\square$

Warning 2.1.28. Remark 2.1.27 and Proposition 2.1.23 are generally false if we do not restrict our attention to $K(n)$-even spaces. More precisely, let $\mathcal{C}$ denote the category of graded modules over $\pi_* K(n)$, and let let $F : \mathcal{S} \to \mathcal{C}$ be the functor of Remark 2.1.19. We will regard $\mathcal{C}$ as a symmetric monoidal category (using the usual sign conventions for the tensor product of graded vector spaces). Then:

- Every $\mathbb{E}_1$-structure on $K(n)$ determines a monoidal structure on the functor F: that is, it determines an isomorphism
$$K(n)_*(X) \otimes_{\pi_* K(n)} K(n)_*(Y) \simeq K(n)_*(X \times Y)$$
for every pair of spaces X and Y. However, these isomorphisms depend on the multiplication chosen on $K(n)$.

- Choose an $\mathbb{E}_1$-algebra structure on $K(n)$, and regard F as a monoidal functor. Then F is symmetric monoidal if and only if the multiplication $m : K(n) \otimes_E K(n) \to K(n)$ is homotopy commutative. If $p = 2$, the latter condition is *never* satisfied, so the functor F is never symmetric monoidal.

Construction 2.1.29. Let E be a Lubin-Tate spectrum with residue field κ, and let $K(n)$ denote the associated Morava K-theory. Every $K(n)$-even space X can be regarded as a commutative coalgebra object of the ∞-category $\mathcal{S}^{\mathrm{e}}$. It follows from Proposition 2.1.23 that the vector space $K(n)_0(X)$ inherits the structure of a commutative coalgebra object of Vect_κ. In particular, the dual space $K(n)^0(X) \simeq \mathrm{Hom}_\kappa(K(n)_0(X), \kappa)$ inherits the structure of a linearly compact topological κ-algebra. We let $\mathrm{KSpec}(X)$ denote the formal scheme $\mathrm{Spf}\, K(n)^0(X)$.

Remark 2.1.30. The construction $X \mapsto \mathrm{KSpec}(X)$ determines a functor from the homotopy category of $K(n)$-even spaces to the category of formal schemes over κ, which preserves finite products. In particular, if X is an $K(n)$-even space which has the structure of a group object in the homotopy category $\mathrm{h}\mathcal{S}$ of spaces, then $\mathrm{KSpec}(X)$ is a formal group over κ. If the multiplication on X is homotopy commutative, then the formal group $\mathrm{KSpec}(X)$ is commutative.

2.2 Cohomology of p-Divisible Groups

Let κ be a perfect field of characteristic $p > 0$, fixed throughout this section. Our goal is to study the cohomology of p-divisible groups over κ. We begin with some general definitions.

Definition 2.2.1. Let A be an associative algebra over κ, equipped with an augmentation $\epsilon : A \to \kappa$. For each $n \geq 0$, we let Ext^n_A denote the Ext-group $\mathrm{Ext}^n_A(\kappa, \kappa)$, where we regard κ as an A-module via ϵ. Then Ext^*_A is a graded algebra over κ, which we will refer to as the *cohomology ring* of A.

Example 2.2.2. For any Hopf algebra A over κ, the unit map $\kappa \to \mathrm{Ext}^0_A$ is an isomorphism.

Remark 2.2.3. For our applications in this paper, we are interested only in the special case of Definition 2.2.1 where A is a commutative and cocommutative Hopf algebra, and the augmentation $\epsilon : A \to \kappa$ is the counit map of A. In this case, the algebra Ext^*_A is graded-commutative: that is, for homogeneous elements $x \in \mathrm{Ext}^m_A$, $y \in \mathrm{Ext}^n_A$, we have $xy = (-1)^{mn} yx \in \mathrm{Ext}^{m+n}_A$. One can think of Ext^*_A as the cohomology of the formal group $\mathbf{G} = \mathrm{Spf}\, A^\vee$ (with coefficients in the trivial representation of $\mathbf{G}$).

Remark 2.2.4. Let A be an augmented κ-algebra, and let $\mathfrak{m}_A$ be its augmentation ideal. Then the Ext-groups Ext^n_A can be computed as the cohomology of the reduced cobar complex C^*_A, where $C^m_A = \mathrm{Hom}_\kappa(\mathfrak{m}_A^{\otimes m}, \kappa)$, and the differential $d : C^m_A \to C^{m+1}_A$ is given by
$$(d\lambda)(a_0, \ldots, a_m) = \sum_{1 \leq i \leq m} (-1)^{i+1} \lambda(a_0, a_1, \ldots, a_{i-1}a_i, a_{i+1}, \ldots, a_m).$$
In particular, the differential $C^0_A \to C^1_A$ vanishes, and the kernel of the differential $C^1_A \to C^2_A$ consists of those functionals on $\mathfrak{m}_A$ which vanish on $\mathfrak{m}_A^2$. In particular, we obtain a canonical isomorphism of Ext^1_A with the dual of $\mathfrak{m}_A / \mathfrak{m}_A^2$. If A is a connected Hopf algebra, this is also the dual of the quotient $\mathrm{DM}(A)/F\,\mathrm{DM}(A)$ (Proposition 1.3.20).

Remark 2.2.5. Let A be a connected Hopf algebra over κ, and assume that the relative Frobenius map $F : A^{(p)} \to A$ is trivial. Then A is annihilated by p, and therefore has dimension p^e over κ for some integer e. Then $\mathrm{DM}(A)/F\,\mathrm{DM}(A) \simeq \mathrm{DM}(A)$ has dimension e as a vector space over κ, so that $e = \dim_\kappa \mathrm{Ext}^1_A$.

Definition 2.2.6. Let A be a (commutative and cocommutative) Hopf algebra over κ. We will say that A is *F-divisible* if the following conditions are satisfied:

(a) The relative Frobenius map $F : A^{(p)} \to A$ is an epimorphism of Hopf algebras over κ.

(b) Let $A[F]$ denote the kernel of the map $F : A^{(p)} \to A$ (formed in the abelian category $\mathbf{Hopf}_\kappa$ of Hopf algebras over κ). Then $A[F]$ is finite-dimensional (as a vector space over κ.

Example 2.2.7. Let A be a Hopf algebra over κ. We will say that A is *p-divisible* if it satisfies the following variants of conditions (a) and (b) of Definition 2.2.6:

(a') The Hopf algebra homomorphism $[p] : A \to A$ is an epimorphism (see Notation 1.3.2).

(b') Let $A[p]$ denote the kernel of $[p]$ (formed in the abelian category $\mathbf{Hopf}_\kappa$). Then $A[p]$ is a finite-dimensional vector space over κ.

Every p-divisible Hopf algebra is F-divisible, in the sense of Definition 2.2.6.

Construction 2.2.8. Given a short exact sequence of Hopf algebras

$$\kappa \to A' \to A \to A'' \to \kappa,$$

we obtain a quasi-isomorphism

$$\mathrm{RHom}_A(\kappa, \kappa) \simeq \mathrm{RHom}_{A''}(\kappa, \mathrm{RHom}_A(A'', \kappa)) \simeq \mathrm{RHom}_{A''}(\kappa, \mathrm{RHom}_{A'}(\kappa, \kappa)).$$

The Postnikov filtration on $\mathrm{RHom}_{A'}(\kappa, \kappa)$ determines a spectral sequence of algebras $\{E_r^{s,t}, d_r\}_{r \geq 2}$ converging to $\mathrm{Ext}^*(A)$, whose second page is given by

$$E_2^{s,t} \simeq \mathrm{Ext}^s_{A''}(\kappa, \mathrm{Ext}^t_{A'}).$$

Here the action of A'' on $\mathrm{Ext}^t_{A'}$ is induced by the conjugation action of A'' on A'. Since the comultiplication on A is cocommutative, this action is trivial, so that the action of A'' on $\mathrm{Ext}^t_{A'}$ factors through the counit map $A'' \to \kappa$. If we further assume that A' is Noetherian, then each $\mathrm{H}^t(A')$ is a finite dimensional vector space over κ, so that the canonical map $\mathrm{Ext}^s_{A''} \otimes_\kappa \mathrm{Ext}^t_{A'} \to \mathrm{Ext}^s_{A''}(\kappa, \mathrm{Ext}^t_{A'})$ is an isomorphism and we obtain a canonical isomorphism

$$E_2^{s,t} \simeq \mathrm{Ext}^s_{A''} \otimes_\kappa \mathrm{Ext}^t_{A'}.$$

In particular, we have an exact sequence of low-degree terms

$$0 \to \mathrm{Ext}^1_{A''} \to \mathrm{Ext}^1_A \to \mathrm{Ext}^1_{A'} \xrightarrow{\psi} \mathrm{Ext}^2_{A''} \to \mathrm{Ext}^2_A.$$

Remark 2.2.9. For later use, it will be helpful to have an explicit description of the map $\psi : \mathrm{Ext}^1_{A'} \to \mathrm{Ext}^2_{A''}$ appearing in Construction 2.2.8. Let $\lambda : \mathfrak{m}_{A'} \to \kappa$ be a vector space homomorphism which annihilates $\mathfrak{m}^2_{A'}$, so that we can identify λ with an element of $\mathrm{Ext}^1_{A'}$ (see Remark 2.2.4). Since A is faithfully flat as an A'-algebra, the quotient $\mathfrak{m}_A/\mathfrak{m}_{A'} \simeq A/A'$ is a flat A'-module. It follows that the sequence

$$0 \to \mathfrak{m}_{A'} \to \mathfrak{m}_{A'} \to \mathfrak{m}_A/\mathfrak{m}_{A'} \to 0$$

remains exact after tensoring with $A'/\mathfrak{m}_{A'}$. In particular, the canonical map $\mathfrak{m}_{A'}/\mathfrak{m}^2_{A'} \to \mathfrak{m}_A/\mathfrak{m}_A\mathfrak{m}_{A'}$ is injective, so that λ can be extended to a linear map $\overline{\lambda} : \mathfrak{m}_A \to \kappa$ which vanishes on $\mathfrak{m}_A\mathfrak{m}_{A'}$. Let $\overline{\mu} = d\overline{\lambda} \in C^2_A$: that is, $\overline{\mu}$ is the linear map $\mathfrak{m}_A \otimes_\kappa \mathfrak{m}_A \to \kappa$ given by $\overline{\mu}(x, y) = \overline{\lambda}(xy)$. Since $\overline{\lambda}$ vanishes on $\mathfrak{m}_A\mathfrak{m}_{A'}$, the map $\overline{\mu}$ factors (uniquely) as a composition

$$\mathfrak{m}_A \otimes_\kappa \mathfrak{m}_A \to \mathfrak{m}_{A''} \otimes_\kappa \mathfrak{m}_{A''} \xrightarrow{\mu} \kappa.$$

Unwinding the definitions, we see that $\mu \in C^2_{A''}$ is a cocycle representing $\psi(\lambda) \in \mathrm{Ext}^2_{A''}$.

We can now state our main result.

Theorem 2.2.10. *Suppose we are given an exact sequence of Hopf algebras over* κ

$$\kappa \to A' \to A \xrightarrow{u} A'' \to \kappa.$$

Assume that A *is connected and* F*-divisible, that* A' *is finite-dimensional, and that the map* u *factors through the relative Frobenius* $A''^{(p)} \to A''$*. Let* $\psi : \mathrm{Ext}^1_{A'} \to \mathrm{Ext}^2_{A''}$ *be defined as in Construction 2.2.8. Then:*

(1) *The Hopf algebra* A'' *is connected and* F*-divisible.*

(2) *The map* ψ *induces an isomorphism* $\mathrm{Sym}^*(\mathrm{Ext}^1_{A'}) \to \mathrm{Ext}^*_{A''}$*.*

(3) *Let* $y_1, \ldots, y_m$ *form a basis for* $\mathrm{Ext}^1_{A'}$*. For each* $I = \{i_1 < \ldots < i_k\} \subseteq \{1, \ldots, m\}$*, let* $y_I = y_{i_1} \cdots y_{i_k} \in \mathrm{Ext}^k_{A'}$*. Then* $\mathrm{Ext}^*_{A'}$ *is freely generated by the elements* y_I *as a module over* Ext^*_A*.*

Remark 2.2.11. In the situation of Theorem 2.2.10, the hypothesis that u factors through the relative Frobenius map $A''^{(p)} \to A''$ is automatically satisfied if, for example, the kernel of the map $u^{(p)} : A^{(p)} \to A''^{(p)}$ contains $A[F] = \ker(F : A^{(p)} \to A)$.

Example 2.2.12. Let A be a connected p-divisible Hopf algebra over κ. Then for every integer $t \geq 1$, the exact sequence

$$\kappa \to A[p^t] \to A \xrightarrow{[p^t]} A \to \kappa$$

satisfies the hypotheses of Theorem 2.2.10. It follows that Ext^*_A is canonically isomorphic to the symmetric algebra on the vector space $\mathrm{Ext}^1_{A[p^t]}$. This statement remains valid without the connectedness hypothesis on A, but the connected case will be sufficient for our applications in this paper.

The proof of Theorem 2.2.10 will require some preliminaries.

Proposition 2.2.13. *Let* A *be a Hopf algebra over* κ*, and let* $F : A^{(p)} \to A$ *denote the relative Frobenius map. Then, for each* $n > 0$*,* F *induces the zero map* $\mathrm{Ext}^n_A \to \mathrm{Ext}^n_{A^{(p)}}$*.*

Proof. Note that Ext^n_A is the κ-linear dual of the nth homotopy group of the $\mathbb{E}_\infty$-algebra given by $\kappa \otimes_A \kappa$. The Frobenius map F on A induces a map from the underlying spectrum of $\kappa \otimes_A \kappa$ to itself which agrees with the power operation P^0 of Construction SAG.8.4.2.6, and therefore annihilates the positive homotopy groups of $\kappa \otimes_A \kappa$. □

Corollary 2.2.14. *Suppose we are given an exact sequence of Hopf algebras over* κ

$$\kappa \to A' \to A \xrightarrow{u} A'' \to \kappa,$$

where A *is* F*-divisible and* A' *is finite-dimensional. Then:*

(1) *The Hopf algebra* A'' *is* F*-divisible.*

(2) *The Hopf algebras* $A[F]$ *and* $A''[F]$ *have the same dimension over* κ*.*

(3) *We have* $\mathrm{Ext}^1_A \simeq 0 \simeq \mathrm{Ext}^1_{A''}$*.*

(4) *Suppose that* u *factors through the relative Frobenius map Frobenius map* $F : A''^{(p)} \to A''$*. Then the map* $\psi : \mathrm{Ext}^1_{A'} \to \mathrm{Ext}^2_{A''}$ *is an isomorphism.*

Proof. We have a commutative diagram

$$\begin{tikzcd} A^{(p)} \arrow[r, "u^{(p)}"] \arrow[d] & A''^{(p)} \arrow[d] \\ A \arrow[r, "u"] & A''. \end{tikzcd}$$

Since the left vertical map and the bottom horizontal maps are Hopf algebra epimorphisms, the right vertical map is likewise a Hopf algebra epimorphism. We have a commutative diagram of short exact sequences

$$\begin{tikzcd}
\kappa \arrow[r] & A'^{(p)} \arrow[r] \arrow[d] & A^{(p)} \arrow[r] \arrow[d] & A''^{(p)} \arrow[r] \arrow[d] & \kappa \\
\kappa \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & \kappa.
\end{tikzcd}$$

Applying the snake lemma, we obtain an exact sequence

$$\kappa \to A'[F] \to A[F] \to A''[F] \to \operatorname{coker}(F : A'^{(p)} \to A') \to \kappa.$$

It follows immediately that $A''[F]$ is finite-dimensional as a vector space over κ, which proves (1). Moreover, we have

$$\dim_\kappa(A''[F]) = \frac{\dim_\kappa(A[F]) \dim_\kappa(\operatorname{coker}(F : A'^{(p)} \to A'))}{\dim_\kappa A'[F]}.$$

The short exact sequence

$$\kappa \to A'[F] \to A'^{(p)} \to A' \to \operatorname{coker}(F : A'^{(p)} \to A') \to \kappa$$

gives an equality

$$\frac{\dim_\kappa \operatorname{coker}(F : A'^{(p)} \to A')}{\dim_\kappa A'[F]} = \frac{\dim_\kappa A'}{\dim_\kappa A'^{(p)}} = 1,$$

so that $\dim_\kappa(A''[F]) \simeq \dim_\kappa(A[F])$, thereby proving (2).

Assertion (3) follows from Remark 2.2.4. To prove (4), we note that the spectral sequence of Construction 2.2.8 yields an exact sequence of low degree terms

$$0 \to \operatorname{Ext}^1_{A''} \xrightarrow{\alpha} \operatorname{Ext}^1_A \to \operatorname{Ext}^1_{A'} \xrightarrow{\psi} \operatorname{Ext}^2_{A''} \xrightarrow{\beta} \operatorname{Ext}^2_A.$$

If u factors through the relative Frobenius map of A'', the maps α and β are zero by Proposition 2.2.13. Since $\operatorname{Ext}^1_A \simeq 0$, we conclude that ψ is an isomorphism. □

Example 2.2.15. Let A be a connected F-divisible Hopf algebra over κ. Then $A^{(p)}$ is also connected and F-divisible, and the exact sequence

$$\kappa \to A[F] \to A^{(p)} \xrightarrow{F} A \to \kappa$$

satisfies the hypotheses of Corollary 2.2.14. It follows that we have a canonical isomorphism $\operatorname{Ext}^2_A \simeq \operatorname{Ext}^1_{A[F]}$. Using Remark 2.2.5, we deduce that $\dim_\kappa A[F] = p^e$, where $e = \dim_\kappa \operatorname{Ext}^2_A$.

Remark 2.2.16. Suppose we are given an exact sequence of connected Hopf algebras

$$\kappa \to A' \to A \to A'' \to \kappa$$

satisfying the hypotheses of Corollary 2.2.14. Then $\dim_\kappa A[F] \simeq \dim_\kappa A''$. It follows from Example 2.2.15 that $\dim_\kappa \operatorname{Ext}^2_A = \dim_\kappa \operatorname{Ext}^2_{A''}$.

Lemma 2.2.17. *Let B be a finite-dimensional Hopf algebra over κ. Assume that B is local and connected. Then, as an algebra over κ, B is isomorphic to a tensor product of algebras of the form $\kappa[T]/(T^{p^e})$.*

Proof. Let $M = \mathrm{DM}(B)$ be the Dieudonne module of B. Since B is local, the action of F on M is locally nilpotent. For each $m \geq 0$, let $W(m)$ denote the quotient $F^m M / F^{m+1} M$. For each $m \geq 0$, let d_m denote the dimension of $W(m)$ as a vector space over κ. Since B is finite dimensional, we have $W(m) \simeq 0$ for $m \gg 0$, and $\dim_\kappa B = p^{\sum d_m}$. Let $\varphi : \kappa \to \kappa$ be the Frobenius map, so that F induces a φ-semilinear surjection of

κ-vector spaces $\theta_m : W(m) \to W(m+1)$ for $m \gg 0$. We may therefore choose a basis $\{v(m)_1, \ldots, v(m)_{d_m}\}$ for each $W(m)$ with the property that

$$\theta_m v(m)_i = \begin{cases} v(m+1)_i & \text{if } 1 \leq i \leq d_{m+1} \\ 0 & \text{otherwise.} \end{cases}.$$

Let $d = d_0$, and let $v_1, \ldots, v_d \in M$ be a collection of representatives for $v(0)_1, \ldots, v(0)_d \in W(0) = M/FM$. For $1 \leq i \leq d$, let e_i be the smallest integer such that $i > d_{e_i}$. Then the image of $F^{e_i} v_i$ vanishes in $W(e_i)$, so that $F^{e_i} v_i \in F^{e_i+1} M$. Altering our choice of v_i, we may assume that $F^{e_i} v_i = 0$. We will identify M with a subset of B, so that each v_i is an element of B satisfying $v_i^{p^{e_i}} = 0$. Consequently, there is a unique map of commutative rings

$$\theta : \bigotimes_{1 \leq i \leq m} \kappa[T_i]/(T_i^{p^{e_i}}) \to B$$

carrying each T_i to the element $v_i \in B$. We will complete the proof by showing that θ is an isomorphism. Since the domain and codomain of θ have the same dimension (as vector spaces over κ), it will suffice to show that θ is surjective. For this, it suffices to show that θ induces a surjection on Zariski cotangent spaces, which follows from Proposition 1.3.20. $\square$

Lemma 2.2.18. *Let B be a finite-dimensional connected Hopf algebra over κ, and let $d = \dim_\kappa \mathrm{Ext}^1_B$. Then, for each integer $m \geq 0$, we have $\dim_\kappa \mathrm{Ext}^m_B = \binom{m+d-1}{m}$ (in other words, the Poincare series of the graded ring Ext^*_B is equal to the Poincare series for a polynomial ring in d-variables over κ). Moreover, Ext^*_B is generated (as a ring) by elements of degree ≤ 2.*

Proof. Since κ is perfect, we can write B as a tensor product of Hopf algebras $B_0 \otimes_\kappa B_1$, where B_0 is a local ring and B_1 is étale over κ. Since there is a canonical isomorphism $\mathrm{Ext}^*_B \simeq \mathrm{Ext}^*_{B_0}$, we may replace B by B_0 and thereby reduce to the case where B is a local ring.

According to Lemma 2.2.17, B is isomorphic to a tensor product of algebras of the form $\kappa[T]/(T^{p^e})$. The result now follows from the observation that $\mathrm{Ext}^*_{\kappa[T]/(T^{p^e})}(\kappa, \kappa) \simeq \mathrm{H}^*(B\,\mathbf{Z}/p^e\,\mathbf{Z}; \kappa)$ is either a polynomial ring on a single class in degree 1 (if $p^e = 2$) or the tensor product of an exterior algebra on a class of degree 1 and a polynomial algebra on a class of degree 2 (if $p^e > 2$). $\square$

Proof of Theorem 2.2.10. Let $\{E_r^{s,t}, d_r\}_{r \geq 2}$ be the spectral sequence associated to the short exact sequence of Hopf algebras

$$\kappa \to A' \to A \xrightarrow{u} A'' \to \kappa,$$

so that $E_2^{s,t} \simeq \mathrm{H}^s(A'') \otimes_\kappa \mathrm{H}^t(A')$. Let $W \subseteq \mathrm{H}^2(A')$ be the image of the restriction map $\mathrm{Ext}^2_A \to \mathrm{Ext}^2_{A'}$, so that we can identify W with the subspace of $E_2^{0,2}$ consisting of permanent cycles. Choose a basis $y_1, \ldots, y_n$ for the vector space $\mathrm{Ext}^1_{A'} \simeq E_2^{0,1}$ over κ. For each subset $I = \{i_1 < \ldots < i_k\} \subseteq \{1, \ldots, n\}$, we let y_I denote the product $y_{i_1} y_{i_2} \cdots y_{i_k} \in \mathrm{Ext}^k_{A'}$.

According to Corollary 2.2.14, the differential d_2 induces an isomorphism $\psi : \mathrm{Ext}^1_{A'} \to \mathrm{Ext}^2_{A''}$, and $\mathrm{Ext}^1_{A''}$ vanishes. It follows that $E_2^{1,1} \simeq E_3^{2,0} \simeq 0$, so that the restriction map $\mathrm{Ext}^2_A \to E_2^{0,2} \simeq \mathrm{Ext}^2_{A'}$ is injective, and therefore induces an isomorphism from Ext^2_A to W. Using Remark 2.2.16, we obtain

$$\dim_\kappa(W) = \dim_\kappa \mathrm{Ext}^2_A = \dim_\kappa \mathrm{Ext}^2_{A''} = \dim_\kappa \mathrm{Ext}^1_{A'} = n.$$

The elements $\psi(y_1), \ldots, \psi(y_n)$ form a basis for $\mathrm{Ext}^2_{A''}$. The differential d_2 also induces a map $\psi' : \mathrm{Ext}^2_{A'} \to \mathrm{Ext}^1_{A'} \otimes_\kappa \mathrm{Ext}^2_{A''}$. Since d_2 is a derivation with respect to the algebra structure on $E_2^{*,*}$, we obtain

$$\psi'(y_i y_j) = y_i \otimes \psi(y_j) - y_j \otimes \psi(y_i).$$

The collection of elements $\{y_i \otimes \psi(y_j) - y_j \otimes \psi(y_i)\}_{1 \leq i < j \leq n}$ are linearly independent in $\mathrm{Ext}^1_{A'} \otimes_\kappa \mathrm{Ext}^2_{A''}$. It follows that the elements $\{y_i y_j\}_{1 \leq i < j \leq n}$ form a basis for a subspace $W' \subseteq \mathrm{Ext}^2_{A'}$, and that $\psi' | W'$ is injective. Since W consists of permanent cycles, we have $W \cap W' = \{0\}$. The dimension of W' is $\frac{n^2 - n}{2}$. Applying

Lemma 2.2.18, we deduce that $\dim_\kappa \mathrm{Ext}^2_{A'} = \frac{n^2+n}{2} = \dim_\kappa W + \dim_\kappa W'$. From this, we deduce that $\mathrm{Ext}^2_{A'}$ is a direct sum of W and W'. Since ψ' is injective when restricted to W', we conclude that $W = \ker(\psi')$.

We next prove the following:

$(*)$ As a module over $\mathrm{Sym}^* W$, $\mathrm{Ext}^*_{A'}$ is freely generated by the elements $\{y_I\}_{I \subseteq \{1,\ldots,n\}}$.

To prove this, let us regard $\mathrm{Sym}^* W$ as a graded ring (with elements of W regarded as homogeneous of degree 2), and let M^* denote the graded $\mathrm{Sym}^* W$-module freely generated by elements $\{Y_I\}_{I \subseteq \{1,\ldots,n\}}$, where we regard Y_I as being of degree $|I|$. There is a unique homomorphism of graded $\mathrm{Sym}^*(W)$-modules $\nu : M^* \to \mathrm{Ext}^*_{A'}$, given on generators by $\nu(Y_I) = y_I$. We wish to show that ν is an isomorphism. Using Lemma 2.2.18, we see that $\dim_\kappa M^m = \dim_\kappa \mathrm{Ext}^*_{A'}$ for each $m \geq 0$. Consequently, it will suffice to show that ν is surjective. It is evident from the construction that ν induces a surjection $M^m \to \mathrm{Ext}^*_{A'}$ for $m \leq 2$. Since $\mathrm{Ext}^*_{A'}$ is generated (as a ring) by elements of degree ≤ 2, we are reduced to proving that the image of ν is a subring of $\mathrm{Ext}^*_{A'}$.

Fix an element $x \in \mathrm{im}(\nu) \subseteq \mathrm{Ext}^*_{A'}$ belonging to the image of ν. We wish to show that for all $x' \in \mathrm{im}(\nu)$, we have $xx' \in \mathrm{im}(\nu)$. It clearly suffices to prove this in the special case where $x' \in \mathrm{Sym}^* W$ or $x' = y_i$ for some i. The first case is obvious (since ν is a $\mathrm{Sym}^* V$-module homomorphism). In the second case, we may assume that $x = x_0 y_I$ for some $x_0 \in \mathrm{Sym}^* W$ and some $I \subseteq \{1,\ldots,n\}$. If $i \notin I$, we have $xx' = \pm x_0 y_{I \cup \{i\}} = \pm \nu(x_0 Y_{I \cup \{i\}})$. We may therefore suppose that $i \in I$, so that $x = x_1 y_i$ for some $x_1 \in \mathrm{im}(\nu)$. Note that $\psi'(y_i^2) = y_i \otimes \psi(y_i) - y_i \otimes \psi(y_i) = 0$, so that $y_i^2 \in \ker(\psi') = W$. Since the image of ν is stable under multiplication by $\mathrm{Sym}^* W$, we conclude that $xx' = x_1 y_i^2 \in \mathrm{im}(\nu)$, as desired. This completes the proof of $(*)$.

We now construct some auxiliary spectral sequences. We let $\{E(0)^{s,t}_r, d_r\}$ be the spectral sequence given by

$$E(0)^{s,t}_r = \begin{cases} \mathrm{Sym}^m(W) & \text{if } s = 0, t = 2m \\ 0 & \text{otherwise,} \end{cases}$$

with all differentials trivial. For $1 \leq i \leq n$, let $\{E(i)^{s,t}_r, d_r\}$ be the spectral sequence given by

$$E(i)^{s,t}_r = \begin{cases} \kappa & \text{if } s = t = 0 \\ \kappa Y_i & \text{if } s = 0, t = 1, r = 2 \\ \kappa Z_i & \text{if } s = 2, t = 0, r = 2 \\ 0 & \text{otherwise,} \end{cases}$$

where the differential d_2 carries Y_i to Z_i. The inclusion $W \hookrightarrow E_2^{0,2}$ induces a map of spectral sequences $E(0)^{s,t}_r \to E^{s,t}_r$. Similarly, for $1 \leq i \leq n$ we have a map of spectral sequences $E(i)^{s,t}_r \to E^{s,t}_r$, given by $Y_i \mapsto y_i$ and $Z_i \mapsto \psi(y_i)$. Since $\{E^{s,t}_r, d_r\}$ is a spectral sequence of algebras, we can tensor these maps together to obtain a map of spectral sequences $\xi : \{E'^{s,t}_r, d_r\} \to \{E^{s,t}_r, d_r\}$, where $E'^{s,t}_r$ is the tensor product of the spectral sequences $E(i)^{s,t}_r$. Note that $E'^{*,0}_2$ is a polynomial algebra on the classes $\{Z_i\}_{1 \leq i \leq n}$. Consequently, to prove (1), it will suffice to show that ξ induces an isomorphism $E'^{*,0}_2 \to E^{*,0}_2 \simeq \mathrm{Ext}^*_{A''}$. In fact, we will show that ξ is an isomorphism of spectral sequences. For this, it suffices to verify the following assertion for each $m \geq 0$:

$(\star_m)$ The map ξ induces an isomorphism $E'^{s,t}_2 \to E^{s,t}_2$ when $s+t < m$, and a monomorphism when $s+t = m$.

Note that $(\star_m)$ is equivalent to the apparently weaker assertion that $E'^{s,0}_2 \to E^{s,0}_2 \simeq \mathrm{Ext}^s_{A''}$ is bijective for $s < m$ and injective for $s = m$. This condition is evidently satisfied for $m \leq 3$. We will prove it in general using induction on m. Assume that $m \geq 3$ and that condition $(\star_m)$ holds; we wish to verify $(\star_{m+1})$. We first show that the map $E'^{m,0}_2 \to E^{m,0}_2$ is surjective. Suppose otherwise: then there exists a class $\eta \in E^{m,0}_2 \simeq \mathrm{Ext}^m_{A''}$ which does not belong to the image of ξ. Using the inductive hypothesis, we see that the image of η in $E^{m,0}_r$ cannot be a coboundary for any $r \geq 2$, so that η has nontrivial image $\overline{\eta} \in E^{m,0}_\infty$. It follows that η has nonzero image under the pullback map $\mathrm{Ext}^m_{A''} \to \mathrm{Ext}^m_A$, contradicting Proposition 2.2.13.

We now show that ξ induces an injection $E_2'^{m+1,0} \to E_2^{m+1,0}$. If m is even, then $E_2'^{m+1,0} \simeq 0$ and there is nothing to prove. Assume therefore that m is odd, and let $\eta \in E_2'^{m+1,0}$ lies in the kernel of ξ. Note that the differential $d_2 : E_2'^{m-1,1} \to E_2'^{m+1,0}$ is surjective, so we can write $\eta = d_2(\alpha)$ for some $\alpha \in E_2'^{m-1,1}$. Then $\xi(\alpha) \in E_2^{m-1,1}$ lies in the kernel of the map $d_2 : E_2^{m-1,1} \to E_2^{m+1,0}$, and is therefore a permanent cycle. Since the map $0 \simeq E_2'^{m,0} \to E_2^{m,0}$ is surjective, we have $\mathrm{Ext}^m_{A''} \simeq 0$. It follows that the image of α must be trivial in $E_\infty^{m-1,1}$. Note that condition $(\star_m)$ implies ξ induces a surjection $E_r'^{s,t} \to E_r^{s,t}$ for $s+t < m$. Since the differential d_r on $E_r'^{s,t}$ is trivial for $r > 2$, we conclude that the differential $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$ vanishes for $s+t < m$, so that $E_\infty^{m-1,1} \simeq E_3^{m-1,1} = \mathrm{coker}(d_2 : E_2^{m-3,2} \to E_2^{m-1,1})$. Then $\xi(\alpha) = d_2(\overline{\beta})$ for some $\overline{\beta} \in E_2^{m-3,2}$. Using $(\star_m)$, we can write $\overline{\beta} = \xi(\beta)$ for some $\beta \in E_2'^{m-3,2}$. Then $\xi(d_2(\beta)) = \xi(\alpha)$, so that condition $(\star_m)$ implies that $\alpha = d_2(\beta)$. Then $\eta = d_2(d_2(\beta)) = 0$, as desired. This completes the proof of (1). Assertion (2) follows from (1) and $(*)$. □

2.3 The Spectral Sequence of a Filtered Spectrum

Let $K(n)$ denote the Morava K-theory spectrum associated to a formal group $\mathbf{G}_0$ of finite height over a perfect field κ of characteristic $p > 0$. Let X be an Eilenberg-MacLane space $K(\mathbf{Z}/p\,\mathbf{Z}, m)$. Our goal in §2.4 is to compute the Morava K-theory $K(n)^*(X)$. The basic strategy is to use induction on m. Let $G = K(\mathbf{Z}/p\,\mathbf{Z}, m-1)$, so that (without loss of generality) we can regard G as a topological abelian group whose classifying space BG is homotopy equivalent to X. Then X is equipped with the corresponding Milnor filtration, given by partial realizations of the standard simplicial topological space with geometric realization BG. This filtration determines a spectral sequence converging to $K(n)^*(X)$, whose second page can be calculated in terms of the Morava K-homology groups $K(n)_*(G)$. For our applications, we will need to know not only that such a spectral sequence exists, but the exact details of its construction. Our goal in this section is to review the relevant details.

We begin with a more general construction: the spectral sequence associated to a filtered spectrum. Our exposition will be somewhat terse; for a more detailed account (with more proofs and slightly different notational conventions), we refer the reader to §HA.1.2.2.

Definition 2.3.1. Let Sp denote the ∞-category of spectra, and let $\mathbf{Z}$ denote the linearly ordered set of integers (regarded as a category). A *filtered spectrum* is a functor $X : \mathrm{N}(\mathbf{Z})^{\mathrm{op}} \to \mathrm{Sp}$.

In other words, a filtered spectrum is a diagram of spectra

$$\cdots \to X(2) \to X(1) \to X(0) \to X(-1) \to X(-2) \to \cdots$$

Notation 2.3.2. Let X be a filtered spectrum. We let $X(\infty)$ denote the limit $\varprojlim_n X(n)$. For $m \leq n \leq \infty$, we let $X(n,m)$ denote the fiber of the canonical map $X(n) \to X(m)$.

Construction 2.3.3. Given a filtered spectrum X and integers s, t, and r with $r \geq 1$, we define subgroups

$$B_r^{s,t}(X) \subseteq Z_r^{s,t}(X) \subseteq \pi_{t-s} X(s, s-1)$$

as follows:

- $Z_r^{s,t}(X)$ is the image of the map $\pi_{t-s} X(s+r-1, s-1) \to \pi_{t-s} X(s, s-1)$
- $B_r^{s,t}(X)$ is the kernel of the composite map

$$Z_r^{s,t}(X) \hookrightarrow \pi_{t-s} X(s, s-1) \to \pi_{t-s} X(s, s-r).$$

We let $E_r^{s,t}(X)$ denote the quotient $Z_r^{s,t}(X)/B_r^{s,t}(X)$. The fiber sequence of spectra

$$X(s+r, s+r-1) \to X(s+r, s-1) \to X(s+r-1, s-1).$$

determines a boundary map $\delta : \pi_{s-t}X(s+r-1, s-1) \to Z_r^{s+r,t+r-1}(X) \subseteq \pi_{s-t}X(s+r, s+r-1)$. There is a unique map $d_r : E_r^{s,t}(X) \to E_r^{s+r,t+r-1}(X)$ which fits into a commutative diagram

$$\begin{tikzcd}
\pi_{s-t}X(s+r-1,s-1) \arrow[r] \arrow[d, "\delta"] & Z_r^{s,t}(X) \arrow[r] & E_r^{s,t}(X) \arrow[d, "d_r"] \\
Z_r^{s+r,t+r-1}(X) \arrow[rr] & & E_r^{s+r,t+r-1}(X).
\end{tikzcd}$$

The collection $\{E_r^{s,t}, d_r\}_{r\geq 1}$ is a spectral sequence of abelian groups, which depends functorially on the filtered spectrum X.

Note that we can identify $E_r^{s,t}(X)$ with the image of the map $\pi_{t-s}X(s+r-1, s-1) \to \pi_{t-s}X(s, s-r)$. In particular, if $X(i) \simeq 0$ for $i < 0$, then for $r > s$ we can identify $E_r^{s,t}(X)$ with the subgroup of $\pi_{t-s}X(s)$ given by the image of $\pi_{t-s}X(s+r-1, s-1)$. We therefore have canonical monomorphisms

$$\cdots \hookrightarrow E_{s+2}^{s,t}(X) \hookrightarrow E_{s+1}^{s,t}(X).$$

We let $E_\infty^{s,t}(X)$ denote the inverse limit of this diagram.

Let X be a filtered spectrum. In good cases, one can show that the spectral sequence of Construction 2.3.3 converges to the homotopy groups of the spectrum $X(\infty)$. For example, we have the following result:

Proposition 2.3.4. *Let $X : \mathrm{N}(\mathbf{Z})^{\mathrm{op}} \to \mathrm{Sp}$ be a filtered spectrum. Suppose that the following conditions are satisfied:*

(a) *The spectrum $X(s)$ vanishes for $s < 0$.*

(b) *For every pair of integers $s, t \in \mathbf{Z}$, we have $E_r^{s,t} \simeq E_{r+1}^{s,t}$ for $r \gg 0$.*

For each $s \geq 0$, let $F^s\pi_n X(\infty)$ denote the kernel of the map $\pi_n X(\infty) \to \pi_n X(s)$. Then:

(1) *For each integer n, the abelian group $\varprojlim^1\{\pi_n X(s)\}_{s\geq 0}$ is trivial.*

(2) *The canonical map $\pi_n X(\infty) \to \varprojlim\{\pi_n X(s)\}_{s\geq 0}$ is an isomorphism.*

(3) *The canonical map $\pi_n X(\infty) \to \varprojlim \pi_n X(\infty)/F^s\pi_n X(\infty)$ is an isomorphism.*

(4) *We have canonical isomorphisms $F^s\pi_n X(\infty)/F^{s+1}\pi_n X(\infty) \simeq E_\infty^{s,n+s}(X)$.*

Proof. Fix an integer $n \geq 0$. For $a \leq b$, let $\phi_{a,b} : \pi_n X(b) \to \pi_n X(a)$ be induced by the spectrum map $X(b) \to X(a)$. We first prove the following:

($*$) For each $s \in \mathbf{Z}$, there exists another integer $s' \geq s$ with the following property: for $s'' \geq s'$, the maps $\phi_{s,s'}$ and $\phi_{s,s''}$ have the same image in $\pi_n X(s)$.

The proof of ($*$) proceeds by induction on s, the case $s < 0$ being vacuous by virtue of assumption (a). To handle the inductive step, choose $s' \geq s$ with the property that $\mathrm{im}(\phi_{s-1,s'}) = \mathrm{im}(\phi_{s-1,s''})$ for $s'' \geq s'$. Condition (b) implies that there exists $r > s$ such that $E_r^{s,n-s} \simeq E_{r'}^{s,n-s}$ for $r' \geq r$. Enlarging s' if necessary, we may suppose that $s' \geq s + r - 1$. We now claim that s' satisfies the requirements of ($*$). To prove this, suppose that $\eta \in \pi_n X(s)$ lies in the image of the map $\phi_{s,s'}$, and let $s'' \geq s'$. We wish to show that $\eta \in \mathrm{im}(\phi_{s,s''})$. Note that $\phi_{s-1,s}(\eta)$ belongs to the image of $\phi_{s-1,s'}$, so we can write $\phi_{s-1,s}(\eta) = \phi_{s-1,s''}(\overline{\eta})$ for some $\overline{\eta} \in \pi_n X(s'')$. Replacing η by $\eta - \phi_{s,s''}(\overline{\eta})$, we can reduce to the case where $\phi_{s-1,s}(\eta) = 0$. Write $\eta = \phi_{s,s'}(\eta')$. Then η' belongs to the kernel of the map $\pi_n X(s') \to X(s-1)$, and therefore to the image of the map $\pi_n X(s', s-1) \to \pi_n X(s')$. It follows that $\eta \in E_{r'}^{s,n-s} \subseteq \pi_n X(s)$ for $r' = s' + 1 - s$. Since $r' \geq r$, we have $E_{r'}^{s,n-s} = E_{r''}^{s,n-s}$, where $r'' = s'' + 1 - s$. It follows that η belongs to the image of the map $\pi_n X(s'', s-1) \to \pi_n X(s)$, and in particular to the image of $\phi_{s,s''}$.

For each $s \geq 0$, let T_s denote the intersection $\bigcap_{s' \geq s} \mathrm{im}(\phi_{s,s'}) \subseteq \pi_n X(s)$. We claim that for $a \leq b$, the map $\phi_{a,b}$ induces a surjection $T_b \to T_a$. To prove this, suppose that $\eta \in T_a$. Using $(*)$, we deduce that $T_b = \mathrm{im}(\phi_{b,b'})$ for some $b' \geq a$. Then $\eta = \phi_{a,b'}(\overline{\eta})$ for some $\overline{\eta} \in \pi_n X(b')$. Then $\eta = \phi_{a,b}(\phi_{b,b'}(\overline{\eta}) \in \phi_{a,b}(T_b)$.

For each $s \geq 0$, condition $(*)$ implies that there exists $s' \geq s$ such that the map $\phi_{s,s'}$ carries $\pi_n X(s')$ into T_s. It follows that $\phi_{s,s'}$ induces the zero map from $\pi_n X(s')/T_{s'}$ to $\pi_n X(s)/T_s$. We have a short exact sequence of towers of abelian groups

$$0 \to \{T_s\}_{s \geq 0} \to \{\pi_n X(s)\}_{n \geq 0} \to \{\pi_n X(s)/T_s\}_{s \geq 0} \to 0,$$

giving rise to exact sequences

$$0 \to \varprojlim \{T_s\}_{s \geq 0} \to \varprojlim \{\pi_n X(s)\}_{s \geq 0} \to \varprojlim \{\pi_n X(s)/T_s\}_{s \geq 0}$$

$$\varprojlim \{\pi_n X(s)/T_s\}_{s \geq 0} \to \varprojlim{}^1 \{T_s\}_{s \geq 0} \to \varprojlim{}^1 \{\pi_n X(s)\}_{s \geq 0} \to \varprojlim{}^1 \{\pi_n X(s)/T_s\}_{s \geq 0}.$$

The above argument shows that $\{\pi_n X(s)/T_s\}_{s \geq 0}$ is a zero object in the category of pro-abelian groups, so that $\varprojlim{}^i \{\pi_n X(s)/T_s\}_{s \geq 0} \simeq 0$ for $i = 0, 1$. We therefore obtain isomorphisms

$$\varprojlim \{T_s\}_{s \geq 0} \to \varprojlim \{\pi_n X(s)\}_{s \geq 0} \qquad \varprojlim{}^1 \{T_s\}_{s \geq 0} \to \varprojlim{}^1 \{\pi_n X(s)\}_{s \geq 0}.$$

Since each of the maps $T_{s+1} \to T_s$ is surjective, we have $\varprojlim{}^1 \{T_s\}_{s \geq 0} \simeq 0$. This proves (1), which immediately implies (2).

Note that $\pi_n X(\infty)/F^s \pi_n X(\infty)$ can be identified with the image of the map $\phi_{s,\infty} : \pi_n X(\infty) \to \pi_n X(s-1)$. To prove (3), it will therefore suffice to show that $T_s = \mathrm{im}(\phi_{s,\infty})$. The containment $\mathrm{im}(\phi_{s,\infty}) \subseteq \bigcap_{s' \geq s} \mathrm{im}(\phi_{s,s'}) = T_s$ is clear. To prove the converse, it suffices to prove that the map

$$\pi_n X(\infty) = \varprojlim \{T_a\}_{a \geq 0} \to T_s$$

is surjective, which follows from the surjectivity of the maps $T_{a+1} \to T_a$.

We now prove (4). Note that $\phi_{s,\infty}$ induces a bijection from $F^s \pi_n X(\infty)/F^{s+1} \pi_n X^{\infty}$ to the kernel of the map $T_s \to \pi_n X(s-1)$. On the other hand, $E_\infty^{s,n-s}$ can be identified with the subgroup of $\pi_n X(s)$ given by the intersection of the images of the maps $\pi_n X(s', s-1) \to \pi_n X(s)$. The inclusion $E_\infty^{s,n-s} \subseteq \ker(T_s \to \pi_n X(s-1))$ is clear. To prove the reverse inclusion, suppose that $\eta \in T_s$ and $\phi_{s-1,s}(\eta) = 0$; we wish to prove that $\eta \in E_\infty^{s,n-s}$. Choose $s' \geq s$; we will show that η belongs to the image of the map $\pi_n X(s', s-1) \to \pi_n X(s)$. Since $\eta \in T_s$, we can write $\eta = \phi_{s,s'}(\eta')$ for some $\eta' \in X(s')$. Then $\eta' \in \ker(\phi_{s-1,s'})$, so that η' belongs to the image of the map $\pi_n X(s', s-1) \to \pi_n X(s')$, from which we conclude that η belongs to the image of the composite map

$$\pi_n X(s', s-1) \to \pi_n X(s') \to \pi_n X(s).$$

$\square$

Example 2.3.5. Let G be a topological group, let BG be its classifying space, and let $K(n)$ be a Morava K-theory. The classifying space BG can be described as the geometric realization of a simplicial space $Y_\bullet$, with $Y_s = G^s$. The construction $[s] \mapsto K(n)^{Y_s}$ determines a cosimplicial spectrum with totalization $K(n)^{BG}$. In particular, we can identify $K(n)^{BG}$ with the limit of a tower of spectra

$$\cdots \to X(2) \to X(1) \to X(0)$$

with $X(s) = \mathrm{Tot}_s\, K(n)^{Y_\bullet}$. Let $\{E_r^{s,t}, d_r\}_{r \geq 1}$ be the spectral sequence associated to this tower of spectra (see Construction 2.3.3). The second page of this spectral sequence is given by the cohomology of the (normalized or unnormalized) cochain complex associated to the cosimplicial graded abelian group $\pi_* K(n)^{Y_\bullet}$.

Now suppose that the topological group G is $K(n)$-even, in the sense of Definition 2.1.20, and let $A = K(n)_0(G)$. Then the group structure on G exhibits A as a cocommutative (but generally non-commutative) Hopf algebra over κ (Remark 2.1.30). For each $s \geq 0$, Remark 2.1.19 supplies a canonical equivalence

$$\pi_* K(n)^{Y_s} \simeq \pi_* K(n) \otimes_\kappa \mathrm{Hom}_\kappa(A^{\otimes s}, \kappa).$$

Consequently, we obtain a canonical isomorphism

$$E_2^{s,t} \simeq \pi_t K(n) \otimes_\kappa \mathrm{Ext}_A^s,$$

where Ext_A^s is as in Definition 2.2.1 (see Remark 2.2.4).

Remark 2.3.6. In the situation of Example 2.3.5, the spectral sequence $\{E_r^{s,t}, d_r\}_{r\geq 1}$ is automatically convergent, since it can be identified with the κ-linear dual of the Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_s^{K(n)_* G}(\pi_* K(n), \pi_* K(n)) \Rightarrow K(n)_{*+s} BG$$

obtained from the equivalence of spectra

$$K(n) \otimes_{K(n)[G]} K(n) \simeq K(n)[BG].$$

We will not need this observation: in the applications we describe in §2.4, it is easy to verify that the hypotheses of Proposition 2.3.4 are satisfied.

Remark 2.3.7. In the situation of Example 2.3.5, $\{E_r^{s,t}, d_r\}_{r\geq 2}$ is a spectral sequence of algebras. That is, each $E_r^{*,*}$ has the structure of a bigraded ring, each differential d_r satisfies the Leibniz rule, and each identification of $E_{r+1}^{*,*}$ with the cohomology of the differential d_r is an isomorphism of bigraded algebras. Moreover, the identification

$$E_2^{s,t} \simeq \pi_t K(n) \otimes_\kappa \mathrm{Ext}_A^s$$

is an isomorphism of bigraded rings.

2.4 The Main Calculation

Throughout this section, we fix a perfect field κ of characteristic $p > 0$, and a one-dimensional formal group $\mathbf{G}_0$ over κ of height $n < \infty$. Let E denote the associated Lubin-Tate spectrum and $K(n)$ the associated Morava K-theory. Our goal is to compute the Morava K-groups $K(n)_*(X)$, where X is an Eilenberg-MacLane space of the form $K(\mathbf{Z}/p^t\,\mathbf{Z}, m)$ or $K(\mathbf{Q}_p/\mathbf{Z}_p, m)$, for $m > 0$.

Definition 2.4.1. Let X be a topological abelian group. We will say that X is $K(n)$-*good* if the following conditions are satisfied:

- (a) The space X is $K(n)$-even, so that $K(n)_0(X)$ can be regarded as a Hopf algebra over κ (Remark 2.1.30).
- (b) The action of p is locally nilpotent on the Hopf algebra $K(n)_0(X)$. That is, $K(n)_0(X)$ can be written as the colimit of a sequence of Hopf algebras $H(t)$ which are annihilated by p^t (when regarded as objects of the abelian category $\mathbf{Hopf}_\kappa$).

If X is $K(n)$-good, we let $D(X)$ denote the Dieudonne module $\mathrm{DM}_+(K(n)_0(X))$, where DM_+ is the functor of Corollary 1.4.15. Then $D(X)$ is a left module over the Dieudonne ring $\mathrm{D}_\kappa = W(\kappa)[F, V]$, and in particular a module over the ring $W(\kappa)$ of Witt vectors of κ. We will refer to $D(X)$ as the *Dieudonne module* of X.

Warning 2.4.2. The condition that a topological abelian group X be $K(n)$-good depends on the group structure of X, and not only on the underlying topological space.

If X is a topological abelian group which is $K(n)$-good, then the following algebraic data are interchangeable:

- The Morava K-theory $K(n)_*(X)$, regarded as a graded Hopf algebra over $\pi_* K(n)$.
- The homology group $K(n)_0(X)$, regarded as a Hopf algebra over κ.

- The formal group $\operatorname{Spf} K(n)^0(X)$ over κ.
- The Dieudonne module $D(X)$.

We will describe the relevant calculations using the language of Dieudonne modules, where the answer seems to admit the cleanest formulation.

Notation 2.4.3. Let H denote the Hopf algebra $K(n)_0 K(\mathbf{Q}_p / \mathbf{Z}_p, 1) \simeq K(n)_0 \mathbf{CP}^\infty$. For each $t \geq 0$, we let $H[p^t]$ denote the kernel of the map $[p^t] : H \to H$ (formed in the abelian category $\mathbf{Hopf}_\kappa$). Since $\mathbf{G}_0 = \operatorname{Spf} H^\vee$ is a smooth 1-dimensional formal group of height n, each $H[p^t]$ is a Hopf algebra of dimension $p^{nt} < \infty$ over κ. We let M denote the D_κ-module given by the inverse limit of the tower

$$\cdots \to \mathrm{DM}(H[p^2]) \to \mathrm{DM}(H[p]) \to \mathrm{DM}(H[p^0]) \simeq 0,$$

so that M is a free module of rank n over the ring of Witt vectors $W(\kappa)$.

When $m = 1$, the calculation of the Morava K-groups $K(n)_* K(\mathbf{Z} / p^t \mathbf{Z}, m)$ is contained in the following standard result:

Proposition 2.4.4. *Let $t \geq 0$ be an integer. The space $K(\mathbf{Z} / p^t \mathbf{Z}, 1)$ is $K(n)$-good. Moreover, the fiber sequence of topological abelian groups*

$$K(\mathbf{Z} / p^t \mathbf{Z}, 1) \to K(\mathbf{Q}_p / \mathbf{Z}_p, 1) \xrightarrow{p^t} K(\mathbf{Q}_p / \mathbf{Z}_p, 1)$$

induces a short exact sequence of Hopf algebras

$$\kappa \to K(n)_0 K(\mathbf{Z} / p^t \mathbf{Z}, 1) \to H \xrightarrow{[p^t]} H \to \kappa.$$

In particular, we obtain an isomorphism of Hopf algebras $K(n)_0 K(\mathbf{Z} / p^t \mathbf{Z}, 1) \simeq H[p^t]$ and of Dieudonne modules $D(K(\mathbf{Z} / p^t \mathbf{Z}, 1)) \simeq M / p^t M$.

Proof. Let BS^1 denote the Kan complex $K(\mathbf{Z}, 2)$, so that there is a canonical map $K(\mathbf{Q}_p / \mathbf{Z}_p, 1) \to BS^1$ which induces an isomorphism on $K(n)$-homology. Fix an invertible element $\beta \in \pi_{-2} K(n)$, and let $\eta \in K(n)^2_{\mathrm{red}} BS^1 \subseteq K(n)^2 BS^1$ be a complex orientation of $K(n)$, so that $H^\vee = K(n)^0 BS^1$ is isomorphic to a power series ring $\kappa[[x]]$, where $x = \beta^{-1}\eta \in K(n)^0 BS^1$.

For every Kan complex X, let $\underline{K(n)}_X$ denote the constant local system of spectra on X with value $K(n)$. Choose a contractible space ES^1 equipped with a Kan fibration $q : ES^1 \to BS^1$. We can identify the complex orientation of $K(n)$ with a class $\eta \in K(n)^2_{\mathrm{red}}(BS^1)$. Any such class determines a map of local systems $\Sigma^{-2} \underline{K(n)}_{BS^1} \to \underline{K(n)}_{BS^1}$ together with a nullhomotopy of the composite map

$$\Sigma^{-2} \underline{K(n)}_{BS^1} \to \underline{K(n)}_{BS^1} \to q_* \underline{K(n)}_{ES^1}.$$

The assumption that η is a complex orientation guarantees that the above maps form a fiber sequence. We have a homotopy pullback diagram

$$\begin{tikzcd} K(\mathbf{Z} / p^t \mathbf{Z}, 1) \arrow[r] \arrow[d, "q'"] & ES^1 \arrow[d, "q"] \\ K(\mathbf{Z}, 2) \arrow[r, "r"] & BS^1, \end{tikzcd}$$

where r is induced by multiplication by p^t. It follows that

$$r^*\eta = r^*(\beta x) = \beta r^*(x) = \beta [p^t](x) \in K(n)^2 K(\mathbf{Z}, 2).$$

Pulling back, we obtain a fiber sequence

$$\Sigma^{-2}\underline{K(n)}_{K(\mathbf{Z},2)} \to \underline{K(n)}_{K(\mathbf{Z},2)} \to q'_*\underline{K(n)}_{K(\mathbf{Z}/p^t\mathbf{Z},1)}$$

of local systems of spectra on $K(\mathbf{Z},2)$. Taking global sections, we obtain a long exact sequence

$$K(n)^{*-2}K(\mathbf{Z},2) \xrightarrow{\phi} K(n)^*K(\mathbf{Z},2) \to K(n)^*K(\mathbf{Z}/p^t\mathbf{Z},2) \to K(n)^{*-1}K(\mathbf{Z},2) \xrightarrow{\phi} K(n)^{*+1}K(\mathbf{Z},2),$$

where ϕ is given by multiplication by $\beta[p^t](x)$.

Since the formal group $\mathbf{G}_0$ has finite height, $[p^t](x)$ is not a zero divisor in $K(n)^0K(\mathbf{Z},2)$. It follows that our long exact sequence reduces to a short exact sequence

$$0 \to K(n)^{*-2}K(\mathbf{Z},2) \xrightarrow{\beta[p^t](x)} K(n)^*K(\mathbf{Z},2) \to K(n)^*K(\mathbf{Z}/p^t\mathbf{Z},1) \to 0.$$

In particular, we deduce that $K(n)^*K(\mathbf{Z}/p^t\mathbf{Z},1)$ vanishes in odd degrees, and that $K(n)^0K(\mathbf{Z}/p^t\mathbf{Z},1)$ is isomorphic to the the quotient $\kappa[[x]]/([p^t](x))$. Passing to κ-linear duals, we obtain the desired result. $\square$

We will attempt to understand other Eilenberg-MacLane spaces by relating them to classifying spaces of cyclic p-groups.

Remark 2.4.5. Let $X_1, X_2, \ldots, X_m$, and Y be $K(n)$-good topological abelian groups, and suppose we are given a multilinear map

$$\phi : X_1 \times \cdots \times X_m \to Y.$$

Then ϕ induces a multilinear map of Hopf algebras, which we can identify with a Hopf algebra homomorphism

$$\psi : K(n)_0(X_1) \boxtimes \cdots \boxtimes K(n)_0(X_m) \to K(n)_0(Y).$$

If we assume that the spaces $X_1, \ldots, X_m$, and Y are $K(n)$-good, we can use Corollary 1.4.15 to identify ψ with a map of Dieudonne modules

$$D(X_1)\widetilde{\otimes}\cdots\widetilde{\otimes}D(X_m) \to D(Y).$$

In particular, we obtain a $W(\kappa)$-multilinear map

$$D(X_1) \times \cdots \times D(X_m) \to D(Y).$$

Construction 2.4.6. Fix $m \geq 0$, and let $Y = K(\mathbf{Q}_p/\mathbf{Z}_p, m)$. Assume that Y is good. Writing Y as a filtered colimit of spaces of the form $K(p^{-t}\mathbf{Z}_p/\mathbf{Z}_p, m)$, we deduce that Y is $K(n)$-good, so that the Hopf algebra $K(n)_0(Y)$ is determined by its Dieudonne module $D(Y)$.

For each $t \geq 0$, consider the map

$$\phi_t : K(\mathbf{Z}/p^t\mathbf{Z},1)^m \to K(\mathbf{Z}/p^t\mathbf{Z},m) \to Y$$

where the first map is given by the iterated cup product, and the second is given by the inclusion $\mathbf{Z}/p^t\mathbf{Z} \simeq p^{-t}\mathbf{Z}_p/\mathbf{Z}_p \subseteq \mathbf{Q}_p/\mathbf{Z}_p$. Using Remark 2.4.5, we see that ϕ_t induces a $W(\kappa)$-multilinear map

$$\theta^m_t : M/p^tM \times \cdots \times M/p^tM \to D(Y).$$

Remark 2.4.7. In the situation of Construction 2.4.6, the antisymmetry of the cup product shows that the map

$$\theta^m_t : (M/p^tM)^m \to D(Y)$$

is antisymmetric: that is, for any permutation σ of $\{1, \ldots, m\}$, we have

$$\theta^m_t(x_1, \ldots, x_m) = \mathrm{sn}(\sigma)\theta^m_t(x_{\sigma(1)}, \ldots, x_{\sigma(m)}),$$

where $\mathrm{sn}(\sigma)$ denotes the sign of the permutation σ.

Remark 2.4.8. Let Y be as in Construction 2.4.6, and let $\xi : K(\mathbf{Z}/p^t\mathbf{Z}, 1) \to K(\mathbf{Z}/p^{t+1}\mathbf{Z}, 1)$ denote the map induced by multiplication by p. The diagram of spaces

$$\begin{tikzcd} K(\mathbf{Z}/p^{t+1},1)^{m-1} \times K(\mathbf{Z}/p^t) \arrow[r] \arrow[d, "\mathrm{id}\times\xi"] & K(\mathbf{Z}/p^t,1)^{m-1} \times K(\mathbf{Z}/p^t,1) \arrow[d, "\phi_t"] \\ K(\mathbf{Z}/p^{t+1},1)^{m-1} \times K(\mathbf{Z}/p^{t+1},1) \arrow[r] & Y \end{tikzcd}$$

commutes up to homotopy. It follows that if we are given elements $x_1, \ldots, x_{m-1} \in M/t^{p+1}M$ with images $\overline{x}_1, \ldots, \overline{x}_{m-1} \in M/t^pM$, and $y \in M/t^pM$, then we have

$$\theta_t^m(\overline{x}_1, \ldots, \overline{x}_{m-1}, y) = \theta_{t+1}^m(x_1, \ldots, x_{m-1}, ty).$$

Lemma 2.4.9. *In the situation of Construction 2.4.6, the multilinear map*

$$\theta_t^m : (M/p^tM) \times \cdots \times (M/p^tM) \to D(Y)$$

is strictly alternating. That is, we have $\theta_t^m(x_1, \ldots, x_m) = 0$ whenever $x_i = x_j$ for $i \neq j$.

Proof. Suppose that we are given a sequence of elements $x_1, \ldots, x_m \in M/p^tM$ such that $x_i = x_j$ for some $i \neq j$. Choose elements $y_1, \ldots, y_m \in M/p^{t+1}M$ lifting the elements x_i, so that $y_i = y_j$. The antisymmetry of Remark 2.4.7 guarantees that $2\theta_{t+1}^m(y_1, \ldots, y_m) = 0$. Using Remark 2.4.8, we obtain

$$\theta_t^m(x_1, \ldots, x_m) = \theta_{t+1}^m(y_1, \ldots, y_{m-1}, py_m) = p\theta_{t+1}^m(y_1, \ldots, y_m).$$

This completes the proof when $p = 2$. If p is odd, the desired result follows immediately from Remark 2.4.7. $\square$

Let $\bigwedge^m M$ denote the mth exterior power of M, regarded as a module over $W(\kappa)$. Then each quotient $\mathbf{Z}/p^t\mathbf{Z} \otimes \bigwedge^m M$ can be identified with the mth exterior power of M/p^tM as a module over $W(\kappa)/p^tW(\kappa)$. It follows from Lemma 2.4.9 that θ_t^m induces a map of $W(\kappa)$-modules $\mathbf{Z}/p^t\mathbf{Z} \otimes \bigwedge^m M \to D(Y)$, which we will also denote by θ_t^m. Remark 2.4.8 guarantees the commutativity of the diagrams

$$\begin{tikzcd} \mathbf{Z}/p^t\mathbf{Z} \otimes \bigwedge^m M \arrow[r, "\theta_t^m"] \arrow[d, "p"] & D(Y) \arrow[d, "\mathrm{id}"] \\ \mathbf{Z}/p^t\mathbf{Z} \otimes \bigwedge^m M \arrow[r, "\theta_{t+1}^m"] & D(Y). \end{tikzcd}$$

Together, these assemble to give a map

$$\theta^m : \mathbf{Q}_p/\mathbf{Z}_p \otimes \bigwedge^m M \to D(Y).$$

We can now state the main result of this section:

Theorem 2.4.10 (Ravenel-Wilson). *Let $m > 0$ be an integer, and let $Y = K(\mathbf{Q}_p/\mathbf{Z}_p, m)$. Then:*

(a) *The ring $K(n)^0(Y)$ is isomorphic to a power series algebra over κ on $\binom{n-1}{m-1}$ variables.*

(b) *The groups $K(n)^i(Y)$ vanish when i is odd. In particular, Y is $K(n)$-good.*

(c) *The map $\theta^m : \mathbf{Q}_p/\mathbf{Z}_p \otimes \bigwedge^m M \to D(Y)$ is an isomorphism.*

(d) *The formal group* $\operatorname{Spf} K(n)^0(Y)$ *is p-divisible of height $\binom{n}{m}$ and dimension $\binom{n-1}{m-1}$.*

(e) *For each $t \geq 0$, the space $Y_t = K(\mathbf{Z}/p^t\mathbf{Z}, m)$ is good. Moreover, the canonical map*

$$Y_t \simeq K(p^{-t}\mathbf{Z}_p/\mathbf{Z}_p, m) \to Y$$

induces a monomorphism of Hopf algebras $K(n)_0 Y_t \to K(n)_0 Y$, which exhibits $K(n)_0 Y_t$ as the kernel of the map $[p^t] : K(n)_0 Y \to K(n)_0 Y$ (in the abelian category of Hopf algebras over κ).

We will prove Theorem 2.4.10 using induction on m. In the case $m = 1$, the desired results follow from Proposition 2.4.4 (and from the definition of Morava K-theory and the Dieudonne module M). To carry out the inductive step, it will suffice to prove the following three results:

Proposition 2.4.11. *Let $m > 1$ be an integer. Suppose that $K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)$ is $K(n)$-even and that the formal group*

$$\operatorname{Spf} K(n)^0 K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)$$

is connected and p-divisible of height $\binom{n}{m-1}$ and dimension $\binom{n-1}{m-1}$. Then $K(\mathbf{Q}_p/\mathbf{Z}_p, m)$ is $K(n)$-even, and $K(n)^0 K(\mathbf{Q}_p/\mathbf{Z}_p, m)$ is isomorphic to a formal power series ring on $\binom{n-1}{m}$ variables.

Proposition 2.4.12. *Let $m > 1$ be an integer. Suppose that $K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)$ and $K(\mathbf{Z}/p\mathbf{Z}, m-1)$ are $K(n)$-good, that $\operatorname{Spf} K(n)^0 K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)$ is a connected p-divisible group of height $\binom{n}{m-1}$ and dimension $\binom{n-1}{m-1}$, that the sequence of Hopf algebras*

$$\kappa \to K(n)_0 K(p^{-1}\mathbf{Z}_p/\mathbf{Z}_p, m-1) \to K(n)_0 K(\mathbf{Q}_p/\mathbf{Z}_p, m-1) \to K(n)_0 K(\mathbf{Q}_p/p^{-1}\mathbf{Z}_p, m-1) \to \kappa$$

is exact, and that the map θ^{m-1} is an isomorphism. Then:

(i) *The map $\theta^m : \mathbf{Q}_p/\mathbf{Z}_p \otimes \bigwedge^m M \to D(K(\mathbf{Q}_p/\mathbf{Z}_p, m))$ is an isomorphism.*

(ii) *The formal group $\operatorname{Spf} K(n)^0 K(\mathbf{Q}_p/\mathbf{Z}_p, m)$ is p-divisible, with height $\binom{n}{m}$ and dimension $\binom{n-1}{m-1}$.*

Proposition 2.4.13. *Let $m > 1$ be an integer. Suppose that $K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)$ and $K(\mathbf{Z}/p^c\mathbf{Z}, m-1)$ are $K(n)$-even, that $\operatorname{Spf} K(n)^0 K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)$ is a connected p-divisible group of height $\binom{n}{m-1}$ and dimension $\binom{n-1}{m-1}$, and that the sequence of Hopf algebras*

$$\kappa \to K(n)_0 K(p^{-c}\mathbf{Z}_p/\mathbf{Z}_p, m-1) \to K(n)_0 K(\mathbf{Q}_p/\mathbf{Z}_p, m-1) \to K(n)_0 K(\mathbf{Q}_p/p^{-c}\mathbf{Z}_p, m-1) \to \kappa$$

is exact. Then $K(p^{-c}\mathbf{Z}_p/\mathbf{Z}_p, m)$ is $K(n)$-even, and the sequence of Hopf algebras

$$\kappa \to K(n)_0 K(p^{-c}\mathbf{Z}_p/\mathbf{Z}_p, m) \to K(n)_0 K(\mathbf{Q}_p/\mathbf{Z}_p, m) \to K(n)_0 K(\mathbf{Q}_p/p^{-c}\mathbf{Z}_p, m) \to \kappa$$

is exact.

Proof of Proposition 2.4.11. Let $G = K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)$. Then G can be realized as a topological abelian group, whose classifying space BG is a model for $K(\mathbf{Q}_p/\mathbf{Z}_p, m)$, and set $A = K(n)_0(G)$. Then the function spectrum $K(n)^{BG}$ can be written as the limit of a tower of spectra

$$\cdots \to X(2) \to X(1) \to X(0),$$

where each $X(s)$ is given by the sth partial totalization of the cosimplicial spectrum $[j] \mapsto K(n)^{G^j}$. Let $\{E_r^{s,t}, d_r\}_{r\geq 1}$ be the spectral sequence of Example 2.3.5, whose second page is given by

$$E_2^{s,*} = \operatorname{Ext}_A^s \otimes_\kappa \pi_* K(n).$$

Since the formal group $\operatorname{Spf} A^\vee$ is p-divisible, we have an exact sequence of Hopf algebras

$$\kappa \to A[p] \to A \xrightarrow{[p]} A \to \kappa,$$

where $A[p]$ is finite-dimensional as a vector space over κ. Note that the kernel and cokernel of the Frobenius map on $A[p]$ have the same rank, which is the rank of the kernel of the Frobenius map $\varphi : A^{(p)} \to A$. Since $\operatorname{Spf} A^\vee$ has height $\binom{n}{m-1}$ and dimension $\binom{n-1}{m-1}$, we conclude that $\dim_\kappa \operatorname{Ext}^1_{A[p]} = \binom{n}{m-1} - \binom{n-1}{m-1} = \binom{n-1}{m} < \infty$. Applying Theorem 2.2.10, we see that $E_2^{s,t}$ vanishes when s is odd, and $E_2^{2s,t}$ is given by $\operatorname{Sym}^s_\kappa(\operatorname{Ext}^1_{A[p]}) \otimes_\kappa \pi_t K(n)$. Since $E_2^{s,t}$ vanishes unless s and t are both even, we conclude that the differentials d_r vanish for $r \geq 2$, so that $E_2^{s,t} \simeq E_\infty^{s,t}$. In particular, the filtered spectrum $\{X(s)\}$ satisfies the hypotheses of Proposition 2.3.4. It follows that each $\pi_m K(n)^{BG} = K(n)^{-m} BG$ admits a filtration

$$K(n)^{-m} BG = F^0 K(n)^{-m} BG \supseteq F^1 K(n)^{-m} BG \supseteq \cdots,$$

where $K(n)^{-m} BG \simeq \varprojlim_s K(n)^{-m} BG / F^s K(n)^{-m} BG$, whose associated graded is given by

$$F^s K(n)^{-m} BG / F^{s+1} K(n)^{-m} BG \simeq E_\infty^{s,m+s} \simeq \operatorname{Ext}^s_A \otimes_\kappa \pi_{m+s} K(n).$$

It follows immediately that the space BG is $K(n)$-even. Let $B = K(n)^0 BG$, and let $I(s) = F^s K(n)^0 BG$ for $s \geq 0$. Note that this filtration is multiplicative: that is, we have $I(s)I(s') \subseteq I(s+s')$. In particular, each $I(s)$ is an ideal in B. We have $B \simeq \varprojlim_s B/I(s)$; let us regard B as endowed with the inverse limit topology. There is an isomorphism of associated graded rings

$$\operatorname{gr}(B) \simeq \operatorname{Sym}^*_\kappa(\operatorname{Ext}^1_{A[p]} \otimes_\kappa \pi_2 K(n)).$$

Let $t_1, \ldots, t_k$ be a basis for the vector space $\operatorname{Ext}^1_{A[p]} \otimes_\kappa \pi_2 K(n)$, where $k = \binom{n-1}{m}$. For $1 \leq i \leq k$, choose $\bar{t}_i \in I(2)$ having image t_i in $\operatorname{gr}^2(B) = I(2)/I(3)$. We then have a unique continuous ring homomorphism $\kappa[[T_1, \ldots, T_k]] \to B$, carrying each T_i to $\bar{t}_i$. This map induces an isomorphism of associated graded rings, and is therefore an isomorphism. $\square$

Remark 2.4.14. In the situation of Proposition 2.4.11, let $B = K(n)^0 K(\mathbf{Q}_p / \mathbf{Z}_p, m)$, and let $\mathfrak{m}_B$ denote its augmentation ideal. The proof of Proposition 2.4.12 shows that the filtration

$$B = I(0) \supseteq I(1) \supseteq I(2) \supseteq \cdots$$

is a reindexing of the $\mathfrak{m}_B$-adic filtration. More precisely, we have

$$I(s) = \begin{cases} \mathfrak{m}_B^k & \text{if } s = 2k \\ \mathfrak{m}_B^k & \text{if } s = 2k-1. \end{cases}$$

In particular, we have an isomorphism $\mathfrak{m}_B / \mathfrak{m}_B^2 \simeq I(2)/I(3)$.

Let $H = K(n)_0 K(\mathbf{Q}_p / \mathbf{Z}_p, m)$, so that B is the κ-linear dual of H and the sequence of ideals $\{I(s)\}_{s \geq 0}$ is dual to an increasing filtration

$$\kappa \simeq F^0 H \subseteq F^1 H \subseteq \cdots.$$

Dualizing the above reasoning, we conclude that $\operatorname{Prim}(H) \subseteq F^2 H$, and that the composite map $\operatorname{Prim}(H) \hookrightarrow F^2 H \to F^2 H / F^1 H$ is an isomorphism of κ-vector spaces.

Proof of Proposition 2.4.12. We will prove assertion (i); assertion (ii) is an immediate consequence of (i) and Proposition 2.4.11. Replacing κ by its algebraic closure if necessary, we may assume that M is generated over D_κ by an element x satisfying $Fx = V^{n-1}x$, so that M is freely generated as a $W(\kappa)$-module by $x, Vx, V^2x, \ldots, V^{n-1}x$ with $V^n x = px$. Given a subset $I = \{i_1 < \ldots < i_m\} \subseteq \{0, \ldots, n-1\}$, we let $V^I x = V^{i_1} x \wedge \ldots \wedge V^{i_m} x \in \bigwedge^m M$. Note that the action of V on M induces a Verschiebung map $V : \bigwedge^m M \to \bigwedge^m M$, given on generators by

$$V(\lambda V^{i_1} x \wedge \ldots \wedge V^{i_m} x) = \begin{cases} \varphi^{-1}(\lambda) V^{i_1+1} x \wedge \ldots \wedge V^{i_m+1} x & \text{if } i_m < n-1 \\ (-1)^{m-1} p \varphi^{-1}(\lambda) x \wedge V^{i_1+1} \wedge \ldots \wedge V^{i_{m-1}+1} x & \text{otherwise.} \end{cases}$$

Applying the snake lemma to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigwedge^m M \arrow[r] \arrow[d, "V"] & \mathbf{Q} \otimes \bigwedge^m M \arrow[r] \arrow[d, "V"] & \mathbf{Q}_p / \mathbf{Z}_p \otimes \bigwedge^m M \arrow[r] \arrow[d, "V"] & 0 \\
0 \arrow[r] & \bigwedge^m M \arrow[r] & \mathbf{Q} \otimes \bigwedge^m M \arrow[r] & \mathbf{Q}_p / \mathbf{Z}_p \otimes \bigwedge^m M \arrow[r] & 0,
\end{tikzcd}$$

we deduce that V induces a surjection from $\bigwedge^m M \otimes \mathbf{Q}_p / \mathbf{Z}_p$ to itself, and obtain an isomorphism

$$\ker(V : \mathbf{Q}_p / \mathbf{Z}_p \otimes \bigwedge^m M \to \mathbf{Q}_p / \mathbf{Z}_p \otimes \bigwedge^m M) \simeq \operatorname{coker}(V : \bigwedge^m M \to \bigwedge^m M).$$

By inspection, the right hand side has basis (as a κ-vector space) given by elements of the form $V^{i_1}x \wedge \ldots \wedge V^{i_m}x$ with $i_1 = 0$. It follows that the left hand side has dimension $\binom{n-1}{m-1}$ as a vector space over κ.

Set $Y = K(\mathbf{Q}_p / \mathbf{Z}_p, m)$, and note that the map $\theta^m : \bigwedge^m M \otimes \mathbf{Q}_p / \mathbf{Z}_p \to D(Y)$ is V-linear. We will prove the following:

($*$) The map θ^m induces a surjection

$$\ker(V : \mathbf{Q}_p / \mathbf{Z}_p \otimes \bigwedge^m M \to \mathbf{Q}_p / \mathbf{Z}_p \otimes \bigwedge^m M) \to \ker(V : D(Y) \to D(Y)).$$

Assume ($*$) for the moment. Proposition 2.4.11 implies that the map $V : D(Y) \to D(Y)$ is surjective, and that its kernel has dimension $\binom{n-1}{m-1}$ over κ. It follows that θ induces an isomorphism $\ker(V : \mathbf{Q}_p / \mathbf{Z}_p \otimes \bigwedge^m M \to \mathbf{Q}_p / \mathbf{Z}_p \otimes \bigwedge^m M) \to \ker(V : D(Y) \to D(Y))$, so that θ^m is injective when restricted to the kernel of V. We claim that θ^m is injective. To prove this, suppose that $0 \neq z \in \mathbf{Q}_p / \mathbf{Z}_p \otimes \bigwedge^m M$. Note that $V^t z = 0$ for sufficiently large values of t. Choose a minimal value of t such that $V^t z = 0$. Since $z \neq 0$, we have $t > 0$. Then $0 \neq V^{t-1} z \in \ker(V)$, so that $V^{t-1}\theta(z) = \theta(V^{t-1}z) \neq 0$ and therefore $\theta(z) \neq 0$.

We now prove the surjectivity of θ^m. Let $y \in D(Y)$; we wish to show that y belongs to the image of θ^m. Using Proposition 2.4.11, we see that the formal group $\operatorname{Spf} K(n)^0(Y)$ is connected so that the action of V on $D(Y)$ is locally nilpotent. It follows that there exists t such that $V^t y = 0$. We now proceed by induction on t. If $t = 0$, then $y = 0$ and there is nothing to prove. Otherwise, $V^{t-1}y \in \ker(V)$. Using ($*$), we deduce that there exists $z \in \mathbf{Q}_p / \mathbf{Z}_p \otimes \bigwedge^m M$ with $\theta^m(z) = V^{t-1}y$. Since V is surjective on $\mathbf{Q}_p / \mathbf{Z}_p \otimes \bigwedge^m M$, we can write $y = V^{t-1}\overline{y}$ for some $\mathbf{Q}_p / \mathbf{Z}_p \otimes \overline{y} \in M$. Then $z - \theta^m(\overline{y})$ is annihilated by V^{t-1}. Using the inductive hypothesis, we see that $z - \theta^m(\overline{y})$ belongs to the image of θ^m, so that z also belongs to the image of θ.

It remains to prove ($*$). Let $c : K(\mathbf{Z} / p\, \mathbf{Z}, 1) \times K(\mathbf{Z} / p\, \mathbf{Z}, m-1) \to Y$ be the composition of the cup product map with the inclusion $K(\mathbf{Z} / p\, \mathbf{Z}, m) \simeq K(p^{-1}\, \mathbf{Z}_p / \mathbf{Z}_p, m) \to K(\mathbf{Q}_p / \mathbf{Z}_p, m)$. Set $B = K(n)_0 K(\mathbf{Z} / p\, \mathbf{Z}, 1)$ and $B' = K(n)_0 K(\mathbf{Z} / p\, \mathbf{Z}, m-1)$. Since θ^{m-1} is an isomorphism and B' is the kernel of $[p]$ on the Hopf algebra $K(n)_0 K(\mathbf{Q}_p / \mathbf{Z}_p, m-1)$, the canonical map $(M/pM)^{\otimes m-1} \to \operatorname{DM}(B')$ induces an isomorphism $\mathbf{Z} / p\, \mathbf{Z} \otimes \bigwedge^{m-1} M \simeq \operatorname{DM}(B')$. The map c induces a bilinear map of Hopf algebras

$$\mu : B \otimes_\kappa B' \to K(n)_0(Y),$$

hence a pairing of Dieudonne modules $\lambda : M/pM \times (\mathbf{Z} / p\, \mathbf{Z} \otimes \bigwedge^{m-1} M) \to D(Y)$. Consider the map

$$\overline{\lambda} : \ker(V : M/pM \to M/pM) \otimes_\kappa \operatorname{DM}(B')/F \operatorname{DM}(B') \to \ker(V : D(Y) \to D(Y))$$

introduced in Notation 1.3.31. To prove ($*$), it will suffice to show that $\overline{\lambda}$ is surjective.

Let $A = K(n)_0 K(\mathbf{Q}_p / \mathbf{Z}_p, m-1)$, let $\mathfrak{m}_{B'}$ denote the augmentation ideal of B', and let $\overline{\mu} : \operatorname{Prim}(B) \times \mathfrak{m}_{B'}/\mathfrak{m}^2_{B'} \to \operatorname{Prim}(K(n)_0(Y))$ be as in Remark 1.3.32. According to Proposition 1.3.33, we have a commutative diagram

$$\begin{tikzcd}
\ker(V : M/pM \to M/pM) \otimes_\kappa \mathrm{DM}(B')/F\,\mathrm{DM}(B') \arrow[r, "\overline{\lambda}"] \arrow[d, "\sim"] & \ker(V : \mathrm{DM}(A) \to \mathrm{DM}(A)) \arrow[d, "\sim"] \\
\mathrm{Prim}(B) \otimes_\kappa \mathfrak{m}_{B'}/\mathfrak{m}_{B'}^2 \arrow[r, "\overline{\mu}"] & \mathrm{Prim}(A).
\end{tikzcd}$$

It will therefore suffice to show that $\overline{\mu}$ is surjective.

Let $X_\bullet$ denote the simplicial space given by the Čech nerve of the base point inclusion $* \to K(\mathbf{Z}/p\,\mathbf{Z}, 1)$ (so that $X_\bullet$ can be identified with the simplicial set given by $K_t = (\mathbf{Z}/p\,\mathbf{Z})^t$). Associated to this simplicial space is a spectral sequence $\{E_r'^{s,t}, d_r\}_{r\geq 1}$ converging to $K(n)_* K(\mathbf{Z}/p\,\mathbf{Z}, 1)$. From the second page onward, this spectral sequence can be identified with the Atiyah-Hirzebruch spectral sequence for the spectrum $K(n)$: in particular, we have a canonical isomorphism $E_2'^{s,t} \simeq \mathrm{H}_s(K(\mathbf{Z}/p\,\mathbf{Z}, 1); \pi_t K(n))$. We have an associated filtration

$$0 = \mathrm{gr}^{-1} B \hookrightarrow \mathrm{gr}^0 B \hookrightarrow \mathrm{gr}^1 \hookrightarrow \cdots,$$

with $\mathrm{gr}^s B / \mathrm{gr}^{s-1} B \simeq E_\infty'^{s,-s}$. Note that $\mathrm{Prim}(B)$ is contained in $\mathrm{gr}^2 B$, and that the projection map $\mathrm{Prim}(B) \to \mathrm{gr}^2 B / \mathrm{gr}^1 B$ is an isomorphism.

Let $Y_\bullet$ denote the simplicial space appearing in the proof of Proposition 2.4.11. Associated to $Y_\bullet$ is another spectral sequence $\{E_r^{s,t}, d_r\}_{r\geq 1}$ converging to a filtration of $K(n)_* K(\mathbf{Q}_p/\mathbf{Z}_p, m)$, which is a κ-linear predual of the spectral sequence appearing in the proof of Proposition 2.4.12. We have a homotopy commutative diagram

$$\begin{tikzcd}
* \times K(\mathbf{Z}/p\,\mathbf{Z}, m-1) \arrow[r] \arrow[d] & * \arrow[d] \\
K(\mathbf{Z}/p\,\mathbf{Z}, 1) \times K(\mathbf{Z}/p\,\mathbf{Z}, m-1) \arrow[r, "c"] & K(\mathbf{Q}_p/\mathbf{Z}_p, m).
\end{tikzcd}$$

which induces a map of simplicial spaces $\alpha : X_\bullet \times K(\mathbf{Z}/p\,\mathbf{Z}, m-1) \to Y_\bullet$ and therefore a map

$$\mathrm{gr}^s B \otimes_\kappa B' \to \mathrm{gr}^s K(n)_0(Y).$$

We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Prim}(B) \otimes_\kappa \mathfrak{m}_{B'} \arrow[r] \arrow[d] & \mathrm{Prim}(K(n)_0(Y)) \arrow[d] \\
\mathrm{gr}^2 B / \mathrm{gr}\, 1B \otimes_\kappa \mathfrak{m}_{B'} \arrow[r] & \mathrm{gr}^2 K(n)_0(Y) / \mathrm{gr}^1 K(n)_0(Y),
\end{tikzcd}$$

where the vertical maps are isomorphisms (see Remark 2.4.14). Consequently, to show that $\overline{\mu}$ is surjective, it will suffice to show that the induced map of spectral sequences $\{E_r'^{s,t} \otimes_\kappa B', d_r\} \to \{E_r^{s,t}, d_r\}$ induces a surjection $E_\infty'^{2,-2} \otimes_\kappa \mathfrak{m}_{B'} \to E_\infty^{2,-2}$. Because $E_2'^{2,-2}$ consists of permanent cycles, it suffices to show that the map of second pages $E_2'^{2,-2} \otimes_\kappa \mathfrak{m}_{B'} \to E_2^{2,-2}$ is surjective. Using the 2-periodicity of the graded ring $\pi_* K(n)$, we are reduced to showing that the map

$$\psi : E_2'^{2,0} \otimes_\kappa \mathfrak{m}_{B'} \simeq \mathrm{H}_2(K(\mathbf{Z}/p\,\mathbf{Z}, 1); \mathbf{Z}/p\,\mathbf{Z}) \otimes_{\mathbf{Z}/p\,\mathbf{Z}} \mathfrak{m}_{B'} \to \mathrm{Tor}_2^A(\kappa, \kappa) \simeq E_2^{2,0}$$

is surjective.

Our assumptions allow us to identify B' with the kernel (in the abelian category $\mathbf{Hopf}_\kappa$) of the map $[p] : A \to A$. Note that $\mathrm{H}_2(K(\mathbf{Z}/p\,\mathbf{Z}, 1); \mathbf{Z}/p\,\mathbf{Z})$ is a free module of rank one over $\mathbf{Z}/p\,\mathbf{Z}$ with a canonical generator (dual to the generator of $\mathrm{H}^2(K(\mathbf{Z}/p\,\mathbf{Z}, 1); \mathbf{Z}/p\,\mathbf{Z})$ classifying the extension $0 \to \mathbf{Z}/p\,\mathbf{Z} \to$

$\mathbf{Z}/p^2\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0$). Using this generator, we can identify ψ with a map $\psi' : \mathfrak{m}_{B'} \to \mathrm{Tor}_2^A(\kappa,\kappa)$. The surjectivity of ψ now follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathfrak{m}_{B'} \arrow[r, "\psi'"] \arrow[d] & \mathrm{Tor}_2^A(\kappa,\kappa) \arrow[d] \\
\mathfrak{m}_{B'}/\mathfrak{m}_{B'}^2 \arrow[r, "\sim"] & \mathrm{Tor}_1^{B'}(\kappa,\kappa),
\end{tikzcd}$$

since the right vertical map is the κ-linear dual of the isomorphism $\mathrm{Ext}^1_{B'} \to \mathrm{Ext}^2_A$ of Theorem 2.2.10. $\square$

Proof of Proposition 2.4.13. Set

$$G' = K(\mathbf{Z}_p/p^c\mathbf{Z}_p, m-1) \qquad G = K(\mathbf{Q}_p/p^c\mathbf{Z}_p, m-1) \qquad G'' = K(\mathbf{Q}_p/\mathbf{Z}_p, m-1),$$

so that we have a fiber sequence of topological abelian groups $G' \to G \to G''$. Set $A' = K(n)_0 G'$ and $A = K(n)_0 G$, so that the map $G' \to G$ induces a Hopf algebra homomorphism $A' \to A$. Moreover, we can also identify $K(n)_0 G'$ with A, so that the map $G \to G'$ induces the Hopf algebra homomorphism $[p^c] : A \to A$.

We will identify $K(\mathbf{Z}/p^c\mathbf{Z}, m)$ with the classifying space of the group G', and let $\{E_r^{s,t}, d_r\}_{r \geq 1}$ be the spectral sequence of Example 2.3.5, whose second page is given by

$$E_2^{s,*} = \mathrm{Ext}^s_{A'} \otimes_\kappa \pi_* K(n).$$

Let $R = K(n)^0 K(\mathbf{Q}_p/\mathbf{Z}_p, m)$, and let $\mathfrak{m}_R$ denote its maximal ideal, so that the proof of Proposition 2.4.10 supplies canonical isomorphisms $\mathfrak{m}_R^s/\mathfrak{m}_R^{s+1} \simeq \mathrm{Ext}^{2s}_A \otimes_\kappa \pi_{2s} K(n)$ (see Remark 2.4.14). By assumption, the fiber sequence

$$G' \to G \to G''$$

induces an exact sequence of connected Hopf algebras

$$\kappa \to A' \to A \xrightarrow{[p^c]} A \to \kappa.$$

Let $\psi : \mathrm{Ext}^1_{A'} \to \mathrm{Ext}^2_A$ be the isomorphism of Theorem 2.2.10, and let $v \in \pi_2 K(n)$ be a nonzero element. The main ingredient in our proof is the following assertion:

$(*)$ Suppose we are given an element $x \in E_2^{1,0}$ and an element $y \in \mathfrak{m}_R$ representing

$$\psi(x) \otimes v \in \mathrm{Ext}^2_A \otimes_\kappa \pi_2 K(n) \simeq \mathfrak{m}_R/\mathfrak{m}_R^2$$

Suppose furthermore that the Hopf algebra homomorphism $[p^c] : R \to R$ carries y to an element $y' \in \mathfrak{m}_R^s$, and let $x' \in E_2^{2s,2s-2}$ denote the image of y' under the composite map

$$\mathfrak{m}_B^s/\mathfrak{m}_B^{s+1} \simeq \mathrm{Ext}^{2s}_A \otimes_\kappa \pi_{2s} K(n) \to \mathrm{Ext}^{2s}_{A'} \otimes_\kappa \pi_{2s} K(n) = E_2^{2s,2s} \xrightarrow{v^{-1}} E_2^{2s,2s-2}$$

Then x and x' survive to the $(2s-1)$st page of the spectral sequence $\{E_r^{s,t}, d_r\}_{r\geq 1}$. That is, there exist elements $\{x_r \in E_r^{1,0}, x'_r \in E_r^{2s,2s-2}\}_{2 \leq r \leq 2s-1}$ such that $x_2 = x$, $x'_2 = y$, $d_r x_r = d_r x'_r = 0$ for $2 \leq r < 2s-1$, each x_r is a cycle representing x_{r+1}, and each x'_r is a cycle representing x'_{r+1}. Moreover, we have $v d_{2s-1}(x_{2s-1}) = x'_{2s-1}$ in $E_{2s-1}^{2s,2s-2}$.

Let us now explain how to complete the proof, assuming $(*)$. We first treat the case where $m = n$. Then $\mathrm{Spf}\, R$ is a 1-dimensional p-divisible group of height 1 over κ. Replacing κ by its algebraic closure if necessary, we may assume that $\mathrm{Spf}\, R$ is the formal multiplicative group, so that there exists an isomorphism $R \simeq \kappa[[y]]$, where the p-series $[p^c]$ is given by $[p^c](y) = y^{p^c}$. Then the image of y in $\mathfrak{m}_R/\mathfrak{m}_R^2 \simeq \mathrm{Ext}^2_A \otimes_\kappa \pi_2 K(n)$ has the form $\psi(x) \otimes v$, for a unique element $x \in E_2^{1,0}$. It follows from $(*)$ that x survives to the $(2p^c - 1)$st page of the spectral sequence, and that the differential d_{2p^c-1} carries (the residue class of) x to (the residue class of) $v^{-1}\overline{y}^{p^c}$, where $\overline{y}$ denotes the image of y in the graded ring $\mathrm{gr}(R) = \bigoplus \mathfrak{m}_R^s / \mathfrak{m}_R^{s+1}$.

We now define an auxiliary spectral sequence $\{E_r'^{s,t}, d_r'\}_{r \geq 2}$ as follows: for $r \leq 2p^c - 1$, we let $E_r'^{*,*}$ denote the free $(\pi_* K(n))[Y]$-module on generators 1 and X, where X has bidegree $(1,0)$ and Y has bidegree $(2,2)$. For $r \geq 2p^c$, we let $E'^{*,*_r}$ denote the quotient $(\pi_* K(n))[Z]/(Z^{p^t})$. Finally, the differentials d_r' vanish unless $r = p^t - 1$, in which case d_r' is the unique $(\pi_* K(n))[Y]$-linear map satisfying

$$d_r'(1) = 0 \qquad d_r'(X) = -v^{-1} Y^{p^c}.$$

Using $(*)$, we deduce that there is a map of spectral sequences $\{E_r'^{s,t}, d_r'\} \to \{E_r^{s,t}, d_r\}$, given on the second page by

$$Y^a \mapsto \overline{y}^a \qquad XY^a = x\overline{y}^a.$$

By inspection, this map induces an isomorphism on the second page, and is therefore an isomorphism of spectral sequences. In particular, the spectral sequence $\{E_r^{s,t}, d_r\}_{r \geq 2}$ stabilizes after a finite number of steps, and its final page is given by

$$E_\infty^{*,*} \simeq E_{2p^c}^{*,*} \simeq (\pi_* K(n))[\overline{y}]/(\overline{y}^{p^c}),$$

where each $\overline{y}_I$ has bidegree $(2,2)$.

Using Proposition 2.3.4, we deduce that each group $K(n)^i K(\mathbf{Z}/p^c\mathbf{Z}, n)$ admits a filtration

$$K(n)^i K(\mathbf{Z}/p^c\mathbf{Z}, n) = F^0 K(n)^i K(\mathbf{Z}/p^c\mathbf{Z}, n) \supseteq F^1 K(n)^i K(\mathbf{Z}/p^c\mathbf{Z}, n) \supseteq \cdots$$

with

$$K(n)^i K(\mathbf{Z}/p^c\mathbf{Z}, n) \simeq \varprojlim K(n)^i K(\mathbf{Z}/p^c\mathbf{Z}, n)/F^s K(n)^i K(\mathbf{Z}/p^c\mathbf{Z}, n)$$

and $F^s K(n)^i K(\mathbf{Z}/p^c\mathbf{Z}, n)/F^{s+1} K(n)^i K(\mathbf{Z}/p^c\mathbf{Z}, n) \simeq E_\infty^{s,s-i}$. Since $E_\infty^{s,t} \simeq 0$ unless both s and t are even, we immediately deduce that $K(n)^i K(\mathbf{Z}/p^c\mathbf{Z}, n)$ vanishes when i is odd: that is, the space $K(\mathbf{Z}/p^c\mathbf{Z}, n)$ is $K(n)$-even. Moreover, the associated graded ring $\mathrm{gr}^* K(n)^0 K(\mathbf{Z}/p^c\mathbf{Z}, m)$ is isomorphic to a truncated polynomial algebra $\kappa[\overline{y}]/\overline{y}^c$.

Since multiplication by p^c on $K(\mathbf{Z}/p^c\mathbf{Z}, m)$ is nullhomotopic, the canonical map

$$\theta : R \to K(n)^0 K(\mathbf{Z}/p^c\mathbf{Z}, n)$$

annihilates $y^{p^c} = [p^c](y)$. We now complete the proof by observing that θ determines an isomorphism $R/(y^{p^c}) \to K(n)^0 K(\mathbf{Z}/p^c\mathbf{Z}, n)$ (since the induced map of associated graded rings is an isomorphism).

We now treat the case $m \neq n$. Here the argument is similar, but the details are more complicated because we do not have a simple formula for the map $[p^c] : R \to R$. Let $\mathfrak{m}_R$ denote the maximal ideal of R and let N denote the dual of the Dieudonne module $\mathrm{DM}(K(n)_0 K(\mathbf{Q}_p/\mathbf{Z}_p, m))$. As a $W(\kappa)$-module, we will identify N with

$$\mathrm{Hom}_{W(\kappa)}(\mathrm{DM}(K(n)_0 K(\mathbf{Q}_p/\mathbf{Z}_p, m)), W(\kappa)[p^{-1}]/W(\kappa))$$

equipped with the action of D_κ described in Remark 1.4.18. Using Proposition 2.4.12, we can identify N with the dual $\mathrm{Hom}_{W(\kappa)}(\wedge^m M, W(\kappa))$. Replacing κ by its algebraic closure if necessary, we may assume that M is generated (as a D_κ-module) by an element γ satisfying $F\gamma = V^{n-1}\gamma$, so that M is freely generated as a $W(\kappa)$-module by the elements $\gamma, V\gamma, \ldots, V^{n-1}\gamma$ with $V^n\gamma = p\gamma$. For every subset $I = \{i_1, \ldots, i_m\} \subseteq \{0, \ldots, n-1\}$, we let $V^I(\gamma) = V^{i_1}\gamma \wedge \ldots \wedge V^{i_m}\gamma \in \wedge^m M$. The elements $V^I\gamma$ form a basis for $\bigwedge^m M$ as a $W(\kappa)$-module. We let $\{\delta_I\}_{I \subseteq \{0,\ldots,n-1\}}$ denote the dual basis for N. Unwinding the definitions, we see that the action of F on N is given on this basis by the formula

$$F\delta_{\{i_1 < \ldots, < i_m\}} = \begin{cases} \delta_{\{i_1-1,\ldots,i_m-1\}} & \text{if } i_1 > 0 \\ (-1)^{m-1} p \delta_{\{i_2-1,\ldots,i_m-1,n-1\}} & \text{if } i_1 = 0. \end{cases}$$

Since $m \neq n$, the action of V on N is topologically nilpotent. We may therefore write R as an inverse limit of finite-dimensional connected Hopf algebras H_α over κ. Then $N \simeq \varprojlim \mathrm{DM}(H_\alpha)$, so that the inclusions $\mathrm{DM}(H_\alpha) \hookrightarrow H_\alpha$ determine a map $\rho_R : N \to R$. Let $\mathfrak{J}$ denote the collection of all subsets of $\{0, \ldots, n-1\}$ which have cardinality m and contain the element $n-1$. For each $I \in \mathfrak{J}$, let $y_I = \rho_R(\delta_I) \in R$. The elements $\{\delta_I\}_{I \in \mathfrak{J}}$ form a basis for N/FN, so that the images of the elements $\{y_I\}_{I \in \mathfrak{J}}$ form a basis for $\mathfrak{m}_R/\mathfrak{m}_R^2$ (see Proposition 1.3.20). It follows that R can be identified with the formal power series ring generated by the elements y_I. For each $I = \{i_1 < \cdots < i_m\} \in \mathfrak{J}$, let $e_I = i_{m-1} - i_{m-2}$, and let $I^+ = \{e_I - 1 < e_I + i_1 < \ldots, < e_I + i_{m-2}\}$, so that $p\delta_I = (-1)^{m-1} F^{e_I} \delta_{I^+}$. We define $I(k)$ for $k \geq 0$ by the recursion

$$I(0) = 0 \qquad I(k+1) = I(k)^+,$$

and we let $e_I^{(k)} = e_I + e_{I(1)} + \cdots + e_{I(k-1)}$, so that $p^k \delta_I = (-1)^{k(m-1)} F^{e_I^{(k)}} \delta_{I(k)}$. Let $\sigma : R \to R$ denote the antipodal map if $k(m-1)$ is odd, and the identity map otherwise. Then the map $[p^c] : R \to R$ carries $\sigma(y_I)$ to

$$[p^c]\rho_R((-1)^{c(m-1)}\delta_I) = \rho_R((-1)^{c(m-1)} p^c \delta_I) = (F^{e_I^{(c)}} \delta_{I(t)}) = \sigma y_{I(t)}^{p^{e_I^{(c)}}}.$$

For each $I \in \mathfrak{J}$, we define a spectral sequence $\{E_r^{s,t}(I), d_r\}_{r \geq 2}$ as follows. For $r \leq 2p^{e_I} - 1$, we let $E_r^{*,*}(I)$ denote the free $(\pi_* K(n))[Z_I]$-module on generators 1 and X_I, where X_I has bidegree $(1,0)$ and Z_I has bidegree $(2,2)$. For $r \geq 2p^{e_I}$, we let $E_r^{*,*}(I)$ be the quotient $(\pi_* K(n))[Z_I]/(Z_I^{p^{e_I^{(c)}}})$. Finally, the differential d_r vanishes unless $r = 2p^{e_I^{(c)}} - 1$, in which case d_r is the unique $(\pi_* K(n))[Z_I]$-linear map given by

$$d_{2p^{e_I^{(c)}}-1}(1) = 0 \qquad d_{2p^{e_I^{(c)}}-1}(X_I) = v^{-1}(-1)^{t(m-1)} Z_I^{p^{e_I^{(c)}}}.$$

Let $\overline{y}_I$ denote the image of y_I in $\mathfrak{m}_R/\mathfrak{m}_R^2 \simeq E_2^{2,2}$, and choose an element $x_I \in E_2^{1,0}$ such that $\psi(x_I) = \overline{y}_I$. Using $(*)$, we obtain a map of spectral sequences $E_r^{s,t}(I) \to E_r^{s,t}$, given on the second page by

$$Z_I^a \mapsto \overline{y}_{I(t)}^a \qquad X_I Z_I^a \mapsto x_I \overline{y}_{I(t)}^a.$$

Since $\{E_r^{s,t}, d_r\}_{r \geq 2}$ is a spectral sequence of algebras, we can tensor these maps together (over the graded ring $\pi_* K(n)$) to obtain a map of spectral sequences

$$\bigotimes_{I \in \mathfrak{J}} \{E_r^{s,t}(I), d_r\}_{r \geq 2} \to \{E_r^{s,t}, d_r\}_{r \geq 2}.$$

By inspection, this map is an isomorphism on the second page, and therefore an isomorphism. In particular, the spectral sequence $\{E_r^{s,t}, d_r\}_{r \geq 2}$ stabilizes after a finite number of steps, and its final page is given by

$$E_\infty^{*,*} \simeq (\pi_* K(n))[\overline{y}_I]/(\overline{y}_{I(t)}^{e_I^{(c)}}),$$

where each $\overline{y}_I$ has bidegree $(2,2)$.

Using Proposition 2.3.4, we deduce that each group $K(n)^i K(\mathbf{Z}/p^c \mathbf{Z}, m)$ admits a filtration

$$K(n)^i K(\mathbf{Z}/p^c \mathbf{Z}, m) = F^0 K(n)^i K(\mathbf{Z}/p^c \mathbf{Z}, m) \supseteq F^1 K(n)^i K(\mathbf{Z}/p^c \mathbf{Z}, m) \supseteq \cdots$$

with

$$K(n)^i K(\mathbf{Z}/p^c \mathbf{Z}, m) \simeq \varprojlim K(n)^i K(\mathbf{Z}/p^c \mathbf{Z}, m)/F^s K(n)^i K(\mathbf{Z}/p^c \mathbf{Z}, m)$$

and $F^s K(n)^i K(\mathbf{Z}/p^c \mathbf{Z}, m)/F^{s+1} K(n)^i K(\mathbf{Z}/p^c \mathbf{Z}, m) \simeq E_\infty^{s,s-i}$. Since $E_\infty^{s,t} \simeq 0$ unless both s and t are even, we immediately deduce that $K(n)^i K(\mathbf{Z}/p^c \mathbf{Z}, m)$ vanishes when i is odd: that is, the space $K(\mathbf{Z}/p^c \mathbf{Z}, m)$ is $K(n)$-even. Moreover, $K(n)^0 K(\mathbf{Z}/p^c \mathbf{Z}, m)$ is a vector space over κ of dimension $\prod_{I \in \mathfrak{J}} p^{e_I^{(c)}}$, and its associated graded ring $\mathrm{gr}^* K(n)^0 K(\mathbf{Z}/p^c \mathbf{Z}, m)$ is isomorphic to a truncated polynomial algebra $\kappa[\overline{y}_I]/(\overline{y}_{I(t)}^{e_I^{(c)}})$.

Let $K \subseteq R$ denote the ideal defining the kernel of the map $[p^c] : \mathrm{Spf}\, R \to \mathrm{Spf}\, R$. Since multiplication by p^c on $K(\mathbf{Z}/p^c\mathbf{Z}, m)$ is nullhomotopic, the canoncial map $\theta : R \to K(n)^0 K(\mathbf{Z}/p^c\mathbf{Z}, m)$ annihilates the ideal K. To complete the proof, it will suffice to show that θ induces an isomorphism $R/K \to K(n)^0 K(\mathbf{Z}/p^c\mathbf{Z}, m)$. It is clear that the map θ is surjective (in fact, it is surjective at the level of associated graded rings, where R is equipped with the filtration appearing in the proof of Proposition 2.4.11). It will therefore suffice to show that $\dim_\kappa R/K \leq \prod_{I \in \mathcal{J}} p^{e_I^{(c)}}$. This is clear, since $\prod_{I \in \mathcal{J}} p^{e_I^{(c)}}$ is the dimension of the truncated power series ring $\kappa[[y_I]]/(y_{I(t)}^{p^{e_I^{(c)}}})$, and each $y_{I(t)}^{p^{e_I^{(c)}}}$ belongs to the ideal K.

It remains to prove $(*)$. Let $Y_\bullet$ and $Y''_\bullet$ be the simplicial spaces obtained by applying the bar construction to the topological abelian groups $G = K(\mathbf{Q}_p/p^c\mathbf{Z}_p, m-1)$ and $G'' = K(\mathbf{Q}_p/\mathbf{Z}_p, m-1)$, and let X and X'' denote the filtered spectra given by the partial totalizations

$$X(s) = \mathrm{Tot}_s\, K(n)^{Y_\bullet} \qquad X''(s) = \mathrm{Tot}_s\, K(n)^{Y''_\bullet}.$$

Finally, let W denote the constant tower of spectra

$$\cdots \to W(2) \to W(1) \to W(0)$$

where each $W(s)$ is equal to $K(n)$. The pullback square of topological abelian groups

$$\begin{tikzcd} G' & G \arrow[l] \\ * \arrow[u] & G'' \arrow[l] \arrow[u] \end{tikzcd}$$

determines a commutative diagram of filtered spectra

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X \arrow[d] \\ W \arrow[r] & X'. \end{tikzcd}$$

We let $X(\infty)$ denote the inverse limit $\varprojlim_s X(s) \simeq K(n)^{K(\mathbf{Q}_p/p^c\mathbf{Z}_p, m)}$, and define $X'(\infty)$, $X''(\infty)$, and $W(\infty)$ similarly. For $0 \leq a \leq b \leq \infty$ we let $X(b,a)$ denote the fiber of the map $X(b) \to X(a)$, and define $X'(b,a)$, $X''(b,a)$, and $W(b,a)$ similarly (so that $W(b,a) \simeq 0$).

Let $x \in E_2^{1,0}$ be as in $(*)$. Remark 2.4.14 supplies an isomorphism $\mathrm{Ext}^2_A \simeq \mathrm{im}(\pi_{-2}X''(\infty,0) \to \pi_{-2}X''(2,0))$. We will abuse notation by identifying $\psi(x)$ with the element of $\pi_{-2}X''(2,0)$ given by its image under this isomorphism, and x with an element of $\pi_{-1}X'(1)$ belonging to the image of the map $\pi_{-1}X'(2,0) \to \pi_{-1}X'(1)$. We will need the following assertion:

- $(*')$ There exists an element $z \in \pi_{-2}\,\mathrm{fib}(X''(2,0) \to X(2,0))$, whose image in $\pi_{-2}X''(2,0)$ coincides with $-\psi(x)$, and whose image in under the composite map

$$\pi_{-2}\,\mathrm{fib}(X''(2,0) \to X(2,0)) \to \pi_{-2}\,\mathrm{fib}(W(2,0) \to X'(2,0)) \simeq \pi_{-1}X'(2,0) \to \pi_{-1}X'(1)$$

coincides with x.

Assuming $(*')$ for the moment, let us prove $(*)$. Choose $y \in \mathfrak{m}_R \simeq \pi_0 X''(\infty,0)$ representing $v\psi(x)$, and let z be as in $(*)$. Since the map

$$\pi_{-2}\,\mathrm{fib}(X''(\infty,0) \to X(2,0)) \to \pi_{-2}X''(\infty,0) \times_{\pi_{-2}X''(2,0)} \pi_{-2}\,\mathrm{fib}(X''(2,0) \to X(2,0))$$

is surjective, we can choose an element $\overline{z} \in \pi_{-2}\,\mathrm{fib}(X''(\infty,0) \to X(2,0))$ whose image in $\pi_{-2}X''(\infty,0)$ is $v^{-1}y$, and whose image in $\pi_{-2}\,\mathrm{fib}(X''(2,0) \to X(2,0))$ coincides with z. By assumption, we have $[p^c](y) \in$

$\mathfrak{m}_R^s$, so that the image of $v^{-1}y$ in $\pi_{-2}X(\infty,0)$ lifts to an element $\overline{y} \in X(\infty, 2s-1)$. Let $\overline{y}'$ and $\overline{z}'$ denote the images of $\overline{y}$ and $\overline{z}$ in $\pi_{-2}\,\mathrm{fib}(X(\infty,0) \to X(2,0)) \simeq \pi_{-2}X(\infty,2)$. The $\overline{y}$ and $\overline{z}$ have the same image in $\pi_{-2}X(\infty,0)$, so that the difference $\overline{y} - \overline{z}$ belongs to the image of the boundary map $\pi_{-1}X(2,0) \to \pi_0\,\mathrm{fib}(X(\infty,0) \to X(2,0))$. The proof of Proposition 2.4.11 shows that the map $\pi_{-1}X(2,0) \to \pi_{-1}X(1,0)$ vanishes. Using the commutativity of the diagram

$$\begin{tikzcd}
\pi_{-1}X(2,0) \arrow[r] \arrow[d] & \pi_{-2}\,\mathrm{fib}(X(\infty,0) \to X(2,0)) \arrow[d] \\
\pi_{-1}X(1,0) \arrow[r] & \pi_{-2}\,\mathrm{fib}(X(\infty,0) \to X(1,0)),
\end{tikzcd}$$

we see that $\overline{y}$ and $\overline{z}$ have the same image in $\pi_{-2}X(\infty,1)$. Let z_0 denote the image of $\overline{z}$ in $\pi_{-2}\,\mathrm{fib}(X''(\infty,0) \to X(1))$, so that the pair $(z_0, \overline{y})$ lifts to an element w of

$$\pi_{-2}(\mathrm{fib}(X''(\infty,0) \to X(1,0)) \times_{\mathrm{fib}(X(\infty,0)\to X(1,0))} \mathrm{fib}(X(\infty,0) \to X(2s-1,0)),$$

which we will identify with $\pi_{-2}(X''(\infty) \times_{X(\infty)} X(\infty, 2s-1))$. Let $\overline{x}$ denote the image of w in

$$\pi_{-2}(W(\infty) \times_{X'(\infty)} X'(\infty, 2s-1)) \simeq \pi_{-1}X'(2s-1,0).$$

By construction, $\overline{x}$ is a preimage of x under the canonical map $\pi_{-1}X'(2s-1,0) \to \pi_{-1}X'(1)$. This proves that x survives to the $(2s-1)$st page of our spectral sequence. Moreover, $d_{2s-1}(x)$ can be identified with the image of $\overline{x}$ under the composite map

$$\pi_{-1}X'(2s-1,0) \simeq \pi_{-2}\,\mathrm{fib}(0 \to X'(2s-1,0)) \to \pi_{-2}\,\mathrm{fib}(X'(2s,0) \to X'(2s-1,0)) \simeq \pi_{-2}X'(2s,2s-1).$$

Equivalently, $d_{2s-1}(x)$ is represented by the image of w under the composite map

$$\pi_{-2}(X''(\infty) \times_{X(\infty)} X(\infty,2s-1)) \to \pi_{-2}X(\infty,2s-1) \to \pi_{-2}X(2s,2s-1) \to \pi_{-2}X'(2s,2s-1),$$

which coincides with the element x' appearing in $(*)$.

We now prove $(*')$. If Z is a pointed space, we let $K(n)^*_{\mathrm{red}}(Z)$ denote the reduced $K(n)$-cohomology of Z: that is, the kernel of the map $K(n)^*(Z) \to K(n)^*(\mathrm{red})$ given by evaluation at the base point. Unwinding the definitions, we obtain canonical isomorphisms

$$\pi_*X(d,d-1) \simeq K(n)^{-*-d}_{\mathrm{red}}(G^{\wedge d}) \simeq \mathrm{Hom}_\kappa(\mathfrak{m}_A^{\otimes d}, \kappa) \otimes_\kappa \pi_{d+*}K(n)$$

$$\pi_*X'(d,d-1) \simeq K(n)^{-*-d}_{\mathrm{red}}(G'^{\wedge d}) \simeq \mathrm{Hom}_\kappa(\mathfrak{m}_{A'}^{\otimes d}, \kappa) \otimes_\kappa \pi_{d+*}K(n).$$

Using the fiber sequences

$$X(1,0) \to \Sigma X(2,1) \to \Sigma X(2,0)$$

$$X'(1,0) \to \Sigma X'(2,1) \to \Sigma X'(2,0),$$

we deduce the existence of exact sequences

$$0 \to \pi_{-1}X(2,0) \to \mathrm{Hom}_\kappa(\mathfrak{m}_A, \kappa) \xrightarrow{\nu} \mathrm{Hom}_\kappa(\mathfrak{m}_A^{\otimes 2}, \kappa) \to \pi_{-2}X(2,0) \to 0,$$

where ν is dual to the multiplication map $\mathfrak{m}_A^{\otimes 2} \to \mathfrak{m}_A$. The kernel of ν can be identified with Ext^1_A (Remark 2.2.4), which vanishes by Theorem 2.2.10. It follows that the homotopy groups of $X(2,0)$ are concentrated in even degrees. We have a diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_\kappa(\mathfrak{m}_A, \kappa) \arrow[r] \arrow[d, "\alpha"] & \mathrm{Hom}_\kappa(\mathfrak{m}_A^{\otimes 2}, \kappa) \arrow[r] \arrow[d, "\alpha'"] & \pi_{-2}X''(2,0) \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_\kappa(\mathfrak{m}_A, \kappa) \arrow[r] & \mathrm{Hom}_\kappa(\mathfrak{m}_A^{\otimes 2}, \kappa) \arrow[r] & \pi_{-2}X(2,0) \arrow[r] & 0.
\end{tikzcd}$$

Here the maps α and α' are induced by the Hopf algebra homomorphism $[p^c] : A \to A$. Since $\operatorname{Spf} A^\vee$ is p-divisible, these maps are injective. The snake lemma provides an exact sequence

$$0 \to \ker(\beta) \xrightarrow{\mu} \operatorname{coker}(\alpha) \to \operatorname{coker}(\alpha') \to \operatorname{coker}(\beta) \to 0$$

Since $\pi_{-1}X(2,0) \simeq 0$, we the canonical map $\pi_{-2}\operatorname{fib}(X''(2,0) \to X(2,0)) \to \ker(\beta)$ is an isomorphism. Moreover, the canonical map $\pi_{-2}\operatorname{fib}(X''(2,0) \to X(2,0)) \to \pi_{-1}X'(1,0)$ is given by the composition

$$\pi_{-2}\operatorname{fib}(X''(2,0) \to X(2,0)) \simeq \ker(\beta) \xrightarrow{-\mu} \operatorname{coker}(\alpha) \to \pi_{-1}X'(1,0).$$

Consequently, to prove $(*')$, it will suffice to show that for each $x \in E_2^{1,0}$, the element $\psi(x) \in \pi_{-2}X(2,0)$ belongs to the kernel of β, and $\mu(\psi(x)) \in \operatorname{coker}(\alpha)$ is a preimage of x under the canonical map $\operatorname{coker}(\alpha) \to X'(1,0)$. This follows immediately from the description of $\psi(x)$ supplied by Remark 2.2.4. □

3 Alternating Powers of p-Divisible Groups

Let κ be a perfect field of characteristic $p > 0$, let $\mathbf{G}_0$ be a smooth connected 1-dimensional formal group of height $n < \infty$ over κ, and let $K(n)$ denote the associated Morava K-theory spectrum. In §2.4, we reviewed the Ravenel-Wilson calculation of the groups $K(n)_* K(\mathbf{Z}/p^t\,\mathbf{Z}, d)$. In the language of Dieudonne modules, this calculation can be summarized as follows: $K(n)_0 K(\mathbf{Z}/p^t\,\mathbf{Z}, d)$ is a connected Hopf algebra over κ, whose Dieudonne module can be identified with the dth exterior power of the Dieudonne module of $K(n)_0 K(\mathbf{Z}/p^t\,\mathbf{Z}, 1)$.

Our goal in this section is to describe the passage from $\mathbf{G}_0$ to $K(n)_0 K(\mathbf{Z}/p^t\,\mathbf{Z}, d)$ in a purely algebro-geometric way, which does not make reference to Dieudonne theory. To this end, we introduce a general construction, which associates to each finite flat commutative group scheme G over a commutative ring R a collection of group schemes $\{\operatorname{Alt}^{(d)}_G\}_{d \geq 1}$. Our main results can be summarized as follows:

- (a) Let R be a perfect field of characteristic $p > 0$, let G be a truncated connected p-divisible group of dimension 1 over κ (see Definition 3.1.1), and write $G = \operatorname{Spec} H^\vee$ for some Hopf algebra H over κ. Then $\operatorname{Alt}^{(d)}_G = \operatorname{Spec} A$, where the Dieudonne module of A is the dth exterior power of the Dieudonne module of H (Theorem 3.3.1).
- (b) In the special case where $R = \kappa$, we will show that Theorem 2.4.10 supplies a canonical isomorphism $\operatorname{Spec} K(n)_0 K(\mathbf{Z}/p^t\,\mathbf{Z}, d) \simeq \operatorname{Alt}^{(d)}_{\mathbf{G}_0[p^t]}$ (Corollary 3.3.3).
- (c) Let E denote the Lubin-Tate spectrum determined by κ and $\mathbf{G}_0$, and let $\mathbf{G}$ be the formal group $\operatorname{Spf} E^0(\mathbf{CP}^\infty)$ over $R = \pi_0 E$. Then the isomorphism appearing in (b) lifts to an identification R-schemes
$$\operatorname{Spec} E_0^\wedge K(\mathbf{Z}/p^t\,\mathbf{Z}, d) \simeq \operatorname{Alt}^{(d)}_{\mathbf{G}[p^t]}$$
(see Theorem 3.4.1).
- (d) Let G be a truncated p-divisible group of dimension 1 over an arbitrary commutative ring R. Then each $\operatorname{Alt}^{(d)}_G$ is also a truncated p-divisible group over R (Theorem 3.5.1). In particular, it is a finite flat group scheme over R.

If G is a finite flat group scheme which is annihilated by some odd integer n, then the group schemes $\operatorname{Alt}^{(d)}_G$ are easy to describe: they classify skew-symmetric multilinear maps from the d-fold product $G \times_{\operatorname{Spec} R} \cdots \times_{\operatorname{Spec} R} G$ into the multiplicative group $\mathbf{G}_m$. In the general case, the appropriate definition was suggested to us in a correspondence with Johan de Jong, and will be explained in detail in §3.2. We will discuss (a) and (b) in §3.3, (c) in §3.4, and (d) in §3.5. We note that for $p \neq 2$, these results appear elsewhere in the literature. We refer the reader to [17] for a proof of (c), and to [8] for proofs of (a) and (d).

Note that assertions (a) and (d) make reference the theory of truncated p-divisible groups. For the reader's convenience, we summarize the relevant definitions in §3.1.

3.1 Review of p-Divisible Groups

In this section, we briefly review the theory of p-divisible groups and of truncated p-divisible groups, emphasizing those aspects which will be needed in this paper. For more detailed accounts, we refer the reader to [3], [16], and [11].

Definition 3.1.1. Let R be a commutative ring, let $\mathbf{CAlg}_R$ denote the category of (discrete) commutative R-algebras, let p be a prime number, and let $\mathcal{A}b$ denote the category of abelian groups. A *p-divisible group of height n* over R is a functor $G : \mathbf{CAlg}_R \to \mathcal{A}b$ satisfying the following conditions:

(a) For every integer $t \geq 0$, the functor $A \mapsto \{x \in G(A) : p^t x = 0\}$ is representable by a finite flat group scheme of rank p^{nt} over R, which we will denote by $G[p^t]$.

(b) The union $\bigcup_{t \geq 0} G[p^t]$ is equal to G: that is, for every $A \in \mathbf{CAlg}_R$, the action of p on $G(A)$ is locally nilpotent.

It follows from (a) and (b) that the functor $A \mapsto G(A)$ is a sheaf for the fpqc topology on $\mathbf{CAlg}_R$.

(c) The map $[p] : G \to G$ is an epimorphism of sheaves for the fpqc topology. That is, we have a short exact sequence

$$0 \to G[p] \to G \to G \to 0$$

of fpqc sheaves.

Remark 3.1.2. Let κ be a field, and let H be a p-divisible Hopf algebra over κ (Definition 2.2.6). Then the functor

$$\operatorname{Spf} H^\vee : \mathbf{CAlg}_\kappa \to \mathcal{A}b$$

$$A \mapsto \operatorname{GLike}(H \otimes_\kappa A)$$

is a p-divisible group of height n over R. The construction $H \mapsto \operatorname{Spf} H^\vee$ determines an equivalence from the category of p-divisible Hopf algebras over κ to the category of p-divisible groups over κ.

Definition 3.1.3. Let R be a commutative ring and let G be a finite flat commutative group scheme over R. We will say that G is a *truncated p-divisible group of height n and level t over R* if the following conditions are satisfied:

(a) The rank of G is equal to p^{nt}.

(b) The finite flat group scheme G is annihilated by p^t.

(c) Suppose we are given a ring homomorphism $\phi : R \to \kappa$, where κ is an algebraically closed field of characteristic different from p. Then the finite abelian group $G(\kappa)$ is isomorphic to $(\mathbf{Z}/p^t\,\mathbf{Z})^n$.

(d) Suppose we are given a ring homomorphism $R \to \kappa$, where κ is a perfect field of characteristic p. Write $G \times_{\operatorname{Spec} R} \operatorname{Spec} \kappa = \operatorname{Spec} H$ for some finite-dimensional Hopf algebra H over κ, and let $M = \mathrm{DM}_+(H)$ be its Dieudonne module. Then

$$\dim_\kappa \ker(V : M \to M) + \dim_\kappa \ker(F : M \to M) = n.$$

Proposition 3.1.4. *Let R be a commutative ring and let G be a p-divisible group over R of height n. For each $t \geq 0$, the subgroup $G[p^t]$ is a truncated p-divisible group over R of height n and level t.*

Proof. We will show that G satisfies conditions (a) through (d) of Definition 3.1.3. Conditions (a) and (b) are obvious. To prove (c), let κ be an algebraically closed field of characteristic different from p. Then $G[p^t](\kappa) = \{x \in G(\kappa) : p^t x = 0\}$ is a $\mathbf{Z}/p^t\,\mathbf{Z}$-module of cardinality p^{nt}. Since G is p-divisible, we have short exact sequences

$$0 \to G[p^a](\kappa) \to G[p^{a+1}](\kappa) \to G[p](\kappa) \to 0,$$

so the cardinality of $G[p^t](\kappa)$ is the tth power of the cardinality of $G[p](\kappa)$. It follows that $G[p](\kappa)$ has cardinality p^n. Using the structure theory for finitely generated abelian groups, we deduce that $G[p^t](\kappa)$ is a direct sum of exactly n cyclic groups, each of which has order at most p^t. Since $G[p^t](\kappa)$ has cardinality p^{nt}, each of these groups must have order p^t, so that $G[p^t](\kappa)$ is a free $\mathbf{Z}/p^t\,\mathbf{Z}$-module of rank n.

It remains to verify condition (d). Suppose we are given a map $R \to \kappa$, where κ is a perfect field of characteristic $p > 0$. Write $G = \operatorname{Spf} H^\vee$ for some Hopf algebra H over κ, and let $M = \mathrm{DM}_+(H)$. Let $M[F]$, $M[V]$, and $M[p]$ denote the kernels of F, V, and p on M, respectively. We have an exact sequence

$$0 \to M[F] \to M[p] \xrightarrow{F} M[V],$$

which yields an inequality

$$\dim_\kappa M[F] + \dim_\kappa M[V] \geq \dim_\kappa M[p] = n.$$

To show that equality holds, it will suffice to verify that the above sequence is exact on the right. Choose an element $x \in M[V]$; we wish to show that $x = Fy$ for some $y \in M$ satisfying $py = 0$. Since G is p-divisible, we can write $x = px'$ for some $x' \in M$. Then $y = Vx'$ has the desired properties. $\square$

Proposition 3.1.5. *Let R be a commutative ring, and let G be a finite flat commutative group scheme of rank p^{nt} over R, and suppose that G is annihilated by p^t. Then:*

(1) *If $t \geq 2$, then G is a truncated p-divisible group of height n and level t if and only if the map $[p] : G \to G[p^{n-1}]$ is a surjection of fpqc sheaves. Moreover, if this condition is satisfied, then $G[p^k]$ is a finite flat group scheme for $0 \leq k \leq n$.*

(2) *Suppose that $t = 1$, let $R' = R/pR$, let $G' = G \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ be the reduction of G modulo p, let $G'^{(p)}$ denote the pullback of G along the Frobenius map $\varphi : R' \to R'$, and let*

$$F : G' \to G'^{(p)} \qquad V : G'^{(p)} \to G'$$

denote the relative Frobenius and Verschiebung maps, respectively. Then the following conditions are equivalent:

- (i) *The group scheme G is a truncated p-divisible group of height n and level 1 over R.*
- (ii) *The map F induces an epimorphism $G' \to \ker(V)$ of fpqc sheaves.*
- (iii) *The map V induces an epimorphism $G'^{(p)} \to \ker(F)$ of fpqc sheaves.*

Moreover, if these conditions are satisfied, then $\ker(F)$ and $\ker(V)$ are finite flat group schemes over R'.

The proof of Proposition 3.1.5 will require the following observation:

Lemma 3.1.6. *Let R be a commutative ring, let $\phi : G \to H$ be a map of group schemes which are finite and of finite presentation over R, and assume that G is flat over R. Then there exists a quasi-compact open subset $U \subseteq \operatorname{Spec} R$ with the following property: a map of schemes $X \to \operatorname{Spec} R$ factors through U if and only if the induced map $X \times_{\operatorname{Spec} R} G \to X \times_{\operatorname{Spec} R} H$ is faithfully flat. Moreover, $H \times_{\operatorname{Spec} R} U$ is flat over U.*

Proof. Since G and H are of finite presentation over R, we can find a finitely generated subring $R_0 \subseteq R$, group schemes G_0 and H_0 which are of finite presentation over R_0, and a map of group schemes $\phi_0 : G_0 \to H_0$ such that ϕ is isomorphic to the induced map $\operatorname{Spec} R \times_{\operatorname{Spec} R_0} G_0 \to \operatorname{Spec} R \times_{\operatorname{Spec} R_0} H_0$. Enlarging R_0 if necessary, we may suppose that G_0 and H_0 are finite over R_0, and that G_0 is flat over R_0. Replacing R by R_0, we can reduce to the case where R is a finitely generated commutative ring, and in particular Noetherian.

For each point $x \in \operatorname{Spec} R$, let $\kappa(x)$ denote the associated residue field. Let U denote the subset of $\operatorname{Spec} R$ consisting of those points x for which the induced map $\operatorname{Spec} \kappa(x) \times_{\operatorname{Spec} R} G \to \operatorname{Spec} \kappa(x) \times_{\operatorname{Spec} R} H$ is faithfully flat. The main point is to establish the following:

($*$) Suppose that $x \in U$. Then there exists an open subset $V \subseteq \operatorname{Spec} R$ containing x, such that $V \times_{\operatorname{Spec} R} H$ is flat over V.

Assume that ($*$) is satisfied. Let x be a point of $\operatorname{Spec} R$ and choose V satisfying ($*$). Note that the fiber-by-fiber flatness criterion (Corollary 11.3.11 of [6]) implies that the map $V \times_{\operatorname{Spec} R} G \to V \times_{\operatorname{Spec} R} H$ is flat. In particular, its image is an open subset of $V \times_{\operatorname{Spec} R} H$, with closed complement K. Since the projection map $V \times_{\operatorname{Spec} R} H \to V$ is finite, the image of K is a closed subset of V, which does not contain the point x. We may therefore shrink V if necessary to reduce to the case where $K = \emptyset$, so that the map $V \times_{\operatorname{Spec} R} G \to V \times_{\operatorname{Spec} R} H$ is faithfully flat. It follows that $V \subseteq U$. It follows that U is open (since it contains an open neighborhood of each point $x \in U$), that the fiber product $U \times_{\operatorname{Spec} R} H$ is flat over U (since this can be tested locally on U), and that the induced map $U \times_{\operatorname{Spec} R} G \to U \times_{\operatorname{Spec} R} H$ is faithfully flat. It is clear that any map of schemes $X \to \operatorname{Spec} R$ for which the induced map $X \times_{\operatorname{Spec} R} G \to X \times_{\operatorname{Spec} R} H$ much factor through U, so that U has the desired properties (the quasi-compactness of U is automatic, since R is Noetherian).

It remains to prove ($*$). Choose a point $x \in U$, and let R_x denote the corresponding localization of R. We will prove that $\operatorname{Spec} R_x \times_{\operatorname{Spec} R} H$ is flat over R_x. Writing R_x as a direct limit of R-algebras of the form $R[a^{-1}]$, we will then deduce that $\operatorname{Spec} R[a^{-1}] \times_{\operatorname{Spec} R} H$ is flat over $R[a^{-1}]$ for some element $a \in R$ which does not vanish in $\kappa(x)$, so that $V = \operatorname{Spec} R[a^{-1}]$ has the desired properties.

Replacing R by R_x, we may reduce to the case where R is a local Noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $\kappa = R/\mathfrak{m}$, and that the map $\operatorname{Spec} \kappa \times_{\operatorname{Spec} R} G \to \operatorname{Spec} \kappa \times_{\operatorname{Spec} R} H$ is faithfully flat. We will complete the proof of by showing that H is flat over R.

Write $H = \operatorname{Spec} A$ and $G = \operatorname{Spec} B$, for some finite R-algebras A and B. We wish to prove that A is flat over R. Choose elements $a_1, \ldots, a_n \in A$ whose images form a basis for $A/\mathfrak{m}A$ as a vector space over κ. This determines a map of finitely generated R-modules $\theta : R^n \to A$; we claim that θ is an isomorphism. Since R is a Noetherian local ring, it will suffice to show that θ induces an isomorphism after $\mathfrak{m}$-adic completion. In fact, we claim that θ induces an isomorphism

$$(R/\mathfrak{m}^t)^n \to A/\mathfrak{m}^t A$$

for each $t \geq 0$. This is equivalent to the requirement that $A/\mathfrak{m}^t A$ is flat as an $R/\mathfrak{m}^t$-module. We may therefore replace R by $R/\mathfrak{m}^t$, and thereby reduce to the case where R is a local Artin ring.

For every finitely generated R-module M, let $l(M)$ denote the length of M. By Nakayama's lemma, we have a surjection of R-modules $R^n \to A$, and we wish to show that this map is an isomorphism. Let K denote the kernel of the epimorphism of group schemes $\operatorname{Spec} \kappa \times_{\operatorname{Spec} R} G \to \operatorname{Spec} \kappa \times_{\operatorname{Spec} R} H$, and let r denote the rank of K over κ. Then $\operatorname{Spec} \kappa \times_{\operatorname{Spec} R} G$ has rank nr over κ.

Since A is a finite R-algebra, it is an Artinian ring. It therefore admits a finite filtration by ideals

$$0 = I(0) \subset I(1) \subset \cdots \subset I(k) = A,$$

where each quotient $I(j)/I(j-1)$ is isomorphic (as a A-module) to some residue field κ_j of A, which is a finite extension of κ. Then $I(j)B/I(j-1)B$ is a quotient of the tensor product $I(j)/I(j-1) \otimes_A B \simeq \kappa_j \otimes_A B$. Note that $\kappa_j \otimes_A B$ is a torsor for the group scheme $\operatorname{Spec} \kappa_j \times_{\operatorname{Spec} \kappa} K$ over κ, so that $\kappa_j \otimes_A B$ has dimension r as a vector space over κ_j, and length $r \dim_\kappa(\kappa_j)$ as an R-module. Since G is flat over R, B is a free R-module of rank nr, so we have

$$nrl(R) = l(B) = \sum_{1 \leq j \leq k} l(I(j)B/I(j-1)B) \leq \sum_{1 \leq j \leq k} r \dim_\kappa(\kappa_j) = rl(A).$$

Dividing by r, we deduce that $nl(R) \leq l(A)$, so that any surjection of R-modules $R^n \to A$ is automatically an isomorphism. $\square$

Corollary 3.1.7. *Let R be a commutative ring and let $\phi : G \to H$ be a map of group schemes which are of finite presentation over R. Assume that G and H are finite over R, that G is flat over R, and that the induced map $\operatorname{Spec} \kappa(x) \times_{\operatorname{Spec} R} G \to \operatorname{Spec} \kappa(x) \times_{\operatorname{Spec} R} H$ is faithfully flat, for each point $x \in \operatorname{Spec} R$. Then ϕ is faithfully flat, and H is flat over R.*

Proof of Proposition 3.1.5. Suppose first that G is a truncated p-divisible group over R of height n and level $t \geq 2$. We will show that the map $[p] : G \to G[p^{t-1}]$ is faithfully flat, and that $G[p^{t-1}]$ is flat over R (repeating this argument, we may deduce that $G[p^a]$ is flat over R for $0 \leq a \leq t$). Using Corollary 3.1.7, we can reduce to the case where R is a field κ. Without loss of generality, we may assume that κ is algebraically closed. If the characteristic of κ is different from p, the desired result follows immediately from condition (c) of Definition 3.1.3.

Suppose therefore that κ has characteristic p. For each $0 \leq a \leq t$, let r_a denote the rank of the group scheme $G[p^a]$ over κ. Using the exact sequence

$$0 \to G[F] \to G \xrightarrow{F} G[V]^{(p)},$$

and condition (d) of Definition 3.1.3, we deduce that $r_1 \leq p^n$. Using the exact sequence

$$0 \to G[p] \to G[p^a] \xrightarrow{p} G[p^{a-1}],$$

we obtain the inequality $r_a \leq r_1 r_{a-1}$, so by induction we have $r_a \leq r_1^a \leq p^{an}$. Since G has rank p^{nt}, each of these inequalities must be an equality. In particular, the maps $p : G[p^a] \to G[p^{a-1}]$ are surjective for $0 < a \leq t$, as desired.

Conversely, suppose that $t \geq 2$ and that the map $p : G \to G[p^{t-1}]$ is an epimorphism for the flat topology; we wish to prove that G is a truncated p-divisible group of height n and level t. Conditions (a) and (b) of Definition 3.1.3 are automatic. To verify (c), let κ be an algebraically closed field of characteristic different from p. The surjectivity of the map $p : G \to G[p^{t-1}]$ implies the surjectivity of the maps $G[p^a] \to G[p^{a-1}]$ for $0 < a \leq t$, so that G is a successive extension of t copies of $G[p]$. It follows that $G[p](\kappa)$ has order p^n, and is therefore an n-dimensional vector space over $\mathbf{Z}/p\,\mathbf{Z}$. It follows that the group $G(\kappa)$ is a direct sum of n cyclic groups of order $\leq p^t$. Since $G(\kappa)$ has order p^{nt} by assumption, we conclude that $G(\kappa)$ is a free $\mathbf{Z}/p^t\,\mathbf{Z}$-module of rank n.

Now suppose that $t = 1$. We will show that conditions (i) and (ii) are equivalent; the equivalence of (i) and (iii) then follows by the same argument. Suppose first that (i) is satisfied; we wish to prove that the sequence

$$0 \to \ker(F) \to G' \to \ker(V)$$

is exact on the right (and that, in this case, $\ker(V)$ and $\ker(F)$ are flat over R/pR). Using Corollary 3.1.7, we can reduce to the case where $R = \kappa$ is a field of characteristic p. In this case, it suffices to show that the rank of G' is equal to the sum of the ranks of $\ker(F)$ and $\ker(V)$, which follows immediately from condition (d) of Definition 3.1.3.

Now suppose that (ii) is satisfied; we will show that G is a truncated p-divisible group of height n and level 1. By assumption, G satisfies conditions (a) and (b) of Definition 3.1.3, and condition (c) is automatic (since every $\mathbf{Z}/p\,\mathbf{Z}$-module is free). It remains to verify condition (d). Without loss of generality, we may assume that $R = \kappa$ is a perfect field of characteristic p. Write $G = \operatorname{Spf} H^{\vee}$ and let $M = \mathrm{DM}(H)$. Condition (ii) gives an exact sequence

$$0 \to \ker(F : M \to M) \to M \to \ker(V : M \to M) \to 0,$$

so that $\dim_\kappa \ker(F) + \dim_\kappa \ker(V) = \dim_\kappa M = n$. $\square$

Corollary 3.1.8. *Let R be a commutative ring, let G be a finite flat commutative group scheme of rank p^{nt} over R, and suppose that G is annihilated by p^t. Then there exists a quasi-compact open subscheme $U \subseteq \operatorname{Spec} R$ with the following property: for any commutative R-algebra A, the map $\operatorname{Spec} A \to \operatorname{Spec} R$ factors through U if and only if $G_A = G \times_{\operatorname{Spec} R} \operatorname{Spec} A$ is a truncated p-divisible group of height n and level t over A.*

Proof. If $t \geq 2$, the desired result follows by applying Lemma 3.1.6 to the map of group schemes $p : G \to G[p^{t-1}]$ and invoking Proposition 3.1.5. If $t = 1$, we apply Lemma 3.1.6 to the map $F : G' \to \ker(V : G'^{(p)} \to G')$ appearing in Proposition 3.1.5 (note that if U is a quasi-compact open subset of $\operatorname{Spec} R/pR$, then $U \cup \operatorname{Spec} R[p^{-1}]$ is a quasi-compact open subset of $\operatorname{Spec} R$). □

Remark 3.1.9. Let R be a commutative ring, and let G be either a p-divisible group over R, or a truncated p-divisible group of level $t \geq 1$ over R. Let $G_0 = \operatorname{Spec} R/pR \times_{\operatorname{Spec} R} G$ and let $G_0^{(p)}$ denote its pullback along the Frobenius map $\varphi : R/pR \to R/pR$, and let $F : G_0 \to G_0^{(p)}$ denote the relative Frobenius map. Proposition 3.1.5 implies that $G_0[F] = \ker(F) \subseteq G_0[p]$ is a finite flat group scheme over $\operatorname{Spec} R/pR$. It follows that the rank of $G_0[F]$ is a locally constant function on $\operatorname{Spec} R/pR$. We will say that G has *dimension* d if the $G_0[F]$ has rank p^d over R/pR.

Warning 3.1.10. Let G be a p-divisible group (or a truncated p-divisible group of level $t \geq 1$) over a commutative ring R. If p is invertible in R, then the dimension of G is not uniquely determined: according to our definition, G has dimension d for every integer $d \geq 0$.

We will need the following converse of Proposition 3.1.4:

Theorem 3.1.11 (Grothendieck)**.** *Let κ be a perfect field of characteristic $p > 0$, and let R be a complete local Noetherian ring with residue field κ. Let G be a truncated p-divisible group of height n and level t over R. Then there exists a p-divisible group H over R and an isomorphism $G \simeq H[p^t]$*

For a proof, we refer the reader to [11].

3.2 Group Schemes of Alternating Maps

Let G be an abelian group and BG its classifying space. We have canonical isomorphisms

$$\mathrm{H}_1(BG; \mathbf{Z}) \simeq G \qquad \mathrm{H}_2(BG; \mathbf{Z}) \simeq \wedge^2 G.$$

If A is another abelian group, the universal coefficient theorem gives an exact sequence

$$0 \to \operatorname{Ext}^1(G, A) \to \mathrm{H}^2(BG; A) \xrightarrow{\beta} \operatorname{Hom}(\wedge^2 G, A) \to 0. \tag{1}$$

Here $\mathrm{H}^2(G; A)$ can be interpreted as the set of isomorphism classes of central extensions

$$0 \to A \to \widetilde{G} \to G \to 0$$

and the map β assigns to every such extension the associated commutator pairing $G \times G \to A$. From this description, it is obvious that the kernel of β can be identified with the set $\operatorname{Ext}^1(G, A)$ of isomorphism classes of extensions of G by A in the category of abelian groups. However, the exactness of the sequence 1 on the right is more subtle: it depends crucially on the vanishing of the group $\operatorname{Ext}^2(G, A)$.

If we work in the setting of group schemes rather than ordinary groups, the analogue of the sequence (1) need not be exact. For example, let k be a field of characteristic 2, and let α_2 denote the group scheme over k representing the functor $\alpha_2(A) = \{x \in A : x^2 = 0\}$ (regarded as a group with respect to addition). Then there is an alternating bilinear map

$$b : \alpha_2 \times_{\operatorname{Spec} k} \alpha_2 \to \mathbf{G}_m,$$

given on points by the formula

$$(x, y) \mapsto 1 + xy.$$

However, we will see below that b cannot arise as the commutator pairing for any central extension

$$0 \to \mathbf{G}_m \to \widetilde{\alpha}_2 \to \alpha_2 \to 0$$

(see Example 3.2.7).

Definition 3.2.1. Let R be a commutative ring, and let $\mathbf{CAlg}_R$ denote the category of commutative R-algebras. Let G be a commutative group scheme over R and let $c : G \times_{\operatorname{Spec} R} G \to \mathbf{G}_m$ be a map of R-schemes. If A in $\mathbf{CAlg}_R$ and we are given points $x, y \in G(A)$, we let $c(x, y) \in \mathbf{G}_m(A) = A^\times$ denote the image of (x, y) under c. We say that c is:

- *bilinear* if $c(x, y + z) = c(x, y)c(x, z)$ and $c(x + y, z) = c(x, z)c(y, z)$ for all $x, y, z \in G(A)$.
- *skew-symmetric* if c is bilinear and $c(x, y) = c(y, x)^{-1}$ for all $x, y \in G(A)$).
- a 2-*cocycle* if $c(x, y)c(x + y, z) = c(x, y + z)c(y, z)$ for all $x, y \in G(A)$.
- a *symmetric* 2-*cocycle* if it is a 2-cocycle and $c(x, y) = c(y, x)$ for all $x, y \in G(A)$.

For each $A \in \mathbf{CAlg}_R$, we let G_A denote the fiber product $G \times_{\operatorname{Spec} R} \operatorname{Spec} A$, regarded as a group scheme over A. We let $\operatorname{CoCyc}_G(A)$ denote the set of all 2-cocycles $c : G_A \times G_A \to \mathbf{G}_m$, $\operatorname{CoCyc}^s_G(A)$ the set of all symmetric 2-cocycles $c : G_A \times_{\operatorname{Spec} A} G_A \to \mathbf{G}_m$, and $\operatorname{Skew}^{(2)}_G(A)$ the set of all skew-symmetric maps $b : G_A \times_{\operatorname{Spec} A} G_A \to \mathbf{G}_m$. We regard CoCyc_G, CoCyc^s_G, and $\operatorname{Skew}^{(2)}_G$ as functors from $\mathbf{CAlg}_R$ to the category of sets.

If G is a finite flat group scheme over R, then each of these functors is representable by an affine scheme of finite presentation over R (they can be described as a closed subschemes of the affine scheme parametrizing all maps from $c : G \times_{\operatorname{Spec} R} G \to \mathbf{G}_m$). The collection of 2-cocycles (symmetric 2-cocycles, alternating maps) is closed under multiplication, so that we can regard CoCyc_G, CoCyc^s_G, and $\operatorname{Skew}^{(2)}_G$ as commutative group schemes over R.

Every 2-cocycle $c : G \times_{\operatorname{Spec} R} G \to \mathbf{G}_m$ determines an alternating bilinear map $b : G \times_{\operatorname{Spec} R} G \to \mathbf{G}_m$, given on A-valued points by the formula $b(x, y) = c(x, y)c(y, x)^{-1}$. Note that $b(x, y)$ is trivial if and only if c is symmetric. We therefore have an exact sequence of group-valued functors

$$0 \to \operatorname{CoCyc}^s_G \to \operatorname{CoCyc}_G \to \operatorname{Skew}^{(2)}_G .$$

Notation 3.2.2. Let G be a finite flat commutative group scheme over a commutative ring R. For every $A \in \mathbf{CAlg}_R$, we let $\mathbf{G}^G_m(A)$ denote the set of all morphisms of schemes $G_A \to \mathbf{G}_m$. Then $A \mapsto \mathbf{G}^G_m(A)$ is a functor from $\mathbf{CAlg}_A$ to the category of sets. Note that $\mathbf{G}^G_m$ is representable by a group scheme over R. Every map of schemes $\lambda : G \to \mathbf{G}_m$ determines a symmetric 2-cocycle $c : G \times_{\operatorname{Spec} R} G \to \mathbf{G}_m$, given on points by

$$c(x, y) = \lambda(x + y) - \lambda(x) - \lambda(y).$$

This construction determines a map of functors $\mathbf{G}^G_m \to \operatorname{CoCyc}^s_G$. The kernel of this map is the Cartier dual $\mathbf{D}(G)$ of G: that is, the finite flat group scheme parametrizing group homomorphisms from G into $\mathbf{G}_m$.

Proposition 3.2.3. *Let G be a finite flat commutative group scheme over a commutative ring R. Then the complex of group schemes*

$$0 \to \mathbf{D}(G) \to \mathbf{G}^G_m \to \operatorname{CoCyc}^s_G \to 0$$

is exact for the fppf topology.

Proof. The exactness of the sequence

$$0 \to \mathbf{D}(G) \to \mathbf{G}_m^G \to \mathrm{CoCyc}_s^G$$

is clear. To prove the exactness on the right, it will suffice to show that every symmetric 2-cocycle $c : G \times_{\operatorname{Spec} R} G \to \mathbf{G}_m$ arises from a map $\lambda : G \to \mathbf{G}_m$, at least locally in the fppf topology. Let $\widetilde{G}$ denote the product $G \times \mathbf{G}_m$, and equip $\widetilde{G}$ with the structure of a group scheme via the formula

$$(x, y)(x', y') = (xx', yy'c(x, x')).$$

We have a sequence of group schemes

$$0 \to \mathbf{G}_m \to \widetilde{G} \xrightarrow{\pi} G \to 0$$

which is exact (even as a sequence of presheaves). Since c is a symmetric 2-cocycle, $\widetilde{G}$ is a commutative group scheme. We can think of $\widetilde{G}$ as the space of nonzero sections of a line bundle $\mathcal{L}$ over G. Write $G = \operatorname{Spec} A$, so that $\mathcal{L}$ determines an invertible A-module M. The group structure on $\widetilde{G}$ determines a comultiplication $M \to M \otimes_R M$, which determines a commutative ring structure on the R-linear dual $M^\vee$ of M. Unwinding the definitions, we see that the affine scheme $\operatorname{Spec} M^\vee$ parametrizes splittings of the sequence

$$0 \to \mathbf{G}_m \to \widetilde{G} \xrightarrow{\pi} G \to 0.$$

Since $M^\vee$ is faithfully flat over R, we deduce that this sequence splits locally for the fppf topology. We may therefore assume (after changing R if necessary) that there is a map of group schemes $\phi : G \to \widetilde{G}$ such that $\pi \circ \phi = \mathrm{id}_G$. Then we can write $\phi = \mathrm{id}_G \times \lambda$ for some map $\lambda : G \to \mathbf{G}_m$, which clearly satisfies

$$\lambda(x) + \lambda(y) + c(x, y) = \lambda(x + y).$$

□

Corollary 3.2.4. *Let G be a finite flat commutative group scheme over a commutative ring R. Then CoCyc_G^s is a smooth affine group scheme over R.*

Proof. The assertion is local on R; we may therefore assume that $G = \operatorname{Spec} A$ where A is a finite free R-module. Write $A \simeq R^m$. Then $\mathbf{G}_m^G$ can be identified with an open subscheme of the affine space $\mathbf{A}^m$ of dimension m over R. In particular, $\mathbf{G}_m^G$ is a smooth R-scheme. Proposition 3.2.3 implies that the map $\mathbf{G}_m^G \to \mathrm{CoCyc}_G^s$ is faithfully flat. The group scheme $\mathbf{G}_m^G$ is flat over R, so that CoCyc_G^s is likewise flat over R.

The scheme CoCyc_G^s is also of finite presentation over R. Consequently, to verify its smoothness, we may reduce to the case where R is an algebraically closed field κ. Then CoCyc_G^s is an affine group scheme over κ, which is smooth if and only if it is reduced. Because the map $\mathbf{G}_m^G \to \mathrm{CoCyc}_G^s$ is faithfully flat, it suffices to check that $\mathbf{G}_m^G$ is reduced, which follows immediately from the fact that $\mathbf{G}_m^G$ is smooth over κ. □

Definition 3.2.5. Let G be a finite flat commutative group scheme over a commutative ring R. We let $\mathrm{Alt}_G^{(2)}$ denote the quotient of CoCyc_G by the subgroup CoCyc_G^s (in the category of fppf sheaves on $\mathbf{CAlg}_R$). Since CoCyc_G^s is a smooth group scheme, we can regard $\mathrm{Alt}_G^{(2)}$ as an algebraic space over R (which is separated, since CoCyc_G^s is a closed subgroup of CoCyc_G).

In the situation of Definition 3.2.5, the exact sequence

$$0 \to \mathrm{CoCyc}_G^s \to \mathrm{CoCyc}_G \to \mathrm{Skew}_G^{(2)}$$

induces a monomorphism $\mathrm{Alt}_G^{(2)} \hookrightarrow \mathrm{Skew}_G^{(2)}$. In particular, we see that the map $\mathrm{Alt}_G^{(2)} \to \mathrm{Skew}_G^{(2)}$ is quasi-finite and therefore quasi-affine. It follows that $\mathrm{Alt}_G^{(2)}$ is representable by a quasi-affine scheme over R.

Proposition 3.2.6. *Let G be a finite flat group scheme over a commutative ring R. If there exists an odd integer n such that multiplication by n annihilates G, then the map $\mathrm{Alt}^{(2)}_G \to \mathrm{Skew}^{(2)}_G$ is an isomorphism.*

Proof. We wish to prove that the sequence

$$0 \to \mathrm{CoCyc}^s_G \to \mathrm{CoCyc}_G \to \mathrm{Skew}^{(2)}_G \to 0$$

is exact. In fact, we will prove that is exact as a sequence of presheaves: that is, that the sequence of groups

$$0 \to \mathrm{CoCyc}^s_G(A) \to \mathrm{CoCyc}_G(A) \to \mathrm{Skew}^{(2)}_G(A) \to 0$$

is exact for every R-algebra A. Replacing R by A, we are reduced to proving that for every alternating bilinear map $b : G \times_{\mathrm{Spec}\, R} G \to \mathbf{G}_m$ has the form $b(x, y) = c(x, y)c(y, x)^{-1}$, for some 2-cocycle $c : G \times_{\mathrm{Spec}\, R} G \to \mathbf{G}_m$. Write $n = 2m - 1$ for some integer m, and define c by the formula $c(x, y) = b(x, my)$. Since c is bilinear, it is a 2-cocycle. We now compute

$$c(x, y)c(y, x)^{-1} = b(x, my)b(y, mx)^{-1} = b(x, my)b(y, x)^{-m} = b(x, my)b(x, y)^m = b(x, 2my) = b(x, y).$$

□

Example 3.2.7. Let R be a commutative ring in which $2 = 0$ and let $G = \alpha_2$ be the finite flat group scheme over κ given by the functor $G(A) = \{x \in A : x^2 = 0\}$ (regarded as a group under addition). Then $\mathrm{Skew}^{(2)}_G$ is isomorphic to the additive group $\mathbf{G}_a$, where the isomorphism carries a scalar $\lambda \in \mathbf{G}_a(A)$ to the skew-symmetric map

$$G_A \times_{\mathrm{Spec}\, A} G_A \to \mathbf{G}_a$$

given by $(x, y) \mapsto 1 + \lambda xy$. However, the group scheme $\mathrm{Alt}^{(2)}_G$ is trivial. To prove this, it will suffice to show that the map $\mathrm{CoCyc}^s_G \to \mathrm{CoCyc}_G$ is an isomorphism of group schemes. Let $A \in \mathbf{CAlg}_R$ and let c by an A-point of CoCyc_G. Write $G_A = \mathrm{Spec}\, A[x]/(x^2)$, so that we can identify c with an invertible element $c(x, y)$ of the ring $A[x, y]/(x^2, y^2)$ satisfying the equation

$$c(x, y)c(x + y, z) \quad = \quad c(x, y + z)c(y, z). \tag{2}$$

Write $c(x, y) = \lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 xy$. Comparing the coefficients of x in (2), we obtain

$$\lambda_0\lambda_1 + \lambda_1\lambda_0 = \lambda_1\lambda_0.$$

Since λ_0 is invertible, we deduce that $\lambda_1 = 0$. Similarly, comparing the coefficients of z in 2, we obtain $\lambda_2 = 0$. It follows that the cocycle c is symmetric.

Remark 3.2.8. The construction $G \mapsto \mathrm{Skew}^{(2)}_G$ is contravariantly functorial in G. Moreover, for every map $q : G \to G'$ of finite flat commutative group schemes over R, the induced map $\mathrm{Skew}^{(2)}_{G'} \to \mathrm{Skew}^{(2)}_G$ carries $\mathrm{Alt}^{(2)}_{G'}$ into $\mathrm{Alt}^{(2)}_G$. If q is faithfully flat, we can say a bit more: the diagram

$$\begin{tikzcd}
\mathrm{Alt}^{(2)}_{G'} \arrow[r] \arrow[d] & \mathrm{Alt}^{(2)}_G \arrow[d] \\
\mathrm{Skew}^{(2)}_{G'} \arrow[r] & \mathrm{Skew}^{(2)}_G
\end{tikzcd}$$

is a pullback square. To prove this, we must verify the following:

(∗) Suppose that $b : G' \times_{\mathrm{Spec}\, R} G' \to \mathbf{G}_m$ is a skew-symmetric bilinear map and that the induced map $b : G \times_{\mathrm{Spec}\, R} G \to \mathbf{G}_m$ is the commutator pairing associated to a central extension

$$0 \to \mathbf{G}_m \to \widetilde{G} \to G \to 0.$$

Then, locally for the fppf topology, the map b_0 has the same property.

To prove $(*)$, set $G'' = \ker(q)$ and let $\widetilde{G}''$ denote the fiber product $\widetilde{G} \times_G G''$, so that we have a central extension

$$0 \to \mathbf{G}_m \to \widetilde{G}'' \to G'' \to 0.$$

Since the pairing b vanishes on G'', this extension is abelian. Passing to an fppf covering of R, we may suppose that this sequence splits (see the proof of Proposition 3.2.3). A choice of splitting gives a closed embedding of group schemes $\phi : G'' \to \widetilde{G}$. For $x \in G''(A)$, $\widetilde{y} \in \widetilde{G}(A)$, we have

$$\phi(x)\widetilde{y}\phi(x)^{-1}\widetilde{y}^{-1} = b(x, y) = 1,$$

where y denotes the image of $\widetilde{y}$ in $G(A)$. It follows that the image of ϕ is a central subgroup of $\widetilde{G}$. We then have a central extension

$$0 \to \mathbf{G}_m \to \widetilde{G}/\phi(G'') \to G' \to 0,$$

whose commutator pairing is given by b_0.

We now generalize the above discussion to multilinear functions of several variables.

Definition 3.2.9. Let R be a commutative ring and G a finite flat commutative group scheme over R. For each integer $d \geq 1$, we let G^d denote the dth power of G in the category of R-schemes. We will say that a map $b : G^d \to \mathbf{G}_m$ is *skew-symmetric* if it is multilinear and, for every commutative R-algebra A, every tuple of points $x_1, \ldots, x_d \in G(A)$, and every pair $1 \leq i < d$, we have

$$b(x_1, \ldots, x_d) = b(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_d)^{-1} \in \mathbf{G}_m(A).$$

For each $A \in \mathbf{CAlg}_R$, we let $\mathrm{Skew}_G^{(d)}(A)$ denote the set of all skew-symmetric maps $G_A^d \to \mathbf{G}_m$. We regard $\mathrm{Skew}_G^{(d)}$ as a functor from $\mathbf{CAlg}_R$ to the category of abelian groups. It is easy to see that $\mathrm{Skew}_G^{(d)}$ is representable by an affine group scheme of finite presentation over R.

Notation 3.2.10. Let R be a commutative ring, let X be a finite flat R-scheme, and let Y be an arbitrary R-scheme. We let Y^X denote the Weil restriction of $X \times_{\operatorname{Spec} R} Y$ along the projection map $X \to \operatorname{Spec} R$. That is, Y^X denotes the functor $\mathbf{CAlg}_R \to \mathcal{S}\mathrm{et}$ given by

$$Y^X(A) = \mathrm{Hom}(X_A, Y),$$

where the Hom-set on the left hand side is computed in the category of R-schemes. This construction has the following properties:

(a) If Y is affine, then Y^X is representable by an affine R-scheme.

(b) If Y is quasi-affine, then Y^X is representable by a quasi-affine R-scheme.

(c) In cases (a) or (b), if Y is of finite presentation over R, then so is Y^X.

Construction 3.2.11. Let R be a commutative ring and let G be a finite flat commutative group scheme over R. For each integer $d \geq 2$, we have a canonical isomorphism $\mathbf{G}_m^{G^d} \simeq (\mathbf{G}_m^{G^2})^{G^{d-2}}$ which restricts to a closed immersion $\mathrm{Skew}_G^{(d)} \hookrightarrow (\mathrm{Skew}_G^{(2)})^{G^{d-2}}$. We let $\mathrm{Alt}_G^{(d)}$ denote the fiber product

$$(\mathrm{Alt}_G^{(2)})^{G^{d-2}} \times_{(\mathrm{Skew}_G^{(2)})^{G^{d-2}}} \mathrm{Skew}_G^{(d)}$$

If $d = 1$, we simply set $\mathrm{Alt}_G^{(d)} = \mathrm{Skew}_G^{(d)}$.

Remark 3.2.12. In the situation of Construction 3.2.11, $\mathrm{Alt}_G^{(d)}$ is a quasi-affine group scheme over R equipped with a quasi-finite monomorphism $\mathrm{Alt}_G^{(d)} \hookrightarrow \mathrm{Skew}_G^{(d)}$. Note that $\mathrm{Skew}_G^{(d)}$ is a closed subscheme of $\mathbf{G}_m^{G^d}$, and therefore of finite type over R. It follows that $\mathrm{Alt}_G^{(d)}$ is also of finite type over R.

Remark 3.2.13. The isomorphism $\mathbf{G}_m^{G^d} \simeq (\mathbf{G}_m^{G^2})^{G^{d-2}}$ depends on a choice of a pair of elements of the set $\{1, \ldots, d\}$. However, the subscheme $\mathrm{Alt}_G^{(d)}$ of $\mathrm{Skew}_G^{(d)}$ given in Construction 3.2.11 is independent of this choice.

Remark 3.2.14. When $d = 1$, we have $\mathrm{Alt}_G^{(d)} \simeq \mathrm{Skew}_G^{(d)} \simeq \mathbf{D}(G)$, where $\mathbf{D}(G)$ denotes the Cartier dual group scheme of G.

Remark 3.2.15. When $d = 2$, the group scheme $\mathrm{Alt}_G^{(d)}$ of Construction 3.2.11 agrees with the group scheme $\mathrm{Alt}_G^{(2)}$ introduced in Definition 3.2.1.

Remark 3.2.16. Suppose that the finite flat group scheme G is annihilated by multiplication by n, for some odd integer n. Using Proposition 3.2.6, we see that the map $\mathrm{Alt}_G^{(d)} \to \mathrm{Skew}_G^{(d)}$ is an isomorphism.

Example 3.2.17. Let G be a finite flat commutative group scheme over R which is annihilated by some odd number n. Then the monomorphism $\mathrm{Alt}_G^{(d)} \hookrightarrow \mathrm{Skew}_G^{(d)}$ is an isomorphism; this follows immediately from Proposition 3.2.6.

We conclude this section by analyzing the behavior of the construction $G \mapsto \mathrm{Alt}_G^{(d)}$ with respect to products. First, we need to introduce a bit of notation.

Definition 3.2.18. Let G_0 and G_1 be finite flat commutative group schemes over a commutative ring R, and let d_0 and d_1 be nonnegative integers. For every R-algebra A, we let $\mathrm{Skew}_{G_0,G_1}^{(d_0,d_1)}(A)$ denote the set of all maps of A-schemes

$$(G_0)_A^{d_0} \times_{\mathrm{Spec}\, A} (G_1)_A^{d_1} \to \mathbf{G}_m$$

which are multilinear and skew-symmetric in each variable. We regard the construction $A \mapsto \mathrm{Skew}_{G_0,G_1}^{(d_0,d_1)}(A)$ as a functor from $\mathbf{CAlg}_R$ to the category of abelian groups, which we will denote by $\mathrm{Skew}_{G_0,G_1}^{(d_0,d_1)}$. Note that since G_0 and G_1 are finite and flat over R, the functor $\mathrm{Skew}_{G_0,G_1}^{(d_0,d_1)}$ is representable by an affine scheme of finite presentation over R.

There are evident closed immersions

$$(\mathrm{Skew}_{G_0}^{(d_0)})^{G_1^{d_1}} \hookleftarrow \mathrm{Skew}_{G_0,G_1}^{(d_0,d_1)} \hookrightarrow (\mathrm{Skew}_{G_1}^{(d_1)})^{G_0^{d_0}}.$$

We let $\mathrm{Alt}_{G_0,G_1}^{(d_0,d_1)}$ denote the fiber product

$$(\mathrm{Alt}_{G_0}^{(d_0)})^{G_1^{d_1}} \times_{(\mathrm{Skew}_{G_0}^{(d_0)})^{G_1^{d_1}}} \mathrm{Skew}_{G_0,G_1}^{(d_0,d_1)} \times_{(\mathrm{Skew}_{G_1}^{(d_1)})^{G_0^{d_0}}} (\mathrm{Alt}_{G_1}^{(d_1)})^{G_0^{d_0}}.$$

Construction 3.2.19. Let G_0 and G_1 be finite flat commutative group schemes over a commutative ring R, and let $G = G_0 \times_{\mathrm{Spec}\, R} G_1$, which we also regard as a commutative group scheme over R. Let $d_0, d_1 \geq 0$ be nonnegative integers, and let $d = d_0 + d_1$. The inclusion maps $G_0 \hookrightarrow G \hookleftarrow G_1$ induce a closed immersion $j : G_0^{d_0} \times_{\mathrm{Spec}\, R} G_1^{d_1} \to G^d$. Composition with j induces a map of R-schemes $\gamma_{d_0,d_1} : \mathrm{Skew}_G^{(d)} \to \mathrm{Skew}_{G_0,G_1}^{(d_0,d_1)}$.

Proposition 3.2.20. *Let G_0 and G_1 be finite flat commutative group schemes over a commutative ring R, and let d be a positive integer. Then the maps γ_{d_0,d_1} constructed above induce an isomorphism*

$$\gamma : \mathrm{Skew}_G^{(d)} \to \prod_{d=d_0+d_1} \mathrm{Skew}_{G_0,G_1}^{(d_0,d_1)}$$

in the category of R-schemes. Moreover, γ restricts to an isomorphism

$$\mathrm{Alt}_G^{(d)} \to \prod_{d=d_0+d_1} \mathrm{Alt}_{G_0,G_1}^{(d_0,d_1)}.$$

Proof. It is easy to see that γ is an isomorphism which carries $\mathrm{Alt}^{(d)}_G$ into the product $\prod_{d=d_0+d_1} \mathrm{Alt}^{(d_0,d_1)}_{G_0,G_1}$ (which we will identify with a closed subscheme of the product $\prod_{d=d_0+d_1} \mathrm{Skew}^{(d_0,d_1)}_{G_0,G_1}$). To complete the proof, it will suffice to show that if $A \in \mathbf{CAlg}_R$ and b is an A-valued point of $\mathrm{Skew}^{(d)}_G$ such that $\gamma_{d_0,d_1}(b) \in \mathrm{Alt}^{(d_0,d_1)}_{G_0,G_1}(A)$ for all d_0, d_1 with $d_0 + d_1 = d$, then $b \in \mathrm{Alt}^{(d)}_G(A)$. We may assume without loss of generality that $A = R$ and $d = 2$. Since composition with the projection map $G \to G_0$ carries $\mathrm{Alt}^{(2)}_{G_0}$ to $\mathrm{Alt}^{(2)}_G$, we may (after modifying b by a point in the image of this map) assume that $\gamma_{2,0}(b) = 1$. Similarly, we may assume $\gamma_{0,2}(b) = 1$. It will therefore suffice to verify the following:

$(*)$ Let $b : G \times_{\operatorname{Spec} R} G \to \mathbf{G}_m$ be a skew-symmetric bilinear map which vanishes on $G_0 \times_{\operatorname{Spec} R} G_0$ and $G_1 \times_{\operatorname{Spec} R} G_1$. Then b arises as the commutator pairing of a central extension

$$0 \to \mathbf{G}_m \to \widetilde{G} \to G \to 0.$$

To prove this, we take $\widetilde{G}$ to be the product $G_0 \times_{\operatorname{Spec} R} G_1 \times_{\operatorname{Spec} R} \mathbf{G}_m$, equipped with the group structure given on points by the formula

$$(x, y, t)(x', y', t') = (x + x', y + y', tt'b(x, y')).$$

$\square$

Remark 3.2.21. Let R be a commutative ring, and let G denote the commutative group scheme over R associated to the finite abelian group $\mathbf{Z}/n\,\mathbf{Z}$. Then every central extension

$$0 \to \mathbf{G}_m \to \widetilde{G} \to G \to 0.$$

is commutative. It follows that the group scheme $\mathrm{Alt}^{(d)}_G$ is trivial for $d \geq 2$. More generally, if H is a finite flat group scheme over R, we have canonical isomorphisms

$$\mathrm{Alt}^{(d,d')}_{G,H} \simeq \begin{cases} \mathrm{Alt}^{(d')}_H & \text{if } d = 0 \\ \mathrm{Alt}^{(d')}_H[n] & \text{if } d = 1 \\ 0 & \text{if } d \geq 2, \end{cases}$$

where $\mathrm{Alt}^{(d')}_H[n]$ denotes the subscheme of n-torsion points of $\mathrm{Alt}^{(d')}_H$. Consequently, if H is annihilated by n, we Proposition 3.2.20 supplies isomorphisms Proposition 3.2.20 supplies isomorphisms

$$\mathrm{Alt}^{(d)}_{G \times_{\operatorname{Spec} R} H} \simeq \begin{cases} \mathrm{Alt}^{(d)}_H \times_{\operatorname{Spec} R} \mathrm{Alt}^{(d-1)}_H & if d \geq 2. \\ \mathbf{D}(H) \times_{\operatorname{Spec} R} \mu_n & \text{if } d = 1. \end{cases}$$

3.3 The Case of a Field

In this section, we will study the construction $G \mapsto \mathrm{Alt}^{(d)}(G)$ in the case where G is a finite flat group scheme over a perfect field κ of characteristic $p > 0$. Our main result is conveniently stated using the language of Dieudonne modules:

Theorem 3.3.1. *Let κ be a perfect field of characteristic $p > 0$ and let G be a truncated p-divisible group over κ of height n, level t, and dimension* 1. *Write $G = \operatorname{Spec} H^\vee$ and write $\mathrm{Alt}^{(d)}_G = \operatorname{Spec} A$ for some Hopf algebras H and A over κ. Then there is a surjective map of Dieudonne mdoules*

$$\mathrm{DM}_+(H) \widetilde{\otimes} \mathrm{DM}_+(H) \widetilde{\otimes} \cdots \widetilde{\otimes} \mathrm{DM}_+(H) \to \mathrm{DM}_+(A),$$

which induces an isomorphism

$$\wedge^d_{W(\kappa)/p^t W(\kappa)} \mathrm{DM}_+(H) \to \mathrm{DM}_+(A)$$

of modules over $W(\kappa)/p^t W(\kappa)$.

From Theorem 3.3.1, we immediately deduce the following special case of Theorem 3.5.1:

Corollary 3.3.2. *Let κ be a field, and let G be a truncated p-divisible group over κ of height n, level t, and dimension* 1*. Then $\mathrm{Alt}_G^{(d)}$ is a truncated p-divisible group of height $\binom{n}{d}$, level t, and dimension $\binom{n-1}{d}$.*

Proof. Without loss of generality, we may assume that κ is algebraically closed. We proceed by induction on n. Note that if $d = 1$, then $\mathrm{Alt}_G^{(d)}$ is the Cartier dual of G, and the result is obvious. We will therefore assume that $d \geq 2$.

Suppose first that G is disconnected. Then (since κ is algebraically closed) we can write $G = G' \times_{\mathrm{Spec}\,\kappa} \underline{\mathbf{Z}/p^t\mathbf{Z}}$, where $\underline{\mathbf{Z}/p^t\mathbf{Z}}$ denotes the constant group scheme associated to the cyclic group $\mathbf{Z}/p^t\mathbf{Z}$. Since $d \geq 2$, Remark 3.2.21 supplies an isomorphism $\mathrm{Alt}_G^{(d)} \simeq \mathrm{Alt}_{G'}^{(d)} \times_{\mathrm{Spec}\,\kappa} \mathrm{Alt}_{G'}^{(d-1)}$. From the inductive hypothesis, we deduce that $\mathrm{Alt}_{G'}^{(d)}$ and $\mathrm{Alt}_{G'}^{(d-1)}$ are truncated p-divisible groups of heights $\binom{n-1}{d}$ and $\binom{n-1}{d-1}$, level t, and dimensions $\binom{n-2}{d}$ and $\binom{n-2}{d-1}$. It follows that $\mathrm{Alt}_G^{(d)}$ is a truncated p-divisible group of height $\binom{n-1}{d} + \binom{n-1}{d-1} = \binom{n}{d}$, level t, and dimension $\binom{n-2}{d} + \binom{n-2}{d-1} = \binom{n-1}{d}$.

We now treat the case where G is connected. If κ has characteristic different from p, then $G \simeq 0$ and there is nothing to prove. We may therefore assume that κ has characteristic p. Write $G = \mathrm{Spec}\,H^\vee$ for some Hopf algebra H over κ, and let $M = \mathrm{DM}_+(H)$ be its Dieudonne module. Then M is a free $W(\kappa)/p^tW(\kappa)$-module of rank n. Theorem 3.3.1 implies that $\mathrm{Alt}_G^{(d)} = \mathrm{Spec}\,A$, with $\mathrm{DM}_+(A) \simeq \wedge^d_{W(\kappa)/p^tW(\kappa}M$, so that $\mathrm{DM}_+(A)$ is a free $W(\kappa)/p^tW(\kappa)$-module of rank $\binom{n}{d}$. If $t \geq 2$, this implies that $\mathrm{Alt}_G^{(d)}$ is a truncated p-divisible group of height $\binom{n}{d}$ and level t (Proposition 3.1.5). Moreover, to show that it has dimension $\binom{n-1}{d}$, we may replace G by $G[p]$ and thereby reduce to the case where $t = 1$.

Since G has dimension 1, the quotient M/VM is a 1-dimensional vector space over κ. Choose an element $x \in M$ having nonzero image in M/VM. Let m be the largest integer for which the elements $x, Vx, \ldots, V^{m-1}x \in M$ are linearly independent, and let $M' \subseteq M$ be the subspace they span. Note that since we can write $V^m x$ as a linear combination of the elements $V^i x$ for $0 \leq i < m$, the subspace $M' \subseteq M$ is closed under the action of V. We claim that $M' = M$ (so that $m = \dim_\kappa(M) = n$). We will prove that $V^a M \subseteq M'$ for all $a \geq 0$, using descending induction on a. Since G is connected, the action of V on M is locally nilpotent, hence nilpotent (since M is finite-dimensional), so that $V^a M = 0 \subseteq M'$ for $a \gg 0$. Assume now that $a > 0$ and that $V^a M \subseteq M'$; we wish to prove that $V^{a-1}M \subseteq M'$. Fix $y \in M$; we will show that $V^{a-1}y \in M'$. Since the image of x generates M/VM, we can write $y = cx + Vy'$ for some $y' \in M$ and some scalar $c \in \kappa$. Then $V^{a-1}y = cV^{a-1}x + V^a y'$. Since $V^{a-1}x$ and $V^a y'$ belong to M, so does $V^{a-1}y$.

We next claim that $V^n x = 0$. Suppose otherwise, write $V^n x = \sum_{0 \leq i < n} c_i V^i x$, and let k be the smallest integer such that $c_k \neq 0$. Let $N \subseteq M$ be the linear subspace spanned by $V^i x$ for $k \leq i < n$. Then N is stable under the action of V. Moreover, if $y = \sum_{k \leq i < n} a_i V^i x$, then we have

$$Vy = c_k \varphi^{-1}(a_{n-1})V^k x + \sum_{k < i < n} (c_i \varphi^{-1}(a_{n-1}) + \varphi^{-1}(a_{i-1}))V^i x.$$

Suppose that $Vy = 0$. Examining the coefficient of $V^k x$ in the above expression, we deduce that $a_{n-1} = 0$. Examining the coefficient of $V^i x$ for $k < i < n$, we deduce that $a_{i-1} = 0$. It follows that $y = 0$: that is, the restriction of V to N is injective. Since V is locally nilpotent on M, we conclude that $N = 0$ and obtain a contradiction.

Since $V^n x = 0$, the element $V^{n-1}x$ is a nonzero element of $\ker(V)$. Since G is 1-dimensional, $\ker(V)$ is a 1-dimensional vector space over κ, and is therefore spanned by $V^{n-1}x$. In particular, we have $Fx = \lambda V^{n-1}x$ for some scalar $\lambda \in \kappa$. Note that Fx spans the image of F (since F annihilates $V^i x$ for $i > 0$). The equality $\mathrm{im}(F) = \ker(V)$ (Proposition 3.1.5) implies that λ is nonzero.

For each subset $I = \{i_1 < i_2 < \ldots < i_d\} \subseteq \{0, \ldots, n-1\}$, let $V^I x$ denote the image of $V^{i_1}x \wedge V^{i_2}x \wedge \cdots \wedge V^{i_d}x$ in $\mathrm{DM}_+(A) \simeq \wedge^d_\kappa M$, so that the elements $V^I x$ form a basis for $\mathrm{DM}_+(A)$. We then have

$$V(V^I x) = \begin{cases} V^{\{i_1+1 < \ldots < i_d+1\}}x & \text{if } i_d < n \\ 0 & \text{otherwise.} \end{cases}$$

$$F(V^I x) = \begin{cases} \pm\lambda V^{\{i_2-1<i_3-1<\ldots<i_d-1<n-1\}} & \text{if } i_1 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the kernel of V on $\mathrm{DM}_+(A)$ has dimension $\binom{n-1}{d-1}$ over κ, and the kernel of F on $\mathrm{DM}_+(A)$ has dimension $\binom{n-1}{d}$. We have

$$\binom{n-1}{d-1} + \binom{n-1}{d} = \binom{n}{d} = \dim_\kappa \mathrm{DM}_+(A),$$

so that $\operatorname{Spec} A$ is a truncated p-divisible group of height $\binom{n}{d}$ and level 1. The dimension of $\operatorname{Spec} A$ is given by $\dim_\kappa \mathrm{DM}_+(A)[F] = \binom{n-1}{d}$. □

Corollary 3.3.3. *Let $\mathbf{G}_0$ be a smooth connected 1-dimensional formal group over a perfect field κ of characteristic $p > 0$, and let $X = K(\mathbf{Z}/p^t\,\mathbf{Z}, d)$. Then we have a canonical isomorphism*

$$\operatorname{Spec} K(n)_0(X) \simeq \mathrm{Alt}^{(d)}_{\mathbf{G}_0[p^t]}$$

of group schemes over κ.

Proof. Combine Theorem 3.3.1 with Theorem 2.4.10. □

The proof of Theorem 3.3.1 will occupy our attention for the rest of this section. Our strategy is roughly as follows. First, let $M = \mathrm{DM}_+(G)$ and write $\mathrm{Alt}^{(d)}_G = \operatorname{Spec} A$. We will show by explicit construction that the exterior power $\wedge^d_{W(\kappa)/p^tW(\kappa)}$ admits the structure of a Dieudonne module, and construct a surjective map $\mathrm{DM}_+(A) \to \wedge^d_{W(\kappa)/p^tW(\kappa)}$. The main step of the proof is to show that $\mathrm{DM}_+(A)$ is an Artinian $W(\kappa)$-module having length $\leq t\binom{n}{d}$. We can rephrase this assertion as follows:

Proposition 3.3.4. *Let κ be a field of characteristic $p > 0$ and let G be a truncated p-divisible group of height n, level t, and dimension 1 over κ. For each $d \geq 1$, $\mathrm{Alt}^{(d)}_G$ is a finite flat group scheme over κ rank $\leq p^{t\binom{n}{d}}$.*

We first prove Proposition 3.3.4 in the simplest nontrivial case.

Lemma 3.3.5. *Let κ be a field of characteristic $p > 0$ and let G be a connected truncated p-divisible group of height n, level 1, and dimension 1 over κ. For each $d \geq 1$, $\mathrm{Alt}^{(d)}_G$ is a finite flat group scheme over κ rank $\leq p^{\binom{n}{d}}$.*

Proof. We may assume without loss of generality that κ is perfect. Write $G = \operatorname{Spec} H^\vee$ for some Hopf algebra H over κ, and let $M = \mathrm{DM}_+(H)$ denote the Dieudonne module of H. The proof of Corollary 3.3.2 shows that there exists an element $x \in M$ such that the elements $x, Vx, \ldots, V^{n-1}x$ form a basis for M as a vector space over κ. Moreover, we have $V^n x = 0$, and $Fx = \lambda V^{n-1}x$ for some nonzero scalar $\lambda \in \kappa$.

Let X denote the scheme parametrizing multilinear maps $G^d \to \mathbf{G}_m$, so that $X \simeq \operatorname{Spec} H^{\boxtimes d}$ (here $H^{\boxtimes d}$ denotes the dth tensor power of H, with respect to the tensor product $\boxtimes$ of §1.1. Write $\mathrm{Alt}^{(d)}_G = \operatorname{Spec} A$ for some Hopf algebra A over κ. We have a monomorphism of group schemes $\mathrm{Alt}^{(d)}_G \to X$, which induces an epimorphism of Hopf algebras $H^{\boxtimes d} \to A$, hence a surjection of Dieudonne modules

$$\rho : M \widetilde{\otimes} M \widetilde{\otimes} \cdots \widetilde{\otimes} M \to \mathrm{DM}_+(A)$$

(Proposition 1.4.14), which we can identify with a κ-multilinear map

$$\theta : M \times M \times \cdots \times M \to \mathrm{DM}_+(A).$$

Since $\mathrm{Alt}^{(d)}_G \subseteq \mathrm{Skew}^{(d)}_G$, the map θ is antisymmetric in its arguments.

Let $N \subseteq \mathrm{DM}_+(A)$ be the linear subspace spanned by elements of the form $\theta(V^{i_1}x, V^{i_2}x, \ldots, V^{i_d}x)$, where $0 \leq i_1 < i_2 < \ldots < i_d < n$. Using the formulas

$$V\theta(V^{i_1}x, V^{i_2}x, \ldots, V^{i_d}x) = \begin{cases} \theta(V^{i_1+1}x, \ldots, V^{i_d+1}x) & \text{if } i_d < n-1 \\ 0 & \text{if } i_d = n-1. \end{cases}$$

$$F\theta(V^{i_1}x, V^{i_2}x, \ldots, V^{i_d}x) == \begin{cases} 0 & \text{if } i_1 > 0 \\ (-1)^{d-1}\lambda\theta(V^{i_2-1}x, \ldots, V^{i_d-1}x, V^{n-1}x) & \text{if } i_1 = 0. \end{cases}$$

we deduce that N is a D_κ-submodule of $\mathrm{DM}_+(A)$. By construction, N has dimension $\leq \binom{n}{d}$ as a vector space over κ. Consequently, to show that $\mathrm{Alt}^{(d)}_G$ has rank $\leq p^{\binom{n}{d}}$, it will suffice to show that $N = \mathrm{DM}_+(A)$.

Because ρ is surjective, $\mathrm{DM}_+(A)$ is generated by the image of θ as a D_κ-module. It is therefore generated as a D_κ-module by elements of the form $\theta(V^{i_1}x, \ldots, V^{i_d}x)$. It will therefore suffice to show that each of these elements belongs to N. If the integers $i_1, \ldots, i_d$ are distinct, this follows from the definition of N (and the antisymmetry of θ). We will complete the proof by showing that $\theta(V^{i_1}x, \ldots, V^{i_d}x) = 0$ whenever $i_j = i_{j'}$ for $j \neq j'$. Using the antisymmetry of θ, we may assume that $i_1 = i_2$. The vanishing of $\theta(V^{i_1}x, \ldots, V^{i_d}x)$ follows by antisymmetry if $p \neq 2$; let us therefore assume that $p = 2$.

By construction, the map $G^d \times_{\mathrm{Spec}\,\kappa} \mathrm{Alt}^{(d)}_G \to \mathbf{G}_m$ induces a map

$$\nu : G^{d-2} \times_{\mathrm{Spec}\,\kappa} \mathrm{Alt}^{(d)}_G \to \mathrm{Alt}^{(2)}_G .$$

Write $\mathrm{Alt}^{(2)}_G = \mathrm{Spec}\, B$, so that the above construction yields a κ-linear map $\theta' : M \times M \to \mathrm{DM}_+(B)$. Then ν determines a map of Hopf algebras

$$B \boxtimes H^{\boxtimes d-2} \to A,$$

hence a κ-multilinear map

$$\theta'' : \mathrm{DM}_+(B) \times M \times \cdots \times M \to \mathrm{DM}_+(A).$$

satisfying

$$\theta(y_1, \ldots, y_d) = \theta''(\theta'(y_1, y_2), y_3, \ldots, y_d).$$

Consequently, to prove that $\theta(V^{i_1}x, \ldots, V^{i_d}x) = 0$ when $i_1 = i_2$, it will suffice to prove that $\theta'(V^{i_1}x, V^{i_2}x) = 0$. We may therefore reduce to the case where $d = 2$.

Using Corollary 1.4.15, we see that the epimorphism of Dieudonne modules $\mathrm{DM}_+(A) \to \mathrm{DM}_+(A)/N$ induces an epimorphism of Hopf algebras $A \to C$, which classifies a bilinear map of group schemes

$$\mu : G_C \times_{\mathrm{Spec}\, C} G_C \to \mathbf{G}_m$$

over C. We will complete the proof by showing that μ is trivial.

For $0 \leq m \leq n$, let $M(m)$ denote the κ-linear subspace of M spanned by $\{V^i x\}_{m \leq i < n}$, so that

$$0 = M(n) \subset M(n-1) \subset \cdots \subset M(0) = M.$$

Using Corollary 1.4.15, we can write $M(m)$ as $\mathrm{DM}_+(H(m))$ for some Hopf subalgebra $H(m)$, so that $\mathrm{Spec}\, H(m)^\vee$ is a closed subgroup $G(m) \subseteq G$. Let μ_m denote the restriction of μ to the product $G(m)_C \times_{\mathrm{Spec}\, C} G_C$. We will prove that each of the maps μ_m vanishes, using descending induction on m. If $m = n$, the result is obvious. To carry out the inductive step, let us suppose that μ_{m+1} vanishes. We can identify μ_m with a trilinear map of group schemes over κ

$$G(m) \times_{\mathrm{Spec}\,\kappa} G \times_{\mathrm{Spec}\,\kappa} \mathrm{Spec}\, C \to \mathbf{G}_m,$$

which is classified by a map of Hopf algebras $H(m) \boxtimes H \to C$. To show that this map is trivial, it will suffice to show that the composite map

$$M(m) \times M \hookrightarrow M \times M \xrightarrow{\theta} \mathrm{DM}_+(A) \to \mathrm{DM}_+(A)/N$$

vanishes. That is, it suffices to show that $\theta(V^i x, V^j x) \in N$ whenever $i \geq m$. This follows from the inductive hypothesis if $i > m$, and follows from the definition of N if $i \neq j$. We are therefore reduced to proving that $\theta(V^m x, V^m x) = 0$. For this, it suffices to show that the composite map

$$M(m) \times M(m) \hookrightarrow M \times M \xrightarrow{\theta} \mathrm{DM}_+(A) \to \mathrm{DM}_+(A)/N$$

vanishes: that is, that the map μ vanishes when restricted to $G(m)_C \times_{\operatorname{Spec} C} G(m)_C$.

Let μ' denote the restriction of μ to $G(m)_C \times_{\operatorname{Spec} C} G(m)_C$. We have an exact sequence of finite flat group schemes over C

$$0 \to G(m+1)_C \to G(m)_C \to (G(m)/G(m+1))_C \to 0.$$

Since μ' vanishes on $G(m+1)_C \times_{\operatorname{Spec} C} G(m)_C$, it descends to a skew-symmetric pairing

$$\mu'' : (G(m)/G(m+1))_C \times_{\operatorname{Spec} C} (G(m)/G(m+1))_C \to \mathbf{G}_m,$$

classifies by a map of group schemes $\operatorname{Spec} C \to \mathrm{Skew}^{(2)}_{G(m)/G(m+1)}$. Using Remark 3.2.8, we see that this map factors through the subscheme $\mathrm{Alt}^{(2)}_{G(m)/G(m+1)}$. But this map is automatically trivial, because the group scheme $\mathrm{Alt}^{(2)}_{G(m)/G(m+1)}$ is trivial (this follows from Example 3.2.7, since $G(m)/G(m+1)$ is isomorphic to the group scheme $\alpha_2 = \ker(F : \mathbf{G}_a \to \mathbf{G}_a)$). □

We now discuss some general principles which will allow us to reduce Proposition 3.3.4 to Lemma 3.3.5.

Construction 3.3.6. Let G be a finite flat commutative group scheme over a commutative ring R. Let $n > 0$ be an integer, and suppose that the map $n : G \to G$ factors as a composition

$$G \xrightarrow{q} G' \xrightarrow{j} G,$$

where G' is a finite flat group scheme over R, the map q is faithfully flat, and j is a closed immersion. Let $d \geq 1$ be an integer. Given a-skew symmetric map $b : G^d \to \mathbf{G}_m$, we can define a new skew-symmetric map $b' : G'^d \to \mathbf{G}_m$ by the formula

$$b'(q(x_1), q(x_2), \ldots, q(x_d)) = b(x_1, \ldots, x_d)^n$$

for all $x_1, \ldots, x_d \in G(A)$, $A \in \mathbf{CAlg}_R$. The well-definedness of b' follows from the fact that q is a surjection for the fppf topology, and the observation that $q(x_i) = q(x'_i)$ implies that $x_i - x'_i$ is annihilated by n (so that $b(x_1, \ldots, x_n)^n = b(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_d)^n$ for any multilinear b). The construction $b \mapsto b'$ is functorial, and determines a map of schemes $\psi_n : \mathrm{Skew}^{(d)}_G \to \mathrm{Skew}^{(d)}_{G'}$.

Proposition 3.3.7. *Let G be a finite flat group scheme over a commutative ring R and $n > 0$ an integer satisfying the requirements of Construction 3.3.6, so that G fits into a short exact sequence*

$$0 \to G' \xrightarrow{j} G \xrightarrow{u} G'' \to 0$$

where G' is the image of the map $[n] : G \to G$ as an fppf sheaf. Let $d \geq 1$ be an integer. Then:

(1) *We have a short exact sequence*

$$0 \to \mathrm{Skew}^{(d)}_{G''} \xrightarrow{\gamma} \mathrm{Skew}^{(d)}_G \xrightarrow{\psi_n} \mathrm{Skew}^{(d)}_{G'},$$

where γ is determined by the functoriality of the construction $H \mapsto \mathrm{Skew}^{(d)}_H$ and α_n is defined as in Construction 3.3.6.

(2) *The map ψ_n restricts to a morphism $\beta_n : \mathrm{Alt}^{(d)}_G \to \mathrm{Alt}^{(d)}_{G'}$.*

(3) *The exact sequence of* (1) *restricts to an exact sequence*

$$0 \to \mathrm{Alt}^{(d)}_{G'} \to \mathrm{Alt}^{(d)}_{G} \xrightarrow{\beta_n} \mathrm{Alt}^{(d)}_{G'}.$$

Proof. Since $G \to G''$ is an epimorphism of fppf sheaves, it is clear that the restriction map $\gamma : \mathrm{Skew}^{(d)}_{G''} \to \mathrm{Skew}^{(d)}_{G}$ is a monomorphism. Let $b : G''^{d} \to \mathbf{G}_m$ be a skew-symmetric map, and let

$$G \xrightarrow{q} G' \xrightarrow{j} G$$

be the factorization appearing in Construction 3.3.6. We then have

$$\begin{aligned}
(\psi_n \circ \gamma)(b)(q(x_1), \ldots, q(x_d)) &= \gamma(b)(x_1, \ldots, x_d)^n \\
&= \gamma(b)(nx_1, x_2, \ldots, x_d) \\
&= \gamma(b)((j \circ q)x_1, x_2, \ldots, x_d) \\
&= b((u \circ j \circ q)x_1, u(x_2), \ldots, u(x_d)) \\
&= 1
\end{aligned}$$

since $u \circ j = 0$. Since q is an fppf surjection, this proves that $\psi_n \circ \gamma$ is trivial.

To complete the proof of (1), it will suffice to show that the kernel of β_n is contained in the image of γ. To this end, suppose we are given a skew-symmetric map $b : G^d \to \mathbf{G}_m$ such that $\psi_n(b)$ is trivial. Then for each $x_1, \ldots, x_d \in G(A)$, we have

$$b((j \circ q)x_1, x_2, \ldots, x_d) = b(x_1, x_2, \ldots, x)^n = (\psi_n b)(q(x_1), \ldots, q(x_d)) = 1.$$

Since the map q is an fppf surjection, we deduce that

$$b(j(y), x_2, \ldots, x_d) = 1$$

for all $y \in G'(A)$, $x_2, \ldots, x_d \in G(A)$. It follows that the map $b(x_1, x_2, \ldots, x_d)$ depends only on the image of x_1 in $G''(A)$. The same argument shows that $b(x_1, \ldots, x_d)$ depends only on the image of each x_i in $G''(A)$: that is, b is given by the composition

$$G^d \to G''^{d} \xrightarrow{b'} G_m$$

for some map b'. Since b is skew-symmetric, it is easy to see (using the faithful flatness of the map $G \to G''$) that b' is also skew-symmetric, so that $b = \gamma(b')$ lies in the image of γ.

We now prove (2). Suppose that $R' \in \mathbf{CAlg}_R$ and that $b \in \mathrm{Alt}^{(d)}_{G}(R')$, which we will identify with a subset of $\mathrm{Skew}^{(d)}_{G}(A)$. We wish to show that $\psi_n(b) \in \mathrm{Alt}^{(d)}_{G'}(R') \subseteq \mathrm{Skew}^{(d)}_{G'}(R')$. Replacing R by R', we may assume that $R = R'$. If $d = 1$, there is nothing to prove; let us therefore assume that $d \geq 1$. The skew-symmetric map $\psi_n(b)$ determines a map $G'^{d-2} \to \mathrm{Skew}^{(2)}_{G'}$, and we wish to show that this map factors through $\mathrm{Alt}^{(2)}_{G'}$. Since the map $j : G \to G'$, it will suffice to show that the composite map

$$G^{d-2} \to G'^{d-2} \to \mathrm{Skew}^{(2)}_{G'}$$

factors through $\mathrm{Alt}^{(2)}_{G'}$. By construction, this map is given by the composition

$$G^{d-2} \to \mathrm{Skew}^{(2)}_{G} \xrightarrow{\psi'_n} \mathrm{Skew}^{(2)}_{G'},$$

where ψ'_n is obtained by applying Construction 3.3.6 in the case $d = 2$. We are therefore reduced to proving (2) in the case $d = 2$.

Suppose that $b : G \times_{\mathrm{Spec}\, R} G \to \mathbf{G}_m$ is an alternating bilinear map which arises as the commutator pairing of a central extension

$$0 \to \mathbf{G}_m \to \widetilde{G} \to G \to 0.$$

Let us regard μ_n as a closed subgroup of $\widetilde{G}$, and let $\overline{G}$ denote the quotient $\widetilde{G}/\mu_n$. We have another central extension

$$0 \to \mathbf{G}_m \to \overline{G} \to G \to 0,$$

whose commutator pairing is given by b^n. Let K denote the kernel of the multiplication map $n : G \to G$, and let $\overline{K}$ denote the inverse image $K \times_G \overline{G}$. For $x, y \in K(A)$, we have $b(x,y)^n = b(nx, y) = b(0, y) = 1$. It follows that $\overline{K}$ is an abelian extension of K by $\mathbf{G}_m$. Passing to a finite flat covering of R, we may suppose that this extension splits (as in the proof of Proposition 3.2.3). A choice of splitting gives a closed embedding of group schemes $\phi : K \to \overline{G}$. If $x \in K(A)$ and $\overline{y} \in \overline{G}(A)$, then we have

$$\phi(x)\overline{y}\phi(x)^{-1}\overline{y}^{-1} = b(x,y)^n = b(nx,y) = b(0,y) = 1$$

where $y \in G(A)$ denotes the image of $\overline{y}$. It follows that the image of K is a central subgroup over $\overline{G}$. Let $\overline{G}''$ denote the quotient $\overline{G}/\phi(K)$. We then have a central extension

$$0 \to \mathbf{G}_m \to \overline{G}'' \to G'' \to 0,$$

and a simple calculation shows that the commutator pairing of this extension coincides with $\psi_n(b)$.

We now prove (3). Using (1) and (2), we are reduced to proving the following assertion:

(∗) Let $R' \in \mathbf{CAlg}_R$. Then the inverse image of $\mathrm{Alt}^{(d)}_G(R') \subseteq \mathrm{Skew}^{(d)}_G(R')$ under the map $\gamma : \mathrm{Skew}^{(d)}_{G''}(R') \to \mathrm{Skew}^{(d)}_G(R')$ is given by $\mathrm{Alt}^{(d)}_{G''}(R') \subseteq \mathrm{Skew}^{(d)}_{G''}(R')$.

Replacing R by R', we may reduce to the case where $R = R'$. We may also assume $d \geq 2$ (otherwise there is nothing to prove). Let $b_0 \in \mathrm{Skew}^{(d)}_{G''}(R)$ and let $b = \gamma(b_0) \in \mathrm{Skew}^{(d)}_G(R)$, and assume that $b \in \mathrm{Alt}^{(d)}_G(R)$. Note that b and b_0 determine maps

$$\phi : G^{d-2} \to \mathrm{Skew}^{(2)}_G \qquad \phi_0 : G''^{d-2} \to \mathrm{Skew}^{(2)}_{G''}.$$

Moreover, ϕ is given by the composition

$$G^{d-2} \to G''^{d-2} \xrightarrow{\phi_0} \mathrm{Skew}^{(2)}_{G''} \xrightarrow{\gamma'} \mathrm{Skew}^{(2)}_G,$$

where γ' is the map given by composition with u. Since ϕ factors through $\mathrm{Alt}^{(2)}_G$ and the map u is faithfully flat, we conclude that $\gamma' \circ \phi_0$ factors through $\mathrm{Alt}^{(2)}_G$ (see Remark 3.2.8). □

Proof of Proposition 3.3.4. We proceed by induction on t. Since G is a truncated p-divisible group of level t, we have an exact sequence

$$0 \to G[p] \to G \xrightarrow{[p]} G[p^{t-1}] \to 0$$

Applying Proposition 3.3.7, we obtain a short exact sequence

$$0 \to \mathrm{Alt}^{(d)}_{G[p^{t-1}]} \to \mathrm{Alt}^{(d)}_G \to \mathrm{Alt}^{(d)}_{G[p]}.$$

The inductive hypothesis implies that $\mathrm{Alt}^{(d)}_{G[p^{t-1}]}$ is a finite flat group scheme over κ of rank $\leq p^{(t-1)\binom{n}{d}}$. Consequently, to prove that $\mathrm{Alt}^{(d)}_G$ is a finite flat group scheme of rank $\leq p^{t\binom{n}{d}}$, it will suffice to show that $\mathrm{Alt}^{(d)}_{G[p]}$ is a finite flat group scheme of rank $\leq p^{\binom{n}{d}}$. Replacing G by $G[p]$, we may reduce to the case where $t = 1$.

Without loss of generality, we may suppose that κ is algebraically closed. We now proceed by induction on the height of G. If G is connected, the desired result follows from Lemma 3.3.5. Otherwise, we can write $G = G' \times \underline{\mathbf{Z}/p\mathbf{Z}}$, where G' has height $n-1$ and $\underline{\mathbf{Z}/p\mathbf{Z}}$ denotes the constant group scheme over κ associated to the finite group $\mathbf{Z}/p\mathbf{Z}$. In this case, Remark 3.2.21 supplies an isomorphism

$$\mathrm{Alt}^{(d)}_G \simeq \mathrm{Alt}^{(d)}_{G'} \times_{\operatorname{Spec}\kappa} \mathrm{Alt}^{(d-1)}_{G'}.$$

The inductive hypothesis implies that $\mathrm{Alt}^{(d)}_{G'}$ and $\mathrm{Alt}^{(d-1)}_{G'}$ are finite flat group schemes over κ of rank at most $p^{\binom{n-1}{d}}$ and $p^{\binom{n-1}{d-1}}$, respectively. It follows that $\mathrm{Alt}^{(d)}_{G}$ is a finite flat group scheme over κ of rank at most $p^{\binom{n-1}{d}+\binom{n-1}{d-1}} = p^{\binom{n}{d}}$, as desired. □

Lemma 3.3.8. *Let G be a finite flat commutative group scheme over a commutative ring R, and suppose that the map $[2] : G \to G$ factors as a composition*

$$G \xrightarrow{q} G' \xrightarrow{j} G$$

where G' is a finite flat group scheme over R, q is faithfully flat, and j is a closed immersion. For each $d \geq 1$, the map $\psi_2 : \mathrm{Skew}^{(d)}_G \to \mathrm{Skew}^{(d)}_{G'}$ of Construction 3.3.6 factors through the closed subscheme $\mathrm{Alt}^{(d)}_{G'} \subseteq \mathrm{Skew}^{(d)}_{G'}$.

Proof. Fix a skew-symmetric multilinear map $b : G^d \to \mathbf{G}_m$, and let $b' : G'^d \to \mathbf{G}_m$ be defined as in Construction 3.3.6. We wish to prove that b' determines an R-point of $\mathrm{Alt}^{(d)}_{G'}$. If $d = 1$ there is nothing to prove; let us therefore assume that $d \geq 2$. Let $A \in \mathbf{CAlg}_R$ and suppose we are given points $x_3, \ldots, x_d \in G'(A)$; we wish to show that the map

$$b'(\bullet, \bullet, x_3, \ldots, x_d) : G'_A \times_{\mathrm{Spec}\, A} G'_A \to \mathbf{G}_m$$

determines an A-point of $\mathrm{Skew}^{(d)}_{G'}$. The assertion is local on A with respect to the flat topology. We may therefore suppose that $x_i = q(y_i)$ for some A-points $y_i \in G(A)$. Replacing R by A, we may suppose that $A = R$. Replacing b by the map $G \times_{\mathrm{Spec}\, R} G \to \mathbf{G}_m$ given by $(u, v) \mapsto b(u, v, y_3, \ldots, y_d)$, we may reduce to the case $d = 2$.

Let $H = G \times_{\mathrm{Spec}\, R} \mathbf{G}_m$. We regard H as a group scheme, with the multiplication given on points by the formula $(z, t)(z', t') = (z + z', tt'b(z, z'))$. Let $K = G[2]$ denote the kernel of q. We note that K is a central subgroup of H. Let $\widetilde{G}$ denote the quotient H/K. We then have an exact sequence

$$0 \to \mathbf{G}_m \to \widetilde{G} \to G' \to 0$$

which exhibits $\widetilde{G}$ as a central extension of G' by $\mathbf{G}_m$. A simple calculation shows that the commutator pairing of this extension is given by b'. □

Proof of Theorem 3.3.1. Without loss of generality, we may assume that the field κ is algebraically closed. According to Theorem 3.1.11, there exists a p-divisible group $\mathbf{G}$ over κ of height n and an isomorphism $G \simeq \mathbf{G}[p^t]$. For each integer m, write $\mathbf{G}[p^m] = \mathrm{Spec}\, H(m)^\vee$ for some finite-dimensional Hopf algebra $H(m)$ over κ, and let $\overline{M}$ denote the inverse limit $\varprojlim \{\mathrm{DM}_+(H(m))\}_{m \geq 0}$. Then $\overline{M}$ is a left D_κ-module which is free of rank n as a module over $W(\kappa)$, and we can identify M with the quotient $\overline{M}/p^t\overline{M}$.

We first claim the following: for every sequence of elements $y_1, \ldots, y_d \in \overline{M}$, the wedge product

$$Fy_1 \wedge \cdots \wedge Fy_d$$

is divisible by p^{d-1} in $\wedge^d_{W(\kappa)} \overline{M}$. To prove this, we write $\mathbf{G}$ as the product of a connected p-divisible group $\mathbf{G}_0$ with a constant p-divisible group $(\mathbf{Q}_p / \mathbf{Z}_p)^a$, so that we have a corresponding decomposition of D_κ-modules $\overline{M} = \overline{M}_0 \times W(\kappa)^a$. Let $z_1, \ldots, z_a$ be the standard basis for $W(\kappa)^a$, so that the action of F on $W(\kappa)^a$ is given by $Fz_i = pz_i$. The proof of Corollary 3.3.2 shows that there exists an element $x_0 \in \overline{M}/p\overline{M}$ such that $x_0, Vx_0, \ldots, V^{n-a-1}x_0$ form a basis for $\overline{M}_0/p\overline{M}_0$ as a vector space over κ. Let x be an element of $\overline{M}_0$ representing x_0, so that the elements $x, Vx, \ldots, V^{n-a-1}x$ form a basis for $\overline{M}_0$ over $W(\kappa)$. Then the set $S = \{x, Vx, \ldots, V^{n-a-1}x, z_1, \ldots, z_a\}$ freely generates $\overline{M}$ as a $W(\kappa)$-module. We may therefore assume without loss of generality that $y_1, \ldots, y_d$ is a collection of distinct elements of S. Note Fy is divisible by p for every element $y \in S - \{x\}$, so that $Fy_1 \wedge \ldots \wedge Fy_d$ is divisible by p^{d-1}.

Let $\varphi : W(\kappa) \to W(\kappa)$ denote the Frobenius map. It follows from the preceding argument that the construction

$$y_1 \wedge \cdots \wedge y_d \mapsto p^{1-d}Fy_1 \wedge \cdots \wedge Fy_d$$

determines a φ-semilinear endomorphism of $\overline{M}$. Similarly, the construction

$$y_1 \wedge \cdots \wedge y_d \mapsto V y_1 \wedge \cdots \wedge V y_d$$

determines a φ^{-1}-semilinear automorphism of $\overline{M}$. These endomorphisms evidently commute, and their composition is the $W(\kappa)$-linear map given by multiplication by p. It follows that these endomorphisms determine the structure of a D_κ-module on the exterior power $N = \wedge^d_{W(\kappa)} \overline{M}$.

For each integer $m \geq 0$, let $X(m) = \operatorname{Spec} H(m)^{\boxtimes d}$ denote the scheme parametrizing multilinear maps $\mathbf{G}[p^m]^d \to \mathbf{G}_m$. Choose a Hopf algebra $A(m)$ over κ with $\mathrm{DM}_+(A(m)) \simeq N/p^m N$. Using Corollary 1.4.15, we see that the $W(\kappa)$-multilinear map

$$v_m : \overline{M}/p^m\overline{M} \times \cdots \times \overline{M}/p^m\overline{M} \to N/p^m N$$

determines a multilinear map of Hopf algebras

$$H(m) \otimes_\kappa \cdots \otimes_\kappa H(m) \to A(m),$$

hence a map $u_m : \operatorname{Spec} A(m) \to X(m)$. Since v is skew-symmetric, we can regard u_m as a map from $\operatorname{Spec} A(m)$ to the closed subscheme $\mathrm{Skew}^{(d)}_{\mathbf{G}[p^m]} \subseteq X(m)$.

For $m \geq 0$, multiplication by p induces an injective map $\overline{M}/p^m\overline{M} \to \overline{M}/p^{m+1}\overline{M}$, hence a monomorphism of Hopf algebras $A(m) \to A(m+1)$. The induced map of affine group schemes fits into a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} A(m+1) \arrow[r, "u_{m+1}"] \arrow[d] & \mathrm{Skew}^{(d)}_{\mathbf{G}[p^{m+1}]} \arrow[d, "\psi_p"] \\
\operatorname{Spec} A(m) \arrow[r, "u_m"] & \mathrm{Skew}^{(d)}_{\mathbf{G}[p^m]},
\end{tikzcd}$$

where ψ_p is defined as in Construction 3.3.6. When $p = 2$, it follows from Lemma 3.3.8 that the composite map

$$\operatorname{Spec} A(m+1) \to \operatorname{Spec} A(m) \to \mathrm{Skew}^{(d)}_{\mathbf{G}[p^m]}$$

factors through the closed subscheme $\mathrm{Alt}^{(d)}_{\mathbf{G}[p^m]} \subseteq \mathrm{Skew}^{(d)}_{\mathbf{G}[p^m]}$. In the case $p > 2$, the existence of this factorization is automatic (since $\mathrm{Alt}^{(d)}_{\mathbf{G}[p^m]} = \mathrm{Skew}^{(d)}_{\mathbf{G}}[p^m]$; see Example 3.2.17). It follows that we may identify u_m with a map of group schemes from $\operatorname{Spec} A(d)$ to $\mathrm{Alt}^{(d)}_{\mathbf{G}[p^m]}$.

To complete the proof, it will suffice to show that the map $u_t : \operatorname{Spec} A(t) \to \mathrm{Alt}^{(d)}_{\mathbf{G}[p^t]} = \mathrm{Alt}^{(d)}_G$ is an isomorphism. Note that $\wedge^d_{W(\kappa)/p^t W(\kappa)}$ is generated (as a module over $W(\kappa)$) by the image of v_t. Consequently, v_t induces a surjection of D_κ-modules

$$M \widetilde{\otimes} \cdots \widetilde{\otimes} M \to N/p^t N,$$

and therefore an epimorphism of Hopf algebras $H(t)^{\boxtimes d} \to A(t)$. It follows that the map $u_t : \operatorname{Spec} A(t) \to \mathrm{Alt}^{(d)}_G$ is a monomorphism of group schemes over κ. By construction, $\operatorname{Spec} A(t)$ is a finite flat group scheme of rank $p^{t\binom{n}{d}}$ over κ. Consequently, to prove that u_t is an isomorphism, it will suffice to show that $\mathrm{Alt}^{(d)}_G$ is a finite flat group scheme of rank $\leq p^{t\binom{n}{d}}$, which follows from Proposition 3.3.4. □

3.4 Lubin-Tate Cohomology of Eilenberg-MacLane Spaces

Throughout this section, we fix a perfect field κ of characteristic $p > 0$ and a smooth connected 1-dimensional formal group $\mathbf{G}_0$ of height $n < \infty$ over κ. Let E denote Lubin-Tate spectrum corresponding to $(\kappa, \mathbf{G}_0)$, and let $K(n)$ denote the associated Morava K-theory. Corollary 3.3.3 asserts that for each $d \geq 1$, the Morava K-theory $K(n)_0 K(\mathbf{Z}/p^t\,\mathbf{Z}, d)$ can be described as the ring of functions on the (affine) group scheme $\mathrm{Alt}^{(d)}_{\mathbf{G}_0[p^t]}$. Our goal in this section is to prove an analogous result for the Lubin-Tate spectrum E:

Theorem 3.4.1. *Let $d \geq 1$ and $t \geq 0$ be integers, and let $X = K(\mathbf{Z}/p^t\,\mathbf{Z}, d)$. Then $E^0(X)$ and $E^\wedge_0(X)$ are free modules of rank $p^{t\binom{n}{d}}$ over the Lubin-Tate ring $R = \pi_0 E \simeq W(\kappa)[[v_1, \ldots, v_{n-1}]]$. Let $\mathbf{G} = \operatorname{Spf} E^0(\mathbf{CP}^\infty)$ denote formal group over R given by the universal deformation of $\mathbf{G}_0$. Then we have a canonical isomorphism*

$$\operatorname{Spec} E^\wedge_0(X) \simeq \operatorname{Alt}^{(d)}_{\mathbf{G}[p^t]}$$

of group schemes over R. In particular, $\operatorname{Alt}^{(d)}_{\mathbf{G}[p^t]}$ is a finite flat group scheme of rank $p^{t\binom{n}{d}}$.

Though Theorem 3.4.1 is a statement about homotopy theory, it has purely algebraic consequences:

Corollary 3.4.2. *Let A be a complete local Noetherian ring with perfect residue field κ, and let $\mathbf{G}_A$ be a p-divisible group over A lifting $\mathbf{G}_0$. Then, for each $t \geq 0$, $\operatorname{Alt}^{(d)}_{\mathbf{G}_A[p^t]}$ is a finite flat group scheme of rank $p^{t\binom{n}{d}}$ over A.*

Proof. Let $R = \pi_0 E$. Since the formal group $\mathbf{G} = \operatorname{Spf} E^0(\mathbf{CP}^\infty)$ is a universal deformation of $\mathbf{G}_0$, the quotient map $R \to \kappa$ lifts uniquely to a ring homomorphism $R \to A$ such that $\mathbf{G}_A \simeq \operatorname{Spec} A \times_{\operatorname{Spec} R} \mathbf{G}$. It will therefore suffice to show that $\operatorname{Alt}^{(d)}_{\mathbf{G}[p^t]}$ is a finite flat group scheme of rank $p^{t\binom{n}{d}}$ over R, which follows immediately from Theorem 3.4.1. $\square$

The first part of Theorem 3.4.1 is an immediate consequence of Theorem 2.4.10 together with the following standard result:

Proposition 3.4.3. *Let X be a space. Suppose that $K(n)_1 X \simeq 0$ and that $K(n)_0 X$ is a vector space of dimension $m < \infty$ over κ. Then:*

(1) *There is an equivalence of E-module spectra $L_{K(n)} E[X] \simeq E^m$.*

(2) *There is an equivalence of E-module spectra $E^X \simeq E^m$.*

(3) *The modules $E^\wedge_0(X)$ and $E^0(X)$ are free of rank m over the Lubin-Tate ring $R = \pi_0 E$, canonically dual to one another, and we have $E^\wedge_i(X) \simeq 0 \simeq E^i(X)$ when i is odd.*

(4) *The canonical maps*

$$\kappa \otimes_R E^\wedge_0(X) \to K(n)_0 X \qquad \kappa \otimes_R E^0(X) \to K(n)^0 X$$

are isomorphisms.

Proof. Choose a basis $x_1, \ldots, x_m$ for $K(n)_0 X$ as a vector space over κ. According to Lemma 2.1.25, we can lift these to classes $\overline{x}_1, \ldots, \overline{x}_m \in E^\wedge_0(X) = \pi_0(L_{K(n)} E[X])$. The choice of such elements determines a map of E-module spectra $\theta : E^m \to L_{K(n)} E[X]$. After smashing over E with $K(n)$, θ reduces to a map

$$\theta_0 : K(n)^m \to K(n) \otimes_E L_{K(n)} E[X] \simeq K(n) \otimes_E E[X] \simeq K(n)[X].$$

By construction, θ_0 induces an isomorphism on homotopy groups, and is therefore an equivalence. Since the domain and codomain of θ are $K(n)$-local, we conclude that θ is also an equivalence. This proves (1). Moreover, it shows that the elements $\overline{x}_1, \ldots, \overline{x}_m$ freely generate $E^\wedge_0(X)$ as a module over R, so that the reduction map $\kappa \otimes_R E^\wedge_0(X) \to K(n)_0 X$ is an isomorphism. This proves half of (3) and (4); the remaining assertions now follow by duality. $\square$

Notation 3.4.4. Let X be a space satisfying the hypotheses of Proposition 3.4.3. We let $\operatorname{ESpec}(X)$ denote the $\pi_0 E$-scheme $\operatorname{Spec} E^0(X)$.

Remark 3.4.5. Let X and Y be spaces satisfying the hypotheses of Proposition 3.4.3. It follows from Remark 2.1.22 that $X \times Y$ also satisfies the hypotheses of Proposition 3.4.3. In particular, $E^0(X)$, $E^0(Y)$, and $E^0(X \times Y)$ are free modules of finite rank over $R = \pi_0 E$. Moreover, the canonical map

$$E^0(X) \otimes_R E^0(Y) \to E^0(X \times Y)$$

induces an isomorphism after tensoring with the residue field κ of R, and is therefore an isomorphism of R-modules. It follows that the canonical map

$$\mathrm{ESpec}(X \times Y) = \mathrm{ESpec}\, X \times_{\mathrm{Spec}\, R} \mathrm{ESpec}\, Y$$

is an isomorphism of R-schemes.

Now suppose that X is a space satisfying the hypotheses of Proposition 3.4.3, and that X is equipped with a multiplication map $m : X \times X \to X$ endowing it with the structure of a commutative group object in the homotopy category of spaces. The construction $Y \mapsto \mathrm{ESpec}\, Y$ commutes with products, we conclude that $\mathrm{ESpec}\, X$ has the structure of a commutative group object in the category of R-schemes: that is, it is a finite flat commutative group scheme over X.

Example 3.4.6. Let $\mathbf{G} = \mathrm{Spf}\, E^0(\mathbf{CP}^\infty)$ be the universal deformation of $\mathbf{G}_0$, regarded as a formal group over $R = \pi_0 E$. The canonical map $K(\mathbf{Z}/p^t\,\mathbf{Z}, 1) \hookrightarrow \mathbf{CP}^\infty$ induces a map of R-schemes $\mathrm{ESpec}\, K(\mathbf{Z}/p^t\,\mathbf{Z}, 1) \to \mathbf{G}[p^t]$. This map is an isomorphism on special fibers (Proposition 2.4.4), and therefore an isomorphism (since the domain and codomain and finite and flat over R).

Construction 3.4.7. Fix integers $t \geq 0$ and $d \geq 1$, and consider the iterated cup product map

$$K(\mathbf{Z}/p^t\,\mathbf{Z}, 1)^d \to K(\mathbf{Z}/p^t\,\mathbf{Z}, d).$$

Since $K(\mathbf{Z}/p^t\,\mathbf{Z}, 1)$ and $K(\mathbf{Z}/p^t\,\mathbf{Z}, d)$ satisfy the hypotheses of Proposition 3.4.3 (see Theorem 2.4.10), we obtain a map of R-schemes

$$c : (\mathbf{G}[p^t])^d (\mathrm{ESpec}\, K(\mathbf{Z}/p^t\,\mathbf{Z}, 1))^d \to \mathrm{ESpec}\, K(\mathbf{Z}/p^t\,\mathbf{Z}, d).$$

Since the cup product is multilinear and skew-symmetric up to homotopy, the map c has the same properties. Passing to Cartier duals, we obtain a map of R-schemes

$$\theta_t : \mathrm{Spec}\, E_0^\wedge K(\mathbf{Z}/p^t\,\mathbf{Z}, d) \to \mathrm{Skew}^{(d)}_{\mathbf{G}[p^t]}.$$

Proposition 3.4.8. *Let $d \geq 1$ and $t \geq 0$ be integers. Then the map*

$$\theta_t : \mathrm{Spec}\, E_0^\wedge K(\mathbf{Z}/p^t\,\mathbf{Z}, d) \to \mathrm{Skew}^{(d)}_{\mathbf{G}[p^t]}$$

of Construction 3.4.7 factors through the subscheme $\mathrm{Alt}^{(d)}_{\mathbf{G}[p^t]} \subseteq \mathrm{Skew}^{(d)}_{\mathbf{G}[p^t]}$ introduced in Construction 3.2.11.

Proof. If p is odd, there is nothing to prove (Proposition 3.2.17). Assume therefore that $p = 2$, and consider the map $\gamma : \mathrm{Spec}\, E_0^\wedge K(\mathbf{Z}/p^{t+1}\,\mathbf{Z}, d) \to \mathrm{Spec}\, E_0^\wedge K(\mathbf{Z}/p^t\,\mathbf{Z}, d)$ induces by the multiplication-by-p map

$$K(\mathbf{Z}/p^t\,\mathbf{Z}, d) \to K(\mathbf{Z}/p^{t+1}\,\mathbf{Z}, d).$$

We have a commutative diagram of R-schemes

$$\begin{tikzcd}
\mathrm{Spec}\, E_0^\wedge K(\mathbf{Z}/p^{t+1}\,\mathbf{Z}, d) \arrow[r] \arrow[d, "\gamma"] & \mathrm{Skew}^{(d)}_{\mathbf{G}[p^{t+1}]} \arrow[d, "\psi_p"] \\
\mathrm{Spec}\, E_0^\wedge K(\mathbf{Z}/p^t\,\mathbf{Z}, d) \arrow[r] & \mathrm{Skew}^{(d)}_{\mathbf{G}[p^t]}
\end{tikzcd}$$

where ψ_p is defined as in Construction 3.3.6. Using Lemma 3.3.8, we deduce that the composite map

$$\operatorname{Spec} E_0^{\wedge} K(\mathbf{Z}/p^{t+1}\mathbf{Z}, d) \xrightarrow{\gamma} \operatorname{Spec} E_0^{\wedge} K(\mathbf{Z}/p^t\mathbf{Z}, d) \to \operatorname{Skew}^{(d)}_{\mathbf{G}[p^t]}$$

factors through $\operatorname{Alt}^{(d)}_{\mathbf{G}[p^t]}$. To complete the proof, it will suffice to show the $E_0^{\wedge} K(\mathbf{Z}/p^{t+1}\mathbf{Z}, d)$ is faithfully flat over $E_0^{\wedge} K(\mathbf{Z}/p^t\mathbf{Z}, d)$. Using the fiber-by-fiber flatness criterion (Corollary 11.3.11 of [6]) and Proposition 3.4.3, we are reduced to proving that the map $K(n)_0 K(\mathbf{Z}/p^{t+1}\mathbf{Z}, d) \to K(n)_0 K(\mathbf{Z}/p^t\mathbf{Z}, d)$ is faithfully flat, or equivalently that it induces a surjection of Dieudonne modules

$$\mathrm{DM}_+(K(n)_0 K(\mathbf{Z}/p^{t+1}\mathbf{Z}, d) \to K(n)_0 K(\mathbf{Z}/p^t\mathbf{Z}, d).$$

This follows immediately from Theorem 2.4.10. □

The final ingredient we will need for our proof of Theorem 3.4.1 is the following purely algebraic fact:

Proposition 3.4.9. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field κ, and suppose we are given a map of R-schemes $f : X \to Y$ satisfying the following conditions:*

- (a) *The map $X \to \operatorname{Spec} R$ is finite flat of rank r, for some integer $r \geq 0$.*
- (b) *For every map from R to a field k, the fiber product $\operatorname{Spec} k \times_{\operatorname{Spec} R} Y$ is a finite flat k-scheme of rank r.*
- (c) *The map $Y \to \operatorname{Spec} R$ is separated and of finite type.*
- (d) *The map of closed fibers $f_0 : \operatorname{Spec} \kappa \times_{\operatorname{Spec} R} X \to \operatorname{Spec} \kappa \times_{\operatorname{Spec} R} Y$ is an isomorphism.*

Then f is an isomorphism.

Proof. Conditions (b) and (c) imply that Y is quasi-finite over $\operatorname{Spec} R$. Using Zariski's main theorem, we deduce that there is a finite R-scheme $Z = \operatorname{Spec} R'$ and an open immersion $j : Y \to Z$. Condition (a) implies that $X = \operatorname{Spec} A$ for some finite flat R-algebra A of rank r. Let $i = j \circ f$ be the resulting map from X to Z, and let $\phi : R' \to A$ be the corresponding map of R-algebras. Using (d), we see that the induced map $i_0 : \operatorname{Spec} \kappa \times_{\operatorname{Spec} R} X \to \operatorname{Spec} \kappa \times_{\operatorname{Spec} R} Z$ is an open immersion. Since both sides are finite over the field κ, the map i_0 is also a closed immersion. Since i_0 is a closed immersion, ϕ induces a surjection $R'/\mathfrak{m}R' \to A/\mathfrak{m}A$. Using Nakayama's lemma, we deduce that ϕ is surjective: that is, the map i is a closed immersion. It follows that f is a closed immersion.

Let $x \in \operatorname{Spec} R$ be a point and let k denote the residue field of R at x. Then f induces a closed immersion

$$f_x : \operatorname{Spec} k \times_{\operatorname{Spec} R} X \to \operatorname{Spec} k \times_{\operatorname{Spec} R} Y.$$

Using (a) and (b), we see that the domain and codomain of f_x are finite flat κ-schemes of the same rank. It follows that f_x is an isomorphism. It follows that the map f is a bijection at the level of topological spaces.

Let $\mathcal{O}_Y$ denote the structure sheaf of Y, and let $\mathcal{I} \subseteq \mathcal{O}_Y$ denote the quasi-coherent ideal sheaf defining the closed immersion f. Since f is bijective, every local section of $\mathcal{I}$ is nilpotent. Since Y is Noetherian, it follows that $\mathcal{I}$ is a nilpotent ideal sheaf. Since $(Y, \mathcal{O}_{\mathcal{Y}}/\mathcal{I}) \simeq (X, \mathcal{O}_X)$ is affine, we conclude that Y is affine, hence of the form $\operatorname{Spec} B$ for some commutative ring B which is finitely generated over R. The closed immersion f determines a surjective map of commutative rings $B \to A$ having nilpotent kernel $I \subseteq B$.

Since B is Noetherian, I is finitely generated. It follows that each quotient I^k/I^{k+1} is finitely generated as an A-module, and therefore also as an R-module. Because I is nilpotent, we conclude that B admits a finite filtration by finitely generated R-modules, and is therefore finitely generated over R. We have an exact sequence of R-modules

$$0 \to I \to B \to A \to 0.$$

Since A is projective as an R-module, this sequence splits. It follows that the quotient $I/\mathfrak{m}I$ can be identified with the kernel of the map $\kappa \otimes_R B \to \kappa \otimes_R A$. This kernel vanishes by assumption (d), so that $I = \mathfrak{m}I$. Applying Nakayama's lemma, we deduce that $I = 0$. This implies that f is an isomorphism, as desired. □

Proof of Theorem 3.4.1. According to Proposition 3.4.8, the map $\theta_t : \operatorname{Spec} E_0^\wedge K(\mathbf{Z}/p^t\,\mathbf{Z}, d) \to \operatorname{Skew}^{(d)}_{\mathbf{G}[p^t]}$ factors through $\operatorname{Alt}^{(d)}_{\mathbf{G}[p^t]}$, and can therefore be identified with a map $f : \operatorname{Spec} E_0^\wedge K(\mathbf{Z}/p^t\,\mathbf{Z}, d) \to \operatorname{Alt}^{(d)}_{\mathbf{G}}[p^t]$. We will prove that f is an isomorphism of R-schemes by verifying the hypotheses of Proposition 3.4.9:

(a) It follows from Proposition 3.4.3 and Theorem 2.4.10 that $E_0^\wedge K(\mathbf{Z}/p^t\,\mathbf{Z}, d)$ is a finite flat R-module of rank $t\binom{n}{d}$.

(b) Let k be a field and suppose we are given a ring homomorphism $R \to k$. Then we have a canonical isomorphism

$$\operatorname{Spec} k \times_{\operatorname{Spec} R} \operatorname{Alt}^{(d)}_{\mathbf{G}[p^t]} \simeq \operatorname{Alt}^{(d)}_{\mathbf{G}[p^t]_k},$$

where $\mathbf{G}[p^t]_k$ denotes the group scheme over k given by $\operatorname{Spec} k \times_{\operatorname{Spec} R} \mathbf{G}[p^t]$. Since $\mathbf{G}[p^t]_k$ is a truncated p-divisible group of height n, level t, and dimension 1, Theorem 3.3.1 implies that $\operatorname{Alt}^{(d)}_{\mathbf{G}[p^t]_k}$ is a finite flat k-scheme of rank $p^{t\binom{n}{d}}$.

(c) The group scheme $\operatorname{Alt}^{(d)}_{\mathbf{G}}[p^t]$ is separated and of finite type over R by Remark 3.2.12.

(d) The map f induces an isomorphism

$$\operatorname{Spec} \kappa \times_{\operatorname{Spec} R} \operatorname{Spec} E_0^\wedge K(\mathbf{Z}/p^t\,\mathbf{Z}, d) \simeq \operatorname{Spec} K(n)_0 K(\mathbf{Z}/p^t\,\mathbf{Z}, d)) \simeq \operatorname{Alt}^{(d)}_{\mathbf{G}_0[p^t]} \simeq \operatorname{Spec} \kappa \times_{\operatorname{Spec} R} \operatorname{Alt}^{(d)}_{\mathbf{G}[p^t]}$$

by virtue of Proposition 3.4.3 and Corollary 3.3.3.

□

3.5 Alternating Powers in General

Let G be a finite flat commutative group scheme over a commutative ring R, and let $d \geq 1$ be an integer. In §3.2 we introduced an R-scheme $\operatorname{Alt}^{(d)}_G$ parametrizing alternating multilinear maps from G^d into the multiplicative group $\mathbf{G}_m$. In this section, we will show that this construction is well-behaved, provided that G satisfies some reasonably hypotheses. We can state our main result as follows:

Theorem 3.5.1. *Let R be a commutative ring, let p be a prime number, let $d \geq 1$ be a positive integer, and let G be an truncated p-divisible group over R of height n, level t, and dimension 1. Then $\operatorname{Alt}^{(d)}_G$ is a truncated p-divisible group over R of height $\binom{n}{d}$, level t, and dimension $\binom{n-1}{d}$.*

Corollary 3.5.2. *Let G be a truncated p-divisible group of height n, level t, and dimension 1 over a commutative ring R. For $d > n$, we have $\operatorname{Alt}^{(d)}_G \simeq \operatorname{Spec} R$.*

Corollary 3.5.3. *Let G be a truncated p-divisible group of height n, level t, and dimension 1 over a commutative ring R. Then the inclusion map $i : \operatorname{Alt}^{(d)}_G \hookrightarrow \operatorname{Skew}^{(d)}_G$ is a closed immersion.*

Proof. Theorem 3.5.1 implies that $\operatorname{Alt}^{(d)}_G$ is finite and flat over R, hence proper over R. Since $\operatorname{Skew}^{(d)}_R$ is an affine R-scheme of finite presentation, we conclude that the map i is proper. Since i is also a monomorphism, it must be a closed immersion. □

Corollary 3.5.4. *Let R be a commutative ring, let $d \geq 1$ be an integer, and let G be a p-divisible group of height n and dimension 1 over R. For each $t \geq 0$, let $\phi_t : \operatorname{Alt}^{(d)}_{G[p^t]} \to \operatorname{Alt}^{(d)}_{G[p^{t+1}]}$ be the map induced by the epimorphism $[p] : G[p^{t+1}] \to G[p^t]$. Then the colimit of the sequence*

$$\operatorname{Alt}^{(d)}_{G[1]} \xrightarrow{\phi_0} \operatorname{Alt}^{(d)}_{G[p]} \xrightarrow{\phi_1} \operatorname{Alt}^{(d)}_{G[p^2]} \xrightarrow{\phi_2} \cdots$$

is a p-divisible group H over R, having height $\binom{n}{d}$ and dimension $\binom{n-1}{d}$. Moreover, each of the canonical maps $\operatorname{Alt}^{(d)}_{G[p^t]} \to H$ induces an isomorphism from $\operatorname{Alt}^{(d)}_{G[p^t]}$ to $H[p^t]$.

Proof. Since each of the group schemes $\mathrm{Alt}^{(d)}_{G[p^t]}$ is annihilated by p^t, the action of p on H is locally nilpotent: that is, we can write $H = \varinjlim_t H[p^t]$. We next claim that multiplication by p induces an epimorphism $\theta : H \to H$ of sheaves with respect to the flat topology. Note that we can write θ as a filtered colimit of maps $\theta_t : H[p^{t+1}] \to H[p^t]$; it will therefore suffice to show that each θ_t is an epimorphism of flat sheaves. We may further write θ_t as a filtered colimit of maps

$$\theta_{t,s} : \mathrm{Alt}^{(d)}_{G[p^s]}[p^{t+1}] \to \mathrm{Alt}^{(d)}_{G[p^s]}[p^t].$$

It follows from Theorem 3.5.1 that $\mathrm{Alt}^{(d)}_{G[p^s]}$ is a truncated p-divisible group of level s for each $s \geq 0$, so that $\theta_{t,s}$ is an epimorphism whenever $s > t$.

We next show that for each $t \geq 0$, the canonical map $\iota_t : \mathrm{Alt}^{(d)}_{G[p^t]} \to H[p^t]$ is an isomorphism. Using Theorem 3.5.1, we deduce that each $H[p^t]$ is representable by a finite flat group scheme over R, so that H is a p-divisible group. Since Theorem 3.5.1 asserts that the truncated p-divisible group $H[p] \simeq \mathrm{Alt}^{(d)}_{G[p]}$ has height $\binom{n}{d}$ and dimension $\binom{n-1}{d}$ over R, we conclude that H also has height $\binom{n}{d}$ and dimension $\binom{n-1}{d}$.

Note that ι_t can be written as a filtered colimit of maps

$$\iota_{t,s} : \mathrm{Alt}^{(d)}_{G[p^t]} \to \mathrm{Alt}^{(d)}_{G[p^{t+s}]}[p^t].$$

It will therefore suffice to show that each $\iota_{t,s}$ is an isomorphism. Working by induction on s, we are reduced to proving that each of the maps ϕ_t induces an isomorphism $\mathrm{Alt}^{(d)}_{G[p^t]} \to \mathrm{Alt}^{(d)}_{G[p^{t+1}]}[p^t]$, which follows from Proposition 3.3.7. $\square$

Passing to Cartier duals, we obtain the following close relative of Corollary 3.5.4:

Corollary 3.5.5. *Let R be a commutative ring, let $d \geq 1$ be an integer, and let G be a p-divisible group of height n and dimension 1 over R. For each $t \geq 0$, let $\psi_t : \mathbf{D}(\mathrm{Alt}^{(d)}_{G[p^t]}) \to \mathbf{D}(\mathrm{Alt}^{(d)}_{G[p^{t+1}]})$ the map induced by the inclusion $G[p^t] \to G[p^{t+1}]$ Then the colimit of the sequence*

$$\mathbf{D}(\mathrm{Alt}^{(d)}_{G[1]}) \xrightarrow{\psi_0} \mathbf{D}(\mathrm{Alt}^{(d)}_{G[p]}) \xrightarrow{\psi_1} \mathbf{D}(\mathrm{Alt}^{(d)}_{G[p^2]}) \xrightarrow{\psi_2} \cdots$$

is a p-divisible group H over R, having height $\binom{n}{d}$ and dimension $\binom{n-1}{d-1}$. Moreover, each of the canonical maps $\mathbf{D}(\mathrm{Alt}^{(d)}_{G[p^t]}) \to H$ induces an isomorphism from $\mathbf{D}(\mathrm{Alt}^{(d)}_{G[p^t]})$ to $H[p^t]$.

Example 3.5.6. Let E be a Lubin-Tate spectrum of height n. Using Corollary 3.5.5 and Theorem 3.4.1, we deduce that the colimit of the sequence

$$\mathrm{ESpec}\, K(p^{-1}\, \mathbf{Z} \,/\, \mathbf{Z}, d) \to \mathrm{ESpec}\, K(p^{-2}\, \mathbf{Z} \,/\, \mathbf{Z}, d) \to \mathrm{ESpec}\, K(p^{-3}\, \mathbf{Z} \,/\, \mathbf{Z}, d) \to \cdots$$

is a p-divisible group of height $\binom{n}{d}$ and dimension $\binom{n-1}{d-1}$ over the Lubin-Tate ring $R = \pi_0 E$. This p-divisible group can also be described as the formal spectrum of power series ring given by $E^0 K(\mathbf{Q}_p \,/\, \mathbf{Z}_p, d) \simeq E^0 K(\mathbf{Z}, d+1)$.

Proof of Theorem 3.5.1. Let R be a commutative ring and let G be an truncated p-divisible group over R of height n, level t, and dimension 1. We wish to prove that $\mathrm{Alt}^{(d)}_{G}$ is a truncated p-divisible group of height $\binom{n}{d}$, level t, and dimension $\binom{n-1}{d}$. Note that $\mathrm{Alt}^{(d)}_{G}$ is annihilated by p^t. By virtue of Corollaries 3.1.8 and 3.3.2, it will suffice to prove that $\mathrm{Alt}^{(d)}_{G}$ is a finite flat group scheme over R.

Write R as a union of its finitely generated subrings R_α. Then we can write $G = G_\alpha \times_{\mathrm{Spec}\, R_\alpha} \mathrm{Spec}\, R$ for some α and some finite flat group scheme G_α over R_α. Using Corollary 3.1.8 and Remark 3.1.9, we conclude that there exists a quasi-compact open subset $U \subseteq \mathrm{Spec}\, R_\alpha$ such that, for every R_α-algebra B, the fiber product $G_B = \mathrm{Spec}\, B \times_{\mathrm{Spec}\, R_\alpha} G_\alpha$ is a truncated p-divisible group of height n, level t, and dimension

1 over A if and only if the induced map $\operatorname{Spec} B \to \operatorname{Spec} R_\alpha$ factors through U. In other words, there exists a finite sequence of elements $r_1, \ldots, r_k \in R_\alpha$ such that G_B is a truncated p-divisible group of height n, level t, and dimension 1 over B if and only if the elements $r_1, \ldots, r_k$ generate the unit ideal in B. In particular, the elements r_i generate the unit ideal in R. Enlarging α if necessary, we may assume that the r_i generate the unit ideal in R_α, so that $U = \operatorname{Spec} R_\alpha$ and G_α is a truncated p-divisible group of height n, level t, and dimension 1. We may therefore replace R by R_α, and thereby reduce to the case where R is Noetherian.

The map $\operatorname{Alt}_G^{(d)} \to \operatorname{Skew}_G^{(d)}$ is a monomorphism of finite presentation, hence quasi-finite and in particular quasi-affine. It follows that $\operatorname{Alt}_G^{(d)}$ is a quasi-affine scheme. Let A denote the ring of global sections of the structure sheaf of $\operatorname{Alt}_G^{(d)}$, so that the canonical map $j : \operatorname{Alt}_G^{(d)} \to \operatorname{Spec} A$ is a open immersion. It follows that the complement of the image of j is the vanishing locus of some ideal $I \subseteq A$.

We first prove the following:

$(*)$ Let $\mathfrak{p}$ be a prime ideal of $\operatorname{Spec} R$ for which the induced map

$$\operatorname{Spec} R_{\mathfrak{p}} \times_{\operatorname{Spec} R} \operatorname{Alt}_G^{(d)} \to \operatorname{Spec} R_{\mathfrak{p}}$$

is finite flat of degree $p^{t\binom{n}{d}}$. Then there exists an open neighborhood $U \subseteq \operatorname{Spec} R$ containing $\mathfrak{p}$ such that the induced map

$$U \times_{\operatorname{Spec} R} \operatorname{Alt}_G^{(d)} \to U$$

is finite flat of degree $p^{t\binom{n}{d}}$.

Let $\mathfrak{p}$ be as in $(*)$, so that so that the localization $A_{\mathfrak{p}}$ is a finitely generated projective $R_{\mathfrak{p}}$ module of rank $p^{t\binom{n}{d}}$, and $I_{\mathfrak{p}} = A_{\mathfrak{p}}$. In particular, the identity element $1 \in A_{\mathfrak{p}}$ belongs to $I_{\mathfrak{p}}$. It follows that there exists an element $f \in R - \mathfrak{p}$ whose image in A belongs to the ideal I. Replacing R by $R[\frac{1}{f}]$, we may suppose that $1 \in I$: that is, that $\operatorname{Alt}_G^{(d)} = \operatorname{Spec} A$ is affine. Since $\operatorname{Alt}_G^{(d)}$ is an R-scheme of finite presentation, we deduce that A is finitely presented as an R-algebra. Choose a finite set of R-algebra generators $x_1, \ldots, x_q \in A$. Enlarging this list if necessary, we may assume that the images of the x_i generate $A_{\mathfrak{p}}$ as a module over $R_{\mathfrak{p}}$. We may therefore write

$$x_i x_j = \sum c_{i,j}^k x_k$$

in $A_{\mathfrak{p}}$, for some coefficients $c_{i,j}^k \in R_{\mathfrak{p}}$. It follows that there exists $f' \notin \mathfrak{p}$ such that

$$x_i x_j = \sum c_{i,j}^k x_k$$

in $A[\frac{1}{f'}]$, for some coefficients $c_{i,j}^k \in R[\frac{1}{g}]$. Replace R by $R[\frac{1}{f'}]$, and set

$$\overline{A} = R[X_1, \ldots, X_q]/(X_i X_j - \sum c_{i,j}^k X_k).$$

Then $\overline{A}$ is a finitely presented as an R-algebra and finitely generated as an R-module. There is an evident map $\phi : \overline{A} \to A$, carrying X_i to x_i. Since the x_i's form algebra generators for A over R, the map $\overline{A} \to A$ is surjective. Since A is finitely presented as an R-algebra, the ideal $\ker(\phi)$ is finitely generated as an $\overline{A}$-module. Since $\overline{A}$ is finitely generated as an R-module, we conclude that $\ker(\phi)$ is finitely generated as an R-module. It follows that $A \simeq \overline{A}/\ker(\phi)$ is finitely presented as an R-module. Let $D = p^{t\binom{n}{d}}$, so that $A_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module of rank D. Choose a collection of elements $y_1, y_2, \ldots, y_D \in A$ whose images in $A_{\mathfrak{p}}$ form a basis. This choice determines a map of R-modules $\psi : R^D \to A$. By construction, $\operatorname{coker}(\psi)_{\mathfrak{p}} = 0$. Since A is finitely generated as an R-module, it follows that there exists an element $f'' \in R - \mathfrak{p}$ such that $\operatorname{coker}(\psi)$ is annihilated by h. Replacing R by $R[\frac{1}{f''}]$, we may suppose that ψ is surjective. We then have an exact sequence of R-modules

$$0 \to K \to R^D \to A \to 0.$$

Since A is finitely presented as an R-module, we conclude that K is finitely generated. Note that $K_{\mathfrak{p}} \simeq 0$. It follows that there exists $f''' \in R - \mathfrak{p}$ such that $K[\frac{1}{f'''}] \simeq 0$. Replacing R by $R[\frac{1}{f'''}]$, we may reduce to the case where ψ is an isomorphism: that is, A is a free R-module of rank d. This completes the proof of $(*)$.

To complete the proof of Theorem 3.5.1, it will suffice to show that every prime ideal $\mathfrak{p} \subseteq R$ satisfies the hypothesis of $(*)$. Replacing R by $R_{\mathfrak{p}}$, we may suppose that R is a local ring. Suppose first that the residue characteristic of R is different from p. Then the group scheme G is étale. Passing to finite étale covering of R, we may suppose that $G \simeq (\underline{\mathbf{Z}/p^t\,\mathbf{Z}})^n$. Invoking Remark 3.2.21 repeatedly, we obtain an isomorphism

$$\mathrm{Alt}_G^{(d)} \simeq (\mu_{p^t})^{\binom{n}{d}},$$

which is evidently a finite flat R-scheme of rank D.

Now suppose that the residue field κ of R has characteristic p. Using Proposition 10.3.1 of [5], we can choose find a faithfully flat morphism $R \to R'$ of local Noetherian rings, such that the residue field of R' is perfect. Replacing R' by its completion if necessary, we may assume that R' is complete. Using faithfully flat descent, we are reduced to proving that the map $\mathrm{Alt}_G^{(d)} \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, R' \to \mathrm{Spec}\, R'$ is finite flat of degree D. We may therefore replace R by R' and thereby reduce to the case where R is a complete Noetherian local ring whose residue field is perfect. Invoking Theorem 3.1.11, we deduce that there exists a p-divisible group $\mathbf{G}$ over R and an isomorphism $G \simeq \mathbf{G}[p^m]$.

We have an exact sequence of p-divisible groups

$$0 \to \mathbf{G}_{\mathrm{inf}} \to \mathbf{G} \to \mathbf{G}_{\text{ét}} \to 0$$

which determines an exact sequence of finite flat group schemes

$$0 \to \mathbf{G}_{\mathrm{inf}}[p^m] \to G \to \mathbf{G}_{\text{ét}}[p^m] \to 0.$$

Replacing R by a finite flat extension if necessary, we may assume that the latter sequence splits and that $\mathbf{G}_{\text{ét}}[p^t]$ is a constant group scheme. Let $G_0 = \mathbf{G}_{\mathrm{inf}}[p^t]$, so that we have an isomorphism

$$G \simeq G_0 \times_{\mathrm{Spec}\, R} (\underline{\mathbf{Z}/p^t\,\mathbf{Z}})^k$$

for some integer k. Applying Remark 3.2.21 repeatedly, we obtain an isomorphism

$$\mathrm{Alt}_G^{(d)} \simeq \mu_{p^t}^{\binom{k}{d}} \times_{\mathrm{Spec}\, R} \prod_{1 \leq d' \leq d} (\mathrm{Alt}_{G_0}^{(d')})^{\binom{k}{d-d'}}.$$

It will therefore suffice to show that each of the group schemes $\mathrm{Alt}_{G_0}^{(d')}$ is finite flat of rank $p^{t\binom{n-k}{d}}$. We may therefore replace $\mathbf{G}$ by $\mathbf{G}_{\mathrm{inf}}$ and thereby reduce to the case where $\mathbf{G}$ is connected, in which case the desired result follows from Corollary 3.4.2. □

4 Ambidexterity

Let M be a compact oriented manifold of dimension d. Poincare duality asserts that cap product with the fundamental homology class of M induces an isomorphism $\mathrm{H}^*(M; \mathbf{Z}) \to \mathrm{H}_{d-*}(M; \mathbf{Z})$. More generally, if $\mathcal{A}$ is any local system of abelian groups on M, then we obtain an isomorphism $\mathrm{H}^*(M; \mathcal{A}) \simeq \mathrm{H}_{d-*}(M; \mathcal{A})$ between homology and cohomology with coefficients in $\mathcal{A}$. In this section, we will study a somewhat different situation in which an analogous duality phenomenon occurs.

Let $\mathcal{C}$ be an ∞-category and let X be a Kan complex. We define a $\mathcal{C}$*-valued local system* on X to be a map of simplicial sets $X \to \mathcal{C}$. We will typically use the symbol $\mathcal{L}$ to denote a $\mathcal{C}$-valued local system on X, and $\mathcal{L}_x$ to denote the value of $\mathcal{L}$ at a point $x \in X$. The collection of all $\mathcal{C}$-valued local systems can be organized into an ∞-category $\mathcal{C}^X = \mathrm{Fun}(X, \mathcal{C})$. If C is an object of $\mathcal{C}$, we let $\underline{C}_X$ denote the constant map

$X \to \mathcal{C}$ taking the value C. The construction $C \mapsto \underline{C}_X$ determines a functor $\delta : \mathcal{C} \to \mathcal{C}^X$. If we assume that $\mathcal{C}$ admits small limits and colimits, then the functor δ admits both left are right adjoints, which we will denote by $\mathcal{L} \mapsto C_*(X;\mathcal{L})$ and $\mathcal{L} \mapsto C^*(X;\mathcal{L})$, respectively. More concretely, these functors are given by the formulas

$$C_*(X;\mathcal{L}) = \varinjlim_{x \in X} \mathcal{L}_x \qquad C^*(X;\mathcal{L}) = \varprojlim_{x \in X} \mathcal{L}_x\,.$$

Our goal is to describe some special situations (depending on the space X and the ambient ∞-category $\mathcal{C}$) in which the functors $\mathcal{L} \mapsto C_*(X;\mathcal{L})$ and $\mathcal{L} \mapsto C^*(X;\mathcal{L})$ are equivalent. We begin by describing some ways to think about

Construction 4.0.7. Let X be a Kan complex, let $\mathcal{C}$ be an ∞-category which admits small limits and colimits, and let $\mu : C^*(X;\bullet) \to C_*(X;\bullet)$ be a natural transformation. Suppose we are given a pair of objects $C, D \in \mathcal{C}$ and a map of Kan complexes $f : X \to \mathrm{Map}_{\mathcal{C}}(C,D)$. We will abuse notation by identifying f with a morphism from $\underline{C}_X$ to $\underline{D}_X$ in the ∞-category $\mathcal{C}^X$ of $\mathcal{C}$-valued local systems on X. We define another map $\int_X f d\mu \in \mathrm{Map}_{\mathcal{C}}(C,D)$ to be the composition

$$C \to C^*(X;\underline{C}_X) \xrightarrow{f} C^*(X;\underline{D}_X) \xrightarrow{\mu} C_*(X;\underline{D}_X) \to D.$$

We will refer to $\int_X f d\mu$ as the *integral of f with respect to μ.*

Remark 4.0.8. In the situation of Construction 4.0.7, we can reconstruct the natural transformation μ from the collection of maps $\int_X f d\mu$. To see this, suppose we are given a local system $\mathcal{L} \in \mathcal{C}^X$. Then the map $\mu(\mathcal{L}) : C^*(X;\mathcal{L}) \to C_*(X;\mathcal{L})$ is given by $\int_X f d\mu$, where $f : X \to \mathrm{Map}_{\mathcal{C}}(C^*(X;\mathcal{L}), C_*(X;\mathcal{L}))$ is the map which assigns to each point $x \in X$ the composite map

$$C^*(X;\mathcal{L}) \to \mathcal{L}_x \to C_*(X;\mathcal{L}).$$

Suppose now that X is an arbitrary Kan complex. For every pair of points $x, y \in X$, let $P_{x,y} = \{x\} \times_{\mathrm{Fun}(\{0\},X)} \mathrm{Fun}(\Delta^1, X) \times_{\mathrm{Fun}(\{1\},X)} \{y\}$ be the space of paths from x to y in X. Suppose that for each $x, y \in X$, we are given a natural transformation $\mu_{P_{x,y}} : C^*(P_{x,y};\bullet) \to C_*(P_{x,y};\bullet)$. Every local system $\mathcal{L}$ on X determines a map $\phi_{x,y} : P_{x,y} \mapsto \mathrm{Map}_{\mathcal{C}}(\mathcal{L}_x, \mathcal{L}_y)$, so that we obtain a map $\int_{P_{x,y}} \phi_{x,y} d\mu_{P_{x,y}}$ from $\mathcal{L}_x$ to $\mathcal{L}_y$ in $\mathcal{C}$. If the natural transformations $\mu_{P_{x,y}}$ are chosen functorially on x and y, then the maps $\int_{P_{x,y}} \phi_{x,y} d\mu_{P_{x,y}}$ also depend functorially on x and y, and therefore determine a map

$$\mathrm{Nm}_X(\mathcal{L}) : \varinjlim_{x \in X} \mathcal{L}_x \to \varprojlim_{y \in X} \mathcal{L}_y\,.$$

This construction itself depends functorially on $\mathcal{L}$, and can therefore be regarded as a natural transformation

$$\mathrm{Nm}_X : C_*(X;\bullet) \to C^*(X;\bullet).$$

We would like to apply the above construction iteratively to construct equivalences $\mu_X : C^*(X;\bullet) \to C_*(X;\bullet)$. More precisely, we will introduce the following:

- For every ∞-category $\mathcal{C}$ which admits small limits and colimits, we introduce a collection of Kan complexes which we call $\mathcal{C}$-*ambidextrous*.
- For every $\mathcal{C}$-ambidexterous Kan complex X, we define an equivalence $\mu_X : C^*(X;\bullet) \to C_*(X;\bullet)$.

We say that a Kan complex X is $\mathcal{C}$-*ambidextrous* if it satisfies the following three conditions:

(a) The Kan complex X is n-truncated for some integer n.

(b) For every pair of points $x, y \in X$, the path space $P_{x,y}$ is $\mathcal{C}$-ambidextrous.

(c) The natural transformation $\mathrm{Nm}_X : C_*(X;\bullet) \to C^*(X;\bullet)$ constructed above is an equivalence.

If these conditions are satisfied, then we define the natural transformation μ_X to be the inverse of the equivalence Nm_X.

Remark 4.0.9. The construction of the natural transformations $\mu_X : C^*(X;\bullet) \to C_*(X;\bullet)$ is recursive: if X is n-truncated for $n \geq -1$, then each path space $P_{x,y}$ is $(n-1)$-truncated, and so we may assume that the natural transformations $\mu_{P_{x,y}}$ have already been defined. A special case occurs when $n = -2$: that is, when X is contractible. In this case, the functor $C \mapsto \underline{C}_X$ induces an equivalence from $\mathcal{C}$ to $\mathcal{C}^X$, and we take μ_X to be the evident identification between the right and left adjoints of this equivalence.

Let us now outline the contents of this section. We begin in §4.1 by giving a more detailed construction of the natural transformation Nm_X. With an eye towards future applications, we carry out the construction in the context of an arbitrary Beck-Chevalley fibration of ∞-categories (see Definition 4.1.3). The specialization to local systems on a Kan complex will be carried out in §4.3.

In §4.2, we study the naturality properties of the norm map. In particular, our results show that for a Kan fibration $f : X \to Y$, the norm map Nm_X can be reconstructed from the norm map Nm_Y, together with the norm maps Nm_{X_y} associated to the fibers of f (see Propositions 4.2.1 and 4.2.2).

Let X be a Kan complex and let $\mathcal{C}$ be an ∞-category which admits small limits and colimits. The requirement that X be $\mathcal{C}$-ambidextrous imposes conditions on both X and $\mathcal{C}$. On X, it should be regarded as a finiteness condition: it is generally only reasonable to expect ambidexterity in the case where the homotopy groups of X are finite (though there are exceptions for some values of $\mathcal{C}$; see Example 4.3.11). As a condition on $\mathcal{C}$, ambidexterity amounts to a kind of generalized additivity: it implies, in particular, that there is a canonical way to "integrate" a family of morphisms in $\mathcal{C}$ indexed by X. In §4.4, we will make these ideas more precise, and use them to produce some simple examples of pairs $(\mathcal{C}, X)$ which satisfy ambidexterity. For example, we will show that an Eilenberg-MacLane space $X = K(\mathbf{Z}/p\,\mathbf{Z}, d)$ is $\mathcal{C}$-ambidextrous whenever $\mathcal{C}$ is a stable ∞-category having the property that p acts invertible on each object of $\mathcal{C}$ (Proposition 4.4.20).

4.1 Beck-Chevalley Fibrations and Norm Maps

Our goal in this section is to give an account of the theory of ambidexterity, including a precise construction of the associated norm maps (which were described informally in the introduction to §4). With an eye toward future applications, we will work in a somewhat general context:

(a) We will view ambidexterity as a property of maps of spaces, rather than spaces. For every ∞-category $\mathcal{C}$ which admits finite limits and colimits, we will introduce a collection of *$\mathcal{C}$-ambidextrous maps $f : X \to Y$* between Kan complexes, having the following property: if $f : X \to Y$ is $\mathcal{C}$-ambidextrous, then the left and right adjoints to the pullback functor $f^* : \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ are (canonically) equivalent. This does not really result in any additional generality: we will later see that the $\mathcal{C}$-ambidexterity of a map $f : X \to Y$ is really a condition on the homotopy fibers of f (Corollary 4.3.6). However, it is a convenient mechanism for encoding the naturality properties of the norm, which play an essential role in our definition.

(b) If $\mathcal{C}$ is an ∞-category which admits small limits and colimits, then the theory of $\mathcal{C}$-ambidexterity depends on the construction $X \mapsto \mathrm{Fun}(X, \mathcal{C})$ which assigns to each Kan complex X the collection of $\mathcal{C}$-valued local systems on X. We will develop our theory in the more general case of a construction $X \mapsto \mathcal{C}_X$, which assigns an ∞-category $\mathcal{C}_X$ to each object X of an ambient ∞-category $\mathcal{X}$, and a pair of adjoint functors

$$\mathcal{C}_X \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathcal{C}_Y$$

to each morphism $f : X \to Y$ in $\mathcal{X}$.

We begin by introducing some definitions.

Notation 4.1.1. Let $\mathcal{X}$ be an ∞-category and let $q : \mathcal{C} \to \mathcal{X}$ be a map which is both a Cartesian fibration and a coCartesian fibration. For each object $X \in \mathcal{X}$, we let $\mathcal{C}_X$ denote the fiber $q^{-1}\{X\} = \mathcal{C} \times_{\mathcal{X}} \{X\}$. If $f : X \to Y$ is a morphism in $\mathcal{X}$, then f determines a pair of adjoint functors

$$\begin{tikzcd} \mathcal{C}_X \arrow[r, shift left, "f_!"] & \mathcal{C}_Y \arrow[l, shift left, "f^*"] \end{tikzcd}.$$

In this situation, we let

$$\phi_f : f_! f^* \to \mathrm{id}_{\mathcal{C}_Y} \qquad \psi_f : \mathrm{id}_{\mathcal{C}_X} \to f^* f_!$$

denote the associated counit and unit transformations.

Given a commutative diagram σ :

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{X}$, we have a canonical equivalence of functors $g'^* \circ f^* \xrightarrow{\sim} g^* \circ f^*$, which induces a natural transformation $BC[\sigma] : f'_! g'^* \to g^* f_!$. We will refer to $BC[\sigma]$ as the *Beck-Chevalley transformation* associated to σ.

Remark 4.1.2. Given a commutative diagram σ :

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

the Beck-Chevalley transformation $BC[\sigma]$ is given by the composition

$$f'_! g'^* \xrightarrow{\psi_f} f'_! g'^* f^* f_! \simeq f'_! f'^* g^* f_! \xrightarrow{\phi_{f'}} g^* f_!.$$

Definition 4.1.3. Let $\mathcal{X}$ be an ∞-category which admits pullbacks. We will say that a map of simplicial sets $q : \mathcal{C} \to \mathcal{X}$ is a *Beck-Chevalley fibration* if the following conditions are satisfied:

(1) The map q is both a Cartesian fibration and a coCartesian fibration.

(2) For every pullback square σ :

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in the ∞-category $\mathcal{X}$, the Beck-Chevalley transformation $BC[\sigma] : f'_! g'^* \to g^* f_!$ is an equivalence of functors from $\mathcal{C}_{Y'}$ to $\mathcal{C}_X$.

Remark 4.1.4. When condition (2) of Definition 4.1.3 is satisfied, we let $BC[\sigma]^{-1} : g^* f_! \to f'_! g'^*$ denote a homotopy inverse to the Beck-Chevalley transformation $BC[\sigma]$, so that $BC[\sigma]^{-1}$ is well-defined up to homotopy.

Remark 4.1.5. Let $\mathcal{X}$ be an ∞-category which admits pullbacks and let $q : \mathcal{C} \to \mathcal{X}$ be both a Cartesian and a coCartesian fibration. Then condition (2) of Definition 4.1.3 admits either of the following equivalent formulations:

(2′) Let $\overline{\sigma}$:

$$\begin{tikzcd} \overline{X}' \arrow[r, "\overline{f}'"] \arrow[d, "\overline{g}'"] & \overline{Y}' \arrow[d, "\overline{g}"] \\ \overline{X} \arrow[r, "\overline{f}"] & \overline{Y} \end{tikzcd}$$

be a commutative diagram in $\mathcal{C}$ whose image in $\mathcal{X}$ is a pullback square. If $\overline{f}$ is q-coCartesian and both $\overline{g}$ and $\overline{g}'$ are q-Cartesian, then $\overline{f}'$ is q-coCartesian.

(2′′) Let $\overline{\sigma}$:

$$\begin{tikzcd} \overline{X}' \arrow[r, "\overline{f}'"] \arrow[d, "\overline{g}'"] & \overline{Y}' \arrow[d, "\overline{g}"] \\ \overline{X} \arrow[r, "\overline{f}"] & \overline{Y} \end{tikzcd}$$

be a commutative diagram in $\mathcal{C}$ whose image in $\mathcal{X}$ is a pullback square. If $\overline{g}'$ is q-Cartesian and both $\overline{f}$ and $\overline{f}'$ are q-coCartesian, then $\overline{g}$ is q-Cartesian.

Let $q : \mathcal{C} \to \mathcal{X}$ be a Beck-Chevalley fibration. Then every morphism $f : X \to Y$ in $\mathcal{X}$ induces a pair of functors

$$f_! : \mathcal{C}_X \to \mathcal{C}_Y \qquad f^* : \mathcal{C}_Y \to \mathcal{C}_X .$$

The functor $f_!$ is characterized by the fact that it is a left adjoint to f^*. We will be interested in studying situations in which $f_!$ is also a *right* adjoint to f^*.

Notation 4.1.6. Let $q : \mathcal{C} \to \mathcal{X}$ be a map of ∞-categories which is both a Cartesian fibration and a coCartesian fibration. Let $f : X \to Y$ be a morphism in $\mathcal{X}$, and suppose we are given a natural transformation $\mu : \mathrm{id}_{\mathcal{C}_Y} \to f_! f^*$. Let C and D be objects of $\mathcal{C}_Y$. For every morphism $u : f^*C \to f^*D$ in $\mathcal{C}_X$, we let $\int_f u d\mu \in \mathrm{Map}_{\mathcal{C}_Y}(C, D)$ denote the composite map

$$C \xrightarrow{\mu} f_! f^* C \xrightarrow{f_!(u)} f_! f^* D \xrightarrow{\phi_f} D.$$

The construction $u \mapsto \int u d\mu$ determines a map

$$d\mu : \mathrm{Map}_{\mathcal{C}_X}(f^*C, f^*D) \to \mathrm{Map}_{\mathcal{C}_Y}(C, D).$$

Remark 4.1.7. In the situation of Notation 4.1.6, suppose that the functor f^* also denotes a right adjoint f_*. Then giving a natural transformation $\mu : \mathrm{id}_{\mathcal{C}_Y} \to f_! f^*$ is equivalent to giving a natural transformation $f_* \to f_!$. Note that the "integration" procedure of Notation 4.1.6 can be regarded as a generalization of Construction 4.0.7.

Construction 4.1.8. Let $\mathcal{X}$ be an ∞-category which admits pullbacks and let $q : \mathcal{C} \to \mathcal{X}$ be a Beck-Chevalley fibration. We will define the following data for $n \geq -2$:

(a) A collection of morphisms in $\mathcal{X}$, which we call *n-ambidextrous morphisms*.

(b) For each n-ambidextrous morphism $f : X \to Y$ in $\mathcal{X}$, a natural transformation $\mu_f^{(n)} : \mathrm{id}_{\mathcal{C}_Y} \to f_! \circ f^*$, which is well-defined up to homotopy and exhibits $f_!$ as a right adjoint to f^*.

The construction proceeds by induction on n. If $n = -2$, we declare that a morphism $f : X \to Y$ in $\mathcal{X}$ is n-ambidextrous if and only if f is an equivalence. In this case, we define $\mu_f^{(n)}$ to be a homotopy inverse to the counit map $\phi_f : f_! \circ f^* \to \mathrm{id}_{\mathcal{C}_Y}$ (which is an equivalence, since the adjoint functors $f_!$ and f^* are mutually inverse equivalences).

Assume now that the collection of n-ambidextrous morphisms have been defined for some $n \geq -2$, and that the natural transformation $\mu_g^{(n)} : \mathrm{id}_{\mathcal{C}_Y} \to g_! \circ g^*$ has been specified for every n-ambidextrous morphism g. Let $f : X \to Y$ be an arbitrary morphism in $\mathcal{C}$, and let $\delta : X \to X \times_Y X$ be the diagonal map, so that we have a commutative diagram

$$\begin{tikzcd}
X \arrow[dr, "\delta"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times_Y X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y.
\end{tikzcd}$$

Let σ be the square appearing in this diagram. Since σ is a pullback square and $q : \mathcal{C} \to \mathcal{X}$ is a Beck-Chevalley fibration, the Beck-Chevalley transformation $BC[\sigma] : \pi_{1!}\pi_2^* \to f^* f_!$ admits a homotopy inverse $BC[\sigma]^{-1} : f^* f_! \to \pi_{1!}\pi_2^*$. We will say that f is *weakly* $(n+1)$-*ambidextrous* if the diagonal map δ is n-ambidextrous. In this case, we define a natural transformation $\nu_f^{(n+1)} : f^* f_! \to \mathrm{id}_{\mathcal{Z}_X}$ to be the composition

$$f^* f_! \xrightarrow{BC[\sigma]^{-1}} \pi_{1!}\pi_2^* \xrightarrow{\mu_\delta^{(n)}} \pi_{1!}\delta_!\delta^*\pi_1^* \simeq \mathrm{id}_{\mathcal{C}_X} \circ \mathrm{id}_{\mathcal{C}_X} = \mathrm{id}_{\mathcal{C}_X} .$$

We will say that f is a $(n+1)$-*ambidextrous* if the following condition is satisfied:

$(*)$ For every pullback diagram

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

in $\mathcal{X}$, the map f' is weakly $(n+1)$-ambidextrous and the natural transformation $\nu_{f'}^{(n+1)}$ is the counit for an adjunction between f'^* and $f'_!$.

If condition $(*)$ is satisfied, we let $\mu_f^{(n+1)} : \mathrm{id}_{\mathcal{C}_Y} \to f_! f^*$ denote a compatible unit for the adjunction

$$\begin{tikzcd}
\mathcal{C}_X \arrow[r, shift left, "f^*"] & \mathcal{C}_Y \arrow[l, shift left, "f_!"]
\end{tikzcd}$$

determined by $\nu_f^{(n+1)}$.

Remark 4.1.9. In the situation of Construction 4.1.8, let $f : X \to Y$ be a weakly $(n+1)$-ambidextrous morphism. Then we can describe the natural transformation $\nu_f^{(n+1)}$ more informally as follows: for each object $C \in \mathcal{C}_X$, the map $\nu_f^{(n+1)}(C) : f^* f_! C \to C$ is the image of

$$\int_\delta \mathrm{id}_C \, d\mu_\delta^{(n)} \in \mathrm{Map}_{\mathcal{C}_{X \times_Y X}}(\pi_2^* C, \pi_1^* C)$$

under the homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}_{X \times_Y X}}(\pi_2^* C, \pi_1^* C) \simeq \mathrm{Map}_{\mathcal{C}_X}(\pi_{1!}\pi_2^* C, C) \simeq \mathrm{Map}_{\mathcal{C}_X}(f^* f_! C, C).$$

Our first observation is that the natural transformations introduced in Construction 4.1.8 are independent of n, provided that n is sufficiently large.

Proposition 4.1.10. *Let $\mathcal{X}$ be an ∞-category which admits pullbacks and let $q : \mathcal{C} \to \mathcal{X}$ be a Beck-Chevalley fibration. Then:*

(1) *If $f : X \to Y$ is an n-ambidextrous morphism in $\mathcal{X}$, then f is n-truncated (that is, the induced map $\mathrm{Map}_{\mathcal{X}}(Z, X) \to \mathrm{Map}_{\mathcal{X}}(Z, Y)$ has n-truncated homotopy fibers, for each $Z \in \mathcal{X}$).*

(2) *Let $f : X \to Y$ be an n-ambidextrous morphism in $\mathcal{X}$. Then any pullback of f is also n-ambidextrous.*

(3) *Let $f : X \to Y$ be a weakly n-ambidextrous morphism in $\mathcal{X}$. Then any pullback of f is also weakly n-ambidextrous.*

(4) *Let $-1 \leq m \leq n$. If f is a weakly m-ambidextrous morphism in $\mathcal{X}$, then f is weakly n-ambidextrous. Moreover, the natural transformations $\nu_f^{(m)}, \nu_f^{(n)} : f^* f_! \to \mathrm{id}_{\mathcal{C}_X}$ agree up to homotopy.*

(5) *Let $-2 \leq m \leq n$. If f is an m-ambidextrous morphism in $\mathcal{X}$, then f is n-ambidextrous. Moreover, the natural transformations $\mu_f^{(m)}, \mu_f^{(n)} : \mathrm{id}_{\mathcal{C}_Y} \to f_! f^*$ agree up to homotopy.*

(6) *Let $-2 \leq m \leq n$ and let f be a (weakly) n-ambidextrous morphism in $\mathcal{X}$. Then f is (weakly) m-ambidextrous if and only if f is m-truncated.*

Proof. By definition, a morphism $f : X \to Y$ is (-2)-ambidextrous if and only if it is an equivalence, and if $f : X \to Y$ is an n-ambidextrous for $n > -2$ then the diagonal map $\delta : X \to X \times_Y X$ is an $(n-1)$-ambidextrous. Assertion (1) follows immediately by induction on n (see Lemma HTT.5.5.6.15). Assertion (2) is immediate from the definitions, and (3) follows from (2).

To prove (4) and (5), it suffices to treat the case $n = m + 1$. We proceed by a simultaneous induction on m. The implication (4) $\Rightarrow$ (5) is clear. Conversely, if assertion (5) holds for some integer $m \geq -2$, then assertion (4) holds for the integer $m + 1$. It will therefore suffice to prove assertion (4) in the special case $m = -2$, so that f is an equivalence in $\mathcal{X}$. In this case, the diagonal map $\delta : X \to X \times_Y X$ is also an equivalence, and therefore (-2)-ambidextrous. In this case, the functors $f_!$ and f^* are homotopy inverse to one another, and a simple calculation shows that the map $v_f^{(-1)}$ is a homotopy inverse to the unit map $\psi_f : \mathrm{id}_{\mathcal{C}_X} \to f^* f_!$. In particular, this map exhibits f^* as a left adjoint to $f_!$, so that f is (-1)-ambidextrous. Moreover, the maps $\mu_f^{(-1)}$ and $\mu_f^{(-2)}$ are homotopic (both can be described as homotopy inverses to the counit map $f_! f^* \to \mathrm{id}_{\mathcal{C}_Y}$).

We now prove (6), again using induction on m. The "only if" direction follows from (1). To prove the converse, let us assume that f is m-truncated and n-ambidextrous; we wish to show that f is m-ambidextrous (the corresponding assertion for weakly ambidextrous morphisms follows immediately from this). If $m = -2$, then f is an equivalence and there is nothing to prove. Assume therefore that $m > -2$, and let $\delta : X \to X \times_Y X$ be the diagonal map. Replacing f by a pullback if necessary (and invoking (2)), we are reduced to proving that δ is $(m-1)$-ambidextrous and that the map $\nu_f^{(m)} : f^* f_! \to \mathrm{id}_{\mathcal{C}_X}$ is the counit of an adjunction between f^* and $f_!$. Since δ is $(n-1)$-ambidextrous and $(m-1)$-truncated, the inductive hypothesis implies that δ is $(m-1)$-ambidextrous and that the unit maps $\mu_\delta^{(m-1)}$ and $\mu_\delta^{(n-1)}$ are homotopic. It follows that the natural transformations $\nu_f^{(m)}, \nu_f^{(n)} : f^* f_! \to \mathrm{id}_{\mathcal{C}_X}$ are homotopic. Since $\nu_f^{(n)}$ is the counit of an adjunction, the natural transformation $\nu_f^{(m)}$ has the same property. $\square$

Definition 4.1.11. Let $\mathcal{X}$ be an ∞-category which admits pullbacks and let $q : \mathcal{C} \to \mathcal{X}$ be a Beck-Chevalley fibration. We will say that a morphism $f : X \to Y$ in $\mathcal{X}$ is a *weakly ambidextrous* if it is weakly n-ambidextrous for some integer $n \geq -1$. In this case, we let $\nu_f : f^* f_! \to \mathrm{id}_{\mathcal{C}_X}$ denote the natural transformation $\nu_f^{(n)}$ appearing in Construction 4.1.8 (by virtue of Proposition 4.1.10, the homotopy class of this natural transformation is independent of n). We will say that f is *ambidextrous* if it is n-ambidextrous for some n: that is, if every pullback f' of f is weakly ambidextrous and the map $\nu_{f'}$ exhibits f'^* as a left adjoint of $f'_!$. In this case, we let $\mu_f : \mathrm{id}_{\mathcal{C}_Y} \to f_! f^*$ denote a compatible unit for this adjunction (so that $\mu_f = \mu_f^{(n)}$ for any integer n such that f is n-truncated).

Remark 4.1.12. Let $q : \mathcal{C} \to \mathcal{X}$ be a Beck-Chevalley fibration. Let $f : X \to Y$ be a morphism in $\mathcal{X}$. If the pullback functor $f^* : \mathcal{C}_Y \to \mathcal{C}_X$ admits a right adjoint, we will denote that right adjoint by $f_* : \mathcal{C}_X \to \mathcal{C}_Y$. We then have an evident homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_X, \mathcal{C}_X)}(f^* f_!, \mathrm{id}_{\mathcal{C}_X}) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_X, \mathcal{C}_Y)}(f_!, f_*).$$

If f is weakly ambidextrous, we let $\mathrm{Nm}_f : f_! \to f_*$ denote the image of the natural transformation ν_f under this homotopy equivalence. We will refer to Nm_f as the *norm map* associated to f. It follows that f is ambidextrous if and only, for every pullback diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

the following conditions are satisfied:

(a) The map f' is weakly ambidextrous.

(b) The functor f'^* admits a right adjoint f'_*.

(c) The norm map $\mathrm{Nm}_{f'} : f'_! \to f'_*$ is an equivalence.

4.2 Properties of the Norm

Let $q : \mathcal{C} \to \mathcal{X}$ be a Beck-Chevalley fibration of ∞-categories (Definition 4.1.3). In §4.1, we introduced the notion of an *ambidextrous* morphism $f : X \to Y$ in $\mathcal{X}$, and associated to each ambidextrous morphism f a natural transformation $\mu_f : \mathrm{id}_{\mathcal{C}_Y} \to f_! f^*$. Our goal in this section is to establish two naturality properties enjoyed by this construction:

Proposition 4.2.1. *Let $\mathcal{X}$ be an ∞-category which admits pullbacks and let $q : \mathcal{C} \to \mathcal{X}$ be a Beck-Chevalley fibration. Suppose we are given a pullback diagram τ :*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g_X"] & Y' \arrow[d, "g_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in the ∞-category $\mathcal{X}$. Then:

(1) *Assume that f is weakly ambidextrous. Then f' is also weakly ambidextrous. Moreover, the diagram of natural transformations*

$$\begin{tikzcd} f'^* f'_! g_X^* \arrow[r, "BC[\tau]"] \arrow[d, "\nu_{f'}"] & f'^* g_Y^* f_! \arrow[r, "\sim"] & g_X^* f^* f_! \arrow[d, "\nu_f"] \\ g_X^* \arrow[rr, "\mathrm{id}"] & & g_X^* \end{tikzcd}$$

commutes up to homotopy.

(2) *If f is ambidextrous, then the diagram of natural transformations*

$$\begin{tikzcd} g_Y^* \arrow[rr, "\mathrm{id}"] \arrow[d, "\mu_{f'}"] & & g_Y^* \arrow[d, "\mu_f"] \\ f'_! f'^* g_Y^* \arrow[r, "\sim"] & f'_! g_X^* f^* \arrow[r, "BC[\tau]"] & g_Y^* f_! f^* \end{tikzcd}$$

commutes up to homotopy.

Proposition 4.2.2. *Let $\mathcal{X}$ be an ∞-category which admits pullbacks, let $q : \mathcal{C} \to \mathcal{X}$ be a Beck-Chevalley fibration, and suppose we are given a pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{X}$.*

(1) *Assume that f and g are weakly ambidextrous. Then the composition gf is weakly ambidextrous. Moreover, the natural transformation $\nu_{gf} : (gf)^*(gf)_! \to \mathrm{id}_{\mathcal{C}_X}$ is homotopic to the composition*

$$(gf)^*(gf)_! \simeq f^* g^* g_! f_! \xrightarrow{\mu_g} f^* f_! \xrightarrow{\mu_f} \mathrm{id}_{\mathcal{Z}_X} .$$

(2) *Assume that f and g are ambidextrous. Then the composition gf is ambidextrous. Moreover, the natural transformation $\mu_{gf} : \mathrm{id}_{\mathcal{C}_Z} \to (gf)_!(gf)^*$ is homotopic to the composition*

$$\mathrm{id}_{\mathcal{C}_Z} \xrightarrow{\mu_g} g_! g^* \xrightarrow{\mu_f} g_! f_! f^* g^* \simeq (gf)_!(gf)^*.$$

Remark 4.2.3. In the situation of Proposition 4.2.1, assume that f and f' are weakly ambidextrous and that the pullback functors f^* and f'^* admit right adjoints f_* and f'_*. Then assertion (1) reduces to the statement that the composite transformation

$$f'_! g_X^* \xrightarrow{BC[\tau]} g_Y^* f_! \xrightarrow{\mathrm{Nm}_f} g_Y^* f_* \to f'_* g_X^*$$

is homotopic to the map induced by the norm $\mathrm{Nm}_{f'}$.

Remark 4.2.4. In the situation of Proposition 4.2.2, assume that f and g are weakly ambidextrous and that the pullback functors f^* and g^* admit right adjoints f_* and g_*, respectively. Then the pullback functor $(gf)^*$ admits a right adjoint, given by the composition $g_* \circ f_*$. In this case, we can reformulate assertion (1) as follows: the norm map $\mathrm{Nm}_{gf} : (gf)_! \to (gf)_*$ is given by the composition

$$(gf)_! \simeq g_! f_! \xrightarrow{\mathrm{Nm}_g} g_* f_! \xrightarrow{\mathrm{Nm}_f} g_* f_* \simeq (gf)_*.$$

Remark 4.2.5. Propositions 4.2.1 and 4.2.2 barely scratch the surface of the problem of describing all of the coherence and naturality properties enjoyed by the constructions $f \mapsto \mu_f$. To address this problem more completely, it is convenient to use the language of $(\infty, 2)$-categories. Let $q : \mathcal{C} \to \mathcal{X}$ be a Beck-Chevalley fibration. We can associate to q an $(\infty, 2)$-category $\mathcal{Z}$, which we may describe informally as follows:

(*a*) The objects of $\mathcal{Z}$ are the objects of $\mathcal{X}$.

(*b*) Given a pair of objects $X, Y \in \mathcal{X}$, a 1-morphism from X to Y in $\mathcal{Z}$ is given by another object $M \in \mathcal{X}$ equipped with a pair of maps $f : M \to X$, $g : M \to Y$.

(*c*) Composition of 1-morphisms in $\mathcal{Z}$ is given by composition of correspondences: that is, the composition of a morphism $X \leftarrow M \to Y$ with a morphism $Y \leftarrow N \to Z$ is given by $X \leftarrow M \times_Y N \to Z$.

(*d*) Given a pair of objects $X, Y \in \mathcal{Z}$ and a pair of morphisms $X \leftarrow M \to Y$ and $X \leftarrow N \to Y$ in $\mathcal{Z}$, a 2-morphism from M to N is given by a commutative diagram

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ X & P \arrow[l] \arrow[r] \arrow[u, "u"] \arrow[d, "v"] & Y \\ & N \arrow[ul] \arrow[ur] & \end{tikzcd}$$

in the ∞-category $\mathcal{C}$, where u is ambidextrous and v is arbitrary. Composition of 2-morphisms is again given by composition of correspondences.

Then one can construct an essentially unique functor Θ from $\mathcal{Z}$ to the $(\infty, 2)$-category of ∞-categories, given informally as follows:

(a') To each object $X \in \mathcal{Z}$, the functor Θ assigns the ∞-category $\mathcal{C}_X$.

(b') To each 1-morphism $X \xleftarrow{f} M \xrightarrow{g} Y$ in $\mathcal{Z}$, the functor Θ assigns the functor $g_! f^* : \mathcal{C}_X \to \mathcal{C}_Y$.

(c') Given a pair of composable morphisms $X \xleftarrow{f} M \xrightarrow{g} Y$ and $Y \xleftarrow{f'} N \xrightarrow{g'} Z$, consider the commutative diagram σ :

$$\begin{tikzcd} & M \times_Y N \arrow[dl, "f''"'] \arrow[dr, "g''"] & \\ M \arrow[dr, "g"'] & & N \arrow[dl, "f'"] \\ & Y. & \end{tikzcd}$$

Then the compatibility of Θ with composition of 1-morphisms is witnessed by the equivalence

$$(g'g'')_!(ff'')^* \simeq g'_!(g''_! f''^*)f^* \xrightarrow{BC[\sigma]} g'_!(f'^* g_!)f^* = (g'_! f'^*) \circ (g_! f^*).$$

(d') To each 2-morphism in $\mathcal{Z}$, given by a commutative diagram

$$\begin{tikzcd} & M \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & P \arrow[l] \arrow[r] \arrow[u, "u"] \arrow[d, "v"] & Y \\ & N, \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

the functor Θ associates the natural transformation

$$g_! f^* \xrightarrow{\mu_u} g_!(u_! u^*)f^* \simeq g'_!(v_! v^*)f'^* \xrightarrow{\phi_v} g'_! f^*.$$

Proposition 4.2.2 expresses a special case of the compatibility of Θ with composition of 2-morphisms, while Proposition 4.2.1 expresses a special case of the compatibility of Θ with the combination of horizontal and vertical compositions in $\mathcal{Z}$.

We will defer the precise definition of the $(\infty, 2)$-category $\mathcal{Z}$, and the existence of the functor Θ to a future work; the comparatively crude Propositions 4.2.1 and 4.2.2 will be sufficient for our applications in this paper.

From Proposition 4.2.1, we can deduce the following related result:

Corollary 4.2.6. *Let $\mathcal{X}$ be an ∞-category which admits pullbacks and let $q : \mathcal{C} \to \mathcal{X}$ be a Beck-Chevalley fibration. Suppose we are given a pullback diagram τ :*

$$\begin{tikzcd} X' \arrow[r, "g_X"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g_Y"] & Y \end{tikzcd}$$

in the ∞-category $\mathcal{X}$. Then:

(1) *Assume that f is weakly ambidextrous. Then f' is weakly ambidextrous, and the diagram of natural transformations*

$$\begin{tikzcd} g_{X!} f'^* f'_! \arrow[r, "BC[\tau]"] \arrow[d, "\nu_{f'}"] & f^* g_{Y!} f'_! \arrow[r, "\sim"] & f^* f_! g_{X!} \arrow[d, "\nu_f"] \\ g_{X!} \arrow[rr, "\mathrm{id}"] & & g_{X!} \end{tikzcd}$$

commutes up to homotopy.

(2) *If f is ambidextrous, then f' is ambidextrous, and the diagram of natural transformations*

$$\begin{tikzcd} g_{Y!} \arrow[rr, "\mathrm{id}"] \arrow[d, "\mu_{f'}"] & & g_{Y!} \arrow[d, "\mu_f"] \\ g_{Y!} f'_! f'^* \arrow[r, "\sim"] & f_! g_{X!} f'^* \arrow[r, "BC[\tau]"] & f_! f^* g_{Y!} \end{tikzcd}$$

commutes up to homotopy.

Proof. To prove (1), we use the adjointness between $g_{X!}$ and g_X^* to obtain homotopy equivalences

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_{X'}, \mathcal{C}_X)}(g_{X!} f'^* f'_!, g_{X!}) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_X, \mathcal{C}_X}(f'^* f'_!, g_X^* g_{X!}) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_{X'}, \mathcal{C}_X)}(f'^* f'_! g_X^*, g_X^*).$$

Under this homotopy equivalence, the two natural transformations appearing in the statement of (1) correspond to the two natural transformations appearing in the first assertion of Proposition 4.2.1. It then follows from Proposition 4.2.1 that these natural transformations are homotopic. The proof of (2) is similar. $\square$

We now turn to the proofs of Propositions 4.2.1 and 4.2.2.

Proof of Proposition 4.2.1. It follows immediately from our definitions that if f is (weakly) ambidextrous, then f' is also (weakly) ambidextrous. If f is weakly ambidextrous, then f is n-truncated for some integer $n \geq -2$. We proceed by induction on n. In the case $n = -2$, the maps f and f' are equivalences and assertions (1) and (2) are easy. We will therefore assume that $n \geq -1$. To prove (1), let us assume that f and f' are weakly ambidextrous, so that we have a pullback square ρ :

$$\begin{tikzcd} X' \arrow[r, "\delta'"] \arrow[d, "g_X"] & X' \times_{Y'} X' \arrow[d, "g_\pi"] \\ X \arrow[r, "\delta"] & X \times_Y X \end{tikzcd}$$

where δ and δ' are $(n-1)$-truncated and ambidextrous. Let σ denote the pullback square

$$\begin{tikzcd} X \times_Y X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

let σ' denote the pullback square

$$\begin{tikzcd} X' \times_{Y'} X' \arrow[r, "\pi'_1"] \arrow[d, "\pi'_2"] & X' \arrow[d, "f'"] \\ X' \arrow[r, "f'"] & Y', \end{tikzcd}$$

and let ν denote the pullback square

$$\begin{tikzcd}
X' \times_{Y'} X' \arrow[r, "\pi_1"] \arrow[d, "g_\pi"] & X' \arrow[d, "g_X"] \\
X \times_Y X \arrow[r, "\pi_1"] & X,
\end{tikzcd}$$

and consider the diagram of natural transformations

$$\begin{tikzcd}
f'^* f'_! g_X^* \arrow[r, "BC[\sigma']^{-1}"] \arrow[dd, "BC[\tau]"] & \pi'_{1!}\pi_2'^* g_X^* \arrow[r, "\mu_{\delta'}"] \arrow[dd, "\sim"] & \pi'_{1!}\delta'_!\delta'^*\pi_2'^* g_X^* \arrow[r, "\sim"] \arrow[d] & g_X^* \arrow[dd, "\mathrm{id}"] \\
 & & \pi'_{1!}\delta'_!\delta'^* g_\pi^* \pi_2^* \arrow[d, "\sim"] & \\
f'^* g_Y^* f_! \arrow[dd, "\sim"] & \pi'_{1!} g_\pi^* \pi_2^* \arrow[ur, "\mu_{\delta'}"] \arrow[dr, "\mu_\delta"] \arrow[dd, "BC[\nu]"] & \pi'_{1!}\delta'_! g_X^* \delta^* \pi_2^* \arrow[r, "\sim"] \arrow[d, "BC[\rho]"] & g_X^* \arrow[dd] \\
 & & \pi'_{1!} g_\pi^* \delta_! \delta^* \pi_2^* \arrow[d, "BC[\nu]"] & \\
g_X^* f^* f_! \arrow[r, "BC[\sigma]^{-1}"] & g_X^* \pi_{1!}\pi_2^* \arrow[r, "\mu_\delta"] & g_X^* \pi_{1!}\delta_!\delta^*\pi_2^* \arrow[r, "\sim"] & g_X^*.
\end{tikzcd}$$

We wish to prove that the outer rectangle of this diagram commutes up to homotopy. In fact, we claim that the entire diagram commutes up to homotopy. For all parts of the diagram except for the middle and the square on the left, this is routine. The middle part commutes by virtue of the inductive hypothesis. To prove the commutativity of the rectangle on the left, it suffices to show that the diagram

$$\begin{tikzcd}
f'^* f'_! g_X^* \arrow[d, "BC[\tau]"] & \pi'_{1!}\pi_2'^* g_X^* \arrow[l, "BC[\sigma']"'] \arrow[d, "\sim"] \\
f'^* g_Y^* f_! \arrow[d, "\sim"] & \pi'_{1!} g_\pi^* \pi_2^* \arrow[d, "BC[\nu]"] \\
g_X^* f^* f_! & g_X^* \pi_{1!}\pi_2^* \arrow[l, "BC[\sigma]"']
\end{tikzcd}$$

commutes up to homotopy. This is clear, because both compositions can be associated with the Beck-Chevalley transformation associated to the pullback square

$$\begin{tikzcd}
X' \times_{Y'} X' \arrow[r, "\pi'_1"] \arrow[d] & X' \arrow[d] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

We now prove (2). Assume that f is ambidextrous, so that the natural transformations

$$\nu_f : f^* f_! \to \mathrm{id}_{\mathcal{C}_X} \qquad \nu_{f'} : f'^* f'_! \to \mathrm{id}_{\mathcal{C}_{X'}}$$

are counits of adjunctions. It follows that the composite map

$$\alpha : \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_Y, \mathcal{C}_{Y'})}(g_Y^*, g_Y^* f_! f^*) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_X, \mathcal{C}_{Y'})}(g_Y^* f_!, g_Y^* f_! f^* f_!) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_X, \mathcal{C}_{Y'})}(g_Y^* f_!, g_Y^* f_!)$$

is a homotopy equivalence, where the first map is given by precomposition with $f_!$ and the second map is given by composition with μ_f. We note that the natural transformation $g_Y^* \to g_Y^* f_! f^*$ determined by μ_f can be identified with an inverse image of the identity transformation under the equivalence α. Consequently, the commutativity of the diagram appearing in (2) is equivalent to the assertion that the composite map

$$g_Y^* f_! \xrightarrow{\mu_{f'}} f'_! f'^* g_Y^* f_! \simeq f'_! g_X^* f^* f_! \xrightarrow{BC[\tau]} g_Y^* f_! f^* f_! \xrightarrow{\nu_f} g_Y^* f_!$$

is homotopic to the identity. To prove this, consider the diagram

$$\begin{tikzcd}
g_Y^* f_! \arrow[r, "\mu_{f'}"] & f'_! f'^* g_Y^* f_! \arrow[r, "\sim"] & f_! g_X^* f^* f_! \arrow[r, "BC[\tau]"] \arrow[dr, "v_f"] & g_Y^* f_! f^* f_! \arrow[r, "\nu_f"] & g_Y^* f_! \\
f'_! g'_X \arrow[u, "BC[\tau]"] \arrow[r, "\mu_{f'}"] & f'_! f'^* f'_! g_X^* \arrow[u, "BC[\tau]"] \arrow[rr, "\nu_{f'}"] & & f'_! g_X^* \arrow[ur, "BC[\tau]"'] &
\end{tikzcd}$$

The middle trapezoidal diagram commutes up to homotopy by (1), and the outer parts of the diagram obviously commute up to homotopy. Since q is a Beck-Chevalley fibration, the map $BC[\tau]$ is an equivalence. Consequently, to prove that composition appearing in the upper part of the diagram is homotopic to the identity, it suffices to show that the composition appearing in the lower part of the diagram is homotopic to the identity. This follows immediately from the compatibility between the unit $\mu_{f'}$ with the counit $\nu_{f'}$. □

Proof of Proposition 4.2.2. If f and g are weakly ambidextrous, then there exists an integer n such that both f and g are n-truncated. We will prove (1) and (2) by a simultaneous induction on n. If $n = -2$ (so that both f and g are equivalences) then the result is obvious. Let us therefore assume that $n > -2$. We first prove (1). Assume that f and g weakly ambidextrous. Let $\delta(f) : X \to X \times_Y X$ and $\delta(g) : Y \to Y \times_Z Y$ denote the diagonal maps determined by Y and Z. Then $\delta(f)$ and $\delta(g)$ are ambidextrous. We will also consider the ambidextrous maps

$${}_X\delta(g) : X \to X \times_Z Y \qquad {}_X\delta(g)_X : X \times_Y X \to X \times_Z X \qquad \delta(g)_X : X \to Y \times_Z X$$

given by base change of $\delta(g)$. The diagonal map

$$\delta(gf) : X \to X \times_Z X$$

is given by the composition ${}_X\delta(g)_X \circ \delta(f)$. It follows from the inductive hypothesis that $\delta(gf)$ is ambidextrous. Moreover, we deduce that the unit map $\mu_{\delta(gf)} : \mathrm{id} \to \delta(gf)_! \delta(gf)^*$ is given by the composition

$$\mathrm{id} \xrightarrow{\mu_h} {}_X\delta(g)_{X!}\, {}_X\delta(g)_X^* \xrightarrow{\mu_{\delta(f)}} {}_X\delta(g)_{X!}\, \delta(f)_! \delta(f)^*\, {}_X\delta(g)_X^* \simeq \delta(gf)_! \delta(gf)^*$$

where $h = {}_X\delta(g)_X$.

Let ρ denote the pullback diagram

$$\begin{tikzcd}
X \times_Y X \arrow[r, "\pi'_1"] \arrow[d, "\pi'_2"] & X \arrow[d, "f"] \\
X \arrow[r, "f"] & Y,
\end{tikzcd}$$

and consider also the commutative diagram

$$\begin{tikzcd}
X \times_Z X \arrow[r, "{}_X f"] \arrow[d, "f_X"] & X \times_Z Y \arrow[r, "{}_X \pi_1"] \arrow[d, "f_Y"] & X \arrow[d, "f"] \\
Y \times_Z X \arrow[r, "{}_Y f"] \arrow[d, "{}_X \pi_2"] & Y \times_Z Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z.
\end{tikzcd}$$

We will denote the lower right square of this diagram by σ, the upper right square by σ', and the rectangle on the left by τ. Consider the diagram of natural transformations (in the ∞-category $\mathrm{Fun}(\mathcal{C}_X, \mathcal{C}_X)$)

$$\begin{tikzcd}
f^* g^* g_! f_! \arrow[d, "BC[\sigma]^{-1}"] & & & \\
f^* \pi_{1!} \pi_2^* f_! \arrow[r, "\mu_{\delta(g)}"] \arrow[d, "BC[\sigma']^{-1}"] & f^* \pi_{1!} \delta(g)_! \delta(g)^* \pi_2^* f_! \arrow[r, "\sim"] \arrow[d, "BC[\sigma']^{-1}"] & f^* \delta(g)^* \pi_2^* f_! \arrow[r, "\sim"] \arrow[dd, "\sim"] & f^* f_! \arrow[dddd, "BC[\rho]^{-1}"] \\
{}_X\pi_{1!} f_Y^* \pi_2^* f_! \arrow[r, "\mu_{\delta(g)}"] \arrow[dr, "\mu_{X\delta(g)}"] \arrow[dd, "BC[\tau]^{-1}"] & {}_X\pi_{1!} f_Y^* \delta(g)_! \delta(g)^* \pi_2^* f_! \arrow[d, "\sim"] & & \\
& {}_X\pi_{1!X} \delta(g)_{!X} \delta(g)^* f_Y^* \pi_2^* f_! \arrow[r, "\sim"] \arrow[d, "BC[\tau]^{-1}"] & {}_X\delta(g)^* f_Y^* \pi_2^* f_! \arrow[dd, "BC[\rho]^{-1}"] & \\
{}_X\pi_{1!X} f_! f_{XX}^* \pi_2^* \arrow[r, "\mu_{X\delta(g)}"] \arrow[dr, "\mu_{X\delta(g)X}"] & {}_X\pi_{1!X} \delta(g)_{!X} \delta(g)^*_{XX} f_! f_X^* \pi_2^* \arrow[d, "\sim"] & & \\
& {}_X\pi_{1!X} f_{!X} \delta(g)_{X!X} \delta(g)^*_X f_{XX}^* \pi_2^* \arrow[r, "\sim"] & \pi'_{1!} \delta(g)^* f_X^* \pi_2^* \arrow[r] & \pi'_{1!} \pi'^*_2 \arrow[d, "\mu_{\delta(f)}"] \\
& & & \pi'_{1!} \delta(f)_! \delta(f)^* \pi'^*_2 \arrow[d, "\sim"] \\
& & & \mathrm{id}_{\mathcal{C}_X} .
\end{tikzcd}$$

Using the formula for $\mu_{\delta(gf)}$ given by the inductive hypothesis (and the transitivity properties of Beck-Chevalley transformations), we see that ν_{gf} is given by a a counterclockwise circuit around the diagram, from the upper left corner to the lower right corner. On the other hand, the composition

$$(gf)^*(gf)_! \simeq f^* g^* g_! f_! \xrightarrow{v_g} f^* f_! \xrightarrow{\nu_f} \mathrm{id}$$

is given by a clockwise circuit around the same diagram. It therefore suffices to prove that the diagram commutes. This follows by inspection except in the case of the two triangles, which commute by virtue of Proposition 4.2.1 and Corollary 4.2.6.

We now prove (2). Assume that f and g are ambidextrous; we wish to show that $g \circ f$ is ambidextrous. Let $Z' \to Z$ be any morphism in $\mathcal{X}$, and form a pullback diagram

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[r, "g'"] \arrow[d] & Z' \arrow[d] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z.
\end{tikzcd}$$

We wish to prove that $g' \circ f'$ is ambidextrous. Replacing Z by Z', we are reduced to proving that $g \circ f$ is ambidextrous. It follows from (1) that $g \circ f$ is weakly ambidextrous. Consider the composite map

$$\mathrm{id}_{\mathcal{C}_Z} \xrightarrow{\mu_g} g_! g^* \xrightarrow{\mu_f} g_! f_! f^* g^* \simeq (gf)_!(gf)^*.$$

We will complete the proof by showing that μ and ν_{gf} are compatible unit and counits for an adjunction between $(gf)^*$ and $(gf)_!$. In other words, we claim that the composite maps

$$(gf)^* \xrightarrow{\mu} (gf)^*(gf)_!(gf)^* \xrightarrow{\nu_{gf}} (gf)^*$$

$$(gf)_! \xrightarrow{\mu} (gf)_!(gf)^*(gf)_! \xrightarrow{\nu_{gf}} (gf)_!$$

are homotopic to the identity transformations. We will prove that the first composition is homotopic to the identity; the proof in the second case is similar. Unwinding the definitions and using the description of ν_{gf} supplied by (1), we see that the relevant composition is given by a clockwise circuit around the diagram

$$\begin{tikzcd}
f^*g^* \arrow[r, "\mu_g"] \arrow[dr, "\mathrm{id}"'] & f^*g^*g_!g^* \arrow[r, "\mu_f"] \arrow[d, "\nu_g"] & f^*g^*g_!f_!f^*g^* \arrow[d, "\nu_g"] \\
& f^*g^* \arrow[r, "\mu_f"] \arrow[dr, "\mathrm{id}"'] & f^*f_!f^*g^* \arrow[d, "\nu_f"] \\
& & f^*g^*.
\end{tikzcd}$$

The desired result now follows from the fact that this diagram commutes up to homotopy (the commutativity of the triangles follows from the compatibility of the pairs (μ_f, ν_f) and (μ_g, ν_g)). □

4.3 Local Systems

In §4.1, we described the theory of ambidexterity associated to an arbitrary Beck-Chevalley fibration $q : \mathcal{Y} \to \mathcal{X}$. In this section, we will specialize to the case where $\mathcal{X} = \mathcal{S}$ is the ∞-category of spaces, and the fiber of q over an object $X \in \mathcal{X}$ is the ∞-category of local systems on X with values in some ambient ∞-category $\mathcal{C}$. Our main result (Proposition 4.3.5) asserts that in this case, the ambidexterity of a map $f : X \to Y$ can be regarded as a condition on the homotopy fibers of f.

We begin by introducing some notation.

Construction 4.3.1. Let $\mathcal{C}$ be an ∞-category. The construction $X \mapsto \mathrm{Fun}(X, \mathcal{C})$ determines a simplicially enriched functor from the (opposite of the) category of Kan complexes to the category of ∞-categories. Passing to the coherent nerve, we obtain a functor of ∞-categories $\mathrm{Fun}(\bullet, \mathcal{C}) : \mathcal{S}^{\mathrm{op}} \to \mathcal{C}$. We let $q : \mathrm{LocSys}(\mathcal{C}) \to \mathcal{S}$ denote a Cartesian fibration classified by this functor. More informally, we let $\mathrm{LocSys}(\mathcal{C})$ denote the ∞-category whose objects are pairs $(X, \mathcal{L})$, where X is a Kan complex and $\mathcal{L} \in \mathrm{Fun}(X, \mathcal{C})$ is a local system on X with values in $\mathcal{C}$.

For each object $X \in \mathcal{S}$, the inverse image $\mathrm{LocSys}(\mathcal{C})_X = q^{-1}\{X\}$ is canonically equivalent to the ∞-category $\mathrm{Fun}(X, \mathcal{C})$ of $\mathcal{C}$-valued local systems on X. For every map of Kan complexes $f : X \to Y$, the pullback functor $f^* : \mathrm{LocSys}(\mathcal{C})_Y \to \mathrm{LocSys}(\mathcal{C})_X$ can be identified with the functor $\mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ given by composition with f.

Remark 4.3.2. Let $\mathcal{C}$ be an ∞-category which admits small colimits. Then for every functor $f : \mathcal{I} \to \mathcal{J}$ between small ∞-categories, the associated functor $f^* : \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ admits a left adjoint $f_!$, given by left Kan extension along f (see §HTT.4.3.3). It follows in particular that the Cartesian fibration $\mathrm{LocSys}(\mathcal{C}) \to \mathcal{S}$ is also a coCartesian fibration (see Corollary HTT.5.2.2.5).

Proposition 4.3.3. *Let $\mathcal{C}$ be an ∞-category which admits small colimits. Then the forgetful functor*

$$\mathrm{LocSys}(\mathcal{C}) \to \mathcal{S}$$

is a Beck-Chevalley fibration (Definition 4.1.3).

Proof. Suppose we are given a pullback diagram σ :

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g_X"'] & Y' \arrow[d, "g_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

in $\mathcal{S}$. Unwinding the definitions, we must prove the following: if we are given local systems $\mathcal{L}_X : X \to \mathcal{C}$ and $\mathcal{L}_Y : Y \to \mathcal{C}$ and a natural transformation $\alpha : \mathcal{L}_X \to \mathcal{L}_Y \circ f$ which exhibits $\mathcal{L}_Y$ as a left Kan extension of $\mathcal{L}_X$ along f, then the induced natural transformation $\beta : \mathcal{L}_X \circ g_X \to \mathcal{L}_Y \circ g_Y \circ f'$ exhibits $\mathcal{L}_Y \circ g_Y$ as a left Kan extension of $\mathcal{L}_X \circ g_X$ along f'. Fix a point $y' \in Y'$. We wish to prove that the canonical map

$$\theta : \varinjlim((\mathcal{L}_X \circ g_X)|(X' \times_{Y'} Y'_{/y'}) \to (\mathcal{L}_Y \circ g_Y)(y')$$

is an equivalence. Since σ is a pullback diagram, we can identify θ with the canonical map

$$\varinjlim(\mathcal{L}_X \,|X \times_Y Y_{/y}) \to \mathcal{L}_Y(y),$$

where $y = g_Y(y')$ denotes the image of y' in Y. The desired result now follows immediately from the assumption that α exhibits $\mathcal{L}_Y$ as a left Kan extension of $\mathcal{L}_X$ along f. □

Definition 4.3.4. Let $\mathcal{C}$ be an ∞-category which admits small colimits and let $q : \mathrm{LocSys}(\mathcal{C}) \to \mathcal{S}$ denote the Beck-Chevalley fibration of Proposition 4.3.3. We will say that a Kan complex X is *weakly $\mathcal{C}$-ambidextrous* if the projection map $X \to *$ is weakly ambidextrous in the sense of Definition 4.1.11 (for the Beck-Chevalley fibration q). We will say that X is *$\mathcal{C}$-ambidextrous* if it is weakly $\mathcal{C}$-ambidextrous and the natural transformation $\nu_f : f^* f_! \to \mathrm{id}_{\mathrm{Fun}(X,\mathcal{C})}$ is the counit of an adjunction.

We are now ready to state our main result:

Proposition 4.3.5. *Let $\mathcal{C}$ be an ∞-category which admits small colimits, and let $f : X \to Y$ be a Kan fibration between Kan complexes. Then:*

(1) *The map f is ambidextrous (for the Beck-Chevalley fibration $q : \mathrm{LocSys}(\mathcal{C}) \to \mathcal{S}$) if and only if it is n-truncated for some integer n and each fiber X_y of f is $\mathcal{C}$-ambidextrous.*

(2) *The map f is weakly ambidextrous (for the Beck-Chevalley fibration $q : \mathrm{LocSys}(\mathcal{C}) \to \mathcal{S}$) if and only if it is n-truncated for some integer n and each fiber X_y of f is weakly $\mathcal{C}$-ambidextrous.*

Corollary 4.3.6. *Let $\mathcal{C}$ be an ∞-category which admits small colimits, and let $f : X \to Y$ be a Kan fibration between truncated spaces. If Y is $\mathcal{C}$-ambidextrous and each fiber X_y of f is $\mathcal{C}$-ambidextrous, then X is $\mathcal{C}$-ambidextrous.*

Proof. Combine Propositions 4.2.2 and 4.3.5. □

Remark 4.3.7. Let $\mathcal{C}$ be an ∞-category which admits small colimits. Using Proposition 4.3.5, we see that a Kan complex X is weakly $\mathcal{C}$-ambidextrous if and only if, for every pair of vertices $x, y \in X$, the path space $P_{x,y}$ is $\mathcal{C}$-ambidextrous. In particular:

- If X is empty, then it is automatically weakly $\mathcal{C}$-ambidextrous.
- If X is connected, then it is weakly $\mathcal{C}$-ambidextrous if and only if the loop space $\Omega(X)$ (formed with any choice of base point) is $\mathcal{C}$-ambidextrous.
- If X has more than one connected component, then it is weakly $\mathcal{C}$-ambidextrous if and only if the empty space $\emptyset$ is $\mathcal{C}$-ambidextrous (this is equivalent to the assumption that the ∞-category $\mathcal{C}$ is pointed: see Remark 4.4.6) and the loop space $\Omega(X)$ is $\mathcal{C}$-ambidextrous, for every choice of base point $x \in X$.

Our proof of Proposition 4.3.5 will require the following simple observation:

Lemma 4.3.8. *Let $\mathcal{C}$ be an ∞-category which admits small colimits and let X be a Kan complex. Then* $\mathrm{Fun}(X, \mathcal{C})$ *is generated (under small colimits) by objects of the form $i_! C$, where $i : \{x\} \to X$ is the inclusion of a vertex and $C \in \mathcal{C} \simeq$* $\mathrm{Fun}(\{x\}, \mathcal{C})$.

Proof. Let $\delta : X \to X \times X$ denote the diagonal map and let $p : X \times X \to X$ denote the projection onto the second factor. Then left Kan extension along δ and p determine functors

$$\delta_! : \mathrm{Fun}(X, \mathcal{C}) \to \mathrm{Fun}(X \times X, \mathcal{C}) \qquad p_! : \mathrm{Fun}(X \times X, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C}),$$

and the composition $p_! \circ \delta_!$ is homotopic to the identity. Note that we can identify $\mathrm{Fun}(X \times X, \mathcal{C})$ with the ∞-category of diagrams $\mathrm{Fun}(X, \mathrm{Fun}(X, \mathcal{C}))$, and that the functor $p_!$ is given by the formation of colimits of X-indexed diagrams in $\mathrm{Fun}(X, \mathcal{C})$. It follows that every object $\mathcal{L} \in \mathrm{Fun}(X, \mathcal{C})$ is given by the colimit of a diagram $q : X \to \mathrm{Fun}(X, \mathcal{C})$, where $q(x) = (\delta_! X)|(\{x\} \times X)$. Applying Proposition 4.3.3 to the homotopy pullback square

$$\begin{tikzcd}
\{x\} \arrow[r] \arrow[d] & \{x\} \times X \arrow[d] \\
X \arrow[r, "\delta"] & X \times X,
\end{tikzcd}$$

we can identify $q(x)$ with the left Kan extension $i_{x!} \mathcal{L}(x)$, where $i_x : \{x\} \to X$ is the inclusion map and $\mathcal{L}(x) \in \mathcal{C} \simeq \mathrm{Fun}(\{x\}, \mathcal{C})$ is the value of $\mathcal{L}$ at the point x. □

Proof of Proposition 4.3.5. The "only if" directions of (1) and (2) are clear. To prove the reverse implications, we may suppose that f is n-truncated for some integer n. We prove (1) and (2) by a simultaneous induction on n.

If $n = -2$ then f is an equivalence and there is nothing to prove. Suppose now that $n > -2$ and that each fiber of f is weakly $\mathcal{C}$-ambidextrous. Let $\delta : X \to X \times_Y X$ be the diagonal map. We wish to show that δ is ambidextrous. Since δ is $(n-1)$-truncated, it will suffice (by virtue of the inductive hypothesis) to show that each homotopy fiber of δ is $\mathcal{C}$-ambidextrous. Fix a vertex of $X \times_Y X$, which we can identify with a pair of vertices $(x, x') \in X \times X$ having the same image $y \in Y$. Let $P_{x,x'}$ denote the homotopy fiber of δ over the point (x, x'), which we can identify with the space of paths from x to x' in the Kan complex X_y. Since X_y is weakly $\mathcal{C}$-ambidextrous, the diagonal map $X_y \to X_y \times X_y$ is ambidextrous. Using the homotopy pullback diagram

$$\begin{tikzcd}
P_{x,x'} \arrow[r] \arrow[d] & \{(x, x')\} \arrow[d] \\
X_y \arrow[r] & X_y \times X_y,
\end{tikzcd}$$

we deduce that $P_{x,x'}$ is $\mathcal{C}$-ambidextrous as desired.

Now suppose that f is n-truncated and that fibers of f are $\mathcal{C}$-ambidextrous; we wish to show that f is ambidextrous. It follows from the previous step that f is weakly ambidextrous. Replacing f by a pullback of f, we are reduced to showing that the map $\nu_f : f^* f_! \to \mathrm{id}$ is the counit of an adjunction between the functors f^* and $f_!$. Fix a pair of local systems $\mathcal{L}_X : X \to \mathcal{C}$ and $\mathcal{L}_Y : Y \to \mathcal{C}$; we wish to show that the composite map

$$\alpha : \mathrm{Map}_{\mathrm{Fun}(Y,\mathcal{C})}(\mathcal{L}_Y, f_! \mathcal{L}_X) \to \mathrm{Map}_{\mathrm{Fun}(X,\mathcal{C})}(f^* \mathcal{L}_Y, f^* f_! \mathcal{L}_X) \xrightarrow{\nu_f} \mathrm{Map}_{\mathrm{Fun}(X,\mathcal{C})}(f^* \mathcal{L}_Y, \mathcal{L}_X)$$

is a homotopy equivalence. The collection of those local systems $\mathcal{L}_Y$ for which this condition is satisfied is closed under small colimits. We may therefore use Lemma 4.3.8 to reduce to the case where $\mathcal{L}_Y = i_! C$, where $i : \{y\} \to Y$ denotes the inclusion of a vertex and $C \in \mathcal{C}$ is a fixed object. Let $f' : X_y \to \{y\}$ denote the projection map. We have a diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Fun}(Y,\mathcal{C})}(\mathcal{L}_Y, f_! \mathcal{L}_X) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Fun}(X,\mathcal{C})}(f^* \mathcal{L}_Y, f^* f_! \mathcal{L}_X) \arrow[r, "\nu_f"] \arrow[d] & \mathrm{Map}_{\mathrm{Fun}(X,\mathcal{C})}(f^* \mathcal{L}_Y, \mathcal{L}_X) \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(C, (f_! \mathcal{L}_X)(y)) \arrow[r] & \mathrm{Map}_{\mathrm{Fun}(X_y,\mathcal{C})}(f'^* C, f'^*(f_! \mathcal{L}_X)(y)) \arrow[r, "\nu_{f'}"] & \mathrm{Map}_{\mathrm{Fun}(X_y,\mathcal{C})}(f'^* C, \mathcal{L}_X | X_y)
\end{tikzcd}$$

where the commutativity of the right square follows from Proposition 4.2.1. It follows from Proposition 4.3.3 that the vertical map on the right is a homotopy equivalence. We are therefore reduced to showing that the lower horizontal composition is a homotopy equivalence, which follows immediately from our assumption that the fiber X_y is $\mathcal{C}$-ambidextrous. □

We now use Lemma 4.3.8 to give an alternate characterization of $\mathcal{C}$-ambidextrous spaces, which does make explicit mention of the natural transformations ν.

Proposition 4.3.9. *Let $\mathcal{C}$ be an ∞-category which admits small colimits, let X be a truncated Kan complex, and let $f : X \to *$ be the projection from X to a point. Then X is $\mathcal{C}$-ambidextrous if and only if the following conditions are satisfied:*

(1) *The Kan complex X is weakly $\mathcal{C}$-ambidextrous (that is, each path space $P_{x,y} = \mathrm{Fun}(\Delta^1, X) \times_{\mathrm{Fun}(\partial\,\Delta^1, X)} \{(x, y)\}$ is $\mathcal{C}$-ambidextrous: see Remark 4.3.7).*

(2) *The pullback functor $f^* : \mathcal{C} \to \mathrm{Fun}(X, \mathcal{C})$ admits a right adjoint f_*.*

(3) *The functor f_* preserves small colimits.*

Proof. Suppose X is $\mathcal{C}$-ambidextrous. Then condition (1) is obvious, and conditions (2) and (3) follow from the observation that $f_!$ is right adjoint to f^*. Conversely, suppose that conditions (1), (2), and (3) are satisfied. Using condition (2), we can identify $\nu_f : f^* f_! \to \mathrm{id}_{\mathrm{Fun}(X,\mathcal{C})}$ with a norm map $\mathrm{Nm}_X : f_! \to f_*$ (see Remark 4.1.12); we wish to show that Nm_f is an equivalence. For this, it suffices to show that for every local system $\mathcal{L} : X \to \mathcal{C}$, the induced map $f_! \mathcal{L} \to f_* \mathcal{L}$ is an equivalence in $\mathcal{C}$. The collection of those objects $\mathcal{L} \in \mathrm{Fun}(X, \mathcal{C})$ for which this condition is satisfied is closed under small colimits in $\mathrm{Fun}(X, \mathcal{C})$ (by virtue of condition (3)). Using Lemma 4.3.8, we can assume that $\mathcal{L} = i_! C$, where $C \in \mathcal{C}$ is an object and $i : \{x\} \to X$ is the inclusion of a point. Assumption (1) guarantees that i is ambidextrous. Using Remark 4.2.4, we see that the composite map

$$C \simeq (fi)_! C \simeq f_! \mathcal{L} \xrightarrow{\mathrm{Nm}_f} f_* \mathcal{L} \xrightarrow{\mathrm{Nm}_i} f_* i_* C \simeq C$$

is homotopic to the identity. Since i is ambidextrous, the map Nm_i is an equivalence; it follows that Nm_f is an equivalence as well. □

Remark 4.3.10. In the situation of Proposition 4.3.9, suppose that the ∞-category $\mathcal{C}$ admits small limits (this is satisfied, for example, if $\mathcal{C}$ is presentable). Then hypothesis (2) is automatically satisfied. Moreover, we can replace hypothesis (3) by the following variant:

(3′) The functor $f_!$ preserves small limits.

Assuming (3′), the collection of those objects $\mathcal{L} \in \mathrm{Fun}(X, \mathcal{C})$ for which the norm map Nm_f induces an equivalence $f_! \mathcal{L} \to f_* \mathcal{L}$ is closed under small limits. Invoking the dual of Lemma 4.3.8, we can reduce to the case where $\mathcal{L} = i_* C$ where i is the inclusion of a point $x \in X$ and $C \in \mathcal{C}$ is some object. We have a commutative diagram

$$\begin{tikzcd} f_! i_! C \arrow[r, "\mathrm{Nm}_f"] \arrow[d, "\mathrm{Nm}_i"] & f_* i_! C \arrow[d, "\mathrm{Nm}_i"] \\ f_! i_* C \arrow[r] & f_* i_* C. \end{tikzcd}$$

Since i is ambidextrous, the vertical maps are equivalences, and the diagonal map is an equivalence by Remark 4.2.4. It follows that the lower horizontal map is also an equivalence.

Example 4.3.11. Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the ∞-category whose objects are presentable ∞-categories and whose morphisms are functors which preserve small colimits. Let X be an arbitrary simplicial set equipped with a map $\chi : X \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$, classifying a coCartesian fibration $q : \overline{X} \to X$ with presentable fibers. Using Propositions HTT.5.5.3.13 and HTT.3.3.3.1, we can identify $\varprojlim \chi$ with the full subcategory of $\mathrm{Fun}_X(X, \overline{X})$ spanned by those sections of q which carry each edge of X to a q-coCartesian edge of $\overline{X}$. In the special case where X is a Kan complex, this condition is automatic so that $\varprojlim \chi \simeq \mathrm{Fun}_X(X, \overline{X})$. In this case, we can also view χ as a functor $\chi' : X \to \mathcal{P}\mathrm{r}^{\mathrm{R}}$, where $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ denotes the ∞-category of presentable ∞-categories and functors which are accessible and preserve small limits. Using Corollary HTT.5.5.3.4, Theorem HTT.5.5.3.18, and Proposition HTT.3.3.3.1, we obtain an equivalence of ∞-categories $\varinjlim \chi \simeq \varprojlim \chi' \simeq \mathrm{Fun}_X(X, \overline{X})$. Combining these observations, we obtain a canonical equivalence $\varinjlim \chi \simeq \varprojlim \chi$. This equivalence is natural in χ. If $f : X \to *$ is the projection map, we obtain an equivalence α between the functors $f_*, f_! : \mathrm{Fun}(X, \mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$. In particular, the functor f_* preserves small colimits and the functor $f_!$ preserves small limits. Repeatedly applying Proposition 4.3.9, we deduce that every truncated space X is $\mathcal{P}\mathrm{r}^{\mathrm{L}}$-ambidextrous. With more effort, one can show that the equivalence $\alpha : f_! \simeq f_*$ constructed above is homotopic to the norm map Nm_f of Remark 4.1.12.

4.4 Examples

Let $\mathcal{C}$ be an ∞-category which admits small limits. In §4.3, we introduced the notion of a $\mathcal{C}$-ambidextrous space. In this section, we will study conditions on $\mathcal{C}$ which guarantee the existence of a good supply of $\mathcal{C}$-ambidextrous spaces.

Definition 4.4.1. Let X be a space. We will say that X is a *finite n-type* if the following conditions are satisfied:

(1) The space X is n-truncated: that is, the homotopy groups $\pi_m(X, x)$ vanish for $m > n$ (for every choice of base point $x \in X$).

(2) For every point $x \in X$ and every integer m, the set $\pi_m(X, x)$ is finite.

Definition 4.4.2. Let $\mathcal{C}$ be an ∞-category which admits small colimits, and let $n \geq -2$ be an integer. We will say that $\mathcal{C}$ is *n-semiadditive* if every finite n-type X is $\mathcal{C}$-ambidextrous.

Remark 4.4.3. Let $n \geq -2$ be an integer, and let $\mathcal{C}$ be an n-semiadditive ∞-category. For every finite n-type X, let $p : X \to *$ denote the projection map from X to a point, and let μ_X denote the natural transformation $\mu_p : \mathrm{id}_{\mathcal{C}} \to p_! p^*$ of Definition 4.1.11. For every pair of objects $C, D \in \mathcal{C}$, the construction described in Notation 4.1.6 determines a map of spaces

$$d\mu_X : \mathrm{Fun}(X, \mathrm{Map}_{\mathcal{C}}(C, D)) \to \mathrm{Map}_{\mathcal{C}}(C, D),$$

which we will denote by $f \mapsto \int_X f d\mu_X$. This is our motivation for the terminology introduced in Definition 4.4.2: an n-semiadditive ∞-category is an ∞-category $\mathcal{C}$ in which is it possible to "add" a collection of morphisms parametrized by a finite n-type.

Remark 4.4.4. Let X be a finite $(n+1)$-type. For every pair of vertices $x, y \in X$, the path space $P_{x,y}$ is a finite n-type. It follows from Remark 4.3.7 that if $\mathcal{C}$ is an n-semiadditive ∞-category, then X is weakly $\mathcal{C}$-ambidextrous.

Example 4.4.5. A Kan complex X is a finite (-2)-type if and only if it is contractible. It follows that every ∞-category $\mathcal{C}$ which admits small colimits is (-2)-semiadditive.

Remark 4.4.6. Let $\mathcal{C}$ be an ∞-category which admits small colimits. The empty Kan complex $\emptyset$ is always weakly $\mathcal{C}$-ambidextrous (since the diagonal map $\emptyset \to \emptyset \times \emptyset$ is an isomorphism). Note that $\mathrm{Fun}(\emptyset, \mathcal{C}) \simeq \Delta^0$ has a unique object, which we will denote by E. Let $f : \emptyset \to *$ be the inclusion, so that the pullback functor

$$f^* : \mathcal{C} \simeq \mathrm{Fun}(*, \mathcal{C}) \to \mathrm{Fun}(\emptyset, \mathcal{C}) \simeq \Delta^0$$

is the constant map and its left adjoint $f_! : \mathrm{Fun}(\emptyset, \mathcal{C}) \to \mathcal{C}$ carries E to an initial object of $\mathcal{C}$. Unwinding the definitions, we see that the empty Kan complex is $\mathcal{C}$-ambidextrous if and only if the identification

$\nu_f : f^* \circ f_! \simeq \mathrm{id}$ is the counit of an adjunction. This is equivalent to the assertion that for every object $C \in \mathcal{C}$, the canonical map

$$\mathrm{Map}_{\mathcal{C}}(C, f_!(E)) \to \mathrm{Map}_{\mathrm{Fun}(\emptyset,\mathcal{C})}(f^*C, f^*f_!(E)) \xrightarrow{\nu_f} \mathrm{Map}_{\mathrm{Fun}(\emptyset,\mathcal{C})}(f^*C, E) \simeq *$$

is a homotopy equivalence. In other words, $\emptyset$ is $\mathcal{C}$-ambidextrous if and only if $\mathcal{C}$ is a pointed ∞-category: that is, the initial object of $\mathcal{C}$ is also a final object.

Example 4.4.7. Note that a Kan complex X is a finite (-1)-type if and only if it is either empty or contractible. Consequently, if $\mathcal{C}$ is an ∞-category which admits small colimits, then $\mathcal{C}$ is (-1)-semiadditive if and only if it is pointed. In this case, for every pair of objects $C, D \in \mathcal{C}$, Remark 4.4.3 produces a canonical map

$$d\mu_\emptyset : \mathrm{Fun}(\emptyset, \mathrm{Map}_{\mathcal{C}}(C, D)) \to \mathrm{Map}_{\mathcal{C}}(C, D),$$

which we can identify with a morphism from C to D. Unwinding the definitions, we see that this is the zero morphism from C to D: that is, it is given by the composition $C \to 0 \to D$, where 0 denotes a zero object of $\mathcal{C}$.

Example 4.4.8. Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits and let X be a set, which we regard as a discrete space. Then we can identify objects of $\mathrm{LocSys}(\mathcal{C})_X$ with sequences of objects $\{C_x\}_{x\in X}$ of the ∞-category $\mathcal{C}$, indexed by the set X. If $p : X \to *$ denotes the projection from X to a point, then the functors

$$p_! : \mathrm{Fun}(X, \mathcal{C}) \to \mathcal{C} \qquad p_* : \mathrm{Fun}(X, \mathcal{C}) \to \mathcal{C}$$

are given by

$$\{C_x\}_{x\in X} \mapsto \coprod_{x\in X} C_x \qquad \{C_x\}_{x\in X} \mapsto \prod_{x\in X} C_x.$$

Assume that $\mathcal{C}$ is pointed, and therefore (-1)-semiadditive. It follows from Remark 4.3.7 that X is weakly $\mathcal{C}$-ambidextrous. Unwinding the definitions, the norm map $\mathrm{Nm}_p : p_! \to p_*$ associates to each $\{C_x\}_{x\in X} \in \mathrm{Fun}(X, \mathcal{C})$ the map

$$\theta : \coprod_{x\in X} C_x \to \prod_{y\in X} C_y,$$

whose (x, y)-component is given by id_{C_x} for $x = y$, and is otherwise given by the zero map (see Example 4.4.7). It follows that X is $\mathcal{C}$-ambidextrous if and only if θ is an equivalence, for every collection of objects $\{C_x\}_{x\in X}$.

Proposition 4.4.9. *Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits. Then $\mathcal{C}$ is 0-semiadditive if and only if it is pointed and, for every pair of objects $C, D \in \mathcal{C}$, the matrix $\begin{bmatrix} \mathrm{id}_C & 0 \\ 0 & \mathrm{id}_D \end{bmatrix}$ induces an equivalence $C \amalg D \to C \times D$.*

Proof. We will prove the "if" direction; the converse is an immediate consequence of Examples 4.4.7 and 4.4.8. Assume that $\mathcal{C}$ satisfies the conditions of the Proposition; we wish to show that every finite discrete space X is $\mathcal{C}$-ambidextrous. We proceed by induction on the number of elements of X. If X is empty, then the desired result follows from our assumption that $\mathcal{C}$ is pointed (Example 4.4.7). Otherwise, choose an element $x \in X$. Set $Y = \{x, y\}$, and define a map $p : X \to Y$ which carries x to itself and $X - \{x\}$ to y. The fiber $p^{-1}\{x\}$ is contractible (hence $\mathcal{C}$-ambidextrous), and the fiber $p^{-1}\{y\}$ has cardinality $< |X|$ and is therefore $\mathcal{C}$-ambidextrous by the inductive hypothesis. It follows from Example 4.4.8 that Y is $\mathcal{C}$-ambidextrous, so that X is $\mathcal{C}$-ambidextrous by Corollary 4.3.6. □

Remark 4.4.10. In the situation of Proposition 4.4.9, it is not really necessary to assume that $\mathcal{C}$ admits small limits. If $\mathcal{C}$ is a pointed ∞-category which admits small colimits, then for every pair of objects $C, D \in \mathcal{C}$ we obtain a canonical pair of maps

$$C \leftarrow C \amalg D \to D.$$

The 0-semiadditivity of $\mathcal{C}$ is equivalent to the requirement that this pair of maps exhibits $C \amalg D$ as a product of C with D in the ∞-category $\mathcal{C}$.

Remark 4.4.11. Let $\mathcal{C}$ be a 0-semiadditive ∞-category. For every pair of objects $C, D \in \mathcal{C}$ and every finite set X, the construction of Remark 4.4.3 determines a map

$$d\mu_X : \mathrm{Fun}(X, \mathrm{Map}_{\mathcal{C}}(C, D)) \to \mathrm{Map}_{\mathcal{C}}(C, D).$$

In particular, taking X to be a set with two elements, we obtain an addition map

$$+ : \mathrm{Map}_{\mathcal{C}}(C, D) \times \mathrm{Map}_{\mathcal{C}}(C, D) \to \mathrm{Map}_{\mathcal{C}}(C, D).$$

It is not difficult to see that this addition map is commutative and associative up to homotopy, with unit given by the zero map $0 \in \mathrm{Map}_{\mathcal{C}}(C, D)$. With more effort, one can show that this addition is commutative and associative up to coherent homotopy: that is, it underlies an $\mathbb{E}_\infty$-structure on the space $\mathrm{Map}_{\mathcal{C}}(C, D)$.

Remark 4.4.12. Let $\mathcal{A}$ be an ordinary category. Recall that $\mathcal{A}$ is said to be *additive* if the following conditions are satisfied:

(1) The category $\mathcal{A}$ is pointed.

(2) The category $\mathcal{A}$ admits finite products and coproducts.

(3) For every pair of objects $A, B \in \mathcal{A}$, the matrix $\begin{bmatrix} \mathrm{id}_A & 0 \\ 0 & \mathrm{id}_B \end{bmatrix}$ determines an isomorphism from the coproduct $A \amalg B$ to the product $A \times B$.

(4) For every pair of objects A, B, the set of maps $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is an abelian group under the addition law which carries a pair of maps $f, g : A \to B$ to the sum

$$f + g : A \to A \times A \xrightarrow{(f,g)} B \times B \simeq B \amalg B \to B.$$

Note that if $\mathcal{A}$ admits small colimits, then conditions (1), (2), and (3) are equivalent to the 0-semiadditivity of the nerve $\mathrm{N}(\mathcal{A})$. We may therefore regard the theory of 0-semiadditive ∞-categories as a generalization of the theory of additive categories (modulo the assumption of the existence of small colimits, which is mostly a matter of convenience).

Remark 4.4.13. Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits. The characterization of Proposition 4.4.9 shows that the 0-semiadditivity of $\mathcal{C}$ is really a condition on the homotopy category $\mathrm{h}\mathcal{C}$. Namely, the homotopy category $\mathrm{h}\mathcal{C}$ automatically satisfies condition (2) of Remark 4.4.12, and $\mathcal{C}$ is 0-semiadditive if and only if $\mathrm{h}\mathcal{C}$ also satisfies conditions (1) and (3).

Example 4.4.14. Let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits. Then $\mathcal{C}$ is 0-semiadditive. Consequently, for every finite group G, the classifying space BG is weakly $\mathcal{C}$-ambidextrous. Let $p : BG \to *$ denote the projection map. Then the norm map $\mathrm{Nm}_p : p_! \to p_*$ determines a map $C_G \to C^G$, for every G-equivariant object C of $\mathcal{C}$. We denote the cofiber of this norm map by C^{tG}, and refer to it as the *Tate construction* for the action of G on C (see Definition HA.7.1.6.24). Note that the classifying space BG is $\mathcal{C}$-ambidextrous if and only if, for every G-equivariant object C of $\mathcal{C}$, the Tate construction C^{tG} vanishes.

We now turn our attention to 1-semiadditivity.

Notation 4.4.15. Let $\mathcal{C}$ be a 0-semiadditive ∞-category. For every integer $n \geq 0$ and every object $C \in \mathcal{C}$, we let $[n] : C \to C$ denote the n-fold sum of the identity map id_C with itself, under the addition on $\mathrm{Map}_{\mathcal{C}}(C, C)$ described in Remark 4.4.11. More explicitly, we let $[n]$ denote the composite map

$$C \xrightarrow{\delta} \prod_{1 \leq i \leq n} C \simeq \coprod_{1 \leq j \leq n} C \xrightarrow{\delta'} C.$$

Here δ and δ' denote the diagonal and codiagonal, respectively, and the middle equivalence is supplied by Example 4.4.8.

Proposition 4.4.16. *Let $\mathcal{C}$ be a 0-semiadditive ∞-category which admits small limits and colimits. Assume that there is a prime number p with the following property:*

(∗) *For every integer $n \geq 1$ which is relatively prime to p and every object $C \in \mathcal{C}$, the map $n : C \to C$ is an equivalence.*

Then $\mathcal{C}$ is 1-semiadditive if and only if the Eilenberg-MacLane space $K(\mathbf{Z}/p\mathbf{Z}, 1)$ is $\mathcal{C}$-ambidextrous.

Proof. The "only if" direction is obvious. For the converse, suppose that $K(\mathbf{Z}/p\mathbf{Z}, 1)$ is $\mathcal{C}$-ambidextrous. Let X be any finite 1-type; we wish to show that X is $\mathcal{C}$-ambidextrous. Applying Corollary 4.3.6 to the map $X \to \pi_0 X$, we can reduce to the case where X is connected, so that $X \simeq BG$ for some finite group G. Let $P \subseteq G$ be a p-Sylow subgroup, and consider the maps

$$BP \xrightarrow{g} BG \xrightarrow{f} *.$$

We wish to show that the norm map $\mathrm{Nm}_f : f_! \to f_*$ is an equivalence. Let $\mathcal{L} \in \mathrm{Fun}(BG, \mathcal{C})$ be an arbitrary object. The map g is equivalent to a covering space with finite fibers, and is therefore ambidextrous. Let $\alpha : \mathcal{L} \to \mathcal{L}$ denote the composition

$$\mathcal{L} \to g_* g^* \mathcal{L} \simeq g_! g^* \mathcal{L} \to \mathcal{L}.$$

We claim that α is an equivalence. To prove this, it suffices to show that α induces an equivalence $x^* \mathcal{L} \to x^* \mathcal{L}$ for every point $x \in BG$. Unwinding the definitions, we see that $x^*(\alpha)$ is given by the multiplication map $[n] : \mathcal{L}_x \to \mathcal{L}_x$, where n is the cardinality of the quotient G/P. Since P is a p-Sylow subgroup of G, the number n is relatively prime to p so that $[n]$ is an equivalence. It follows that $\mathcal{L}$ is a retract of $g_! g^* \mathcal{L}$. Consequently, to prove that Nm_p induces an equivalence $p_! \mathcal{L} \to f_* \mathcal{L}$, we are free to replace $\mathcal{L}$ by $f_! f^* \mathcal{L}$, and may therefore assume that $\mathcal{L} = f_! \mathcal{L}'$ for some $\mathcal{L}' \in \mathrm{Fun}(BP, \mathcal{C})$. Consider the composite map

$$f_! g_! \mathcal{L}' \xrightarrow{\mathrm{Nm}_f} f_* g_! \mathcal{L}' \xrightarrow{\mathrm{Nm}_g} f_* g_* \mathcal{L}'.$$

Since g ambidextrous, the second map is an equivalence. To prove that the first map is an equivalence, it suffices to show the composite map is an equivalence. According to Remark 4.2.4, the composite map can be identified with the norm for the composition fg. We may therefore replace G by P and reduce to the case where G is a p-group. We now proceed by induction on the cardinality of G. If G is trivial there is nothing to prove. Otherwise, we can choose a normal subgroup $G' \subseteq G$ of order p. It follows from the inductive hypothesis that $B(G/G')$ is $\mathcal{C}$-ambidextrous. We have a fibration $BG \to B(G/G')$, whose fibers are homotopy equivalent to the Eilenberg-MacLane space $BG' \simeq K(\mathbf{Z}/p\mathbf{Z}, 1)$. Applying Corollary 4.3.6, we deduce that BG is $\mathcal{C}$-ambidextrous as desired. □

Proposition 4.4.17. *Let $\mathcal{C}$ be a 0-semiadditive ∞-category, let p be a prime number. Assume that for each $C \in \mathcal{C}$, the map $[p] : C \to C$ is an equivalence. Then, for every finite p-group G, the Eilenberg-MacLane space BG is $\mathcal{C}$-ambidextrous.*

Proof. Arguing as in the proof of Proposition 4.4.16, we can reduce to the case where $G = \mathbf{Z}/p\mathbf{Z}$. Consider the maps

$$* \xrightarrow{g} BG \xrightarrow{f} *.$$

We wish to show that the norm map $\mathrm{Nm}_f : f_! \to f_*$ is an equivalence. Let $\mathcal{L} \in \mathrm{Fun}(BG, \mathcal{C})$ be an arbitrary object. The map g is equivalent to a covering space with finite fibers, and therefore ambidextrous. Let $\alpha : \mathcal{L} \to \mathcal{L}$ denote the composition

$$\mathcal{L} \to g_* g^* \mathcal{L} \simeq g_! g^* \mathcal{L} \to \mathcal{L}.$$

As in the proof of Proposition 4.4.16, we see that for each $x \in BG$, the map $\mathcal{L}_x \to \mathcal{L}_x$ determined by α is homotopic to $p : \mathcal{L}_x \to \mathcal{L}_x$, and therefore an equivalence. It follows that $\mathcal{L}$ is a retract of $g_! \mathcal{L}'$, for $\mathcal{L}' = g^* \mathcal{L}$.

It will therefore suffice to show that Nm_f induces an equivalence $f_! g_! \mathcal{L}' \to f_* g_! \mathcal{L}'$. Remark 4.2.4 implies that the composite map

$$f_! g_! \mathcal{L}' \xrightarrow{\mathrm{Nm}_f} f_* g_! \mathcal{L}' \xrightarrow{\mathrm{Nm}_g} f_* g_* \mathcal{L}' .$$

is an equivalence. Using the two-out-of-three property, we are reduced to proving that Nm_g induces an equivalence $f_* g_! \mathcal{L}' \to f_* g_* \mathcal{L}'$. This follows from our assumption that $\mathcal{C}$ is 0-semiadditive. □

Corollary 4.4.18. *Let $\mathcal{C}$ be a 0-semiadditive ∞-category. Assume that for each integer $n \geq 1$ and each object $C \in \mathcal{C}$, the map $[n] : C \to C$ is an equivalence. Then $\mathcal{C}$ is 1-semiadditive.*

Proof. Combine Propositions 4.4.16 and 4.4.17. □

Proposition 4.4.19. *Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits, and let $n \geq 2$ be an integer. Then $\mathcal{C}$ is n-semiadditive if and only if the following conditions are satisfied:*

(1) *The ∞-category $\mathcal{C}$ is $(n-1)$-semiadditive.*

(2) *For every prime number p, the Eilenberg-MacLane space $K(\mathbf{Z}/p\mathbf{Z}, n)$ is $\mathcal{C}$-ambidextrous.*

Proof. It is clear that conditions (1) and (2) are necessary. To prove sufficiency, assume that (1) and (2) are satisfied and let X be a finite n-type; we wish to show that X is $\mathcal{C}$-ambidextrous. Using Corollary 4.3.6, we are reduced to proving that the homotopy fibers of the truncation map $X \to \tau_{\leq n-1} X$ are $\mathcal{C}$-ambidextrous. We may therefore assume that X is an Eilenberg-MacLane space $K(A, n)$ for some finite abelian group A. We proceed by induction on the cardinality of A. If A is trivial there is nothing to prove, and if A is isomorphic to $\mathbf{Z}/p\mathbf{Z}$ for some prime number p then the conclusion follows from (2). Otherwise, we can choose a short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

where A' and A'' are smaller than A. Then there is a fibration $X \to K(A'', n)$ whose homotopy fibers are equivalent to $K(A', n)$. The inductive hypothesis implies that $K(A'', n)$ and $K(A', n)$ are $\mathcal{C}$-ambidextrous. It follows from Corollary 4.3.6 that X is $\mathcal{C}$-ambidextrous, as desired. □

Proposition 4.4.20. *Let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits. Let p be a prime number with the following property: for every object $C \in \mathcal{C}$, the multiplication map $[p] : C \to C$ is an equivalence. Then the Eilenberg-MacLane spaces $K(\mathbf{Z}/p\mathbf{Z}, m)$ are $\mathcal{C}$-ambidextrous for $m \geq 1$.*

Proof. We proceed by induction on m. The case $m = 1$ follows from Proposition 4.4.17. Assume that $m \geq 2$. The inductive hypothesis implies that $K(\mathbf{Z}/p\mathbf{Z}, m-1)$ is $\mathcal{C}$-ambidextrous, so that $K(\mathbf{Z}/p\mathbf{Z}, m)$ is weakly $\mathcal{C}$-ambidextrous (Remark 4.3.7). Let $X = K(\mathbf{Z}/p\mathbf{Z}, m)$ and let $f : X \to *$ denote the projection map. To complete the proof, it will suffice to show that the functor $f_* : \mathrm{Fun}(X, \mathcal{C}) \to \mathcal{C}$ preserves small colimits. We will complete the proof by showing that f_* is an equivalence of ∞-categories. Equivalently, we will show that the diagonal embedding $\mathcal{C} \to \mathrm{Fun}(X, \mathcal{C})$ is an equivalence of ∞-categories. For this, it suffices to show that for every simplicial set K, the induced map

$$\mathrm{Fun}(K, \mathcal{C})^{\simeq} \to \mathrm{Fun}(K, \mathrm{Fun}(X, \mathcal{C}))^{\simeq}$$

is a homotopy equivalence of Kan complexes. Replacing $\mathcal{C}$ by $\mathrm{Fun}(K, \mathcal{C})$, we are reduced to proving that the diagonal map $\mathcal{C}^{\simeq} \to \mathrm{Fun}(X, \mathcal{C})^{\simeq} \simeq \mathrm{Fun}(X, \mathcal{C}^{\simeq})$ is a homotopy equivalence.

We will prove by induction on n that the diagonal map

$$\delta_n : \tau_{\leq n} \mathcal{C}^{\simeq} \to \mathrm{Fun}(X, \tau_{\leq n} \mathcal{C}^{\simeq})$$

is a homotopy equivalence (the desired result then follows by passing to the homotopy limit in n). When $n = 1$, this follows immediately from the fact X is 2-connective (since $m \geq 2$). Assume that δ_n is a homotopy equivalence and consider the diagram

$$\begin{tikzcd} \tau_{\leq n+1} \mathcal{C}^{\simeq} \arrow[r, "\delta_{n+1}"] \arrow[d] & \mathrm{Fun}(X, \tau_{\leq n+1} \mathcal{C}^{\simeq}) \arrow[d] \\ \tau_{\leq n} \mathcal{C}^{\simeq} \arrow[r, "\delta_n"] & \mathrm{Fun}(X, \tau_{\leq n} \mathcal{C}^{\simeq}). \end{tikzcd}$$

To prove that δ_{n+1} is a homotopy equivalence, it will suffice to show that δ_{n+1} induces a homotopy equivalence between the homotopy fibers taken over any chosen vertex of $\tau_{\leq n} \mathcal{C}^{\simeq}$. Such a vertex depends on a choice of object $C \in \mathcal{C}$. Unwinding the definitions, we are reduced to proving that the diagonal map

$$K(\mathrm{Ext}_{\mathcal{C}}^{-n}(C, C), n+1) \to \mathrm{Fun}(X, K(\mathrm{Ext}_{\mathcal{C}}^{-n}(C, C), n+1))$$

vanishes. This is equivalent to the vanishing of the reduced cohomology groups $\mathrm{H}^*_{\mathrm{red}}(X; A)$, where $A = \mathrm{Ext}_{\mathcal{C}}^{-n}(C, C)$. The vanishing now follows from the observation that the abelian group A is a module over $\mathbf{Z}[\frac{1}{p}]$. $\square$

Corollary 4.4.21. *Let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits. Assume that for each object $C \in \mathcal{C}$, the endomorphism ring $\mathrm{Ext}^0_{\mathcal{C}}(C, C)$ is an algebra over the field $\mathbf{Q}$ of rational numbers. Then $\mathcal{C}$ is n-semiadditive for every integer n.*

Proof. Combine Corollary 4.4.18, Proposition 4.4.20, and Proposition 4.4.19. $\square$

Example 4.4.22. Let R be an $\mathbb{E}_1$-ring, and suppose that $\pi_0 R$ is a vector space over the field $\mathbf{Q}$ of rational numbers. Then the ∞-category LMod_R of left R-module spectra is n-semiadditive for every integer n.

Corollary 4.4.23. *Let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits, and let p be a prime number. Assume that for each object $C \in \mathcal{C}$, the endomorphism ring $\mathrm{Ext}^0_{\mathcal{C}}(C, C)$ is a module over the local ring $\mathbf{Z}_{(p)}$. Then $\mathcal{C}$ is n-semiadditive if and only if the Eilenberg-MacLane spaces $K(\mathbf{Z}/p\,\mathbf{Z}, m)$ are $\mathcal{C}$-ambidextrous for $1 \leq m \leq n$.*

Proof. Combine Propositions 4.4.19, 4.4.16, and 4.4.20. $\square$

5 Ambidexterity of $K(n)$-Local Stable Homotopy Theory

Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits. In §4.3, we introduced the notion of a $\mathcal{C}$-ambidextrous space. If X is a $\mathcal{C}$-ambidextrous space, then for any $\mathcal{C}$-valued local system $\mathcal{L}$ on X, we have an equivalence

$$\mathrm{Nm}_X : C_*(X; \mathcal{L}) \to C^*(X; \mathcal{L}).$$

In §4.4, we studied some situations in which the $\mathcal{C}$-ambidexterity of a space X can be proven by purely formal arguments. However, the cases considered in §4.4 are not particularly interesting: our arguments required assumptions which rule out the existence of interesting local systems (see the proof of Proposition 4.4.20), and the similarity between $C_*(X; \mathcal{L})$ with $C^*(X; \mathcal{L})$ reflects the vanishing of both sides.

Our goal in this section is to establish ambidexterity in a more interesting setting. Our main result (Theorem 5.2.1) asserts that if $\mathcal{C}$ is the ∞-category of $K(n)$-local spectra (Definition 2.1.13) and X is a Kan complex with finitely many finite homotopy groups, then X is $\mathcal{C}$-ambidextrous. The proof is not a formal exercise: it depends crucially on the Ravenel-Wilson calculation of the $K(n)$-cohomology of Eilenberg-MacLane spaces discussed in §2.

Suppose that $\mathcal{C} = \mathrm{Sp}_{K(n)}$ is the ∞-category of $K(n)$-local spectra, and that X is a Kan complex which is known to be weakly $\mathcal{C}$-ambidextrous. Then X is $\mathcal{C}$-ambidextrous if and only if, for every $\mathcal{C}$-valued local system $\mathcal{L}$ on X, the norm construction of Remark 4.1.12 determines an equivalence $\mathrm{Nm}_X : C_*(X; \mathcal{L}) \to C^*(X; \mathcal{L})$. In §5.1 we show that it suffices to prove this in the case where $\mathcal{L}$ is the trivial local system on X (Example

5.1.10). In this case, we can think of Nm_X as a kind of bilinear form on the $K(n)$-local suspension spectrum $L_{K(n)}\Sigma^\infty_+(X)$, and ambidexterity is equivalent to the *nondegeneracy* of this bilinear form.

We carry out the proof of Theorem 5.2.1 in §5.2. Roughly speaking, the idea is to reduce to the case where X is an Eilenberg-MacLane space, so that the Lubin-Tate homology $E_0^\wedge(X)$ is well-understood by means of the calculations of §3.4. The map Nm_X determines a bilinear form on $E_0^\wedge(X)$, and this formal properties of the norm (specifically, Remark 5.1.12 and Proposition 5.1.18 from §5.1) are sufficient to determine this bilinear form explicitly. The desired nondegeneracy is then a consequence of a general algebraic fact about p-divisible groups (Proposition 5.2.2).

In §5.4, we study the structure of the ∞-category $\mathrm{Fun}(X, \mathrm{Sp}_{K(n)})$ of $K(n)$-local spectra on a Kan complex X. It follows from Theorem 5.2.1 that if X has finitely many finite homotopy groups, then the construction $\mathcal{L} \mapsto C^*(X; \mathcal{L})$ determines a colimit-preserving functor from $\mathrm{Fun}(X, \mathrm{Sp}_{K(n)})$ to $\mathrm{Sp}_{K(n)}$. Note that for any $\mathcal{L} \in \mathrm{Fun}(X, \mathrm{Sp})$, the spectrum $C^*(X; \mathcal{L})$ is naturally a module over the function spectrum $C^*(X; S)$ (here S denotes the $K(n)$-local sphere spectrum). Our main result asserts that if X is n-truncated and its homotopy groups are p-groups, then every $K(n)$-local module over $C^*(X; S)$ arises via this construction: more precisely, the global sections functor determines an equivalence of ∞-categories

$$C^*(X; \bullet) : \mathrm{Fun}(X; \mathrm{Sp}_{K(n)}) \to \mathrm{Mod}_{C^*(X;S)}(\mathrm{Sp}_{K(n)}).$$

Our proof relies on a duality phenomenon enjoyed by the Morava K-theory of Eilenberg-MacLane spaces, which we review in §5.3.

5.1 Ambidexterity and Duality

Let $\mathcal{C}$ be an ∞-category which admits small limits and colimits. Our goal in this section is to develop some tools for proving that a Kan complex X is $\mathcal{C}$-ambidextrous. Assume that X is weakly $\mathcal{C}$-ambidextrous. Then the construction of Remark 4.1.12 supplies a map Nm_X from the homology of X to the cohomology of X, taken with respect to an arbitrary $\mathcal{C}$-valued local system on X. Roughly speaking, we can think of a map from homology to cohomology as something like a bilinear pairing on the homology of X. The $\mathcal{C}$-ambidexterity of X is then equivalent to the nondegeneracy of this pairing. Our first goal is to make this idea more precise. We begin by reviewing some general facts about duality in monoidal ∞-categories (for a more detailed discussion, we refer the reader to [14]).

Definition 5.1.1. Let $\mathcal{C}$ be a monoidal category with unit object $\mathbf{1}$. A map

$$e : X \otimes Y \to \mathbf{1}$$

in $\mathcal{C}$ is said to be a *duality datum* if there exists a map $c : \mathbf{1} \to Y \otimes X$ such that the composite maps

$$X \xrightarrow{\mathrm{id} \otimes c} X \otimes Y \otimes X \xrightarrow{e \otimes \mathrm{id}} X$$

$$Y \xrightarrow{c \otimes \mathrm{id}} Y \otimes X \otimes Y \xrightarrow{\mathrm{id} \otimes e} Y$$

coincide with the identity maps id_X and id_Y, respectively. In this case, we say that e and c are *compatible* with one another.

If $\mathcal{C}$ is a monoidal ∞-category, we say that a map $e : X \otimes Y \to \mathbf{1}$ is a *duality datum* if it is a duality datum when regarded as a morphism in the homotopy category of $\mathcal{C}$.

Definition 5.1.2. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We say that an object $X \in \mathcal{C}$ is *dualizable* if there exists another object $Y \in \mathcal{C}$ and a duality datum $e : X \otimes Y \to \mathbf{1}$. In this case, choose $c : \mathbf{1} \to Y \otimes X$ to be compatible with e. We define $\dim(X) \in \pi_0 \mathrm{Map}_{\mathcal{C}}(X, X) = \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, X)$ to be the morphism given by the composition

$$\mathbf{1} \xrightarrow{c} Y \otimes X \simeq X \otimes Y \xrightarrow{e} \mathbf{1}.$$

Remark 5.1.3. In the situation of Definition 5.1.2, the object Y and the morphisms e and c are determined by X up to a contractible space of choices. In particular, $\dim(X)$ depends only on the object X.

Remark 5.1.4. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $e : X \otimes X \to \mathbf{1}$ be a duality datum in $\mathcal{C}$. Assume that e is symmetric up to homotopy (that is, e is homotopic to its composition with the self-equivalence of $X \otimes X$ given by swapping the two factors). Then $\dim(X)$ is given by the composition

$$\mathbf{1} \xrightarrow{c} X \otimes X \xrightarrow{e} \mathbf{1},$$

where c is compatible with e.

Example 5.1.5. Let R be a commutative ring, and let $\mathcal{C}$ denote the ordinary category of R-modules. An object $M \in \mathcal{C}$ is dualizable if and only if it is finitely generated and projective as an R-module. If M is a projective R-module of rank n, then $\dim(M) \in \operatorname{Hom}(R, R) \simeq R$ coincides with the image of n in the commutative ring R.

Example 5.1.6. Let R be an $\mathbb{E}_\infty$-ring. We say that an R-module spectrum M is *projective* if $\pi_0 M$ is a projective module over $\pi_0 R$ and the canonical map

$$\pi_m R \otimes_{\pi_0 R} \pi_0 M \to \pi_m M$$

is an isomorphism for every integer m. Let $\operatorname{Mod}_R^{\mathrm{proj}}$ denote the full subcategory of Mod_R spanned by the projective R-modules. According to Corollary HA.8.2.2.19, the construction $M \mapsto \pi_0 M$ determines an equivalence from the homotopy category of $\operatorname{Mod}_R^{\mathrm{proj}}$ to the category of projective modules over the commutative ring $\pi_0 R$. In particular, we see that if M and N are projective R-modules, then a map

$$e : M \otimes_R N \to R$$

is a duality datum if and only if the induced map

$$\pi_0 M \otimes_{\pi_0 R} \pi_0 N \to \pi_0 R$$

is a duality datum in the category of $\pi_0 R$-modules.

Notation 5.1.7. Let $q : \mathcal{C} \to \mathcal{X}$ be a Beck-Chevalley fibration. For every morphism $f : X \to Y$ in $\mathcal{X}$, we let $[X/Y]$ denote the functor $f_! \circ f^* : \mathcal{C}_Y \to \mathcal{C}_Y$. If f is weakly ambidextrous, we let $\mathrm{TrFm}_f : [X/Y] \circ [X/Y] \to \mathrm{id}$ denote the natural transformation given by

$$(f_! f^*)(f_! f^*) \simeq f_! (f^* f_!) f^* \xrightarrow{\nu_f} f_! (\mathrm{id}_{\mathcal{C}_X}) f^* \simeq f_! f^* \xrightarrow{\phi_f} \mathrm{id}\,.$$

We will refer to TrFm_f as the *trace form* of f.

Proposition 5.1.8. *Let $q : \mathcal{C} \to \mathcal{X}$ be a Beck-Chevalley fibration and let $f : X \to Y$ be a weakly ambidextrous morphism in $\mathcal{X}$. The following conditions are equivalent:*

(1) *The map $\nu_f : f^* f_! \to \mathrm{id}_{\mathcal{C}_X}$ is the counit of an adjunction between f^* and $f_!$.*

(2) *The trace form $\mathrm{TrFm}_f : [X/Y] \circ [X/Y] \to \mathrm{id}_{\mathcal{C}_Y}$ exhibits the functor $[X/Y]$ as its own dual in the monoidal ∞-category $\operatorname{Fun}(\mathcal{C}_Y, \mathcal{C}_Y)$.*

Proof. Suppose first that (1) is satisfied, and let $\mu_f : \mathrm{id}_{\mathcal{C}_Y} \to [X/Y]$ be a unit map which is compatible with ν_f. We define a natural transformation $c : \mathrm{id}_{\mathcal{C}_Y} \to [X/Y] \circ [X/Y]$ given by the composition

$$\mathrm{id} \xrightarrow{\mu_f} [X/Y] = f_! \,\mathrm{id}\, f^* \xrightarrow{\mathrm{id} \times \psi_f \times \mathrm{id}} f_! f^* f_! f^* = [X/Y] \circ [X/Y].$$

We claim that c and TrFm_f exhibit $[X/Y]$ as a self-dual object of $\operatorname{Fun}(\mathcal{C}_Y, \mathcal{C}_Y)$: in other words, the composite maps

$$[X/Y] \xrightarrow{\mathrm{id} \times c} [X/Y] \circ [X/Y] \circ [X/Y] \xrightarrow{\mathrm{TrFm}_f \times \mathrm{id}} [X/Y]$$

$$[X/Y] \xrightarrow{c \times \mathrm{id}} [X/Y] \circ [X/Y] \circ [X/Y] \xrightarrow{\mathrm{id} \times \mathrm{TrFm}_f} [X/Y]$$

are homotopic to the identity. We will prove that the first composition is homotopic to the identity; the proof in the second case is similar. We have a commutative diagram

$$\begin{tikzcd}
{[X/Y]} \arrow[r, "\mathrm{id} \times \mu_f"] \arrow[dr, "\mathrm{id} \times c"'] & {[X/Y] \circ [X/Y]} \arrow[r, "\mathrm{id} \times \nu_f \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \psi_f \times \mathrm{id}"] & {[X/Y]} \arrow[d, "\mathrm{id} \times \psi_f \times \mathrm{id}"] \\
& {[X/Y] \circ [X/Y] \circ [X/Y]} \arrow[r, "\mathrm{id} \times \nu_f \times \mathrm{id}"] \arrow[dr, "\mathrm{TrFm}_f \times \mathrm{id}"'] & {[X/Y] \circ [X/Y]} \arrow[d, "\phi_f \times \mathrm{id}"] \\
& & {[X/Y]}
\end{tikzcd}$$

The upper horizontal and right vertical compositions are homotopic to the identity, by virtue of the compatibilities between the pairs (ϕ_f, ψ_f) and (μ_f, ν_f).

Now suppose that (2) is satisfied, so that there exists a coevaluation map $c : \mathrm{id}_{\mathcal{C}_Y} \to [X/Y] \circ [X/Y]$ compatible with TrFm_f. We define natural transformations $\mu, \mu' : \mathrm{id} \to [X/Y]$ to be the compositions

$$\mu : \mathrm{id} \xrightarrow{c} [X/Y] \circ [X/Y] \xrightarrow{\phi_f \times \mathrm{id}} [X/Y]$$

$$\mu' : \mathrm{id} \xrightarrow{c} [X/Y] \circ [X/Y] \xrightarrow{\mathrm{id} \times \phi_f} [X/Y]$$

We will prove:

(i) The composite transformation

$$f_! \xrightarrow{\mu \times \mathrm{id}} f_! f^* f_! \xrightarrow{\mathrm{id} \times \nu_f} f_!$$

is homotopic to the identity.

(ii) The composite transformation

$$f^* \xrightarrow{\mathrm{id} \times \mu'} f^* f_! f^* \xrightarrow{\nu_f \times \mathrm{id}} f^*$$

is homotopic to the identity.

Assuming (i) and (ii), we deduce that the composite transformation

$$\mathrm{id} \xrightarrow{\mu \times \mu'} f_! f^* f_! f^* \xrightarrow{\mathrm{id} \times \nu_f \times \mathrm{id}} f_! f^*$$

is homotopic to both μ and μ'. It follows that $\mu \simeq \mu'$ is the unit for an adjunction compatible with the counit map ν_f.

It remains to prove (i) and (ii). We will prove (i); the proof of (ii) is similar. Since ϕ_f is the counit for an adjunction between $f_!$ and f^*, the composite map

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_X, \mathcal{C}_Y)}(f_!, f_!) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_Y, \mathcal{C}_Y)}(f_! f^*, f_! f^*) \xrightarrow{\phi_f \circ} \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_Y, \mathcal{C}_Y)}(f_! f^*, \mathrm{id}_{\mathcal{C}_Y})$$

is a homotopy equivalence. It will therefore suffice to show that the composition

$$f_! f^* \xrightarrow{\mu \times \mathrm{id}} f_! f^* f_! f^* \xrightarrow{\mathrm{id} \times \nu_f \times \mathrm{id}} f_! f^* \xrightarrow{\phi_f} \mathrm{id}_{\mathcal{C}_Y}$$

is homotopic to ϕ_f. Using the definitions of TrFm_f and μ, we can rewrite this composition as

$$[X/Y] \xrightarrow{c \times \mathrm{id}} [X/Y] \circ [X/Y] \circ [X/Y] \xrightarrow{\phi_f \times \mathrm{id}} [X/Y] \circ [X/Y] \xrightarrow{e} \mathrm{id}_{\mathcal{C}_Y}.$$

The desired result now follows from the commutativity of the diagram

$$\begin{tikzcd}
{[X/Y]} \arrow[r, "c\times\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & {[X/Y]\circ[X/Y]\circ[X/Y]} \arrow[r, "\phi_f\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times e"] & {[X/Y]\circ[X/Y]} \arrow[d, "e"] \\
 & {[X/Y]} \arrow[r, "\phi_f"] & \mathrm{id}\,.
\end{tikzcd}$$

□

We now restrict our attention to the Beck-Chevalley fibrations arising in the study of local systems.

Notation 5.1.9. Let $\mathcal{C}$ be an ∞-category which admits small colimits, let X be a Kan complex, and let $f : X \to *$ be the projection map. We let $[X]$ denote the functor from $\mathcal{C}$ to itself given by $[f] = f_! \circ f^*$ (see Notation 5.1.7). Unwinding the definitions, we see that $[X]$ is simply the functor given by tensoring with X (since $\mathcal{C}$ admits small colimits, it is naturally tensored over the ∞-category of spaces). In particular, it depends functorially on X. If we are given another map $g : Y \to X$, then the counit map $g_! g^* \to \mathrm{id}$ induces a natural transformation $[Y] = f_! g_! g^* f^* \to f_! f^* = [X]$, which we will denote by α_g.

If X is weakly $\mathcal{C}$-ambidextrous, then we let TrFm_X denote the natural transformation $\mathrm{TrFm}_f : [X]\circ[X] \to \mathrm{id}_{\mathcal{C}}$ described in Notation 5.1.7.

Example 5.1.10. Let X be a Kan complex, and let $\mathcal{C}$ be the image of an accessible exact localization functor $L : \mathrm{Sp} \to \mathrm{Sp}$ (see Proposition 2.1.1). Then the smash product monoidal structure on Sp induces a symmetric monoidal structure on $\mathcal{C}$ (Corollary 2.1.3), and the action of $\mathcal{C}$ on itself determines a monoidal equivalence from $\mathcal{C}$ to the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ spanned by those functors which preserves small colimits (Proposition 2.1.5). Under this equivalence, the functor $[X] : \mathcal{C} \to \mathcal{C}$ corresponds to the object $[X](L(S)) = L(\Sigma^\infty_+ X) \in \mathcal{C}$.

If X is weakly $\mathcal{C}$-ambidextrous, then Notation 5.1.7 determines a map

$$\mathrm{TrFm}_X : L(\Sigma^\infty_+ X) \hat{\otimes} L(\Sigma^\infty_+ X) \to L(S),$$

which is a duality datum if and only if X is $\mathcal{C}$-ambidextrous (Proposition 5.1.8).

It will be useful to have a more explicit description of the trace form TrFm_X assocated to a weakly $\mathcal{C}$-ambidextrous Kan complex X.

Notation 5.1.11. Let $\mathcal{C}$ be an ∞-category which admits small colimits, and let X be a Kan complex equipped with a base point $e : * \to X$. Assume that e is ambidextrous (this is automatic, for example, if the Kan complex X is weakly $\mathcal{C}$-ambidextrous). We let $\mathrm{Tr}_e : [X] \to \mathrm{id}_{\mathcal{C}}$ denote the natural transformation given by

$$[X] = f_! f^* \xrightarrow{\mu_e} f_! e_! e^* f^* \simeq \mathrm{id}_{\mathcal{C}}\,.$$

Remark 5.1.12. Let $\mathcal{C}$ be an ∞-category which admits small colimits, let X be a Kan complex, let $f : X \to *$ be the projection map, and let $e : * \to X$ be an ambidextrous map. Then the composite transformations

$$f_! f^* \xrightarrow{\mathrm{Tr}_e} \mathrm{id}_{\mathcal{C}} \simeq f_! e_! e^* f^* \xrightarrow{\phi_e} f_! f^* = [X]$$

$$f_! f^* \xrightarrow{\mathrm{id}\times\psi_f\times\mathrm{id}} f_! f^* f_! f^* \xrightarrow{\mathrm{Tr}_e\times\mathrm{id}} f_! f^*$$

are homotopic to one another. This follows from the commutativity of the diagram

$$\begin{tikzcd}
f_! f^* \arrow[r, "\mathrm{id} \times \psi_f \times \mathrm{id}"] \arrow[d, "u_e"] & f_! f^* f_! f^* \arrow[d, "u_e"] \\
f_! e_! e^* f^* \arrow[r, "\mathrm{id} \times \psi_f \times \mathrm{id}"] \arrow[d, "\phi_e"] & f_! e_! e^* f^* f_! f^* \arrow[d, "\phi_e"] \\
f_! f^* \arrow[r, "\mathrm{id} \times \psi_f \times \mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & f_! f^* f_! f^* \arrow[d, "\phi_f \times \mathrm{id}"] \\
 & f_! f^*.
\end{tikzcd}$$

Proposition 5.1.13. *Let $\mathcal{C}$ be an ∞-category which admits small colimits, let X be a weakly $\mathcal{C}$-ambidextrous space, and suppose that X admits the structure of a simplicial group. Let $f : X \to *$ be the projection map, $e : * \to X$ in the inclusion of the identity, and $s : X \times X \to X$ denote the subtraction map (given on simplices by $(x, y) \mapsto x^{-1}y$). Then the trace form $\mathrm{TrFm}_X : [X] \circ [X] \to \mathrm{id}_{\mathcal{C}}$ is given by the composition*

$$[X] \circ [X] \simeq [X \times X] \xrightarrow{\alpha_s} [X] = f_! f^* \xrightarrow{u_e} f_! e_! e^* f^* \simeq \mathrm{id}\,.$$

Proof. For every map of Kan complexes $g : Y \to Z$, we let $\phi_g : g_! g^* \to \mathrm{id}$ denote the counit for the natural adjunction between $g_!$ and g^*. Consider the diagram

$$\begin{tikzcd}
X \arrow[dr, "\delta"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & X \times X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X \arrow[d, "f"] \\
 & X \arrow[r, "f"] & *.
\end{tikzcd}$$

Unwinding the definitions, we see that the trace form TrFm_X is given by the composition

$$f_! f^* f_! f^* \simeq f_! \pi_{1!} \pi_2^* f^* \xrightarrow{\mu_\delta} f_! \pi_{1!} \delta_! \delta^* \pi_2^* f^* \simeq f_! f^* \xrightarrow{\phi_f} \mathrm{id}\,.$$

Note that $f\pi_1$ and $f\pi_2$ are both homotopic to fs. We can therefore identify the functor $f_! f^* f_! f^* \simeq f_! \pi_{1!} \pi_2^* f^*$ with $f_! s_! s^* f^*$. Under this identification, TrFm_X is given by

$$f_! s_! s^* f^* \xrightarrow{\mu_\delta} f_! s_! \delta_! \delta^* s^* f^* \xrightarrow{\phi_{fs\delta}} \mathrm{id}\,.$$

Consider the pullback diagram σ :

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\delta"] & * \arrow[d, "e"] \\
X \times X \arrow[r, "s"] & X.
\end{tikzcd}$$

Using this diagram, we can identify $\phi_{fs\delta}$ with the composition

$$f_! s_! \delta_! \delta^* s^* f^* \simeq f_! s_! s^* e_! e^* f_* \xrightarrow{\phi_s} f_! e_! e^* f^* \xrightarrow{\phi_e} f_! f^* \xrightarrow{\phi_f} \mathrm{id}_{\mathcal{C}}\,.$$

Since $e \circ f$ is homotopic to the identity, the composition of the last two of these natural transformations is homotopic to the identity. It follows that TrFm_X is given by the composition

$$f_! s_! s^* f^* \xrightarrow{\mu_\delta} f_! s_! \delta_! \delta^* s^* f^* \simeq f_! s_! s^* e_! e^* f^* \xrightarrow{\phi_s} f_! e_! e^* f^* \simeq \mathrm{id}_{\mathcal{C}}\,.$$

The desired result therefore now follows from the diagram

$$\begin{tikzcd}
f_! s_! s^* f^* \arrow[r, "\mu_\delta"] \arrow[dr, "\mu_e"] \arrow[dd, "\phi_s"] & f_! s_! \delta_! \delta^* s^* f^* \arrow[d, "\sim"] \\
& f_! s_! s^* e_! e^* f^* \arrow[d, "\phi_s"] \\
f_! f^* \arrow[r, "\mu_e"] & f_! e_! e^* f^*,
\end{tikzcd}$$

which commutes up to homotopy by virtue of Proposition 4.2.1. □

Corollary 5.1.14. *Let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits and let p be a prime number. Assume $\mathcal{C}$ is $(n-1)$-semiadditive and that for each object $C \in \mathcal{C}$, the endomorphism ring $\mathrm{Ext}^0_{\mathcal{C}}(C, C)$ is a module over the local ring $\mathbf{Z}_{(p)}$. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is n-semiadditive.*

(2) *The composition*

$$[K(\mathbf{Z}/p\,\mathbf{Z}, n)] \times [K(\mathbf{Z}/p\,\mathbf{Z}, n)] \to [K(\mathbf{Z}/p\,\mathbf{Z}, n) \times K(\mathbf{Z}/p\,\mathbf{Z}, n)] \xrightarrow{\alpha} [K(\mathbf{Z}/p\,\mathbf{Z}, n)] \xrightarrow{\mathrm{Tr}_e} \mathrm{id}_{\mathcal{C}}$$

exhibits the functor $[K(\mathbf{Z}/p\,\mathbf{Z}, n)]$ as a self-dual object of the monoidal ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$. Here Tr_e is defined as in Notation 5.1.11 (where e denotes the base point of $K(\mathbf{Z}/p\,\mathbf{Z}, n)$), and α is induced by the subtraction map

$$K(\mathbf{Z}/p\,\mathbf{Z}, n) \times K(\mathbf{Z}/p\,\mathbf{Z}, n) \to K(\mathbf{Z}/p\,\mathbf{Z}, n)$$

(where we view $K(\mathbf{Z}/p\,\mathbf{Z}, n)$ as a simplicial abelian group).

Proof. Combine Corollary 4.4.23, Proposition 5.1.13, and Proposition 5.1.8. □

Let X be a pointed connected Kan complex which is weakly $\mathcal{C}$-ambidextrous, and let $\delta : X \to X \times X$ be the diagonal map. Then the trace form $\mathrm{TrFm}_X : [X] \circ [X] \to \mathrm{id}_{\mathcal{C}}$ is defined using the natural transformation μ_δ, whose existence is a reflection of the nondegeneracy of the trace form $\mathrm{TrFm}_{\Omega(X)}$. Consequently, it is natural to expect a relationship between the trace form TrFm_X of X and the trace form $\mathrm{TrFm}_{\Omega(X)}$ of the loop space of X. We close this section by establishing such a relationship. First, we need to introduce a bit of terminology.

Notation 5.1.15. Let $\mathcal{C}$ be an ∞-category which admits small colimits and let X be a $\mathcal{C}$-ambidextrous Kan complex, so that the trace form $\mathrm{TrFm}_X : [X] \circ [X] \to \mathrm{id}_{\mathcal{C}}$ exhibits $[X]$ as its own dual in the monoidal ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ (Proposition 5.1.8). We let $\dim(X)$ denote the dimension of the functor $[X]$, as in Definition 5.1.2. That is, we let $\dim(X)$ denote the endomorphism of $\mathrm{id}_{\mathcal{C}}$ given by the composition

$$\mathrm{id}_{\mathcal{C}} \xrightarrow{c} [X] \circ [X] \xrightarrow{\mathrm{TrFm}_X} \mathrm{id}_{\mathcal{C}},$$

where c denotes a coevaluation natural transformation which is compatible with TrFm_X.

Remark 5.1.16. Let $f : X \to *$ denote the projection map. Using the definition of TrFm_X and the proof of Proposition 5.1.8, we see that $\dim(X)$ is given by the composition

$$\mathrm{id}_{\mathcal{C}} \xrightarrow{\mu_f} f_! f^* \simeq f_! \,\mathrm{id}_{\mathcal{C}}\, f^* \xrightarrow{\psi_f} f_! f^* f_! f^* \xrightarrow{\nu_f} f_! \,\mathrm{id}_{\mathcal{C}}\, f^* \simeq f_! f^* \xrightarrow{\phi_f} \mathrm{id}_{\mathcal{C}}\,.$$

Notation 5.1.17. Let $\mathcal{C}$ be an ∞-category which admits small colimits, let X be a Kan complex, and let $\beta : [X] \to \mathrm{id}_{\mathcal{C}}$ be a natural transformation of functors from $\mathcal{C}$ to $\mathcal{C}$. Given a point $e : * \to X$, we let $\beta(e)$ denote the natural transformation from $\mathrm{id}_{\mathcal{C}}$ to itself given by composing α with the map $\mathrm{id}_{\mathcal{C}} \simeq [*] \to [X]$ induced by e. More concretely, if $f : X \to *$ denotes the projection map, then $\beta(e)$ is given by the composition

$$\mathrm{id}_{\mathcal{C}} \simeq f_! e_! e^* f^* \stackrel{\phi_e}{\to} f_! f^* \stackrel{\beta}{\to} \mathrm{id}_{\mathcal{C}} .$$

We will be particularly interested in the case where e is ambidextrous and $\beta = \mathrm{Tr}_e$. In this case, $\mathrm{Tr}_e(e)$ is given by

$$\mathrm{id}_{\mathcal{C}} \simeq f_! e_! e^* f^* \stackrel{\phi_e}{\to} f_! f^* \stackrel{\mu_e}{\to} f_! e_! e^* f^* \simeq \mathrm{id}_{\mathcal{C}} .$$

Our goal in this section is to prove the following:

Proposition 5.1.18 (Product Formula)**.** *Let $\mathcal{C}$ be an ∞-category which admits small colimits, let G be a simplicial group which is $\mathcal{C}$-ambidextrous. Let $e : * \to G$ denote the inclusion of the identity element, and let $E : * \to BG$ be the inclusion of the base point. Then there is a canonical homotopy*

$$\dim(G) \simeq \mathrm{Tr}_e(e) \circ \mathrm{Tr}_E(E)$$

of natural transformations from the identity functor $\mathrm{id}_{\mathcal{C}}$ to itself.

Proof. Let f denote the projection map from G to a point, so that we have a pullback square σ :

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "f"] & * \arrow[d, "E"] \\
* \arrow[r, "E"] & BG.
\end{tikzcd}$$

Consider the diagram of natural transformations

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{C}} \arrow[r, "u_f"] & f_! f^* \arrow[r, "\sim"] \arrow[d, "\psi_f"] & f_! f^* \arrow[r, "\phi_f"] \arrow[d, "\psi_E"] & \mathrm{id}_{\mathcal{C}} \arrow[d, "\psi_E"] \\
& f_!(f^* f_!) f^* \arrow[r, "BC[\sigma]"] \arrow[d, "\nu_f"] & f_! f^* E^* E_! \arrow[r, "\phi_f"] \arrow[d, "\nu_E"] & E^* E_! \arrow[d, "\upsilon_E"] \\
& f_! f^* \arrow[r, "\sim"] & f_! f^* \arrow[r, "\phi_f"] & \mathrm{id}_{\mathcal{C}} .
\end{tikzcd}$$

This diagram commutes up to homotopy (this is obvious except for the lower left square, which commutes by virtue of Proposition 4.2.1). Remark 5.1.16 implies that $\dim(G)$ is given by traversing this diagram via a counterclockwise circuit. It follows that we can write $\dim(G) = \alpha \circ \beta$, where α denotes the composite map $\mathrm{id}_{\mathcal{C}} \stackrel{\psi_E}{\to} E^* E_! \stackrel{\nu_E}{\to} \mathrm{id}_{\mathcal{C}}$ and β the composite map $\mathrm{id}_{\mathcal{C}} \stackrel{\mu_f}{\to} f_! f^* \stackrel{\phi_f}{\to} \mathrm{id}_{\mathcal{C}}$. To complete the proof, it will suffice to prove the following:

(a) The natural transformation α is homotopic to $\mathrm{Tr}_e(e)$.

(b) The natural transformation β is homotopic to $\mathrm{Tr}_E(E)$.

We begin by proving (a). Unwinding the definitions, we see that α and $\mathrm{Tr}_e(e)$ are given by the compositions

$$\alpha : \mathrm{id}_{\mathcal{C}} \xrightarrow{\psi_E} E^* E_! \stackrel{BC[\sigma]^{-1}}{\longrightarrow} f_! f^* \xrightarrow{\mu_e} f_! e_! e^* f^* \simeq \mathrm{id}_{\mathcal{C}}$$

$$\mathrm{Tr}_e(e) : \mathrm{id}_{\mathcal{C}} \simeq f_! e_! e^* f^* \xrightarrow{\phi_e} f_! f^* \xrightarrow{\mu_e} f_! e_! e^* f^* \simeq \mathrm{id} .$$

It will therefore suffice to show that the composite map

$$\mathrm{id}_{\mathcal{C}} \simeq f_! e_! e^* f^* \xrightarrow{\phi_e} f_! f^* \xrightarrow{BC[\sigma]} E^* E_!$$

is homotopic to ψ_E. Note that the diagram σ determines a (nontrivial) homotopy h from the constant map $(E \circ f) : G \to BG$ to itself, which determines a homotopy γ from the functor $f^* E^*$ to itself. The desired result now follows by inspecting the diagram

$$\begin{tikzcd}
f_! e_! e^* f^* \arrow[r, "\phi_e"] \arrow[d, "\psi_E"] & f_! f^* \arrow[d] \\
f_! e_! e^* f^* E^* E_! \arrow[r, "\phi_e"] \arrow[d, "\mathrm{id}"] & f_! f^* E^* E_! \arrow[d, "\gamma"] \\
f_! e_! e^* f^* E^* E_! \arrow[r, "\phi_e"] \arrow[dr, "\sim"] & f_! f^* E^* E_! \arrow[d, "\phi_f"] \\
 & E^* E_!
\end{tikzcd}$$

(note that the composition of the column on the right is homotopic to the Beck-Chevalley transformation $BC[\sigma]$; see Remark 4.1.2).

We now prove (b). Let $F : BG \to *$ denote the projection map. We can write β as a composition

$$\mathrm{id}_{\mathcal{C}} \simeq E^* F^* \xrightarrow{\mu_f} f_! f^* E^* F^* \xrightarrow{\phi_f} E^* F^* \simeq \mathrm{id}_{\mathcal{C}}.$$

Using Proposition 4.2.1, we can rewrite this composition as

$$\mathrm{id}_{\mathcal{C}} \simeq F_! E_! E^* F^* \xrightarrow{\mu_E} F_! E_! E^* (E_! E^*) F^* \xrightarrow{\phi_E} F_! E_! E^* F^* \simeq \mathrm{id}_{\mathcal{C}}.$$

We have a commutative diagram

$$\begin{tikzcd}
F_! E_! E^* E_! E^* F^* \arrow[r, "\mathrm{id} \times \phi_E \times \mathrm{id}"] & F_! E_! E^* F^! \arrow[d, "\phi_{FE}"] \\
F_! E_! f_! f^* E^* F^* \arrow[u, "BC[\sigma]"] \arrow[d, "BC[\sigma]"] \arrow[r, "\phi_{FEf}"] & \mathrm{id}_{\mathcal{C}} \\
F_! E_! E^* E_! E^* F^* \arrow[r, "\mathrm{id} \times \phi_E \times \mathrm{id}"] & F_! E_! E^* F^* \arrow[u, "\phi_{FE}"]
\end{tikzcd}$$

from which it follows that the upper and lower horizontal maps are homotopic to one another. It follows that β is equivalent to the composition

$$\begin{aligned}
\mathrm{id}_{\mathcal{C}} & \simeq F_! E_! E^* F^* \\
& \xrightarrow{\mathrm{id} \times u_E \times \mathrm{id}} F_! E_! E^* E_! E^* F^* \\
& \xrightarrow{\mathrm{id} \times \phi_E \times \mathrm{id}} F_! E_! E^* F^* \\
& \simeq \mathrm{id}_{\mathcal{C}}
\end{aligned}$$

Assertion (b) now follows from the commutativity of the diagram

$$\begin{tikzcd}
F_! E_! E^* F^* \arrow[r, "\phi_E"] \arrow[d, "\mathrm{id} \times u_E \times \mathrm{id}"] & F_! F^* \arrow[d, "u_E"] \\
F_! E_! E^* E_! E^* F^* \arrow[r, "\mathrm{id} \times \phi_E \times \mathrm{id}"] & F_! E_! E^* F^*.
\end{tikzcd}$$

$\square$

5.2 The Main Theorem

Our goal in this section is to prove the following result:

Theorem 5.2.1. *Let $K(n)$ be a Morava K-theory spectrum (Notation 2.1.10), let $\mathrm{Sp}_{K(n)}$ denote the ∞-category of $K(n)$-local spectra, and let X be a Kan complex which is a finite m-type for some integer m (Definition 4.4.1). Then X is $\mathrm{Sp}_{K(n)}$-ambidextrous.*

The proof of Theorem 5.2.1 will occupy our attention throughout this section. Here is an outline of our basic strategy:

(a) Using Corollary 4.4.23, we can reduce to proving Theorem 5.2.1 in the special case where X is an Eilenberg-MacLane space $K(\mathbf{Z}/p\,\mathbf{Z}, m)$.

(b) Using Example 5.1.10, we are reduced to proving that the trace form TrFm_X exhibits $L_{K(n)}\Sigma^\infty_+(X)$ as a self-dual object of the ∞-category $\mathrm{Sp}_{K(n)}$.

(c) Let E denote the Lubin-Tate spectrum corresponding to the Morava K-theory $K(n)$. Using the fact that all $K(n)$-local spectra can be built from $K(n)$-local E-modules (Proposition 5.2.6), we are reduced to proving that TrFm_X exhibits $L_{K(n)}E[X]$ as a self-dual object of the ∞-category $\mathrm{Mod}_E(\mathrm{Sp}_{K(n)})$.

(d) Since $L_{K(n)}E[X]$ is free as an E-module spectrum (Theorem 2.4.10 and Proposition 3.4.3), the claim that TrFm_X is a perfect pairing reduces to a purely algebraic assertion: namely, that TrFm_X exhibits $E^\wedge_0(X)$ as a self-dual module over the commutative ring $\pi_0 E$ (Example 5.1.6).

(e) Using Remark 5.1.12 and Proposition 5.1.18, we see that the pairing on $E^\wedge_0(X)$ determined by TrFm_X can be identified with a certain multiple of the usual trace pairing on $E^\wedge_0(X)$.

(f) According to Theorem 3.4.1, we can identify $E^\wedge_0(X)$ with the ring of functions on the p-torsion subgroup of a p-divisible group over $\pi_0 E$. The nondegeneracy of TrFm_X can now be deduced from a general algebraic fact (Proposition 5.2.2).

Our starting point is the following result, which we prove using an argument of Tate (see [20]).

Proposition 5.2.2. *Let R be commutative ring, p a prime number which is not a zero-divisor in R, and let G be a truncated p-divisible group over R of level 1 and dimension d. Write $G = \operatorname{Spec} A$ where A is a finite flat R-algebra. Then:*

(1) *The trace map $\mathrm{tr} : A \to R$ is divisible by p^d.*

(2) *The construction $(x, y) \mapsto \frac{\mathrm{tr}(xy)}{p^d}$ determines a duality datum $A \otimes_R A \to R$ in the category of R-modules.*

Proof. Writing R as a direct limit of its finitely generated subrings, we can choose a finitely generated subalgebra $R_0 \subseteq R$ such that $G \simeq G_0 \times_{\operatorname{Spec} R_0} \operatorname{Spec} R$, for some finite flat group scheme G_0 over R_0. Enlarging R_0 if necessary, we may suppose that G_0 is a p-divisible group of dimension d and level 1. Note that since $R_0 \subseteq R$, p is not a zero divisor in R_0. We may therefore replace R by R_0 and thereby reduce to the case where R is Noetherian.

Assertion (1) is equivalent to the statement that the map $A \xrightarrow{\mathrm{tr}} R \to R/p^d R$ is zero. If this condition is satisfied, then $(x, y) \mapsto p^{-d}\,\mathrm{tr}(xy)$ determines a map from A to its R-linear dual $A^\vee$, and assertion (2) says that this map is an isomorphism. To show that a map $f : M \to N$ between finitely generated R-modules is zero or an isomorphism, it suffices to show that the induced map of localizations $M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is an isomorphism, for every prime ideal $\mathfrak{p} \subseteq R$. To prove this we may replace R by its localization $R_{\mathfrak{p}}$, and thereby reduce to the case where R is local. Using Proposition 10.3.1 of [5], we can choose find a faithfully flat morphism

$R \to R'$ of local Noetherian rings, such that the residue field of R' is perfect. Replacing R' by its completion if necessary, we may assume that R' is complete. Using faithfully flat descent, we can replace R by R', and thereby reduce to the case where R is a complete local Noetherian ring whose residue field κ is perfect.

If κ is a field of characteristic different from p, then p is invertible in R. In this case, the group scheme G is automatically étale, so that A is an étale R-algebra and the trace pairing $(x, y) \mapsto \mathrm{tr}(xy)$ is nondegenerate. Let us therefore assume that κ has characteristic p. Applying Theorem 3.1.11, we conclude that there exists a p-divisible group $\mathbf{G}$ over R and an isomorphism $G \simeq \mathbf{G}[p]$.

Let $\mathbf{G}_{\mathrm{inf}}$ and $\mathbf{G}_{\text{ét}}$ denote the connected and étale parts of $\mathbf{G}$, respectively, so that we have an exact sequence of p-divisible groups

$$0 \to \mathbf{G}_{\mathrm{inf}} \to \mathbf{G} \to \mathbf{G}_{\text{ét}} \to 0$$

and therefore an exact sequence of finite flat group schemes

$$0 \to \mathbf{G}_{\mathrm{inf}}[p] \to \mathbf{G}[p] \to \mathbf{G}_{\text{ét}}[p] \to 0.$$

Passing to a finite flat covering of $\operatorname{Spec} R$, we may assume that this second sequence splits. It will therefore suffice to prove assertions (1) and (2) after replacing $\mathbf{G}$ by either $\mathbf{G}_{\mathrm{inf}}$ or $\mathbf{G}_{\text{ét}}$.

Assume first that $\mathbf{G}$ is étale. Then $d = 0$ so that assertion (1) is vacuous. To prove assertion (2), we can pass to a finite étale cover of $\operatorname{Spec} R$ and thereby reduce to the case where $\mathbf{G}[p]$ is the constant group scheme associated to a $\mathbf{Z}/p\,\mathbf{Z}$-module M. Then $A \simeq \prod_{x \in M} R$, and assertion (2) is clear.

Now suppose that $\mathbf{G}$ is connected, and can therefore be identified with smooth formal group of dimension d over R. We have a pullback diagram (of formal schemes)

$$\begin{tikzcd} \mathbf{G}[p] \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ \mathbf{G} \arrow[r, "p"] & \mathbf{G}. \end{tikzcd}$$

Let B and B' denote the ring of functions on the formal scheme $\mathbf{G}$, and regard the map $p : \mathbf{G} \to \mathbf{G}$ as determining a finite flat map from B to B'. It will then suffice to prove the following:

$(1')$ The trace map $\mathrm{tr} : B' \to B$ is divisible by p^d.

$(2')$ The construction $(x, y) \mapsto \frac{\mathrm{tr}(xy)}{p^d}$ determines a duality datum $B' \otimes_B B' \to B$ in the category of B-modules.

Let Ω denote the B-module of sections of the cotangent bundle of $\mathbf{G}$ over R, and let Ω' denote the same abelian group regarded as a B'-module. Then the top exterior powers $\wedge^d_B \Omega$ and $\wedge^d_{B'} \Omega'$ are invertible modules over B and B', respectively. There is a canonical trace map $\mathrm{tr}' : \wedge^d_{B'} \Omega' \to \wedge^d_B \Omega$, and the construction $(f, \omega) \mapsto \mathrm{tr}'(f\omega)$ determines a perfect pairing

$$B' \otimes_B (\wedge^d_{B'} \Omega') \to \wedge^d_B \Omega.$$

Tensoring with the inverse of $\wedge^d_B \Omega$, we can regard this as a duality datum

$$\lambda : B' \otimes_B ((\wedge^d_{B'} \Omega') \otimes_B (\wedge^d_B \Omega)^{-1}) \to B.$$

Unwinding the definitions, we see that the trace pairing $(x, y) \mapsto \mathrm{tr}(xy)$ is given by the composition

$$B' \otimes_B B' \xrightarrow{\mathrm{id} \otimes \alpha} B' \otimes_B ((\wedge^d_{B'} \Omega') \otimes_B (\wedge^d_B \Omega)^{-1}) \xrightarrow{\lambda} B,$$

where α corresponds to the map of B'-modules

$$B' \otimes_B \wedge^d_B \Omega \to \wedge^d_{B'} \Omega'$$

given by pullback of differential forms. To complete the proof, it will suffice to prove the following:

(1″) The map α is divisible by p^d.

(2″) The quotient $\frac{\alpha}{p^d}$ is an isomorphism.

Since α is given by the dth exterior power of a map $\beta : B' \otimes_B \Omega \to \Omega'$, we are reduced to proving the following:

(1″) The map β is divisible by p.

(2″) The quotient $\frac{\beta}{p}$ is an isomorphism.

Let Ω_0 denote the R-module of translation-invariant differential forms on $\mathbf{G}$. Then Ω_0 is a projective R-module of rank d, and we have canonical isomorphisms

$$\Omega \simeq B \otimes_R \Omega_0 \qquad \Omega' \simeq B' \otimes_R \Omega_0.$$

Under these isomorphisms, we see that β corresponds to the map

$$B' \otimes_R \Omega_0 \xrightarrow{\mathrm{id} \otimes \gamma} B' \otimes_R \Omega_0,$$

where $\gamma : \Omega_0 \to \Omega_0$ is given by differentiating the multiplication-by-p map from $\mathbf{G}$ to itself. It follows that γ is just given by multiplication by p, from which assertions (1″) and (2″) immediately follow. □

To apply Proposition 5.2.2 in practice, it will be useful to have a criterion for recognizing the trace map.

Proposition 5.2.3. *Let R be a commutative ring, let p be a prime number which is not a zero-divisor on R, let G be a p-divisible group over R of height h and level 1, and write $G = \operatorname{Spec} A$, so that A is a finite flat R-algebra of rank p^h. Suppose that $\lambda : A \to R$ is an R-linear map satisfying the following conditions:*

(1) *The map λ carries $1 \in A$ to $p^h \in R$.*

(2) *Let $\Delta : A \to A \otimes_R A$ be the ring homomorphism classifying the multiplication on the group scheme G. Then the composite map*
$$A \xrightarrow{\Delta} A \otimes_R A \xrightarrow{\lambda \otimes \mathrm{id}} A$$
is given by $a \mapsto \lambda(a)$.

Then λ coincides with the trace map $\mathrm{tr} : A \to R$.

Proof. Replacing R by $R[\frac{1}{p}]$, we may assume that p is invertible in R, so that the group scheme G is étale. Passing to a finite flat cover, we may assume that G is the constant group scheme associated to some finite abelian group M. We can then identify A with the ring A^M of functions $f : M \to A$. Then λ is given by $\lambda(f) = \sum_{x \in M} c_x f(x)$ for some constants $c_x \in R$; we wish to show that each c_x is equal to 1. For each $x \in M$, let $e_x \in A$ be given by

$$e_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Delta(e_x) = \sum_{x = x' + x''} e_{x'} \otimes e_{x''}$, so assertion (2) gives

$$c_x = \sum_{x = x' + x''} c_{x'} e_{x''}$$

in the ring A. It follows that the function $x \mapsto c_x$ is constant. Now $\lambda(1) = \sum_{x \in M} c_x = p^h c_1$. Since p is invertible, condition (1) implies that $c_1 = 1$, so that each c_x is equal to 1. □

Corollary 5.2.4. *Let R be a commutative ring, p a prime number which is not a zero-divisor of R, and G a truncated p-divisible group over R of dimension d, height h, and level 1. Write $G = \operatorname{Spec} A$ where A is a finite flat R-algebra (of rank p^h). Let $\sigma : A \to A$ denote the antipodal map (that is, the ring homomorphism which induces the map $[-1] : G \to G$), and let $\lambda : A \to R$ be an R-linear map with the following properties:*

(1) *The map λ carries $1 \in A$ to $p^{h-d} \in R$.*

(2) *Let $\Delta : A \to A \otimes_R A$ be the ring homomorphism classifying the multiplication on the group scheme G. Then the composite map*

$$A \xrightarrow{\Delta} A \otimes_R A \xrightarrow{\lambda \otimes \mathrm{id}} A$$

is given by $a \mapsto \lambda(a)$.

Then the map $(a, b) \mapsto \lambda(a^\sigma b)$ determines a duality datum

$$A \otimes_R A \to R$$

in the category of R-modules.

Proof. Since σ is an R-module automorphism of A, it will suffice to show that the map $(a, b) \mapsto \lambda(ab)$ determines a duality datum. Using Proposition 5.2.2, we are reduced to proving that $p^d \lambda$ coincides with the trace map of A, which follows from Proposition 5.2.3. □

We now return to Theorem 5.2.1. The main step in the proof is to establish the following:

Corollary 5.2.5. *Let E be a Lubin-Tate spectrum of height n, let p be the characteristic of the residue field of E, and let $K(n)$ denote the associated Morava K-theory spectrum. Assume that $m > 0$ and that $K(\mathbf{Z}/p\mathbf{Z}, m-1)$ is $\mathrm{Sp}_{K(n)}$-ambidextrous. Let $X = K(\mathbf{Z}/p\mathbf{Z}, m)$, and consider the map*

$$\beta : L_{K(n)} E[X] \otimes L_{K(n)} E[X] \to E$$

determined by the trace form TrFm_X of Example 5.1.10, where $\otimes$ denotes the tensor product on the symmetric monoidal ∞-category $\mathrm{Mod}_E(\mathrm{Sp}_{K(n)})$. Then β is a duality datum.

Proof. We have already seen that $L_{K(n)} E[X]$ is a a projective E-module of finite rank (Theorem 2.4.10 and Proposition 3.4.3). Let $R = \pi_0 E$. Using Example 5.1.6, we are reduced to proving that β induces a duality datum

$$\beta : E_0^\wedge(X) \otimes_R E_0^\wedge(X) \to R$$

in the category of R-modules. Let $e : * \to X$ denote the inclusion of the base point, so that the map of spectra $\mathrm{Tr}_e : L_{K(n)}(\Sigma_+^\infty X) \to L_{K(n)}(S)$ induces a map of R-modules $\lambda : E_0^\wedge(X) \to R$. Let $\sigma : E_0^\wedge(X) \to E_0^\wedge(X)$ denote the antipode for the Hopf algebra structure on $E_0^\wedge(X)$. Using Proposition 5.1.13, we see that β classifies the bilinear map $(x, y) \mapsto \lambda(x^\sigma y)$. Consequently, to show that β is a duality datum, it will suffice to show that λ satisfies conditions (1) and (2) of Corollary 5.2.4.

(1) According to Theorems 3.4.1 and 3.5.1, $\operatorname{Spec} E_0^\wedge K(\mathbf{Z}/p\mathbf{Z}, m)$ is a truncated p-divisible group of height $\binom{n}{m}$, dimension $\binom{n-1}{m}$, and level 1. We must show that $\lambda(1) = p^{\binom{n}{m} - \binom{n-1}{m}} = p^{\binom{n-1}{m-1}}$. We proceed by induction on m. Let $e' : * \to K(\mathbf{Z}/p\mathbf{Z}, m-1)$ denote the inclusion of the base point, so that e' induces a trace map $\mathrm{Tr}_{e'} : L_{K(n)} \Sigma_+^\infty K(\mathbf{Z}/p\mathbf{Z}, m-1) \to L_{K(n)}(S)$ and therefore a map of R-modules $\lambda' : E_0^\wedge K(\mathbf{Z}/p\mathbf{Z}, m-1) \to R$. Note that $\lambda(1) \in R \simeq \pi_0 \mathrm{Map}_{\mathrm{Mod}_E(\mathrm{Sp}_{K(n)})}(E, E)$ is the image of the map $\mathrm{Tr}_e(e) \in \mathrm{Map}_{\mathrm{Sp}_{K(n)}}(L_{K(n)}(S), L_{K(n)}(S))$, and similarly $\lambda'(1) \in R$ is the image of $\mathrm{Tr}_{e'} e'$. Combining Proposition 5.1.18 and Example 5.1.5, we obtain an equality

$$\lambda(1)\lambda'(1) = \dim E_0^\wedge K(\mathbf{Z}/p\mathbf{Z}, m-1) = p^{\binom{n}{m-1}}.$$

We are therefore reduced to proving that

$$\lambda'(1) = p^{\binom{n}{m-1} - \binom{n-1}{m-1}} = \begin{cases} p^{\binom{n-1}{m-2}} & \text{if } m \geq 2 \\ 1 & \text{if } m = 1. \end{cases}$$

This follows from the inductive hypothesis if $m \geq 2$, and from a straightforward calculation when $m = 1$.

(2) Let $\Delta : E_0^\wedge(X) \to E_0^\wedge(X) \otimes_R E_0^\wedge(X)$ denote the comultiplication on $E_0^\wedge(X)$ (induced by the diagonal embedding $X \to X \times X$). Then the composite map

$$E_0^\wedge(X) \xrightarrow{\Delta} E_0^\wedge(X) \otimes_R E_0^\wedge(X) \xrightarrow{\lambda \otimes \mathrm{id}} E_0^\wedge(X)$$

coincides with the composition $E_0^\wedge(X) \xrightarrow{\lambda} R \to E_0^\wedge(X)$. This follows immediately from Remark 5.1.12.

□

We can use Corollary 5.2.5 to show that $L_{K(n)}E[K(\mathbf{Z}/p\mathbf{Z}, m)]$ is self-dual as an E-module spectrum. To deduce consequences for the $K(n)$-localization $L_{K(n)}\Sigma^\infty_+ K(\mathbf{Z}/p\mathbf{Z}, m)$, we need the following fact:

Proposition 5.2.6. *Let E be a Lubin-Tate spectrum of height n, let $K(n)$ denote the associated Morava K-theory spectrum, and let $\mathcal{C}$ be the smallest stable subcategory of $\mathrm{Sp}_{K(n)}$ which is closed under retracts and contains the image of the forgetful functor $\mathrm{Mod}_E(\mathrm{Sp}_{K(n)}) \to \mathrm{Sp}_{K(n)}$. Then $\mathcal{C} = \mathrm{Sp}_{K(n)}$.*

Proof. Let $E(n)$ denote the nth Johnson-Wilson spectrum, and let $\mathcal{C}'$ be the smallest stable subcategory of Sp which contains $E(n) \otimes X$ for every spectrum X and is closed under retracts. Since $L_{K(n)}E(n)$ can be written as a retract of E, the functor $L_{K(n)}$ carries $\mathcal{C}' \subseteq \mathrm{Sp}$ into $\mathcal{C} \subseteq \mathrm{Sp}_{K(n)}$. According to Theorem 5.3 of [10], every E-local spectrum belongs to $\mathcal{C}'$. It follows that for every $K(n)$-local spectrum X, we have $X \simeq L_{K(n)}(X) \in L_{K(n)}(\mathcal{C}') \subseteq \mathcal{C}$. □

Corollary 5.2.7. *Let E be a Lubin-Tate spectrum of height n, let p be the characteristic of the residue field of E, and let $K(n)$ denote the associated Morava K-theory spectrum. Assume that $m > 0$ and that $K(\mathbf{Z}/p\mathbf{Z}, m-1)$ is $\mathrm{Sp}_{K(n)}$-ambidextrous. Then $X = K(\mathbf{Z}/p\mathbf{Z}, m)$ is also $\mathrm{Sp}_{K(n)}$-ambidextrous.*

Proof. Let $\mathrm{TrFm}_X : L_{K(n)}\Sigma^\infty_+(X) \otimes L_{K(n)}\Sigma^\infty_+(X) \to L_{K(n)}S$ be defined as in Example 5.1.10; we wish to show that TrFm_X exhibits $L_{K(n)}\Sigma^\infty_+(X)$ as a self-dual object of the symmetric monoidal ∞-category $\mathrm{Sp}_{K(n)}$. In other words, we must show that for every pair of objects $Y, Z \in \mathrm{Sp}_{K(n)}$, the map $\theta_{Y,Z}$ given by the composition

$$\begin{aligned} \mathrm{Map}_{\mathrm{Sp}_{K(n)}}(Y, L_{K(n)}(\Sigma^\infty_+ X) \otimes Z) &\to \mathrm{Map}_{\mathrm{Sp}_{K(n)}}(L_{K(n)}(\Sigma^\infty_+ X) \hat{\otimes} Y, L_{K(n)}(\Sigma^\infty_+ X) \otimes L_{K(n)}(\Sigma^\infty_+ X) \hat{\otimes} Z) \\ &\to \mathrm{Map}_{\mathrm{Sp}_{K(n)}}(L_{K(n)}(\Sigma^\infty_+ X) \otimes Y, Z) \end{aligned}$$

is a homotopy equivalence. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Sp}_{K(n)}$ spanned by those objects Z for which the map $\theta_{Y,Z}$ is a homotopy equivalence for every $Y \in \mathrm{Sp}_{K(n)}$. It is easy to see that $\mathcal{C}$ is a stable subcategory of $\mathrm{Sp}_{K(n)}$ which is closed under retracts. Using Proposition 5.2.6, we are reduced to proving that $\theta_{Y,Z}$ is an equivalence whenever Z admits the structure of an E-module. In this case, we can identify $\theta_{Y,Z}$ with a map

$$\mathrm{Map}_{\mathrm{Mod}_E(\mathrm{Sp}_{K(n)})}(E \otimes Y, (E \otimes L_{K(n)}(\Sigma^\infty_+ X)) \otimes_E Z) \to \mathrm{Map}_{\mathrm{Mod}_E(\mathrm{Sp}_{K(n)})}((E \otimes L_{K(n)}(\Sigma^\infty_+ X)) \otimes_E (E \hat{\otimes} Y), Z),$$

which is a homotopy equivalence by Corollary 5.2.5. □

Proof of Theorem 5.2.1. Let $K(n)$ be a Morava K-theory, and let p denote the characteristic of the field $\pi_0 K(n)$. Applying Corollary 5.2.7 repeatedly, we deduce that each Eilenberg-MacLane space $K(\mathbf{Z}/p\mathbf{Z}, m)$ is $\mathrm{Sp}_{K(n)}$-ambidextrous. Note that if l is an integer not divisible by p, then multiplication by l induces a homotopy equivalence from $K(n)$ to itself. It follows that for every spectrum X, multiplication by l induces a $K(n)$-homology equivalence, so that l acts invertibly on $L_{K(n)}(X)$. Invoking Corollary 4.4.23, we deduce that every finite m-type is $\mathrm{Sp}_{K(n)}$-ambidextrous. □

5.3 Cartier Duality

Throughout this section, we fix a perfect field κ of characteristic $p > 0$, and a smooth connected 1-dimensional formal group $\mathbf{G}_0$ of height $n < \infty$ over κ. Let E denote the Lubin-Tate spectrum determined by the pair $(\kappa, \mathbf{G}_0)$, let $R = \pi_0 E \simeq W(\kappa)[[v_1, \ldots, v_{n-1}]]$ be the Lubin-Tate ring classifying deformations of $\mathbf{G}_0$, and let $\mathbf{G}$ denote the universal deformation of $\mathbf{G}_0$ (regarded as a p-divisible group over R). Let A be a finite abelian p-group. It follows from Theorem 2.4.10 that for $0 \leq d \leq n$, $\operatorname{Spec} K(n)^0 K(A, d)$ is a finite flat commutative group scheme over κ. We begin this section by reviewing a result of Buchstaber and Lazarev, which gives a topological description of the Cartier dual of $\operatorname{Spec} K(n)^0 K(A, d)$ (Theorem 5.3.6). Using this result, we construct an analogous duality in the setting of finite flat commutative group schemes over the Lubin-Tate spectrum E itself (Theorem 5.3.25). From this, we reprove a result of Bauer concerning the convergence properties of the Eilenberg-Moore spectral sequence in Morava K-theory (see Corollary 5.3.27 and Remark 5.3.28), which will play an important role in §5.4.

We begin by introducing some terminology.

Definition 5.3.1. Let A be a commutative ring, and let G be a p-divisible group over A of height n and dimension 1. According to Corollary 3.5.4, the direct limit $\varinjlim_t \operatorname{Alt}^{(n)}_{G[p^t]}$ is again a p-divisible group over R, of height 1 and dimension 0. A *normalization* of G is an isomorphism of p-divisible groups

$$\underline{\mathbf{Q}_p / \mathbf{Z}_p} \simeq \varinjlim_t \operatorname{Alt}^{(n)}_{G[p^t]}$$

Here $\underline{\mathbf{Q}_p / \mathbf{Z}_p}$ denotes the constant p-divisible group associated to the abelian group $\mathbf{Q}_p / \mathbf{Z}_p$.

Remark 5.3.2. In the situation of Definition 5.3.1, we can think of a normalization as given by a compatible family of isomorphisms

$$\underline{\mathbf{Z} / p^t \mathbf{Z}} \simeq \operatorname{Alt}^{(n)}_{G[p^t]}.$$

Such an isomorphism is determined by an A-valued point of $\operatorname{Alt}^{(n)}_{G[p^t]}$: that is, by an alternating multilinear map

$$G[p^t]^n \to \mu_{p^t} \subseteq \mathbf{G}_m.$$

Remark 5.3.3. Let G be a p-divisible group over a commutative ring A, which has height n and dimension 1. Then $\varinjlim_t \operatorname{Alt}^{(n)}_{G[p^t]}$ is an étale p-divisible group of height 1 over A. It follows that a normalization of G always exists after replacing A by a direct limit of étale A-algebras. In particular, if A is a separably closed field, then G always admits a normalization.

Remark 5.3.4. Let A be a Henselian local ring with residue field κ, and let κ denote the residue field of A. Then category of étale local systems on $\operatorname{Spec} A$ is equivalent to the category of étale local systems on $\operatorname{Spec} \kappa$. In particular, if G is a p-divisible group of height n and dimension 1 over A and G_0 is the associated p-divisible group over κ, there is a bijective correspondence between normalizations of G and normalizations of G_0.

Let $\mathbf{G}_0$ be a smooth connected formal group of height $n < \infty$ over a perfect field κ, and let $K(n)$ denote the associated Morava K-theory. According to Proposition 2.4.10, $\operatorname{KSpec}(K(\mathbf{Q}_p / \mathbf{Z}_p, n))$ is a connected p-divisible group of dimension 1 over κ, which is Cartier dual to $\varinjlim_t \operatorname{Alt}^{(n)}_{\mathbf{G}_0[p^t]}$. In this situation, there is a bijective correspondence between normalizations $\mathbf{G}_0$ and isomorphisms $\nu : \operatorname{KSpec}(K(\mathbf{Q}_p / \mathbf{Z}_p, n)) \simeq \widehat{\mathbf{G}}_m$, where $\widehat{\mathbf{G}}_m$ denotes the formal multiplicative group over κ. In what follows, we will generally identify ν with the corresponding normalization of $\mathbf{G}_0$.

Notation 5.3.5. Let A be a finite abelian p-group. We let A^* denote the Pontryagin dual of A, which we will identify with the set $\operatorname{Hom}(A, \mathbf{Q}_p / \mathbf{Z}_p)$ of all homomorphisms from A to $\mathbf{Q}_p / \mathbf{Z}_p$. If $G = \operatorname{Spec} H$ is a finite flat commutative group scheme over a commutative ring A, we let $\mathbf{D}(G) = \operatorname{Spec} H^\vee$ denote the Cartier dual of G.

Let A be a finite abelian p-group and let $0 \leq d \leq n$ be an integer. The bilinear map $A \times A^* \to \mathbf{Q}_p / \mathbf{Z}_p$ determines map of spaces

$$c : K(A, d) \times K(A^*, n-d) \to K(\mathbf{Q}_p / \mathbf{Z}_p, n),$$

classifying the cup product operation $\cup : \mathrm{H}^d(\bullet; A) \times \mathrm{H}^{n-d}(\bullet; A^*) \to \mathrm{H}^n(\bullet; \mathbf{Q}_p / \mathbf{Z}_p)$. Using Theorem 2.4.10 and Remark 2.1.21, we deduce that the spaces $K(A, d)$ and $K(A^*, n-d)$ are $K(n)$-good, so that we obtain a map of formal schemes

$$c : \mathrm{KSpec}(K(A, d)) \times_{\mathrm{Spec}\,\kappa} \mathrm{KSpec}(K(A^*, n-d)) \to \mathrm{KSpec}(K(\mathbf{Q}_p / \mathbf{Z}_p, n)).$$

It follows from the bilinearity of the cup product that c is bilinear. If $\nu : \mathrm{KSpec}(K(\mathbf{Q}_p / \mathbf{Z}_p, n)) \simeq \widehat{\mathbf{G}}_m$ is a normalization of $\mathbf{G}_0$, then c is classified by a map

$$\mathrm{KSpec}(K(A^*, n-d)) \to \mathbf{D}(\mathrm{KSpec}(K(A, d))).$$

of group schemes over κ.

Theorem 5.3.6 (Buchstaber-Lazarev [2]). *Let A be a finite abelian p-group and let $0 \leq d \leq n$ be an integer, and fix a normalization ν of the p-divisible group $\mathbf{G}_0$ over κ. Then the above construction produces an isomorphism*

$$\theta_A : \mathrm{KSpec}(K(A^*, n-d)) \to \mathbf{D}(\mathrm{KSpec}(K(A, d)))$$

of finite flat group schemes over κ.

Proof. Since the cup product of cohomology classes is bilinear, we obtain a map of Hopf algebras

$$K(n)_0 K(A, d) \boxtimes K(n)_0 K(A^*, n-d) \to K(n)_0 K(\mathbf{Q}_p / \mathbf{Z}_p, n).$$

Using Proposition 1.4.14, we can identify this with a map of Dieudonne modules

$$\beta : \mathrm{DM}_+(K(n)_0 K(A, d)) \times \mathrm{DM}_+(K(n)_0 K(A^*, n-d)) \to \mathrm{DM}_+(K(n)_0 K(\mathbf{Q}_p / \mathbf{Z}_p, n)).$$

Using the normalization ν of $\mathbf{G}_0$, we can identify $\mathrm{Spf}\, K(n)^0 K(\mathbf{Q}_p / \mathbf{Z}_p, n)$ with the formal multiplicative group over κ, so that the codomain of β can be identified with $W(\kappa)[p^{-1}]/W(\kappa)$ (where the actions of F and V on $W(\kappa)[p^{-1}]/W(\kappa)$ are induced by the maps $\lambda \mapsto \varphi(\lambda), p\varphi^{-1}(\lambda)$ from $W(\kappa)$ to itself). Using Example 1.4.18, we see that θ_A is an isomorphism if and only if β determines an isomorphism from $\mathrm{DM}_+(K(n)_0 K(A^*, n-d))$ to $\mathrm{Hom}_{W(\kappa)}(\mathrm{DM}_+(K(n)_0 K(A, d)), W(\kappa)[p^{-1}]/W(\kappa))$. The collection of those finite abelian p groups A which satisfy this condition is closed under products. We may therefore assume without loss of generality that A is a cyclic group of the form $\mathbf{Z}/p^t\,\mathbf{Z}$. Let M denote the Dieudonne module of $K(n)_0 K(\mathbf{Z}/p^t\,\mathbf{Z}, 1)$. Using Theorem 2.4.10, we can identify ν with the composition

$$\bigwedge^d_{W(\kappa)/p^t W(\kappa)} M \times \bigwedge^{n-d}_{W(\kappa)/p^t W(\kappa)} M \to \bigwedge^n_{W(\kappa)/p^t W(\kappa)} M \simeq W(\kappa)/p^t W(\kappa) \overset{p^{-t}}{\hookrightarrow} W(\kappa)[p^{-1}]/W(\kappa).$$

To prove that ν is a perfect pairing, it suffices to show that M is a free module of rank n over $W(\kappa)/p^t W(\kappa)$, which follows immediately from our assumption that $\mathbf{G}_0$ is a p-divisible group of height n over κ. $\square$

The remainder of this section is devoted to proving an analogue of Theorem 5.3.6 in the setting of spectral algebraic geometry. That is, we want a version of Theorem 5.3.6 which gives an identification of finite flat group schemes over the Lubin-Tate spectrum E, rather than over the residue field κ. We begin by reviewing some definitions.

Notation 5.3.7. For every ∞-category $\mathcal{C}$ which admits finite products, we let $\mathrm{CMon}(\mathcal{C})$ denote the ∞-category of commutative monoid objects of $\mathcal{C}$ (see §HA.2.4.2). In particular, we let $\mathrm{CMon}(\mathcal{S})$ denote the ∞-category of commutative monoid objects of the ∞-category $\mathcal{S}$ of spaces (that is, the ∞-category of $\mathbb{E}_\infty$-spaces). We will generally abuse notation by identifying an object $X \in \mathrm{CMon}(\mathcal{S})$ with its image under the forgetful functor $\mathrm{CMon}(\mathcal{S}) \to \mathcal{S}$. Note that if $X \in \mathrm{CMon}(\mathcal{S})$, then $\pi_0 X$ inherits the structure of a commutative monoid. We say that X is *grouplike* if $\pi_0 X$ is an abelian group. We let $\mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$ denote the full subcategory of $\mathrm{CMon}(\mathcal{S})$ spanned by the grouplike $\mathbb{E}_\infty$-spaces. The passage to zeroth spaces defines an equivalence of ∞-categories $\Omega^\infty : \mathrm{Sp}^{\mathrm{cn}} \to \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$, where $\mathrm{Sp}^{\mathrm{cn}}$ denotes the ∞-category of connective spectra.

Definition 5.3.8. Let A be an $\mathbb{E}_\infty$-ring, and let CAlg_A denote the ∞-category of $\mathbb{E}_\infty$-algebras over A. A *finite flat commutative group scheme* over A is a functor $G : \mathrm{CAlg}_A \to \mathrm{Sp}^{\mathrm{cn}}$ with the following property: the composite functor $\Omega^\infty \circ G : \mathrm{CAlg}_A \to \mathcal{S}$ is corepresentable by an A-algebra $B \in \mathrm{CAlg}_A$ which is finite flat over A. We let FF_A denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_A, \mathrm{Sp}^{\mathrm{cn}})$ spanned by the finite flat commutative group schemes over R.

Remark 5.3.9. Let A be an $\mathbb{E}_\infty$-ring, and let $\mathrm{CAlg}^{\mathrm{ff}}_A$ denote the full subcategory of CAlg_A spanned by those $\mathbb{E}_\infty$-algebras which are finite and flat over A. Combining the equivalence $\Omega^\infty : \mathrm{Sp}^{\mathrm{cn}} \simeq \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$ with the Yoneda embedding $(\mathrm{CAlg}^{\mathrm{ff}}_A)^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}_A, \mathcal{S})$, we obtain an equivalence of ∞-categories

$$\mathrm{FF}_A \simeq \mathrm{CMon}^{\mathrm{gp}}((\mathrm{CAlg}^{\mathrm{ff}}_A)^{\mathrm{op}}).$$

Put more informally: we can identify finite flat group schemes over A with commutative and cocommutative Hopf algebras in the ∞-category of finite flat A-module spectra.

Notation 5.3.10. The functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$ is lax symmetric monoidal, where we endow Sp with the smash product symmetric monoidal structure, and $\mathcal{S}$ with the usual symmetric monoidal structure. In particular, if A is an $\mathbb{E}_\infty$-ring, then the multiplication on A determines a commutative monoid structure on $\Omega^\infty A$. We let $\mathrm{GL}_1(A)$ denote the union of those connected components of $\Omega^\infty A$ which are invertible in $\pi_0 A$. The construction $A \mapsto \mathrm{GL}_1(A)$ determines a functor $\mathrm{GL}_1 : \mathrm{CAlg} \to \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$. We will generally abuse notation by identifying GL_1 with the corresponding functor $\mathrm{CAlg} \to \mathrm{Sp}^{\mathrm{cn}}$. We will also abuse notation by identifying GL_1 with its restriction to CAlg_A, where A is an arbitrary $\mathbb{E}_\infty$-ring.

Definition 5.3.11. Let A be an $\mathbb{E}_\infty$-ring, and let $G, H : \mathrm{CAlg}_A \to \mathrm{Sp}^{\mathrm{cn}}$ be finite flat commutative group schemes over A. A *Cartier pairing* between G and H is a natural transformation

$$G \otimes H \to \mathrm{GL}_1$$

of functors from CAlg_A to Sp. We let

$$\mathrm{Cart}(G, H) = \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_A, \mathrm{Sp}^{\mathrm{cn}})}(G \otimes H, \mathrm{GL}_1)$$

denote the space of Cartier pairings of G with H. The construction $(G, H) \mapsto \mathrm{Cart}(G, H)$ determines a functor $(\mathrm{FF}_A \times \mathrm{FF}_A)^{\mathrm{op}} \to \mathcal{S}$.

Let $\beta : G \otimes H \to \mathrm{GL}_1$ be a Cartier pairing of finite flat commutative group schemes over A. We say that β is *left universal* if, for every $H' \in \mathrm{FF}_A$, evaluation on β induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{FF}_A}(H', H) \to \mathrm{Cart}(G, H').$$

In this case, we say that H is the *Cartier dual* of G, and write $H = \mathbf{D}(G)$.

Remark 5.3.12. Let G be a finite flat commutative group scheme over an $\mathbb{E}_\infty$-ring A. It follows immediately from the definitions that a Cartier dual of G is determined uniquely (up to a contractible space of choices) if it exists. For existence, we refer the reader to [15]. Moreover, it is proven in [15] that if the functor $\Omega^\infty \circ G$ is representable by an $\mathbb{E}_\infty$-algebra H, then $\Omega^\infty \circ \mathbf{D}(G)$ is representable by the A-linear dual $H^\vee$ of A, endowed with an $\mathbb{E}_\infty$-algebra structure which is A-linear dual to the comultiplication on H.

Remark 5.3.13. In the situation of Definition 5.3.11, we also have the dual notion of a *right universal* Cartier pairing: a natural transformation $\beta : G \otimes H \to \mathrm{GL}_1$ with the property that, for every finite flat commutative group scheme G' over A, evaluation on β induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{FF}_A}(G', G) \to \mathrm{Cart}(G', H).$$

A Cartier pairing is left universal if and only if it is right universal (see [15]). In other words, a Cartier pairing $\beta : G \otimes H \to \mathrm{GL}_1$ exhibits G as a Cartier dual of H if and only if it exhibits H as a Cartier dual of G.

Remark 5.3.14. Let A be an $\mathbb{E}_\infty$-ring. The formation of Cartier duals determines an equivalence of ∞-categories $(\mathrm{FF}_A)^{\mathrm{op}} \simeq \mathrm{FF}_A$.

Our next goal is to produce some examples of finite flat commutative group schemes over ring spectra. In what follows, we fix a perfect field κ of characteristic $p > 0$ and a connected p-divisible group $\mathbf{G}_0$ of finite $n < \infty$ over κ. Let E denote the Lubin-Tate spectrum determined by $(\kappa, \mathbf{G}_0)$, and $K(n)$ the associated Morava K-theory spectrum.

Definition 5.3.15. We say that a space X is $K(n)$-*perfect* if $K(n)_0 X$ is a finite-dimensional vector space over κ, and $K(n)_i X \simeq 0$ when i is odd. We let $\mathcal{S}^{\mathrm{pf}}$ denote the full subcategory of $\mathcal{S}$ spanned by the $K(n)$-perfect spaces. It follows from Remark 2.1.21 that $\mathcal{S}^{\mathrm{pf}}$ is closed under finite products.

Notation 5.3.16. For every Kan complex X, we let $C^*(X; E)$ denote a limit of the constant diagram $X \to \mathrm{Sp}$ taking the value E, and $C_*(X; E)$ the colimit of the same diagram (so that $C^*(X; E)$ is the function spectrum of maps from X to E, and $C_*(X; E)$ is obtained by tensoring E with X). We will regard $C^*(X; E)$ as an $\mathbb{E}_\infty$-ring with the following universal property: for every $\mathbb{E}_\infty$-ring A, there is a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}}(A, C^*(X; E)) \simeq \mathrm{Map}_{\mathcal{S}}(X, \mathrm{Map}_{\mathcal{C}}(A, E)).$$

The construction $X \mapsto C^*(X; E)$ determines a functor from $\mathcal{S}^{op}$ to the ∞-category CAlg_E of $\mathbb{E}_\infty$-algebras over E. In particular, for every pair of spaces X and Y, there is a canonical map

$$C^*(X; E) \otimes_E C^*(Y; E) \to C^*(X \times Y; E)$$

Using Proposition 3.4.3 and Remark 2.1.21, we deduce that this map is an equivalence whenever X and Y are $K(n)$-perfect.

Definition 5.3.17. Let X be a $K(n)$-perfect space. We let ${}^+\mathrm{ESpec}(X)$ denote the functor $\mathrm{CAlg}_E \to \mathcal{S}$ given by the formula

$${}^+\mathrm{ESpec}(X)(A) = \mathrm{Map}_{\mathrm{CAlg}_E}(C^*(X; E), A).$$

Remark 5.3.18. Let X be a $K(n)$-perfect space. Proposition 3.4.3 implies that E^X is a free E-module of finite rank. It follows that ${}^+\mathrm{ESpec}(X)$ is representable by an affine nonconnective spectral Deligne-Mumford stack which is finite flat over E. Moreover, the underlying ordinary scheme of ${}^+\mathrm{ESpec}(X)$ is given by $\mathrm{ESpec}(X) = \mathrm{Spec}\, E^0(X)$.

Remark 5.3.19. The construction $X \mapsto {}^+\mathrm{ESpec}(X)$ determines a functor from $\mathcal{S}^{\mathrm{pf}} \to \mathrm{Fun}(\mathrm{CAlg}_E, \mathcal{S})$ which commutes with finite products. In particular, if X is a grouplike commutative monoid object of $\mathcal{S}$, then we can regard ${}^+\mathrm{ESpec}(X)$ as a finite flat commutative group scheme over E.

Using Theorem 2.4.10 and Remark 5.3.19, we can produce a large class of examples of finite flat commutative group schemes over E.

Definition 5.3.20. Let $\mathbf{Z}$ denote the ring of integers. We will identify $\mathbf{Z}$ with a discrete $\mathbb{E}_\infty$-ring, and let $\mathrm{Mod}_{\mathbf{Z}}$ denote the ∞-category of module spectra over $\mathbf{Z}$ (equivalently, we can described $\mathrm{Mod}_{\mathbf{Z}}$ as obtained from the ordinary category of chain complexes of abelian groups, by inverting quasi-isomorphisms). We will say that an object $M \in \mathrm{Mod}_{\mathbf{Z}}$ is p-*finite* if each homotopy group $\pi_m M$ is a finite abelian p-group, and the groups $\pi_m M$ vanish for $m \gg 0$ and $m \ll 0$. We let $\mathrm{Mod}_{\mathbf{Z}}^{p-\mathrm{fin}}$ denote the full subcategory of $\mathrm{Mod}_{\mathbf{Z}}$ spanned by the p-finite $\mathbf{Z}$-module spectra.

Remark 5.3.21. The construction $M \mapsto \pi_0 M$ induces an equivalence from the ∞-category of discrete $\mathbf{Z}$-module spectra to the ordinary category of abelian groups. We will abuse notation by identifying an abelian group A with its preimage under this equivalence (that is, with the corresponding Eilenberg-MacLane spectrum).

Since the commutative ring $\mathbf{Z}$ has projective dimension ≤ 1, every object $M \in \mathrm{Mod}_{\mathbf{Z}}$ splits (noncanonically) as a direct sum $\bigoplus_{m \in \mathbf{Z}} \Sigma^m(\pi_m M)$. In particular, M is p-finite if and only if it can be obtained as a finite product of $\mathbf{Z}$-module spectra of the form $\Sigma^m(\mathbf{Z}/p^t\,\mathbf{Z})$.

Construction 5.3.22. Let M be a p-finite $\mathbf{Z}$-module spectrum. Using Proposition 2.4.10 and Remark 5.3.21, we see that the space $\Omega^\infty M$ is $K(n)$-perfect. We can therefore view ${}^{+}\mathrm{ESpec}(\Omega^\infty M)$ as a finite flat commutative group scheme over E. We will regard the construction $M \mapsto {}^{+}\mathrm{ESpec}(\Omega^\infty M)$ as a functor from $\mathrm{Mod}_{\mathbf{Z}}^{p-\mathrm{fin}}$ to the ∞-category FF_E of finite flat commutative group schemes over E.

Let M be a p-finite $\mathbf{Z}$-module spectrum. Our next goal is to describe the Cartier dual ${}^{+}\mathrm{ESpec}(\Omega^\infty M)$. First, we need to introduce a bit more terminology.

Definition 5.3.23. Let M be a p-finite $\mathbf{Z}$-module spectrum. We define the *Pontryagin dual* M^* of M to be the mapping object $(\mathbf{Q}_p/\mathbf{Z}_p)^M$, formed in the symmetric monoidal ∞-category $\mathrm{Mod}_{\mathbf{Z}}$. More precisely, M^* is an object of $\mathrm{Mod}_{\mathbf{Z}}$ equipped with a map

$$\beta : M \otimes_{\mathbf{Z}} M^* \to \mathbf{Q}_p/\mathbf{Z}_p,$$

satisfying the following universal property: for every $\mathbf{Z}$-module spectrum N, composition with β induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(N, M^*) \to \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M \otimes_{\mathbf{Z}} N, \mathbf{Q}_p/\mathbf{Z}_p).$$

Remark 5.3.24. Let M be a finite abelian p-group, regarded as a discrete $\mathbf{Z}$-module spectrum. Then the Pontryagin dual M^* of Definition 5.3.23 is also discrete (since $\mathbf{Q}_p/\mathbf{Z}_p$ is an injective $\mathbf{Z}$-module), and agrees with the Pontryagin dual of M defined in Notation 5.3.5.

More generally, the Pontryagin dual M^* of an arbitrary p-finite $\mathbf{Z}$-module spectrum is determined up to equivalence by the existence of natural isomorphisms

$$\pi_m M^* \simeq \mathrm{Hom}(\pi_{-m} M, \mathbf{Q}_p/\mathbf{Z}_p).$$

We are now ready to state our main result.

Theorem 5.3.25. *Let κ be a perfect field of characteristic $p > 0$, let $\mathbf{G}_0$ be a connected p-divisible group of height n over κ, and let E denote the Lubin-Tate spectrum associated to $(\kappa, \mathbf{G}_0)$, and $K(n)$ the associated Morava K-theory. Fix a normalization $\nu : \mathrm{KSpec}\, K(\mathbf{Q}_p/\mathbf{Z}_p, n) \simeq \widehat{\mathbf{G}}_m$ of $\mathbf{G}_0$. Then for every p-finite $\mathbf{Z}$-module spectrum M, ν determines an equivalence of finite flat commutative group schemes*

$$\mathbf{D}(()\,{}^{+}\mathrm{ESpec}(\Omega^\infty M)) \simeq {}^{+}\mathrm{ESpec}(\Omega^{\infty-n} M^*),$$

depending functorially on $M \in \mathrm{Mod}_{\mathbf{Z}}^{p-fin}$.

Before giving the proof of Theorem 5.3.25, let us collect up some consequences. Note that the the construction $X \mapsto L_{K(n)} C_*(X; E)$ determines a symmetric monoidal functor from $\mathcal{S}$ into $\mathrm{Mod}_E(\mathrm{Sp}_{K(n)})$, which carries grouplike commutative monoid objects of $\mathcal{S}$ to $\mathbb{E}_\infty$-algebras over E. In the special case where X is $K(n)$-perfect, the induced comultiplication on the E-linear dual $C^*(X; E)$ of $L_{K(n)} C_*(X; E)$ underlies the Hopf algebra structure on $C^*(X; E)$ which determines the group structure on ${}^{+}\mathrm{ESpec}(X)$. Combining this observation with Theorem 5.3.25, we obtain the following:

Corollary 5.3.26. *In the situation of Theorem 5.3.25, a normalization ν of $\mathbf{G}_0$ determines equivalences*

$$L_{K(n)} C_*(\Omega^\infty M; E) \simeq C^*(\Omega^{\infty-n} M^*; E)$$

of $\mathbb{E}_\infty$-algebras over E, depending functorially on $M \in \mathrm{Mod}_{\mathbf{Z}}^{p-fin}$.

Corollary 5.3.27. *Suppose we are given a pullback diagram σ :*

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M_0 \arrow[d] \\ M_1 \arrow[r] & M_{01} \end{tikzcd}$$

in $\mathrm{Mod}_{\mathbf{Z}}^{p-fin}$. *Assume that each of the objects appearing in this diagram is n-truncated. Then the associated diagram*

$$\begin{tikzcd} C^*(\Omega^\infty M; E) & C^*(\Omega^\infty M_0; E) \arrow[l] \\ C^*(\Omega^\infty M_1; E) \arrow[u] & C^*(\Omega^\infty M_{01}; E) \arrow[l] \arrow[u] \end{tikzcd}$$

is a pushout square in $\mathrm{CAlg}(\mathrm{Mod}_E(\mathrm{Sp}_{K(n)}))$.

Proof. Using Corollary 5.3.26, we are reduced to proving that the diagram

$$\begin{tikzcd} L_{K(n)} C_*(\Omega^{\infty-n} M^*; E) & L_{K(n)} C_*([\Omega^{\infty-n} M_0^*; E) \arrow[l] \\ L_{K(n)} C_*(\Omega^{\infty-n} M^*; E) \arrow[u] & L_{K(n)} C_*(\Omega^{\infty-n} M_{01}^*; E) \arrow[l] \arrow[u] \end{tikzcd}$$

is a pushout square in $\mathrm{CAlg}(\mathrm{Mod}_E(\mathrm{Sp}_{K(n)}))$. Since the functor $X \mapsto L_{K(n)} C_*(X; E)$ is symmetric monoidal, it suffices to show that the diagram τ :

$$\begin{tikzcd} \Omega^{\infty-n} M^* & \Omega^{\infty-n} M_0^* \arrow[l] \\ \Omega^{\infty-n} M_1^* \arrow[u] & \Omega^{\infty-n} M_{01}^* \arrow[l] \arrow[u] \end{tikzcd}$$

is a pushout square in $\mathrm{CAlg}(\mathcal{S}) \simeq \mathrm{CMon}(\mathcal{S})$. Note that the full subcategory $\mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$ is closed under pushouts in $\mathrm{CMon}(\mathcal{S})$; it will therefore suffice to show that τ is a pushout diagram in $\mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$. The functor Ω^∞ is an equivalence from $\mathrm{Sp}^{\mathrm{cn}}$ to $\mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$; we are therefore reduced to proving that the diagram

$$\begin{tikzcd} \Sigma^n M^* & \Sigma^n M_0^* \arrow[l] \\ \Sigma^n M_1^* \arrow[u] & \Sigma^n M_{01}^* \arrow[l] \arrow[u] \end{tikzcd}$$

is a pushout diagram in $\mathrm{Sp}^{\mathrm{cn}}$. This follows immediately from our assumption that σ is a pullback diagram (the connectivity of the spectra appearing in this diagram follows from our assumption that σ is a diagram of n-truncated $\mathbf{Z}$-module spectra). □

Remark 5.3.28. The conclusion of Corollary 5.3.27 is equivalent to the convergence of the Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_*^{K(n)^* \Omega^\infty M_{01}}(K(n)^* \Omega^\infty M_0, K(n)^* \Omega^\infty M_1) \Rightarrow K(n)^* \Omega^\infty M.$$

We refer the reader to [1] for another proof of this convergence result.

Remark 5.3.29. In §5.4 we will prove a generalization of Corollary 5.3.27 (Theorem 5.4.8), where we replace E by an arbitrary $K(n)$-local $\mathbb{E}_\infty$-ring A, and allow Kan complexes which are not generalized Eilenberg-MacLane spaces.

We will reduce Theorem 5.3.25 to Theorem 5.3.6 by means of the following result:

Proposition 5.3.30. *Let κ be a perfect field of characteristic $p > 0$, let $\mathbf{G}_0$ be a connected p-divisible group of height n over κ, and let E denote the Lubin-Tate spectrum associated to $(\kappa, \mathbf{G}_0)$, and $K(n)$ the associated Morava K-theory. Define functors $\theta, \theta' : (\mathrm{Mod}_{\mathbf{Z}}^{p-fin})^{\mathrm{op}} \to \mathrm{FF}_E$ by the formulas*

$$\theta(M) = {}^{+}\mathrm{ESpec}(\Omega^{\infty-n} M^*) \qquad \theta'(M) = \mathbf{D}({}^{+}\mathrm{ESpec}(\Omega^{\infty} M)).$$

Then every normalization ν of $\mathbf{G}_0$ determines a natural transformation $\alpha : \theta \to \theta'$ with the following property: in the special case where A is a finite abelian p-group and $M = \Sigma^d A$, the map of finite flat group schemes

$$^{+}\mathrm{ESpec}(K(n-d, A^*) \to \mathbf{D}({}^{+}\mathrm{ESpec}(K(A, d)))$$

induces a map of ordinary schemes $\mathrm{ESpec}(K(n-d, A^*)) \to \mathbf{D}(\mathrm{ESpec}(K(A, d)))$ which extends the isomorphism $\mathrm{KSpec}\, K(n-d, A^*) \simeq \mathbf{D}(\mathrm{KSpec}\, K(A, d))$ of Theorem 5.3.6.

Proof of Theorem 5.3.25 from Proposition 5.3.30. It will suffice to show that the natural transformation $\alpha : \theta \to \theta'$ is an equivalence. Fix an object $M \in \mathrm{Mod}_{\mathbf{Z}}^{p-\mathrm{fin}}$; we wish to show that the map

$$\alpha(M) : {}^{+}\mathrm{ESpec}(\Omega^{\infty-n} M^*) \to \mathbf{D}({}^{+}\mathrm{ESpec}(\Omega^{\infty} M))$$

is an equivalence of finite flat group schemes over E. To prove this, it suffices to show that $\alpha(M)$ induces an equivalence of $\mathcal{S}$-valued functors: in other words, we must show that it underlies an equivalence of E-module spectra

$$L_{K(n)} C_*(\Omega^{\infty} M; E) \to C^*(\Omega^{\infty-n} M^*; E).$$

Since both sides are finite and flat over E, we are reduced to proving that the underlying map of commutative rings

$$\phi : E_0^{\wedge}(\Omega^{\infty} M) \to E^0(\Omega^{\infty-n} M^*)$$

is an isomorphism. The collection of those objects $M \in \mathrm{Mod}_{\mathbf{Z}}^{p-\mathrm{fin}}$ which satisfy this condition is closed under finite products. We may therefore assume without loss of generality that M has the form $\Sigma^d A$, for some finite abelian p-group A. Note that the domain and codomain of ϕ are free modules of finite rank over the local ring $\pi_0 E$. It follows that ϕ is an isomorphism if and only if the induced map of closed fibers

$$\kappa \otimes_{\pi_0 E} E_0^{\wedge}(\Omega^{\infty} M) \to \kappa \otimes_{\pi_0 E} E^0(\Omega^{\infty-n} M^*)$$

is an isomorphism of vector spaces over κ. If $d < 0$ or $d > n$, then both sides are isomorphic to κ and there is nothing to prove. Otherwise, the desired result follows from Theorem 5.3.6. ☐

The remainder of this section is devoted to the construction of the natural transformation α appearing in the proof of Proposition 5.3.30. Essentially, our goal is mimic the construction that precedes Theorem 5.3.6, working over the Lubin-Tate spectrum E rather than κ. Our first goal is to construct an analogue of the isomorphism $\nu : \mathrm{KSpec}\, K(\mathbf{Q}_p / \mathbf{Z}_p, n) \simeq \widehat{\mathbf{G}}_m$.

Definition 5.3.31. Let A be an $\mathbb{E}_\infty$-ring and let G be a finite flat commutative group scheme over A. We will say that G is *étale* if the functor $\Omega^\infty \circ G$ is corepresentable by an étale A-algebra H. We let $\mathrm{FF}_A^{\text{ét}}$ denote the full subcategory of FF_A spanned by the finite flat commutative group schemes which are étale over A.

Remark 5.3.32. The equivalence of ∞-categories $\mathrm{FF}_A \simeq \mathrm{CMon}^{\mathrm{gp}}((\mathrm{CAlg}_A^{\mathrm{ff}})^{\mathrm{op}})$ restricts to an equivalence $\mathrm{FF}_A^{\text{ét}} \simeq \mathrm{CMon}^{\mathrm{gp}}((\mathrm{CAlg}_A^{\text{fét}})^{\mathrm{op}})$, where $\mathrm{CAlg}_A^{\text{fét}}$ denotes the full subcategory of CAlg_A spanned by those A-algebras which are finite étale over R. Using Theorem HA.8.5.0.6, we see that $\mathrm{CAlg}_A^{\text{fét}}$ is equivalent to (the nerve of) the ordinary category of finite étale $(\pi_0 R)$-algebras. It follows that the ∞-category $\mathrm{FF}_A^{\text{ét}}$ is equivalent to the nerve of the ordinary category of étale local systems of finite abelian groups on the affine scheme $\mathrm{Spec}\, \pi_0 A$. In particular, if M is a finite abelian group, then the constant local system on $\mathrm{Spec}\, \pi_0 A$ taking the value M determines a finite flat group scheme over A, which we will denote by $\underline{M}$.

If A is a Henselian local ring with residue field κ, we further obtain an equivalence of $\mathrm{FF}_A^{\text{ét}} \simeq \mathrm{FF}_\kappa^{\text{ét}}$.

Example 5.3.33. Let A be an $\mathbb{E}_\infty$-ring. The functor $\mathrm{GL}_1 : \mathrm{CAlg}_A \to \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$ admits a left adjoint, which carries a grouplike commutative monoid object $M \in \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$ to the spectrum $C_*(M; A)$, which we can think of as the group algebra of M over A (it is a commutative algebra object of Mod_A, since the construction $X \mapsto C_*(X; A)$ determines a symmetric monoidal functor from $\mathcal{S}$ to Mod_A). In particular, for each $M \in \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$, the $\mathbb{E}_\infty$-ring $C_*(M; A)$ corepresents the functor $\mathrm{GL}_1^M : \mathrm{CAlg}_A \to \mathcal{S}$, given by the formula

$$\mathrm{GL}_1^M(B) = \mathrm{Map}_{\mathrm{CMon}(\mathcal{S})}(M, \mathrm{GL}_1(B)).$$

Note that GL_1^M has the structure of a grouplike commutative monoid in $\mathrm{Fun}(\mathrm{CAlg}_B, \mathcal{S})$. Let us now specialize to the case where M is a finite abelian group (regarded as a discrete space), so that $C_*(M; A)$ is a free A-module of finite rank. Then we can regard GL_1^M as a finite flat commutative group scheme over A. In the special case where $M = \mathbf{Z}/m\,\mathbf{Z}$ for some integer $m > 0$, we will denote GL_1^M by μ_m. Note that the unit element $1 \in \mathbf{Z}/m\,\mathbf{Z}$ determines a natural transformation $\mu_m \to \mathrm{GL}_1$.

Remark 5.3.34. Let A be an $\mathbb{E}_\infty$-ring, let M be a finite abelian group, and let GL_1^M be the finite flat group scheme over A described in Example 5.3.33, so that $\Omega^\infty \circ \mathrm{GL}_1^M$ is corepresented by the group algebra $H = C_*(M; A)$. The dual Hopf algebra $H^\vee$ is étale over R, and is therefore determined up to equivalence by an étale local system of finite abelian groups on the affine scheme $\mathrm{Spec}\,\pi_0 A$ (Remark 5.3.32). By inspection, this is the constant local system with value M. We therefore obtain a canonical equivalence $\mathbf{D}(\mathrm{GL}_1^M) \simeq \underline{M}$ of finite flat commutative group schemes over A. Specializing to the case where $M = \mathbf{Z}/m\,\mathbf{Z}$ is a cyclic group, we obtain equivalences

$$\mathbf{D}(\underline{\mathbf{Z}/m\,\mathbf{Z}}) \simeq \mu_m \qquad \mathbf{D}(\mu_m) \simeq \underline{\mathbf{Z}/m\,\mathbf{Z}}.$$

Remark 5.3.35. Let A be an $\mathbb{E}_\infty$-ring and let G be a finite flat commutative group scheme over A, so that the underlying functor $\Omega^\infty \circ G : \mathrm{CAlg}_A \to \mathcal{S}$ is corepresentable by an A-algebra H which is finite flat over A. Then $\pi_0 A$ is a commutative cocommutative Hopf algebra over the commutative ring $\pi_0 A$, so we can regard $\mathrm{Spec}\,\pi_0 H$ as a finite flat group scheme G_0 over $\pi_0 A$ (in the sense of classical algebraic geometry). Suppose that G_0 is split multiplicative: that is, that there is a finite abelian group M such that $\pi_0 A$ is isomorphic (as a Hopf algebra) to the group algebra $(\pi_0 R)[M]$. Then the Cartier duals of G and GL_1^M are étale and determine the same local system of finite abelian groups on $\mathrm{Spec}\,\pi_0 A$. Applying Remark 5.3.32, we see that the isomorphism $\pi_0 H \simeq (\pi_0 A)[M]$ lifts uniquely (up to a contractible space of choices) to an equivalence $G \simeq \mathrm{GL}_1^M$ of finite flat commutative group schemes over A.

Example 5.3.36. Let κ and $\mathbf{G}_0$ be as in Proposition 5.3.30, and let $\nu : \mathrm{KSpec}\,K(\mathbf{Q}_p/\mathbf{Z}_p, n) \simeq \widehat{\mathbf{G}}_m$ be a normalization of $\mathbf{G}_0$. Using Remark 5.3.35, we see that ν determines a compatible family of equivalences

$$^+\,\mathrm{ESpec}(K(\mathbf{Z}/p^t\,\mathbf{Z}, n)) \simeq \mu_{p^t}$$

of finite flat commutative group schemes over E.

Construction 5.3.37. Note that every p-finite $\mathbf{Z}$-module spectrum M is a compact object of $\mathrm{Mod}_{\mathbf{Z}}$. It follows that the inclusion $\mathrm{Mod}_{\mathbf{Z}}^{p-\mathrm{fin}} \hookrightarrow \mathrm{Mod}_{\mathbf{Z}}$ extends to a fully faithful embedding $\iota : \mathrm{Ind}(\mathrm{Mod}_{\mathbf{Z}}^{p-\mathrm{fin}}) \hookrightarrow \mathrm{Mod}_{\mathbf{Z}}$ which preserves filtered colimits; the essential image of ι is the full subcategory $\mathrm{Mod}_{\mathbf{Z}}^{p-nil}$ spanned by those $\mathbf{Z}$-module spectra M such that $M[p^{-1}] \simeq 0$.

Let $\Psi : \mathrm{Mod}_{\mathbf{Z}}^{p-\mathrm{fin}} \to \mathrm{Fun}(\mathrm{CAlg}_E, \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})) \simeq \mathrm{Fun}(\mathrm{CAlg}_E, \mathrm{Sp}^{\mathrm{cn}})$ denote the functor given by $\Psi(M) = {}^+\,\mathrm{ESpec}(\Omega^\infty M)$. Then Ψ admits an essential unique extension to a functor

$$\widehat{\Psi} : \mathrm{Mod}_{\mathbf{Z}}^{p-nil} \to \mathrm{Fun}(\mathrm{CAlg}_E, \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S}))$$

which commutes with filtered colimits. Using Example 5.3.36, we see that a normalization ν of the p-divisible group $\mathbf{G}_0$ determines a natural transformation

$$\overline{\nu} : \widehat{\Psi}(\Sigma^n\,\mathbf{Q}_p/\mathbf{Z}_p) \simeq \varinjlim_{t \geq 0} \mu_{p^t} \to \mathrm{GL}_1\,.$$

in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}_E, \mathrm{Sp}^{\mathrm{cn}})$.

We now turn to the construction of the natural transformation α appearing in Proposition 5.3.30. For this, it will be convenient to use the language of pairings of ∞-categories developed in §SAG.4.3.1, together with the description of the smash product of spectra given in [15].

Notation 5.3.38. Let VPair denote the ∞-category

$$(\mathrm{Mod}_{\mathbf{Z}}^{p-\mathrm{fin}} \times \mathrm{Mod}_{\mathbf{Z}}^{p-\mathrm{fin}}) \times_{\mathrm{Mod}_{\mathbf{Z}}} (\mathrm{Mod}_{\mathbf{Z}})_{/\Sigma^n \mathbf{Q}_p / \mathbf{Z}_p}$$

whose objects are triples (M, N, α), where M and N are p-finite $\mathbf{Z}$-module spectra and α is a map of $\mathbf{Z}$-modules from $M \otimes_{\mathbf{Z}} N$ into $\Sigma^n \mathbf{Q}_p / \mathbf{Z}_p$.

We let WPair denote the ∞-category

$$(\mathrm{FF}_E \times \mathrm{FF}_E) \times_{\mathrm{Fun}(\mathrm{CAlg}_E, \mathrm{Sp}^{\mathrm{cn}})} \mathrm{Fun}(\mathrm{CAlg}_E, \mathrm{Sp}^{\mathrm{cn}})_{/\,\mathrm{GL}_1}$$

whose objects are triples (G, H, β), where G and H are finite flat commutative group schemes over E, and β is a natural transformation from the pointwise smash product $G \otimes H$ into the functor $\mathrm{GL}_1 : \mathrm{CAlg}_E \to \mathrm{Sp}^{\mathrm{cn}}$.

Remark 5.3.39. The projection map $\lambda : \mathrm{VPair} \to \mathrm{Mod}_{\mathbf{Z}}^{p-\mathrm{fin}} \times \mathrm{Mod}_{\mathbf{Z}}^{p-\mathrm{fin}}$ is a right fibration, which we can regard as a pairing of the ∞-category $\mathrm{Mod}_{\mathbf{Z}}^{p-\mathrm{fin}}$ with itself, in the sense of Definition SAG.4.3.1.1. We note that for $M, N \in \mathrm{Mod}_{\mathbf{Z}}^{p-\mathrm{fin}}$, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M \otimes_{\mathbf{Z}} N, \Sigma^n \mathbf{Q}_p / \mathbf{Z}_p) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \Sigma^n N^*),$$

so that the pairing λ is left and right representable (see Definition SAG.4.3.1.2), with associated duality functor(s) given by $N \mapsto \Sigma^n N^*$.

Similarly, the projection map $\lambda' : \mathrm{WPair} \to \mathrm{FF}_E \times \mathrm{FF}_E$ is a right fibration, which we can regard as a pairing of FF_E with itself. This pairing is also left and right representable, with the associated duality functor(s) given by Cartier duality on FF_E.

Construction 5.3.40. Let ν be a normalization of $\mathbf{G}_0$. Let $\gamma : \mathrm{Mod}_{\mathbf{Z}} \to \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$ denote the composition of the forgetful functor $\mathrm{Mod}_{\mathbf{Z}} \to \mathrm{Sp}$ with the functor $\Omega^\infty : \mathrm{Sp} \to \mathrm{Mon}^{\mathrm{gp}}_{\mathrm{CRing}}(\mathcal{S})$. Suppose we are given an object $(M, N, \alpha) \in \mathrm{VPair}$. Then α determines a natural transformation

$$\alpha_0 : \gamma(M) \boxtimes \gamma(N) \to \gamma(\Sigma^n \mathbf{Q}_p / \mathbf{Z}_p) \circ \wedge$$

of functors from $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ to $\mathcal{S}$. We therefore obtain, for each $A \in \mathrm{CAlg}_E$, a natural transformation

$$\Psi(M)(A) \boxtimes \Psi(N)(A) \to \widehat{\Psi}(\Sigma^n \mathbf{Q}_p / \mathbf{Z}_p)(A) \circ \wedge.$$

Composing with the map $\overline{\nu} : \hat{\Psi}(\Sigma^n \mathbf{Q}_p / \mathbf{Z}_p) \to \mathrm{GL}_1$ of Construction 5.3.37, we obtain a natural transformation $\beta : \Psi(M) \otimes \Psi(N) \to \mathrm{GL}_1$ of functors from CAlg_E to $\mathrm{Sp}^{\mathrm{cn}}$. The construction

$$(M, N, \alpha) \mapsto (\Psi(M), \Psi(N), \beta)$$

determines a functor VPair $\to$ WPair, which fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{VPair} \arrow[r] \arrow[d] & \mathrm{WPair} \arrow[d] \\
\mathrm{Mod}_{\mathbf{Z}}^{p-\mathrm{fin}} \times \mathrm{Mod}_{\mathbf{Z}}^{p-\mathrm{fin}} \arrow[r, "\Psi \times \Psi"] & \mathrm{FF}_E \times \mathrm{FF}_E .
\end{tikzcd}$$

Proof of Proposition 5.3.30. Apply Proposition SAG.4.3.3.4 to the map of pairings given by Construction 5.3.40. $\square$

5.4 The Global Sections Functor

Let $K(n)$ denote a Morava K-theory spectrum, fixed throughout this section. Let A be a $K(n)$-local $\mathbb{E}_\infty$-ring, and let X be a Kan complex. The formation of global sections $\mathcal{F} \mapsto C^*(X;\mathcal{F})$ determines a functor

$$\mathrm{Fun}(X, \mathrm{Mod}_A(\mathrm{Sp}_{K(n)})) \to \mathrm{Mod}_A(\mathrm{Sp}_{K(n)}).$$

It follows from Theorem 5.2.1 that if X is a finite m-type for some integer m, then this functor preserves small colimits. Our first main result in this section asserts that, if X satisfies some slightly stronger conditions, then $\mathcal{F}$ can be recovered from $C^*(X;\mathcal{F})$, regarded as a module over the function spectrum $C^*(X;A)$ (Theorem 5.4.3). First, we need to introduce a bit of terminology:

Definition 5.4.1. Let X be a Kan complex and p a prime number. We will say that X is *p-finite* if the following conditions are satisfied:

(a) The set $\pi_0 X$ is finite.

(b) For each $x \in X$ and each $k > 0$, the homotopy group $\pi_k(X, x)$ is a finite p-group.

(c) The Kan complex X is m-truncated for some integer m.

Remark 5.4.2. The notion of p-finite Kan complex (Definition 5.4.1) is closely related to the notion of p-finite $\mathbf{Z}$-module spectrum (Definition 5.3.20). More precisely, a connective $\mathbf{Z}$-module spectrum M is p-finite if and only if $\Omega^\infty M \in \mathcal{S}$ is p-finite.

Throughout this section, we regard $\mathrm{Sp}_{K(n)}$ as a symmetric monoidal ∞-category with respect to the $K(n)$-localized smash product (Corollary 2.1.3), which we will denote by $\otimes : \mathrm{Sp}_{K(n)} \times \mathrm{Sp}_{K(n)} \to \mathrm{Sp}_{K(n)}$. For any Kan complex X, the symmetric monoidal structure on $\mathrm{Sp}_{K(n)}$ induces a symmetric monoidal structure on $\mathrm{Fun}(X, \mathrm{Sp}_{K(n)})$, given by pointwise tensor product. If $f : X \to Y$ is a map of Kan complexes, then the pullback functor $f^* : \mathrm{Fun}(Y, \mathrm{Sp}_{K(n)}) \to \mathrm{Fun}(X, \mathrm{Sp}_{K(n)})$ is symmetric monoidal. It follows that the pushforward functor $f_* : \mathrm{Fun}(X, \mathrm{Sp}_{K(n)}) \to \mathrm{Fun}(Y, \mathrm{Sp}_{K(n)})$ is lax symmetric monoidal. In particular, f_* carries algebra objects $\mathcal{A} \in \mathrm{Alg}(\mathrm{Fun}(X, \mathrm{Sp}_{K(n)}))$ to algebra objects of $\mathrm{Fun}(Y, \mathrm{Sp}_{K(n)})$, and $\mathcal{A}$-module objects of $\mathrm{Fun}(X, \mathrm{Sp}_{K(n)})$ to $f_* \mathcal{A}$-module objects of $\mathrm{Fun}(Y, \mathrm{Sp}_{K(n)}$.

We can now state our main result:

Theorem 5.4.3. *Let $K(n)$ be a Morava K-theory spectrum of height $n > 0$, let p denote the characteristic of the residue field $\pi_0 K(n)$, let $f : X \to Y$ be a map of Kan complexes, and let $\mathcal{A} \in \mathrm{Alg}(\mathrm{Fun}(X, \mathrm{Sp}_{K(n)}))$ be a local system of $K(n)$-local $\mathbb{E}_1$-rings on X. Then:*

(1) *Assume that the homotopy fibers of f are finite m-types, for some integer m. Then the pushforward functor*
$$G : \mathrm{LMod}_{\mathcal{A}}(\mathrm{Fun}(X, \mathrm{Sp}_{K(n)})) \to \mathrm{LMod}_{f_* \mathcal{A}}(\mathrm{Fun}(Y, \mathrm{Sp}_{K(n)}))$$
has a fully faithful left adjoint.

(2) *Assume that each homotopy fiber of f is p-finite and n-truncated. Then the functor G is an equivalence of ∞-categories.*

Corollary 5.4.4. *In the situation of Theorem 5.4.3, let X be a Kan complex which is n-truncated and p-finite. Then for any $K(n)$-local $\mathbb{E}_1$-ring, the global sections functor*

$$\mathrm{Fun}(X, \mathrm{LMod}_A(\mathrm{Sp}_{K(n)})) \to \mathrm{LMod}_{C^*(X;A)}(\mathrm{Sp}_{K(n)})$$

is an equivalence of ∞-categories.

Corollary 5.4.5. *Let $f : X \to Y$ be a map of Kan complexes.*

(1) *If the homotopy fibers of f are finite m-types and $(n+1)$-connective, then the pullback functor*

$$f^* : \mathrm{Fun}(Y, \mathrm{Sp}_{K(n)}) \to \mathrm{Fun}(X, \mathrm{Sp}_{K(n)})$$

is fully faithful.

(2) *If the homotopy fibers of f are finite m-types and $(n+2)$-connective, then f^* is an equivalence of ∞-categories.*

Proof. Let S denote the $K(n)$-local sphere spectrum. For every space Z, let $\underline{S}_Z$ denote the constant local system on Z taking the value S. We first show that the following properties of a map $f : X \to Y$ are equivalent:

(a) The homotopy fibers of f are finite m-types and the pullback functor f^* is fully faithful.

(b) The homotopy fibers of f are finite m-types and the unit map $u_f : \underline{S}_Y \to f_* f^* \underline{S}_Y \simeq f_* \underline{S}_X$ is an equivalence.

The implication $(a) \Rightarrow (b)$ is obvious, and the converse follows from Theorem 5.4.3. Let us say that a map $f : X \to Y$ is *good* if it satisfies the equivalent conditions (a) and (b). We immediately conclude the following:

(i) A map $f : X \to Y$ is good if and only if, for each point $y \in Y$, the induced map $X \times_Y \{y\} \to \{y\}$ is good (this follows immediately from description (b)).

(ii) If $f : X \to Y$ and $g : Y \to Z$ are good, then the composite map $g \circ f : X \to Z$ is good (this follows immediately from description (a)).

To prove (1), we must show that if the homotopy fibers of $f : X \to Y$ are $(n+1)$-connective finite m-types, then f is good. Using (i), we can reduce to the case where Y is a single point. Then X is a finite m-type for some integer m. We proceed by induction on m. If $m \leq n$, then X is contractible and the result is obvious. Otherwise, we have $m > n$ and the inductive hypothesis implies that $\tau_{\leq m-1} X$ is good. Using (ii), we are reduced to proving that the map $X \to \tau_{\leq m-1} X$ is good. Using (i), we are reduced to showing that for every finite abelian group A, the map $K(A, m) \to *$ is good. We proceed by induction on the order of A. If A is trivial, then $K(A, m)$ is contractible and the result is obvious. Otherwise, there exists a subgroup $A' \subseteq A$ which is cyclic of prime order. Set $A'' = A/A'$. The inductive hypothesis implies that $K(A'', m)$ is good. Using (b), we are reduced to showing that the map $K(A, m) \to K(A'', m)$ is good. Using (a), we are reduced to proving that $K(A', m)$ is good. Write $A' = \mathbf{Z}/l\,\mathbf{Z}$ for some prime number l, and let $X = K(A', m)$. We wish to show that the unit map $S \to C^*(X; S)$ is an equivalence. We note that this unit map is the Spanier-Whitehead dual of canonical map $v : L_{K(n)} \Sigma^\infty_+ (X) \to S$. It will therefore suffice to show that v is an equivalence in $\mathrm{Sp}_{K(n)}$. This is equivalent to the assertion that v induces an isomorphism on $K(n)$-homology groups $K(n)_* X \to \pi_* K(n)$. For $l \neq p$ this is easy (and valid for any $m > 0$); when $l = p$ it follows from Theorem 2.4.10. This completes the proof of (1).

We now prove (2). Suppose that the homotopy fibers of f are $(n+2)$-connective finite m-types; we wish to show that f^* is an equivalence of ∞-categories. We may assume without loss of generality that Y is a point, so that there exists a section $e : Y \to X$ of the map f. Let $M \in \mathrm{LocSys}(\mathrm{Sp}_{K(n)})_X$; we wish to show that f lies in the essential image of f^*. We note that $e^* M \simeq e^* f^* e^* M$. Since the functor e^* is fully faithful by (1), we deduce that M is equivalent to $f^* e^* M$, and therefore belongs to the essential image of f^*, as desired. □

The proof of Theorem 5.4.3 will require some preliminaries.

Proposition 5.4.6 (Push-Pull Formula)**.** *Let $f : X \to Y$ be a map of spaces. Suppose we are given local systems $\mathcal{A} \in \mathrm{Alg}(\mathrm{Fun}(Y, \mathrm{Sp}_{K(n)}))$, $\mathcal{M} \in \mathrm{RMod}_{f^* \mathcal{A}}(\mathrm{Fun}(X, \mathrm{Sp}_{K(n)}))$, and $\mathcal{N} \in \mathrm{LMod}_{\mathcal{A}}(\mathrm{Fun}(Y, \mathrm{Sp}_{K(n)}))$. If the homotopy fibers of f are finite m-types, then the canonical map*

$$\beta_{\mathcal{M}, \mathcal{N}} : (f_* \mathcal{M}) \otimes_{\mathcal{A}} \mathcal{N} \to f_*(\mathcal{M} \otimes_{f^* \mathcal{A}} f^* \mathcal{N})$$

is an equivalence in $\mathrm{Fun}(Y, \mathrm{Sp}_{K(n)})$.

Proof. We may assume without loss of generality that Y consists of a single point, so that we can identify the local system $\mathcal{A}$ with a $K(n)$-local $\mathbb{E}_1$-ring A. Let us regard $\mathcal{M}$ as fixed, and let $\mathcal{C} \subseteq \mathrm{LMod}_A(\mathrm{Sp}_{K(n)})$ denote the full subcategory spanned by those objects $\mathcal{N}$ for which $\beta_{M,N}$ is an equivalence. Since the functor f_* preserves small colimits (Proposition 5.2.1), we conclude that $\mathcal{C}$ is closed under small colimits. Since $\mathrm{LMod}_A(\mathrm{Sp}_{K(n)})$ is generated under small colimits by the objects $\{\Sigma^m A\}_{m \in \mathbf{Z}}$, we are reduced to proving that $\beta_{\mathcal{M},\mathcal{N}}$ is an equivalence when $\mathcal{N}$ has the form $\Sigma^m A$, which is clear. $\square$

Corollary 5.4.7. *Let X be a finite m-type, let A be an $\mathbb{E}_1$-ring which is $K(n)$-local, and let M be a left A-module which is $K(n)$-local. Then the canonical map $C^*(X;A) \otimes M \to C^*(X;M)$ is an equivalence of $K(n)$-local spectra.*

Proof of Theorem 5.4.3. The functor G admits a left adjoint F, given by the formula

$$F(\mathcal{M}) = \mathcal{A} \otimes_{f^* f_* \mathcal{A}} f^* \mathcal{M}.$$

Assume that the homotopy fibers of f are finite m-types; we will show that F is fully faithful. To prove this, we must show that for each $\mathcal{N} \in \mathrm{LMod}_{f_* \mathcal{A}}(\mathrm{Fun}(Y, \mathrm{Sp}_{K(n)}))$, the unit map $u_{\mathcal{N}} : \mathcal{N} \to (G \circ F)(\mathcal{N})$ is an equivalence in $\mathrm{LMod}_{f_* \mathcal{N}}(\mathrm{Fun}(Y, \mathrm{Sp}_{K(n)}))$. To prove this, it suffices to show that for each point $y \in Y$, $u_{\mathcal{N}}$ induces an equivalence $\mathcal{N}_y \to (G \circ F)(\mathcal{N})_y$. Replacing X by the homotopy fiber $X \times_Y \{y\}$, we can reduce to the case where Y consists of a single point. Let $\mathcal{C} = \mathrm{LMod}_{f_* \mathcal{A}}(\mathrm{Sp}_{K(n)})$, and let $\mathcal{C}_0$ denote the full subcategory of $\mathcal{C}$ spanned by those objects $\mathcal{N}$ for which the unit map $u_{\mathcal{N}}$ is an equivalence. Since f is $\mathrm{Sp}_{K(n)}$-ambidextrous (Theorem 5.2.1), the pushforward functor f_* preserves small colimits. It follows that $\mathcal{C}_0$ is closed under small colimits in $\mathcal{C}$. Since $\mathcal{C}$ is generated under small colimits by objects of the form $\Sigma^m f_* \mathcal{A}$ and $\mathcal{C}_0$ is closed under shifts, we are reduced to proving that the unit map $u_{\mathcal{N}}$ is an equivalence in the special case $\mathcal{N} = f_* \mathcal{A}$, in which case the result is clear. This completes the proof of (1).

We now turn to the proof of (2). Assume that the homotopy fibers of f are p-finite and n-truncated; we wish to show that G is an equivalence of ∞-categories. The first part of the proof shows that G admits a fully faithful left adjoint F. It will therefore suffice to show that G is conservative. For this, it suffices to show that the global sections functor

$$f_* : \mathrm{Fun}(X, \mathrm{Sp}_{K(n)}) \to \mathrm{Fun}(Y, \mathrm{Sp}_{K(n)})$$

is conservative. That is, we wish to show that if $\mathcal{M} \in \mathrm{Fun}(X, \mathrm{Sp}_{K(n)})$ is nonzero, then $f_* \mathcal{M}$ is nonzero. In fact, we claim that if $\mathcal{M}_x$ is nonzero for some point $x \in X$, then $(f_* \mathcal{M})_{f(x)}$ is nonzero. To prove this, we can replace f by the map $X \times_Y \{f(x)\} \to \{f(x)\}$ and thereby reduce to the case where Y is a single point.

We now prove by induction on m that if $m \leq n$ and X is m-truncated, then the global sections functor $f_* : \mathrm{Fun}(X, \mathrm{Sp}_{K(n)}) \to \mathrm{Sp}_{K(n)}$ is conservative. If $m = 0$, then X is homotopy equivalent to a discrete space and the result is obvious. To carry out the inductive step, it suffices to show that the pushforward functor $\mathrm{Fun}(X, \mathrm{Sp}_{K(n)}) \to \mathrm{Fun}(\tau_{\leq m-1} X, \mathrm{Sp}_{K(n)})$ is conservative. Repeating the above argument, we can reduce to the case where $\tau_{\leq m-1} X$ is contractible. Then X is an Eilenberg-MacLane space $K(G, m)$, where $1 \leq m \leq n$ and G is a finite p-group (which is abelian if $m > 1$).

We now proceed by induction on the order of G. If G is trivial, then X is contractible and there is nothing to prove. Otherwise, G contains a normal subgroup of order p. We then have an exact sequence of groups

$$0 \to \mathbf{Z}/p\,\mathbf{Z} \to G \to G' \to 0,$$

which gives rise to a fiber sequence of spaces

$$K(\mathbf{Z}/p\,\mathbf{Z}, m) \to K(G, m) \to K(G', m).$$

The inductive hypothesis implies that the global sections functor $\mathrm{Fun}(K(G', m), \mathrm{Sp}_{K(n)}) \to \mathrm{Sp}_{K(n)}$ is conservative. We are therefore reduced to proving that the pushforward functor $\mathrm{Fun}(K(G,m), \mathrm{Sp}_{K(n)} \to \mathrm{Fun}(K(G', m), \mathrm{Sp}_{K(n)})$ is conservative. Working fiberwise, we are reduced to proving that the global sections functor $\mathrm{Fun}(K(\mathbf{Z}/p\,\mathbf{Z}, m), \mathrm{Sp}_{K(n)}) \to \mathrm{Sp}_{K(n)}$ is conservative. To prove this, it will suffice to verify assertion

(2) in the special case where $X = K(\mathbf{Z}/p\,\mathbf{Z}, m)$, Y is a point, and $\mathcal{A}$ is the constant local system taking the value S, where S denotes the $K(n)$-local sphere spectrum (that is, the unit object of the symmetric monoidal ∞-category $\mathrm{Sp}_{K(n)}$).

For $\mathcal{M} \in \mathrm{Fun}(X, \mathrm{Sp}_{K(n)})$, let $v_{\mathcal{M}} : (F \circ G)(\mathcal{M}) \to \mathcal{M}$ denote the counit map. Let $\mathcal{D} \subseteq \mathrm{Fun}(X, \mathrm{Sp}_{K(n)})$ be the full subcategory spanned by those objects $\mathcal{M}$ for which $v_{\mathcal{M}}$ is an equivalence. We wish to show that $\mathcal{D} = \mathrm{Fun}(X, \mathrm{Sp}_{K(n)})$. Since the functor f_* preserves small colimits, the ∞-category $\mathcal{D}$ is closed under small colimits in $\mathrm{LocSys}(\mathrm{Sp}_{K(n)})_X$. Choose a base point $x \in X$, and let $e : \{x\} \to X$ denote the inclusion. We have a canonical equivalence of ∞-categories $\mathrm{Sp}_{K(n)} \simeq \mathrm{Fun}(\{x\}, \mathrm{Sp}_{K(n)})$; we will henceforth identify S with its image in $\mathrm{Fun}(\{x\}, \mathrm{Sp}_{K(n)})$. Since X is connected, Lemma 4.3.8 implies that $\mathrm{Fun}(X, \mathrm{Sp}_{K(n)})$ is generated under small colimits by the essential image of the functor $e_!$. Since $\mathrm{Sp}_{K(n)}$ is generated under small colimits by the objects $\{\Sigma^k S\}_{k \in \mathbf{Z}}$, we are reduced to proving that $e_! S \in \mathcal{D}$. Since the map e is $\mathrm{Sp}_{K(n)}$-ambidextrous (Theorem 5.2.1), we have an equivalence $e_! S \simeq e_* S$. We are therefore reduced to proving that $e_* S \in \mathcal{D}$. In other words, we wish to show that the canonical map

$$\alpha : \underline{S} \otimes_{f^* C^*(X;S)} (f^* f_* e_* S) \to e_* S$$

is an equivalence in $\mathrm{Fun}(X, \mathrm{Sp}_{K(n)})$; here $\underline{S}$ denotes the unit object of $\mathrm{Fun}(X, \mathrm{Sp}_{K(n)})$. Choose a point $y \in X$, and let $P_{x,y} = \{x\} \times_X \{y\}$ denote the space of paths from x to y in X. Unwinding the definitions, we see that α induces an equivalence after evaluation at y if and only if the diagram σ :

$$\begin{tikzcd} C^*(X;S) \arrow[r] \arrow[d] & C^*(\{x\};S) \arrow[d] \\ C^*(\{y\};S) \arrow[r] & C^*(P_{x,y};S) \end{tikzcd}$$

is a pushout square in $\mathrm{CAlg}(\mathrm{Sp}_{K(n)})$. This follows from Corollary 5.3.27.

We may assume without loss of generality that the field $\pi_0 K(n)$ is algebraically closed. Let E denote the Lubin-Tate spectrum associated to , and let $\Phi : \mathrm{CAlg}(\mathrm{Sp}_{K(n)}) \to \mathrm{CAlg}(\mathrm{Mod}_E(\mathrm{Sp}_{K(n)}))$ denote a left adjoint to the forgetful functor. Since Φ is conservative, we are reduced to proving that $\Phi(\sigma)$ is a pushout square in $\mathrm{CAlg}(\mathrm{Mod}_E(\mathrm{Sp}_{K(n)}))$, which follows from Corollaries 5.4.7 and 5.3.27. $\square$

We now use Theorem 5.4.3 to prove a generalization of Corollary 5.3.27.

Theorem 5.4.8. *Let A be a $K(n)$-local $\mathbb{E}_\infty$-ring. Suppose we are given a pullback diagram of spaces.*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ Y' \arrow[r, "f"] & Y. \end{tikzcd}$$

Assume that Y is n-truncated and p-finite and that X is a finite m-type, for some integer m. Then the diagram

$$\begin{tikzcd} C^*(X';A) & C^*(X;A) \arrow[l] \\ C^*(Y';A) \arrow[u] & C^*(Y;A) \arrow[l] \arrow[u] \end{tikzcd}$$

is a pushout square in $\mathrm{CAlg}(\mathrm{Sp}_{K(n)})$.

Remark 5.4.9. Like Corollary 5.3.27, Theorem 5.4.8 can also be deduced from the main result of [1].

We will give the proof of Theorem 5.4.8 at the end of this section. For now, let us deduce some consequences.

Corollary 5.4.10. *Let A be a $K(n)$-local $\mathbb{E}_\infty$-ring, and let $\mathcal{S}^{p-fin}_{\leq n}$ denote the full subcategory of $\mathcal{S}$ spanned by those spaces which are p-finite and n-truncated. The following conditions are equivalent:*

(1) *The functor $\mathcal{S}^{p-fin}_{\leq n} \to \mathrm{CAlg}_A^{\mathrm{op}}$ given by $X \mapsto C^*(X; A)$ is fully faithful.*

(2) *For every n-truncated, p-finite space X, the canonical map $X \to \mathrm{Map}_{\mathrm{CAlg}_A}(C^*(X; A), A)$ is a homotopy equivalence.*

(3) *If $X = K(\mathbf{Z}/p\,\mathbf{Z}, n)$, then the canonical map $X \to \mathrm{Map}_{\mathrm{CAlg}_A}(C^*(X; A), A)$ is a homotopy equivalence.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. We next show that (2) implies the following stronger version of (1):

(1′) Let X and Y be spaces. If X is n-truncated and p-finite, then the canonical map

$$\theta_{X,Y} : \mathrm{Map}_{\mathcal{S}}(Y, X) \to \mathrm{Map}_{\mathrm{CAlg}_A}(C^*(X; A), C^*(Y; A))$$

is a homotopy equivalence.

Note that the collection of those spaces Y for which $\theta_{X,Y}$ is a homotopy equivalence is closed under small colimits in $\mathcal{S}$. Consequently, to show that $\theta_{X,Y}$ is a homotopy equivalence for all $Y \in \mathcal{S}$, it suffices to show that $\theta_{X,*}$ is a homotopy equivalence: that is, that X satisfies condition (2).

We now complete the proof by showing that (3) $\to$ (2). Let $F : \mathcal{S} \to \mathcal{S}$ denote the functor given by $F(X) = \mathrm{Map}_{\mathrm{CAlg}_A}(C^*(X; A), A)$. We have an evident natural transformation $\alpha : \mathrm{id} \to F$. Let $\mathcal{C} \subseteq \mathcal{S}^{p-\mathrm{fin}}_{\leq n}$ denote the full subcategory spanned by those spaces X for which the map $\alpha_X : X \to F(X)$ is an equivalence. We wish to prove that $\mathcal{C} = \mathcal{S}^{p-\mathrm{fin}}_{\leq n}$. We proceed in several steps:

(a) Using assumption (3), we see $K(\mathbf{Z}/p\,\mathbf{Z}, n) \in \mathcal{C}$.

(b) Theorem 5.4.8 implies that the restriction of F to $\mathcal{S}^{p-\mathrm{fin}}_{\leq n}$ preserves finite limits. It follows that $\mathcal{C}$ is closed under finite limits in $\mathcal{S}^{p-\mathrm{fin}}_{\leq n}$.

(c) Combining (a) and (b), we deduce that $K(\mathbf{Z}/p\,\mathbf{Z}, m) \in \mathcal{C}$ for $m \leq n$.

(d) Taking $m = 0$ in (c), we conclude that the finite set $\mathbf{Z}/p\,\mathbf{Z}$ belongs to $\mathcal{C}$. Combining this with (b), we deduce that $\mathcal{C}$ contains every finite discrete space.

(e) Suppose we are given a fibration $f : X \to Y$, where X and Y belong to $\mathcal{S}^{p-\mathrm{fin}}_{\leq n}$. Assume that each fiber X_y belongs to $\mathcal{C}$. Using the left-exactness of F, we deduce that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ F(X) \arrow[r] & F(Y) \end{tikzcd}$$

is a pullback square. If we also assume that $Y \in \mathcal{C}$, it follows that $X \in \mathcal{C}$.

(f) Let $1 \leq m \leq n$, and let G be a p-group which is abelian if $m > 1$. We claim that $K(G, m)$ belongs to $\mathcal{C}$. The proof proceeds by induction on the order of G. If G is trivial, the result is obvious. Otherwise, we can choose an exact sequence

$$0 \to G' \to G \to G'' \to 0,$$

where G' is isomorphic to the cyclic group $\mathbf{Z}/p\,\mathbf{Z}$. Then $K(G', m) \in \mathcal{C}$ by (c), and $K(G'', m) \in \mathcal{C}$ by the inductive hypothesis. Using (e), we deduce that $K(G, m)$ belongs to $\mathcal{C}$.

(g) We now prove that if $X \in \mathcal{S}^{p-\mathrm{fin}}_{\leq m}$ for $m \leq n$, then $X \in \mathcal{C}$. The proof proceeds by induction on m. If $m = 0$, then the desired result follows from (d). Otherwise, the inductive hypothesis implies that $\tau_{\leq m-1} X \in \mathcal{C}$. Since the homotopy fibers of the map $X \to \tau_{\leq m-1} X$ belong to $\mathcal{C}$ by (f), we conclude that $X \in \mathcal{C}$ using (e).

We now complete the proof by taking $m = n$ in step (g). □

Remark 5.4.11. Let A be a $K(n)$-local $\mathbb{E}_\infty$-ring, and let $Z(A) = \mathrm{Map}_{\mathrm{CAlg}_A}(C^*(K(\mathbf{Z}/p\,\mathbf{Z}), n), A)$. The evident map $K(\mathbf{Z}/p\,\mathbf{Z}, n) \to Z(A)$ determines a base point $\eta \in Z(A)$.

Since the functor F appearing in the proof of Corollary 5.4.10 is left exact when restricted to $\mathcal{S}^{p-\mathrm{fin}}_{\leq n}$, we have a canonical homotopy equivalence

$$\Omega^n Z(A) \simeq \mathrm{Map}_{\mathrm{CAlg}_A}(C^*(\mathbf{Z}/p\,\mathbf{Z}; A), A).$$

Since $C^*(\mathbf{Z}/p\,\mathbf{Z}; A)$ is an étale A-algebra, Theorem HA.8.5.0.6 implies that $\mathrm{Map}_{\mathrm{CAlg}_A}(C^*(\mathbf{Z}/p\,\mathbf{Z}; A), A)$ is homotopy equivalent to the discrete space of all $\pi_0 A$-algebra maps from $(\pi_0 A)^{\mathbf{Z}/p\,\mathbf{Z}}$ into $\pi_0 A$. In particular, if the affine scheme $\mathrm{Spec}(\pi_0 A)$ is connected, we obtain isomorphisms

$$\pi_m(Z(A), \eta) \simeq \begin{cases} 0 & \text{if } m > n \\ \mathbf{Z}/p\,\mathbf{Z} & \text{if } m = n. \end{cases}$$

Consequently, A satisfies the equivalent conditions of Corollary 5.4.10 if and only if the space $Z(A)$ is n-connective.

Remark 5.4.12. In the case where A is a Lubin-Tate spectrum, Remark 5.4.11 is a theorem of Sati and Westerland; see [19].

Remark 5.4.13. Let E be a Lubin-Tate spectrum of height n, and suppose that the residue field of E is algebraically closed. In this case, Example 5.3.36 gives an isomorphism of finite flat group schemes ${}^{+}\mathrm{ESpec}\, K(\mathbf{Z}/p\,\mathbf{Z}, n) \simeq \mu_p$. We therefore obtain a canonical homotopy equialence $Z(E) \simeq \mu_p(E) = \mathrm{Map}_{\mathrm{CMon}(\mathcal{S})}(\mathbf{Z}/p\,\mathbf{Z}, \mathrm{GL}_1(E))$.

Conjecture 5.4.14. Let E be a Lubin-Tate spectrum of height n with algebraically closed residue field. Then E satisfies the equivalent conditions of Corollary 5.4.10. Equivalently, the space

$$\mu_p(E) = \mathrm{Map}_{\mathrm{CMon}}(\mathcal{S})(\mathbf{Z}/p\,\mathbf{Z}, \mathrm{GL}_1(E))$$

is n-connective (and therefore homotopy equivalent to an Eilenberg-MacLane space $K(\mathbf{Z}/p\,\mathbf{Z}, n)$).

We now turn to the proof of Theorem 5.2.1. We begin by considering the special case where the fiber product is an ordinary product.

Lemma 5.4.15 (KünnethFormula)**.** *Let X and Y be spaces, and let A be a $K(n)$-local $\mathbb{E}_\infty$-ring. If X is finite m-type, then the canonical map*

$$C^*(X; A) \otimes_A C^*(Y; A) \to C^*(X \times Y; A)$$

is an equivalence of $K(n)$-local $\mathbb{E}_\infty$-rings.

Proof. Let $f : X \to *$ denote the projection map. Using Corollary 5.4.7 we obtain equivalences

$$\begin{aligned} C^*(X \times Y; A) &\simeq f_* f^* C^*(Y; A) \\ &\simeq (f_* f^* A) \otimes_A C^*(Y; A) \\ &\simeq C^*(X; A) \otimes_A C^*(Y; A). \end{aligned}$$

□

Proof of Theorem 5.4.8. For every space Z, let $\underline{A}_Z$ denote the constant local system on Z taking the value A. Since Y is n-truncated and p-finite, Theorem 5.4.3 yields an equivalence of ∞-categories

$$\mathrm{Mod}_{\underline{A}_Y}(\mathrm{LocSys}(\mathrm{Sp}_{K(n)})_Y) \to \mathrm{Mod}_{C^*(Y;A)}(\mathrm{Sp}_{K(n)}).$$

This equivalence is right adjoint to a symmetric monoidal functor, and therefore symmetric monoidal. Consequently, to prove that the diagram $C^*(\sigma; A)$ is a pushout square, it will suffice to show that the diagram

$$\begin{tikzcd} \underline{A}_Y \arrow[r] \arrow[d] & g_* \underline{A}_X \arrow[d] \\ f_* \underline{A}_{Y'} \arrow[r] & q_* \underline{A}_{X'} \end{tikzcd}$$

is a pushout square in $\mathrm{CAlg}(\mathrm{Fun}(Y, \mathrm{Sp}_{K(n)}))$, where $q : X' \to Y$ denotes the map appearing in the diagram σ. This assertion can be checked pointwise on Y. We may therefore reduce to the case where Y is a point, in which case the desired result follows from Lemma 5.4.15. $\square$

References

[1] Bauer, T. *Convergence of the Eilenberg-Moore spectral sequence for generalized cohomology theories.* Available for download at http://arxiv.org/abs/0803.3798.

[2] Buchstaber, V. and A. Lazarev. *Dieudonné modules and p-divisible groups associated with Moarava K-theory of Eilenberg-MacLane spaces.* Algebr. Geom. Topol. 7 (2007), 529-564.

[3] Demazure, M. *Lectures on p-divisible groups.* Lecture notes in Mathematics 302 (1972), Berlin, New York: Springer-Verlag.

[4] Goerss, P. *Hopf rings, Dieudonné modules, and* $E_*\Omega^2 S^3$. Homotopy invariant algebraic structures (Baltimore, MD, 1998), 115174, Contemp. Math., 239, Amer. Math. Soc., Providence, RI, 1999.

[5] Grothendieck, A. *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : III. Étude cohomologique des faisceaux cohérents, Première partie.* Publications Mathématiques de l'IHÈS, 11 (1961), p. 5-167.

[6] Grothendieck, A. *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : IV. Étude locale des schmas et des morphismes de schémas, Troisième partie.* Publications Mathématiques de l'IHÈS, 28 (1966), p. 5-255.

[7] Hazewinkel, M. *Formal groups and applications.* Pure and Applied Mathematics, vol. 78, Academic Press Inc., New York, 1978.

[8] Hedayatzadeh, M. *Exterior Powers of Barsotti-Tate Groups.* Available for download as arXiv:1009.2460v4.

[9] Hovey, M. and H. Sadofsky. *Tate cohomology lowers chromatic Bousfield classes.* Proceedings of the AMS 124, 1996, 3579-3585.

[10] Hovey, M. and H. Sadofsky. *Invertible spectra in the $E(n)$-local stable homotopy category.* Journal of the London Mathematical Society, Volume 60, 1999, 284-302.

[11] Illusie, L. *Déformations de groups de Barsotti-Tate (d'apreès A. Grothendieck)*, Astérisque (1985), no. 127, 151-198.

[12] Johnson, D. and S. Wilson. *The Brown-Peterson homology of elementary p-groups.* Amer. J. Math. 107 (1985), no. 2, 427453.

[13] Lurie, J. *Higher Topos Theory.* Available for download at http://www.math.harvard.edu/ lurie/ .

[14] Lurie, J. *Higher Algebra.* Available for download.

[15] Lurie, J. *Elliptic Cohomology I: Spectral Abelian Varieties.* In preparation.

[16] Messing, W. *The crystals associated to Barsotti-Tate groups with applications to abelian schemes.* Lecture Notes in Mathematics, 264 (1972). Berlin, New York: Springer-Verlag,

[17] Peterson, E. *The Morava E-Theory of Eilenberg-MacLane Spaces.* Available for download as arXiv:1109.5989v2.

[18] Ravenel, D. and S. Wilson. *The Morava K-theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture.* Amer. J. Math. 102 (1980), no. 4, 691748.

[19] Sati, H., and C. Westerland. *Morara K-theory and E-theory.* Available as arXiv:1109.3867 .

[20] Tate, J. *p-Divisible groups.* Proc. Conf. Local Fields (Dribergen, 1966), 158-183.