Uniqueness of *BSO*

J. F. Adams and S. B. Priddy

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Uniqueness of *BSO*

BY J. F. ADAMS AND S. B. PRIDDY
University of Cambridge and Northwestern University

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1. *Introduction.* This paper will show that after localization at any given prime p, the infinite loop space structure on the space BSO is essentially unique. If the word 'localization' is replaced by 'completion', the result continues to hold; and both results continue to hold if the space BSO is replaced by the space BSU.

In order to state this result formally, it is natural to suppose given a connected Ω-spectrum $\mathbf{X}$ whose 0th term X_0 is equivalent to the localization or completion at p of BG, where $G = SO$ or SU according to the case. One should then state and prove that this spectrum $\mathbf{X}$ is equivalent to some fixed spectrum $\mathbf{Y}$. Provided we arrange for $\mathbf{Y}$ to be an Ω-spectrum, this conclusion shows that there is an equivalence of infinite loop spaces from X_0 to the fixed infinite loop space Y_0.

Our proof that $\mathbf{X} \simeq \mathbf{Y}$ falls into two parts. The first step determines the mod p cohomology of $\mathbf{X}$ as a module over the mod p Steenrod algebra. The second step starts from a knowledge of the mod p cohomology of $\mathbf{X}$, and constructs an equivalence of spectra $\mathbf{X} \to \mathbf{Y}$.

The second step is valid not only for the cases $G = SO$ and $G = SU$, but also for the cases $G = O$ and $G = U$; but in the latter cases the first step is not valid in the form we have discussed so far. That is, in these cases, we require information about X_0 not only as a space, but as an H-space.

We therefore begin formal work by considering the second step, and for this purpose we first construct the 'obvious' fixed spectrum $\mathbf{Y}$.

Let $\mathbf{K_R}$ be the spectrum which represents classical (periodic) real K-theory; similarly for $\mathbf{K_C}$ in the complex case. Let d be a fixed integer; let $\mathbf{bg}$ be the spectrum obtained from $\mathbf{K_R}$ or $\mathbf{K_C}$ by killing homotopy groups in degrees $< d$, while retaining the homotopy groups in degrees $\geqslant d$. The spectrum $\mathbf{bg}$ therefore represents $(d-1)$-connected K-theory (real or complex). The notation $\mathbf{bg}$ is chosen to reflect the usual notation for connective K-theory; one obtains $\mathbf{bo}$ and $\mathbf{bso}$ from $\mathbf{K_R}$ by taking $d = 1$ and $d = 2$, while one obtains $\mathbf{bu}$ and $\mathbf{bsu}$ from $\mathbf{K_C}$ by taking $d = 2$ and $d = 4$.

Let p be a fixed prime. Let Λ be either the ring $\mathbf{Z}_p$ of p-adic integers, or the ring $\mathbf{Z}_{(p)}$ of integers localized at p (that is, the ring of fractions a/b with a, b integers and b prime to p). We can introduce coefficients Λ into any spectrum $\mathbf{W}$ by setting $\mathbf{W}_\Lambda = \mathbf{M}\Lambda \wedge \mathbf{W}$, where $\mathbf{M}\Lambda$ is a Moore spectrum for the group Λ. We take our fixed spectrum $\mathbf{Y}$ to be

$$\mathbf{Y} = \mathbf{bg}_\Lambda = \mathbf{M}\Lambda \wedge \mathbf{bg}.$$

So the spectrum $\mathbf{Y}$ represents $(d-1)$-connected K-theory with coefficients in Λ. We write $\mathbf{F}_p$ for the field with p elements.

THEOREM 1·1. *Let* $\mathbf{X}$ *be a spectrum whose homotopy groups are finitely generated modules over* Λ, *and bounded below. Suppose given an isomorphism*

$$\theta\colon H^*(\mathbf{X}; \mathbf{F}_p) \leftarrow H^*(\mathbf{Y}; \mathbf{F}_p)$$

of modules over the mod p *Steenrod algebra. Then there exists an equivalence* $f\colon \mathbf{X} \to \mathbf{Y}$ *of spectra such that* $f^* = \theta$.

If for simplicity we disregard the subsidiary hypotheses, this theorem shows that the 'standard' spectrum $\mathbf{Y}$ is characterized by its cohomology.

Theorem 1·1 uses the assumption that certain groups are finitely generated modules over Λ. It may be reassuring to remark that an abelian group can be made into a finitely generated Λ-module in at most one way (see Lemma 5·3).

We now turn to the first step. Let us arrange for $\mathbf{bg}_\Lambda$ to be an Ω-spectrum, and let us define BO_Λ, BSO_Λ, BU_Λ and BSU_Λ to be the 0th terms of the spectra $\mathbf{bo}_\Lambda$, $\mathbf{bso}_\Lambda$, $\mathbf{bu}_\Lambda$ and $\mathbf{bsu}_\Lambda$. With this definition, Theorem 1·2 can be understood without familiarity with the theory of localization or completion. However, we state the facts. For any spectrum $\mathbf{W}$ we have a map

$$\mathbf{W} \simeq \mathbf{S}^0 \wedge \mathbf{W} = \mathbf{MZ} \wedge \mathbf{W} \to \mathbf{M}\Lambda \wedge \mathbf{W},$$

where $\mathbf{S}^0$ is the sphere-spectrum and the final map is induced by the injection $\mathbf{Z} \to \Lambda$. In particular, we have a map $\mathbf{bg} \to \mathbf{bg}_\Lambda$. If we arrange for both spectra to be Ω-spectra and pass to their 0th terms, we have a map of infinite loop-spaces $BG \to BG_\Lambda$ (for $G = O, SO, U, SU$). If $\Lambda = \mathbf{Z}_{(p)}$, then this map displays BG_Λ as the localization of BG in the sense of Sullivan (13); if $\Lambda = \mathbf{Z}_p$, it displays BG_Λ as the completion of BG. In any case, it is useful to note that the map $BG \to BG_\Lambda$ induces an isomorphism of mod p cohomology.

To state Theorem 1·2, we recall that the equivalence $X_0 \simeq \Omega X_1$ determines an H-space structure on X_0 and a Pontryagin product in $H_*(X_0; \mathbf{F}_p)$.

THEOREM 1·2. *Let* $\mathbf{X}$ *be a connected* Ω*-spectrum. Suppose given a homotopy-equivalence of spaces* $X_0 \simeq Y_0$, *where* $Y_0 = BO_\Lambda$, BSO_Λ, BU_Λ *or* BSU_Λ *according to the case. If* $Y_0 = BO_\Lambda$ *and* $p = 2$, *assume further that the square* x^2 *of the generator* x *in* $H_1(X_0; \mathbf{F}_2)$ *is non-zero; if* $Y_0 = BU_\Lambda$, *assume further that the pth power* x^p *of the generator* x *in* $H_2(X_0; \mathbf{F}_p)$ *is non-zero. Then there is an isomorphism of A-modules*

$$\theta\colon H^*(\mathbf{X}; \mathbf{F}_p) \leftarrow H^*(\mathbf{Y}; \mathbf{F}_p).$$

COROLLARY 1·3. *Under the hypotheses of Theorem* 1·2, *there is an equivalence of spectra* $f\colon \mathbf{X} \to \mathbf{Y}$.

Proof. This follows by combining Theorems 1·1 and 1·2; it is only necessary to check that under the hypotheses of Theorem 1·2, the homotopy groups $\pi_r(\mathbf{X})$ of $\mathbf{X}$ are finitely generated modules over Λ. In fact, they are zero for $r < 0$, while for $r \geqslant 0$ we have

$$\pi_r(\mathbf{X}) \cong \pi_r(X_0) \cong \pi_r(Y_0) \cong \pi_r(\mathbf{Y}) \cong \pi_r(\mathbf{bg}) \otimes \Lambda.$$

We note that certain cases of Theorem 1·1 remain unused in Corollary 1·3, notably the real cases with $d \equiv 0$ or 4 mod 8. For example, one might seek to characterize the infinite loop space structure on $\mathbf{Z} \times BO$ (subject to localization or completion at the prime p, as usual); and it will become clear from our proof that this could be done by specifying rather less than the n-fold loop structure for some definite value of n, which indeed could be fairly small. However we leave such extensions of Theorem 1·2 to those readers who may have a use for them.

We have a natural example of a spectrum $\mathbf{X}$ to which Corollary 1·3 applies. For $G = O$, SO, U or SU we can consider BG as the classifying space for G-bundles of virtual dimension 1; then the tensor product of bundles gives BG the structure of an H-space, which we write $BG_{\otimes}$. Recent work of Segal and May (8, 12) shows that this H-space is the 0th term of a connected Ω-spectrum, which we write $\mathbf{bo}_{\otimes}$, $\mathbf{bso}_{\otimes}$, $\mathbf{bu}_{\otimes}$ or $\mathbf{bsu}_{\otimes}$ according to the case. It is then natural to write $\mathbf{bg}_{\oplus}$ for the spectrum we formerly called $\mathbf{bg}$, to show that there the H-space structure given to BG corresponds to the Whitney sum of bundles.

Corollary 1·4. *After localization at any prime p, the spectra $\mathbf{bso}_{\oplus}$ and $\mathbf{bso}_{\otimes}$ become equivalent; similarly for $\mathbf{bsu}_{\oplus}$ and $\mathbf{bsu}_{\otimes}$.*

This follows immediately from Corollary 1·3.

The corresponding statement clearly fails for $\mathbf{bu}$, and for $\mathbf{bo}$ at the prime 2, because the Pontryagin products in $BU_{\otimes}$ and $BO_{\otimes}$ do not behave as described in the assumptions of Theorem 1·2, and these assumptions are (of course) necessary, as well as sufficient, for the conclusion $\mathbf{X} \simeq \mathbf{Y}$.

Without localization the statement also fails, although this is harder to see.

Another application of Corollary 1·3 arises as follows. According to Boardman and Vogt (5) the spaces F/PL and F/Top are infinite loop spaces; and according to Sullivan ((13), p. 24) the spaces F/PL and F/Top become equivalent to BO upon localization at any odd prime p.

We add some historical remarks. Peterson (11) has proved that at any odd prime p, $BSO_{\oplus}$ and $BSO_{\otimes}$ are equivalent as two-fold loop spaces. Atiyah and Segal (4) have established an algebraic isomorphism

$$\widetilde{KSO}(\ )^{\wedge} \xrightarrow{\cong} (1 + \widetilde{KSO}(\ ))^{\wedge}$$

of p-adic completions. Sullivan (14) has observed that at a regular (odd) prime p, a suitable characteristic class ρ^k induces an equivalence of H-spaces from $BSO_{\oplus}$ to $BSO_{\otimes}$. The second author (10) computed the Dyer–Lashof operations in $H_*(BSO; \mathbf{F}_p)$ which arise from the infinite loop space structures $\oplus$ and $\otimes$; he found that the resulting homology algebras are isomorphic (by an isomorphism very different from the identity). These facts, together with insights of J. P. May, led to the formulation of Corollary 1·4 as a conjecture.

We next attempt to prove this conjecture, in collaboration with J. P. May, by applying May's machinery (7) to specific maps constructed by representation theory. It turns out, however, that in this special case it is better to exploit the special good properties of the spaces BSO and BSU, rather than to rely on machinery which by its nature is adapted to the difficulties of the general case.

After our result was proved, it provided an essential input for the interesting work of Madsen, Snaith and Tornehave (6). In this connexion we note that the methods of the present paper can be pushed further, so as to yield an explicit description of [Y, Y], the ring of endomorphisms of the standard spectrum $\mathbf{Y} = \mathbf{bg}_\Lambda$. However, we omit this from the present paper.

The remainder of this paper is organized as follows. In section 2 we take the 'first step', and prove Theorem 1·2. It remains to prove Theorem 1·1. In order to construct a map $f: \mathbf{X} \to \mathbf{Y}$ of spectra such that $f^* = \theta$, we use the Adams spectral sequence for computing the group $[\mathbf{X}, \mathbf{Y}]$ of maps from X to Y. For this purpose we have to compute the relevant Ext groups

$$\mathrm{Ext}_A^{s,t}(H^*(\mathbf{Y}; \mathbf{F}_p), H^*(\mathbf{X}; \mathbf{F}_p)).$$

This will be done in section 4; it depends on some structure theory for modules over small subalgebras of A, and this will be given in section 3. At this stage we face a difficulty, for the spectra X and Y certainly do not satisfy any standard set of conditions known to be sufficient for the convergence of the Adams spectral sequence. In section 5 we will overcome this difficulty and prove Theorem 1·1.

We would like to thank J. P. May for many conversations and numerous letters; although his name does not appear at the head of this paper, he should be considered as a prime mover in this area. In particular, we owe to him the suggestion that we should cover the p-complete case; and for the adaptation of our proof to this case, we have in the end (after considering a variant of our own) preferred to follow his suggestions.

2. *Cohomology of* **X**. In this section we will determine $H^*(\mathbf{X}; \mathbf{F}_p)$, where X is as in Theorem 1·2; in particular, $X_0 \simeq BG_\Lambda$, where $G = O$, SO, U or SU according to the case.

PROPOSITION 2·1. *If $G = SU$, then*

$$H^*(\mathbf{X}; \mathbf{F}_p) \cong \bigoplus_{r=2}^{p} \Sigma^{2r}(A/(AQ_0 + AQ_1)).$$

We pause to explain the notation. We write A for the mod p Steenrod algebra. We have $Q_0 = \beta_p$ and $Q_1 = P^1\beta_p - \beta_p P^1$, as usual; if $p = 2$ we interpret Q_1 as

$$Sq^{01} = Sq^1Sq^2 + Sq^2Sq^1.$$

The graded module ΣM is defined by regrading M so that an element of degree δ in M appears as an element of degree $\delta + 1$ in ΣM; for example, $\Sigma^{2r}A$ is a free module on one generator of degree $2r$.

PROPOSITION 2·2. *If $G = U$, then*

$$H^*(\mathbf{X}; \mathbf{F}_p) \cong \bigoplus_{r=1}^{p-1} \Sigma^{2r}(A/(AQ_0 + AQ_1)).$$

PROPOSITION 2·3. *If $G = O$ or SO and $p > 2$, then*

$$H^*(\mathbf{X}; \mathbf{F}_p) \cong \bigoplus_{s=1}^{\frac{1}{2}(p-1)} \Sigma^{4s}(A/(AQ_0 + AQ_1)).$$

PROPOSITION 2·4. *If $G = SO$ and $p = 2$, then*

$$H^*(\mathbf{X}; \mathbf{F}_p) \cong \Sigma^2(A/(ASq^3)).$$

PROPOSITION 2·5. *If $G = O$ and $p = 2$, then*

$$H^*(\mathbf{X}; \mathbf{F}_p) \cong \Sigma(A/(ASq^2)).$$

Theorem 1·2 will follow from these propositions, since they all apply to $\mathbf{Y}$ as well as they do to $\mathbf{X}$. The proofs of these propositions all follow the same pattern. We know the homotopy groups of $\mathbf{X}$; in fact, we are given $\pi_r(\mathbf{X}) = 0$ for $r < 0$, and for $r \geqslant 0$ we have

$$\pi_r(\mathbf{X}) \cong \pi_r(X_0) \cong \pi_r(Y_0) \cong \pi_r(BG) \otimes \Lambda.$$

We may now filter $\mathbf{X}$ by considering its Postnikov system. By applying $H^*(\ ;\mathbf{F}_p)$ to this Postnikov system, we obtain a spectral sequence for computing $H^*(\mathbf{X}; \mathbf{F}_p)$, as in (1). The E_1 term of this spectral sequence consists of the cohomology of those Eilenberg–MacLane spectra which appear in the Postnikov system. Thus for every homotopy group $\pi_r(BG)$ isomorphic to $\mathbf{Z}$ we obtain in our E_1 term a module $\Sigma^r(A/A\beta_p)$; and in the case $p = 2$, for every homotopy group $\pi_r(BG)$ isomorphic to $\mathbf{Z}/(2)$ we obtain in our E_1 term a module $\Sigma^r A$. (Of course homotopy groups isomorphic to $\mathbf{Z}/(2)$ arise only for $G = O$ and $G = SO$.) We note explicitly that what we have said applies just as well to the p-complete case $\Lambda = \mathbf{Z}_p$ as to the p-local case $\Lambda = \mathbf{Z}_{(p)}$.

To proceed further we need to know something about the differentials in our spectral sequence; this means that we need to know something about the k-invariants of the spectrum $\mathbf{X}$. We begin with the simplest case, $G = SU$.

LEMMA 2·6. (*a*) *The k-invariant k^{2p+3} of the space BSU_Λ is non-zero.*

(*b*) *If $G = SU$, and $\mathbf{X}$ is as above, then the k-invariant k^{2p+3} of the spectrum $\mathbf{X}$ is non-zero.*

Proof. The k-invariant k^{2i+1} of the spectrum $\mathbf{X}$ gives by suspension the k-invariant k^{2i+1} of the space X_0, that is, of BSU_Λ. So it is clear that part (*a*) of the lemma implies part (*b*). It is also clear that the k-invariant k^{2i+1} of the spectrum $\mathbf{X}$ is zero for $i \leqslant p$, for it lies in a zero group. So the k-invariant k^{2i+1} of X_0 or BSU_Λ is zero for $i \leqslant p$.

We now introduce notation following (1); if W is a space, then $W(m, \ldots, n)$ will mean that term in the Postnikov system of W whose homotopy groups π_r are the same as those of W for $m \leqslant r \leqslant n$, and zero for other values of r. We write $EM(\pi, n)$ for an Eilenberg–MacLane space of type (π, n).

If the k-invariant k^{2p+3} of BSU_Λ were zero, we would have

$$BSU_\Lambda(4, \ldots, 2p+2) \simeq \mathop{\times}_{r=2}^{p+1} EM(\Lambda, 2r);$$

so the operation $Q_1 = P^1\beta_p - \beta_p P^1$ would be non-zero on $H^4(BSU_\Lambda(4, \ldots, 2p+2); \mathbf{F}_p)$, and hence on $H^4(BSU_\Lambda; \mathbf{F}_p)$. This is a contradiction; the operation Q_1 is zero on $H^*(BSU; \mathbf{F}_p)$ since BSU is torsion-free, and hence it is zero on $H^*(BSU_\Lambda; \mathbf{F}_p)$. This contradiction shows that the k-invariant k^{2p+3} of BSU_Λ is non-zero, and proves the lemma.

Proof of Proposition 2·1. Suppose $G = SU$, and consider the spectral sequence mentioned above for computing $H^*(\mathbf{X}; \mathbf{F}_p)$. The differentials are A-module maps, and they are necessarily zero until we come to d_{2p-2}, which must be given on each module

$\Sigma^{2r}(A/A\beta_p)$ by some multiple of $a \mapsto aQ_1$. We claim that this multiple is non-zero. For the lowest differential

$$d_{2p-2}\colon \Sigma^{2p+2}(A/A\beta_p) \to \Sigma^4(A/A\beta_p)$$

this is equivalent to Lemma 2·6(*b*), which we have just proved. For the other differentials we argue as follows. Let us use notation $\mathbf{W}(m, \ldots, n)$ for spectra analogous to that which we have used for spaces. Consider the spectrum $\mathbf{X}(2t+4, \ldots, \infty)$. Its 0th term $\mathbf{X}(2t+4, \ldots, \infty)_0$ is equivalent to the 0th term of $\mathbf{Y}(2t+4, \ldots, \infty)$; for we have

$$\mathbf{X}(2t+4, \ldots, \infty)_0 \simeq X_0(2t+4, \ldots, \infty),$$
$$\mathbf{Y}(2t+4, \ldots, \infty)_0 \simeq Y_0(2t+4, \ldots, \infty),$$

where the right-hand sides are constructed from the equivalent spaces X_0 and Y_0, by the method of killing homotopy groups as applied to spaces. *A fortiori*, the $(-2t)$th term $\mathbf{X}(2t+4, \ldots, \infty)_{-2t}$ is equivalent to the corresponding term $\mathbf{Y}(2t+4, \ldots, \infty)_{-2t}$. But by the Bott periodicity theorem the spectrum $\mathbf{bsu}(2t+4, \ldots, \infty)$ is equivalent, after reindexing its terms, to $\mathbf{bsu}$; this conclusion persists after we introduce coefficients Λ; therefore $\mathbf{Y}(2t+4, \ldots, \infty)_{-2t}$ is equivalent to BSU_Λ. So the work we have already done determines the lowest differential for $\mathbf{X}(2t+4, \ldots, \infty)$, which gives the differential

$$\Sigma^{2t+2p+2}(A/A\beta_p) \to \Sigma^{2t+4}(A/A\beta_p)$$

for $\mathbf{X}$.

Now, the sequence

$$\ldots \to \Sigma^{2t+4p-4}A/A\beta_p \to \Sigma^{2t+2p-2}A/A\beta_p \to \Sigma^{2t}A/A\beta_p,$$

in which every map is given by $a \mapsto aQ_1$, is exact. The simplest way to see this is as follows. Let B be the exterior algebra generated by $Q_0 = \beta_p$ and Q_1; then the sequence

$$\ldots \to \Sigma^{2t+4p-4}B/B\beta_p \to \Sigma^{2t+2p-2}B/B\beta_p \to \Sigma^{2t}B/B\beta_p,$$

in which every map is given by $b \mapsto bQ_1$, is exact. The previous sequence comes from this by applying the functor $A \otimes_B$, and this functor preserves exactness since A is free as a right module over B.

It follows that the spectral sequence we are studying becomes trivial after the differential d_{2p-2}, and we find

$$H^*(\mathbf{X}; \mathbf{F}_p) \cong \bigoplus_{r=2}^{p} \Sigma^{2r}(A/(AQ_0 + AQ_1)).$$

This completes the proof of Proposition 2·1.

We turn to the case $G = U$.

LEMMA 2·7. *If $G = U$, and $\mathbf{X}$ is as in Theorem 1·2, then the k-invariant k^{2p+1} of the spectrum $\mathbf{X}$ is non-zero.*

Proof. If this k-invariant were zero, then we would have an equivalence of spectra

$$\mathbf{X}(2, \ldots, 2p) \simeq \bigtimes_{r=1}^{p} \mathbf{EM}(\Lambda, 2r)$$

(where $\mathbf{EM}(\pi, n)$ means an Eilenberg–MacLane spectrum of type (π, n)). This would yield an equivalence of H-spaces

$$X_0(2, \ldots, 2p) \simeq \bigtimes_{r=1}^{p} EM(\Lambda, 2r).$$

But if x is the generator in $H_2(EM(\Lambda, 2); \mathbf{F}_p)$ then we have $x^p = 0$, and this contradicts the assumption in Theorem 1·2. This contradiction shows that $k^{2p+1} \neq 0$ and proves the lemma.

Proof of Proposition 2·2. We use the same spectral sequence as before. The lowest differential

$$d_{2p-2}\colon \Sigma^{2p} A/A\beta_p \to \Sigma^2 A/A\beta_p$$

is a non-zero multiple of $a \mapsto aQ_1$, by Lemma 2·7. The remaining differentials are determined by what we have already done, for the spectrum $\mathbf{X}(4, \ldots, \infty)$ has 0th term equivalent to BSU_Λ. The rest of the proof goes as for Proposition 2·1.

Proof of Proposition 2·3. If p is odd then on introducing coefficients Λ we have $\mathbf{bo}_\Lambda \simeq \mathbf{bso}_\Lambda$, so that $BO_\Lambda \simeq BSO_\Lambda$; thus the two cases are equivalent. We use the same spectral sequence as before. It behaves like that used in proving Propositions 2·1 and 2·2, except that we now have homotopy groups Λ in degrees $4s$ instead of in degrees $2r$. As for the differential d_{2p-2}, we now dispose of the relevant information about the space BSU_Λ (in any case, the information about BSU was already on record in (1)). We obtain the corresponding information about BSO_Λ by naturality, using either the map $BSO_\Lambda \to BSU_\Lambda$ or the map $BSU_\Lambda \to BSO_\Lambda$. The information about the space X_0 implies the required information about the spectrum $\mathbf{X}$. The rest of the proof goes as for Propositions 2·1 and 2·2.

We turn to the case $G = SO$, $p = 2$.

LEMMA 2·8. *Let n be a positive integer divisible by* 4. *Then the k-invariant k^{n+1} of the space BSO_Λ is non-zero.*

We actually need only the cases $n = 4$ and $n = 8$; but the proof is the same in general.

Proof. For brevity, we write W for BSO_Λ. Suppose (for a contradiction) that $k^{n+1} = 0$. Then we have

$$W(0, \ldots, n) \simeq W(0, \ldots, n-1) \times EM(\Lambda, n).$$

Suppose also that the indecomposable quotient of $H^*(W(0, \ldots, n-1); \mathbf{F}_2)$ in degree n has dimension δ over $\mathbf{F}_2$. Then (by the Künneth formula) the indecomposable quotient of $H^*(W(0, \ldots, n); \mathbf{F}_2)$ in degree n has dimension $\delta + 1$ over $\mathbf{F}_2$, and the same conclusion holds for W. But the indecomposable quotient of $H^*(BSO; \mathbf{F}_2)$ in degree n has dimension 1 over $\mathbf{F}_2$; so we infer that $\delta = 0$, and all elements of $H^n(W(0, \ldots, n-1); \mathbf{F}_2)$ are decomposable.

Since we have

$$W(0, \ldots, n) \simeq W(0, \ldots, n-1) \times EM(\Lambda, n)$$

we can calculate $H^n(W(0, \ldots, n); \mathbf{F}_2)$ by the Künneth formula; we can calculate Sq^1 on it by the Cartan formula; and using the fact that Sq^1 annihilates $H^n(EM(\Lambda, n); \mathbf{F}_2)$, it follows that the image of

$$Sq^1\colon H^n(W(0,\ldots,n);\,\mathbf{F}_2) \to H^{n+1}(W(0,\ldots,n);\,\mathbf{F}_2)$$

consists entirely of decomposable elements. Therefore the same conclusion holds in W. But this contradicts the known relation

$$Sq^1 w_n = w_{n+1}$$

between the Stiefel–Whitney classes in $H^*(BSO;\,\mathbf{F}_2)$. This contradiction proves the lemma.

Proof of Proposition 2·4. We use the same spectral sequence as before. We need to know the first differentials, and we claim that they are as follows.

$$\begin{array}{l}
\Sigma^{8t+8}A/ASq^1 \\
\quad\downarrow a \mapsto aSq^5 \\
\Sigma^{8t+4}A/ASq^1 \\
\quad\downarrow a \mapsto aSq^3 \\
\Sigma^{8t+2}A \\
\quad\downarrow a \mapsto aSq^2 \\
\Sigma^{8t+1}A \\
\quad\downarrow a \mapsto aSq^2 \\
\Sigma^{8t}A/ASq^1 \\
\quad\vdots \\
\Sigma^{8}A/ASq^1 \\
\quad\downarrow a \mapsto aSq^5 \\
\Sigma^{4}A/ASq^1 \\
\quad\downarrow a \mapsto aSq^3 \\
\Sigma^{2}A.
\end{array} \tag{2·9}$$

In fact, for dimensional reasons, each differential must be a multiple of the one shown; we have to check that the multiple is non-zero.

In the cases where the differential is given by Sq^2, the result is easy; for composition with the essential map

$$\eta\colon S^{m+1} \to S^m$$

gives a homomorphism

$$\pi_m(BSO) \to \pi_{m+1}(BSO),$$

which is non-zero if $m \equiv 0$ or $1 \bmod 8$ and $m \geqslant 8$; so the same conclusion holds in the space X_0 and in the spectrum $\mathbf{X}$.

Next we take the final differential

$$\Sigma^4 A/A\,Sq^1 \to \Sigma^2 A.$$

This is zero if and only if the first k-invariant k^5 of the spectrum $\mathbf{X}$ is zero. But if this k-invariant were zero, then the k-invariant k^5 of the space X_0 would be zero, contradicting Lemma 2·8.

Note that in this argument it is essential to use cohomology with coefficients in $\mathbf{Z}_{(2)}$ or $\mathbf{Z}_2$ (as we have implicitly done by using k^5) rather than coefficients $\mathbf{F}_2$; for in the

Postnikov system of X_0, Sq^3 annihilates $H^2(EM(\mathbf{Z}/(2), 2); \mathbf{F}_2)$. The same remark applies to the next argument.

Next we take the penultimate differential

$$\Sigma^8 A/ASq^1 \to \Sigma^4 A/ASq^1.$$

This is related to the second k-invariant k^9 of the spectrum $\mathbf{X}$; here k^9 lies in

$$H^9(\mathbf{X}(2, \ldots, 4); \Lambda).$$

More precisely, consider the following exact sequence.

$$H^9(\mathbf{X}(2); \Lambda) \xrightarrow{j^*} H^9(\mathbf{X}(2, \ldots, 4); \Lambda) \xrightarrow{i^*} H^9(\mathbf{X}(4); \Lambda).$$

The differential in question is zero if and only if $i^*k^9 = 0$. If so, then k^9 lies in $\operatorname{Im} j^*$; in other words, k^9 is a linear combination of

$$\delta_2 Sq^6 b \quad \text{and} \quad \delta_2 Sq^4 Sq^2 b.$$

(Here δ_2: $H^m(\ ; \mathbf{F}_2) \to H^{m+1}(\ ; \Lambda)$ is the Bockstein boundary, and $b \in H^2(\mathbf{X}(2, \ldots, 4); \mathbf{F}_2)$ is the fundamental class.) But we will prove that both these classes suspend to zero in $H^9(X_0(2, \ldots, 4); \Lambda)$. To begin with, the fundamental class in $H^2(X_0(2, \ldots, 4); \mathbf{F}_2)$ can be identified with the Stiefel–Whitney class $w_2 \in H^2(X_0; \mathbf{F}_2) \cong H^2(BSO; \mathbf{F}_2)$, and of course Sq^6 annihilates it for dimensional reasons. To continue, we argue that

$$H^4(X_0(2, \ldots, 4); \mathbf{F}_2) \cong H^4(X_0; \mathbf{F}_2),$$

and that the element $Sq^2 w_2 = (w_2)^2$ in it is the reduction of a class in

$$H^4(X_0(2, \ldots, 4); \Lambda) \cong H^4(X_0; \Lambda).$$

In fact, in BSO the first Pontryagin class P_1 reduces to $(w_2)^2$; so it is sufficient to take the class corresponding to P_1 under the isomorphism

$$H^4(BSO; \Lambda) \leftarrow H^4(BSO_\Lambda; \Lambda).$$

(In the p-local case the fact that this map is an isomorphism in general is well known; in the p-complete case it is perhaps shortest to establish this particular case ad hoc, as is easily done, rather than set up general theory.) In any case, the element $Sq^4 Sq^2 w_2 = (w_2)^4$ in $H^8(X_0(2, \ldots, 4); \mathbf{F}_2)$ is also the reduction of a class defined over Λ, and $\delta_2 Sq^4 Sq^2 w_2 = 0$.

So the hypothesis that the penultimate differential is zero implies that the k-invariant k^9 of the space $X_0 \simeq BSO_\Lambda$ is zero. This contradicts Lemma 2·8.

For the higher differentials we argue as before. Consider the spectrum $\mathbf{X}(8t+2, \ldots, \infty)$. Its 0th term $X_0(8t+2, \ldots, \infty)$ is already equivalent to the 0th term $Y_0(8t+2, \ldots, \infty)$ of $\mathbf{Y}(8t+2, \ldots, \infty)$. *A fortiori*, its $(-8t)$th term $\mathbf{X}(8t+2, \ldots, \infty)_{-8t}$ is equivalent to the corresponding term $\mathbf{Y}(8t+2, \ldots, \infty)_{-8t}$. By the Bott periodicity theorem the spectrum $\mathbf{bso}(8t+2, \ldots, \infty)$ is equivalent, after reindexing its terms, to $\mathbf{bso}$; this conclusion persists after we introduce coefficients Λ; therefore $\mathbf{Y}(8t+2, \ldots, \infty)_{-8t}$ is equivalent to

BSO_Λ. So the work we have already done gives the last two differentials for

$$\mathbf{X}(8t+2, \ldots, \infty);$$

this gives the differentials

$$\Sigma^{8t+8}A/ASq^1 \to \Sigma^{8t+4}A/ASq^1 \to \Sigma^{8t+2}A$$

for $\mathbf{X}$. This completes the proof that the differentials are as shown in (2·9).

Now the sequence (2·9) is exact. To prove it we argue as before. Let B be the subalgebra of A generated by Sq^1 and Sq^2; then it is elementary to check that the following sequence is exact.

$$\begin{array}{c}
\Sigma^{8t+8}B/BSq^1 \\
\downarrow b \mapsto bSq^2Sq^3 \\
\Sigma^{8t+4}B/BSq^1 \\
\downarrow b \mapsto bSq^3 \\
\Sigma^{8t+2}B \\
\downarrow b \mapsto bSq^2 \\
\Sigma^{8t+1}B \\
\downarrow b \mapsto bSq^2 \\
\Sigma^{8t}B/BSq^1 \\
\vdots \\
\downarrow \\
\Sigma^2 B
\end{array}$$

The sequence (2·9) is obtained from this by applying the functor $A \otimes_B$, and this functor preserves exactness since A is free as a right module over B.

We conclude that the spectral sequence becomes trivial after the differentials shown in (2·9), and

$$H^*(\mathbf{X}; \mathbf{F}_2) \simeq \Sigma^2 A/ASq^3.$$

This completes the proof of Proposition 2·4.

We turn to the final case $G = O$, $p = 2$.

LEMMA 2·10. *If $G = O$, $p = 2$ and $\mathbf{X}$ is as in Theorem 1·2, then the k-invariant k^3 of the spectrum $\mathbf{X}$ is non-zero.*

Proof. This is parallel to the proof of Lemma 2·7. If this k-invariant were zero, then we would have an equivalence of spectra

$$\mathbf{X}(1,2) \simeq \mathbf{EM}(\mathbf{Z}/(2), 1) \times \mathbf{EM}(\mathbf{Z}/(2), 2).$$

This would yield an equivalence of H-spaces

$$X_0(1,2) \simeq EM(\mathbf{Z}/(2), 1) \times EM(\mathbf{Z}/(2), 2).$$

But if x is the generator in $H_1(EM(\mathbf{Z}/(2), 1); \mathbf{F}_2)$ then we have $x^2 = 0$, and this contradicts the assumption in Theorem 1·2. This contradiction shows that $k^3 \neq 0$ and proves the lemma.

Proof of Proposition 2·5. We use the same spectral sequence as before. This time, however, we claim that the first differentials are as follows.

$$
\begin{array}{l}
\Sigma^{8t+8}A/ASq^1 \\
\quad\downarrow a \mapsto aSq^5 \\
\Sigma^{8t+4}A/ASq^1 \\
\quad\downarrow a \mapsto aSq^3 \\
\Sigma^{8t+2}A \\
\quad\downarrow a \mapsto aSq^2 \\
\Sigma^{8t+1}A \; a \mapsto aSq^2 \\
\quad\downarrow \\
\Sigma^{8t}A/ASq^1 \\
\quad\vdots \\
\quad\downarrow \\
\Sigma^{8}A/ASq^1 \\
\quad\downarrow a \mapsto aSq^5 \\
\Sigma^{4}A/ASq^1 \\
\quad\downarrow a \mapsto aSq^3 \\
\Sigma^{2}A \\
\quad\downarrow a \mapsto aSq^2 \\
\Sigma A
\end{array}
$$

In fact, the lowest differential $d_1 : \Sigma^2 A \to \Sigma A$ is determined by Lemma 2·10, and all the higher ones are determined by our previous work, since $\mathbf{X}(2, \ldots, \infty)$ is a spectrum whose 0th term is BSO_Λ. The rest of the proof goes as for Proposition 2·4.

3. *Structure theory for modules.* In this section we will record some of the structure theory of modules over small subalgebras of the Steenrod algebra. The results are originally due to the first author.

Let B be a connected graded Hopf algebra of finite dimension over the ground field k. We have in mind the following two examples.

(3·1) $B = E[x, y]$, the exterior algebra over k on two primitive generators x and y of distinct degrees. In our applications, k is $\mathbf{F}_p$, and B is the subalgebra of the mod p Steenrod algebra generated by $x = Q_0$ and $y = Q_1$.

(3·2) $B = A_1$, the subalgebra of the mod 2 Steenrod algebra generated by Sq^1 and Sq^2.

We must now explain that we actually want to discuss stable structure theory rather than structure theory. Let L, M be (say) left B-modules, and let $f_0, f_1 : L \to M$ be B-linear maps; for definiteness we may consider only maps which preserve the grading (leaving maps which change the grading to be introduced later by considering $\mathrm{Hom}_B(\Sigma^t L, M)$ or $\mathrm{Hom}_B(L, \Sigma^t M)$). We say that f_0 and f_1 are *homotopic* if $f_0 - f_1$ factors through a free module F. (If L is finitely generated we may take F to be finitely generated, for $f_0 - f_1$ must map into a finitely generated free submodule of F.) This notion of homotopy corresponds both to projective and to injective homotopy, which in general are distinct. Homotopy is an equivalence relation, for if $f_0 - f_1$ factors through F and $f_1 - f_2$ factors through F', then $f_0 - f_2$ factors through $F \oplus F'$. Composition of maps passes to homotopy classes, so we get a category of homotopy classes. Two modules L, M are *stably equivalent* if they are equivalent in the category of homotopy classes; Lemma 3·4 will show that this term has its usual meaning. We are in fact interested in the classification of modules rather than maps, so we fix on the adjective 'stable'; we speak of 'stable classes of maps' rather than 'homotopy classes', and write '$S\hom_B(L, M)$' for the group of stable classes of maps from L to M. The following results may be taken as justifying these definitions for our purposes.

LEMMA 3·3. (*a*) *For* $s > 0$, $\mathrm{Ext}_B^s(L, M)$ *is a bifunctor on the category of stable maps.*

(*b*) *Let*
$$0 \to L' \xrightarrow{i} L \to L'' \to 0$$
be an exact sequence in which L is free; then
$$\mathrm{Ext}_B^1(L'', M) \cong S\hom_B(L', M).$$

(*c*) *Let*
$$O \to M' \to M \xrightarrow{j} M'' \to O$$
be an exact sequence in which M is free; then
$$\mathrm{Ext}_B^1(L, M') \cong S\hom_B(L, M'').$$

LEMMA 3·4. *L and M are stably equivalent if and only if we have* $L \oplus F \cong M \oplus G$ *for some free modules F and G, which may be taken finitely generated if L and M are so.*

Proof of Lemma 3·3. (*a*) If F is free, then $\mathrm{Ext}_B^s(F, M) = 0$ for $s > 0$ because F is projective, and $\mathrm{Ext}_B^s(L, F) = 0$ for $s > 0$ because (under our hypotheses) F is injective.

(*b*) We have the following exact sequence.
$$0 \leftarrow \mathrm{Ext}_B^1(L'', M) \leftarrow \mathrm{Hom}_B(L', M) \xleftarrow{i^*} \mathrm{Hom}_B(L, M).$$
Here the image of i^* certainly consists of maps $L' \to M$ which factor through L, which is free, so this image maps to zero in $S\hom_B(L', M)$. Conversely, if we have a composite
$$L' \xrightarrow{j} F \xrightarrow{k} M$$
with F free, then since F is injective the map j factors through $i: L' \to L$, and kj lies in the image of i^*. We thus obtain an isomorphism
$$\mathrm{Ext}_B^1(L'', M) \cong S\hom_B(L', M).$$

(*c*) The proof of (*c*) is precisely dual to the proof of (*b*).

Proof of Lemma 3·4. If $L \oplus F \cong M \oplus G$ then L and M are stably equivalent, trivially. We have to prove the converse. Take a map $f: L \to M$ whose stable class is a stable equivalence. By adding to L a suitable free module F we can suppose that f is epi (and here we can suppose that F is finitely generated if M is so). Let K be the kernel of f; then for any module N we have an exact sequence
$$\ldots \leftarrow \mathrm{Ext}_B^{s+1}(M, N) \leftarrow \mathrm{Ext}_B^s(K, N) \leftarrow \mathrm{Ext}_B^s(L, N) \xleftarrow{f^*} \mathrm{Ext}_B^s(M, N) \leftarrow \ldots$$
in which f^* is iso for $s > 0$ by Lemma 3·3(*a*). Thus $\mathrm{Ext}_B^s(K, N) = 0$ for $s > 0$, and this for every module N; so K is projective. Hence K is free (note that under our strong

assumptions on B this follows without assuming K bounded above or below). Since K is free it is injective, and so $L \cong M \oplus K$. Here K is finitely generated if L is so. This proves the lemma.

Various constructions on modules are functorial in the sense that they carry stable maps into stable maps. First, obviously, we have the direct sum $L \oplus M$. Secondly we have the tensor product $L \otimes M$; since B is a Hopf algebra, we can make B act on $L \otimes_k M$ in the usual way; that is, if

$$\psi b = \sum_i b'_i \oplus b''_i,$$

we define

$$b(l \otimes m) = \sum_i (-1)^{|b''_i||l|} b'_i l \otimes b''_i m$$

(where $|l|$ means the degree of l, as usual). In order to see that the tensor product of maps passes to stable classes, we have to remark that if F is free, then $L \otimes F$ and $F \otimes M$ are free. Thirdly, we have the vector-space dual $M^* = \mathrm{Hom}_k(M, k)$. This is graded so that

$$|\langle m^*, m\rangle| = |m^*| + |m|,$$

where k is graded so that all of k is in degree 0. In other words, an element m^* is of degree d if it annihilates all homogeneous elements m except perhaps those of degree $-d$. The obvious way to make M^* a B-module is to make it a right B-module, so that

$$\langle m^* b, m\rangle = \langle m^*, bm\rangle.$$

However, we are willing to assume that B has a conjugation map c, and then we can make M^* into a left B-module by setting

$$bm^* = (-1)^{|b|\,|m^*|}\, m^*(cb).$$

In order to see that the duals of maps pass to stable classes, we have to remark that if F is free, then (under our hypotheses on B) F^* is free.

The module k (graded so that all of k is in degree 0) is a unit for the tensor product. We call a module 'invertible' if its stable equivalence class is invertible (under the tensor product). Such invertible stable equivalence classes form a group, and we will calculate this group when B is as in (3·1) and (3·2); or more precisely, we will calculate the group of invertible classes which can be represented by finitely generated B-modules. First we need to characterize such B-modules, and for this purpose we need invariants of modules.

We can associate to a module M over the exterior algebra $E[x, y]$ the homology groups

$$H(M; x) = \mathrm{Ker}\, x/\mathrm{Im}\, x, \quad H(M; y) = \mathrm{Ker}\, y/\mathrm{Im}\, y.$$

We can also use these homology groups when $B = A_1$, by taking $x = Sq^1$,

$$y = Sq^{01} = Sq^1 Sq^2 + Sq^2 Sq^1.$$

These homology groups are functorial on the category of stable maps, for we have

$$H(F; z) = 0 \quad \text{when } F \text{ is free and } z = x \text{ or } y.$$

(We keep z as a letter which stands for x or y.) We have

$$H(L \oplus M; z) \cong H(L; z) \oplus H(M; z),$$

$$H(L \otimes M; z) \cong H(L; z) \otimes H(M; z)$$

(by the Künneth formula) and

$$H(M^*; z) \cong (H(M; z))^*.$$

So these homology groups commute with the three constructions considered above.

LEMMA 3·5. *Assume* $B = E[x, y]$ *or* $B = A_1$. (*a*) *If* M *is invertible, then* $H(M; x)$ *and* $H(M; y)$ *are of dimension* 1 *over* k.

(*b*) *Suppose* $H(M; x)$ *and* $H(M; y)$ *are of dimension* 1 *over* k, *and* M *is finitely generated; then* M *is invertible, and its inverse is* M^*.

It can be shown by examples that in clause (*b*), the assumption that M is finitely generated cannot be omitted.

Proof. First suppose that M is invertible; say $M \otimes N \simeq k$. Then by the remarks above,

$$H(M; z) \otimes H(N; z) \cong k,$$

so that $H(M; z)$ has dimension 1 over k.

Conversely, assume that M is finitely generated, and $H(M; x)$ and $H(M; y)$ are of dimension 1 over k. Consider the evaluation map

$$M^* \otimes M \to k;$$

by construction, it is a map of B-modules. We see from the Künneth formula that it induces an isomorphism of $H(\ ; x)$ and an isomorphism of $H(\ ; y)$. Therefore

$$M^* \otimes M \simeq k,$$

by the theorem of Adams and Margolis (3) (see theorem 4·2, and note that the proof given remains valid when A is replaced by B). This proves the lemma.

From this point up to and including (3·11) all B-modules are assumed to be finitely generated.

We now give examples of invertible modules.

(i) We define Σ to be the module which is k in degree 1 and zero in other degrees. Its inverse Σ^{-1} is the module which is k in degree -1 and zero in other degrees. Thus multiplying a module M by Σ simply regrades it; this is consistent with our use of the notation ΣM in section 2.

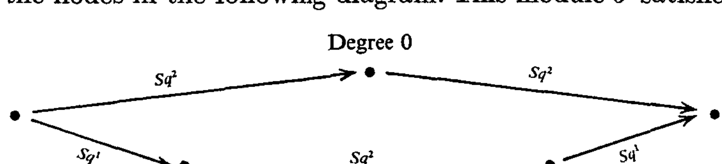
(ii) We define I to be the augmentation ideal of B. Then, we claim, $H(I; z)$ has dimension 1 over k for $z = x$ and for $z = y$; this follows immediately from the exact homology sequence which one obtains from the short exact sequence

$$0 \to I \to B \to k \to 0.$$

Thus I is invertible by Lemma 3·5, provided $B = E[x, y]$ or $B = A_1$.

A proof that I is invertible for more general B is given in (2), pp. 343–44.

(iii) When $B = A_1$, we define J to be the module $\Sigma^{-2}(B/BSq^3)$. It has a base over $\mathbf{F}_2$ indicated by the nodes in the following diagram. This module J satisfies the criteria



of Lemma 3·5; so it is invertible. Moreover, we have $J^* \cong J$; so J will serve as J^{-1} and we have $J^2 \simeq 1$.

For the rest of this section we reserve the letters I and J for the modules just defined.

THEOREM 3·6. *If $B = E[x, y]$, as in* (3·1), *then the group of invertible stable equivalence classes is $\mathbf{Z} \oplus \mathbf{Z}$, generated by Σ and I.*

THEOREM 3·7. *If $B = A_1$, as in* (3·2), *then the group of invertible stable equivalence classes is $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/(2)$, generated by Σ, I and J.*

Proof of Theorems 3·6 *and* 3·7. First we will prove that in Theorem 3·6, the stable equivalence classes $\Sigma^a I^b$ are all distinct. In fact, $H(\Sigma^a I^b; x)$ is k in degree $a + b|x| = c$, say, and zero in other degrees; while $H(\Sigma^a I^b; y)$ is k in degree $a + b|y| = d$, say, and zero in other degrees. Since $|x| \neq |y|$, c and d determine a and b.

Next we will prove that in Theorem 3·7, the stable equivalence classes $\Sigma^a I^b J^c$ (with $c = 0$ or $1 \bmod 2$) are all distinct. In fact, the homology groups of $\Sigma^a I^b J^c$ determine a and b, as above; we need one more invariant to determine c. The simplest is to consider $\dim_k M \bmod 8$, where $k = \mathbf{F}_2$. Using Lemma 3·4, we see that this is an invariant of the stable class of M; it sends the tensor-product of classes to the product of integers mod 8; and we have

$$\begin{aligned} \dim_k(\Sigma) &= 1 \bmod 8, \\ \dim_k(I) &= -1 \bmod 8, \\ \dim_k(J) &= 5 \bmod 8. \end{aligned}$$

So this invariant gives us c (and the residue class of $b \bmod 2$).

It remains to show that every invertible stable class has the form $\Sigma^a I^b$ or $\Sigma^a I^b J^c$, as the case may be.

First we assume $B = E[x, y]$, as in Theorem 3·6. Without loss of generality we may assume that $|x| < |y|$. Let M be an invertible module. Multiplying by some power of Σ, we may assume without loss of generality that $H(M; x)$ is k in degree 0. Let g_0 be a cycle representing the generator, so that $xg_0 = 0$ but $g_0 \notin \operatorname{Im} x$. Consider yg_0. We have $xyg_0 = -yxg_0 = 0$, so $yg_0 \in \operatorname{Ker} x$; since $H(M; x)$ is zero in this degree, we conclude that $yg_0 = xg_1$ for some g_1. Consider yg_1; we have

$$xyg_1 = -yxg_1 = -yyg_0 = 0.$$

Continuing in this way by induction, we find a sequence of elements

$$g_0, g_1, g_2, \ldots, g_n, \ldots$$

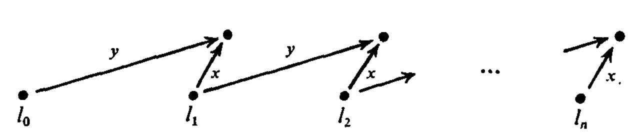
such that $\quad xg_0 = 0 \quad$ and $\quad yg_i = xg_{i+1} \quad$ for all i.

Now M is finitely generated over B, so we must have $g_{n+1} = 0$ for some n. Thus $yg_n = 0$.

As a first case, we consider the possibility that g_n represents a generator of $H(M; y)$. In this case, let $L(n)$ be the module presented by generators $l_0, l_1, l_2, \ldots, l_n$ and relations

$$xl_0 = 0, \quad yl_i = xl_{i+1}, \quad yl_n = 0.$$

$L(n)$:



We have constructed a map $L(n) \to M$ (given by $l_i \mapsto g_i$) which induces an isomorphism of $H(\ ;x)$ and $H(\ ;y)$; so $M \simeq L(n)$ by the theorem of Adams and Margolis. We will show

$$L(n) \simeq (\Sigma^{-|x|}I)^n.$$

The obvious minimal resolution of $L(n)$ over B gives an exact sequence

$$0 \to \Sigma^{|x|} L(n+1) \to F \to L(n) \to 0$$

with F free. On the other hand, by taking the sequence

$$0 \to I \to B \to k \to 0$$

and tensoring with $L(n)$, we get

$$0 \to I \otimes L(n) \to B \otimes L(n) \to L(n) \to 0,$$

and here $B \otimes L(n)$ is free. By Schanuel's Lemma, we get

$$I \otimes L(n) \simeq \Sigma^{|x|} L(n+1).$$

By induction over n, starting with $L(0) \cong k$, we see

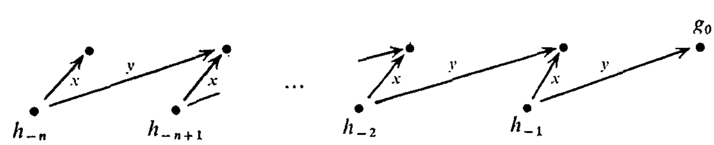
$$L(n) \simeq (\Sigma^{-|x|}I)^n.$$

As a second case, we consider the possibility that g_n represents the zero class in $H(M; y)$. In this case there is an element $h_{n-1} \in M$ such that $g_n = yh_{n-1}$. Replacing g_{n-1} by $g'_{n-1} = g_{n-1} + xh_{n-1}$, we recover our original situation with n replaced by $n-1$.

We may continue this process by induction downwards over n. Either at some stage we encounter the 'first case' and prove that $M \simeq \Sigma^a I^b$ for some $b \geqslant 0$, or else the induction continues right down to $n = 0$. There is no objection to replacing g_0 by $g'_0 = g_0 + xh_0$, for that does not alter its class in $H(M; x)$. We may thus suppose that $xg_0 = 0$, $yg_0 = 0$ and (unless $M \simeq k \simeq \Sigma^0 I^0$) that $g_0 = yh_{-1}$.

Suppose now that we have constructed $h_{-1}, h_{-2}, \ldots, h_{-n}$ with

$$yh_i = xh_{i+1}, \quad yh_{-1} = g_0.$$



Consider xh_{-n}. If $n = 1$ we have

$$yxh_{-1} = -xyh_{-1} = -xg_0 = 0;$$

otherwise we have

$$yxh_{-n} = -xyh_{-n} = -xxh_{-n+1} = 0.$$

As a first case, we consider the possibility that xh_{-n} represents a generator of $H(M; y)$. In this case we can proceed precisely as before, but the module we map to M is now $L(n)^*$, and we obtain

$$M \simeq L(n)^* \simeq (\Sigma^{|x|}I^{-1})^n.$$

As a second case, we consider the possibility that xh_{-n} represents the zero element in $H(M; y)$. In this case there is an element h_{-n-1} in M such that $yh_{-n-1} = xh_{-n}$, and the induction continues. Either at some stage we encounter the 'first case' and prove that $M \simeq \Sigma^a I^b$ for some $b < 0$, or else the induction constructs h_n for all n. Now M is finitely generated over B, so we must have $h_{-n-1} = 0$ for some n. Then $xh_{-n} = 0$. Since $H(M; x)$ is zero in this degree, we conclude that $h_{-n} = xk_{-n}$ for some k_{-n}. Then if $n > 1$, we may replace h_{-n+1} by $h'_{-n+1} = h_{-n+1} + yk_{-n}$; we have $xh'_{-n+1} = 0$, and we continue the induction. This induction finally proves that

$$g_0 = yh_{-1} = yxk_{-1} = -x(yk_{-1}),$$

contradicting the choice of g_0. Therefore the 'first case' must have arisen at some stage, and we have $M \simeq \Sigma^a I^b$ for some a, b. This proves Theorem 3·6.

Next we assume $B = A_1$, as in Theorem 3·7. We will prove that every invertible class is of the form $\Sigma^a I^b J^c$. If M is an invertible module over $B = A_1$, then by neglecting some of its structure we obtain an invertible module over $B' = E[Sq^1, Sq^{01}]$. The module Σ over A_1 yields the module Σ' over B'; the module I over A_1 yields the module $I' \oplus \Sigma^2 B'$ over B'. Assume that over B' we have $M \simeq (\Sigma')^a (I')^b$. Take a B-module N in the stable class $\Sigma^{-a} I^{-b} M$; then over B' we have $N \simeq k$, and we have to prove that over A_1 we have either $N \simeq k$ or $N \simeq J$.

We work simultaneously on N and N^*. Since we know the stable class of N over B', it follows that we can find in N an element g of degree 0 such that $Sq^1 g = 0$, $Sq^{01} g = 0$ and g is simultaneously a generator for $H(N; Sq^1)$ and $H(N; Sq^{01})$. Similarly, in N^* we can find an element g^* of degree 0 such that $Sq^1 g^* = 0$, $Sq^{01} g^* = 0$ and g^* is simultaneously a generator for $H(N^*; Sq^1)$ and $H(N^*; Sq^{01})$. We must have $\langle g^*, g \rangle = 1$.

First suppose that g is indecomposable, so that $g \notin Sq^1 N + Sq^2 N$. Then we can find a map of graded $\mathbf{F}_2$-modules $\theta: N \to k$ which annihilates $Sq^1 N + Sq^2 N$ and maps g to 1. Then θ is an A_1-map from N to k which induces an isomorphism of $H(\ ; Sq^1)$ and $H(\ ; Sq^{01})$, so $N \simeq k$. Similarly, if g^* is indecomposable we find $N^* \simeq k$ and $N \simeq k$.

The only remaining possibility is to suppose

$$g = Sq^1 n + Sq^2 m,$$
$$g^* = Sq^1 n^* + Sq^2 m^*$$

for suitable elements n, m, n^*, m^*. Then

$$Sq^3 m = Sq^1 g + Sq^1 Sq^1 n = 0,$$
$$Sq^3 m^* = Sq^1 g^* + Sq^1 Sq^1 n^* = 0.$$

So m defines an A_1-map $J \to N$, and m^* defines an A_1-map $J \to N^*$, or equivalently $N \to J^* \simeq J$. Also we have

$$\begin{aligned}\langle Sq^2 m^*, Sq^2 m \rangle &= \langle g^* + Sq^1 n^*, g + Sq^1 n \rangle \\ &= \langle g^*, g \rangle + \langle n^*, Sq^1 g \rangle + \langle Sq^1 g^*, n \rangle \\ &= 1.\end{aligned}$$

This shows that the composite $J \to N \to J$ is the identity; so N contains J as a direct summand. Let the complementary direct summand be P; then $H(P; Sq^1) = 0$,

$H(P; Sq^{01}) = 0$ and so P is free. Thus $N \simeq J$. This proves that every invertible class over A_1 is of the form $\Sigma^a I^b J^c$, and completes the proof of Theorem 3·7.

We now turn to the application of these results in computing groups $\mathrm{Ext}_B^s(L, M)$.

LEMMA 3·8. *If $s > 0$ and N is invertible, then*

$$\mathrm{Ext}_B^s(L, M) \cong S\,\mathrm{hom}_B(I^s \otimes L \otimes N, M \otimes N).$$

Proof. If we take the exact sequence

$$0 \to I \to B \to k \to 0$$

and tensor with L, we get an exact sequence

$$0 \to I \otimes L \to B \otimes L \to L \to 0$$

in which $B \otimes L$ is free. So by the classical exact sequence we get

$$\mathrm{Ext}_B^s(L, M) \cong \mathrm{Ext}_B^{s-1}(I \otimes L, M)$$

for $s > 1$, and by induction over s we get

$$\mathrm{Ext}_B^s(L, M) \cong \mathrm{Ext}_B^1(I^{s-1} \otimes L, M).$$

But now using Lemma 3·3 (*b*) we get

$$\mathrm{Ext}_B^1(I^{s-1} \otimes L, M) \cong S\,\mathrm{hom}_B(I^s \otimes L, M).$$

And if N is invertible we clearly have an isomorphism

$$S\,\mathrm{hom}_B(I^s \otimes L, M) \cong S\,\mathrm{hom}_B(I^s \otimes L \otimes N, M \otimes N).$$

This proves the lemma.

The effect of this lemma is that we can read off the groups $\mathrm{Ext}_B^s(L, M)$ for all invertible L and M from comparatively few tables. In the case $B = E[x, y]$ it is sufficient to tabulate

$$S\,\mathrm{hom}_B(I^s, \Sigma^t);$$

in the case $B = A_1$ it is sufficient to tabulate

$$S\,\mathrm{hom}_B(I^s, \Sigma^t) \quad \text{and} \quad S\,\mathrm{hom}_B(I^s J, \Sigma^t).$$

In the case $B = E[x, y]$ we label the table as if the degrees of x and y are 1 and $2p-1$, as in the applications.

Table 3·9. $S\,\mathrm{hom}_{E[x,y]}(I^s, \Sigma^t)$

s														
2	—	—	—	—	—	—	—	—	k	—	k	—	k	—
1	—	—	—	—	—	—	—	—	k	—	k	—	—	—
0	—	—	—	—	—	—	—	—	k	—	—	—	—	—
−1	—	—	—	—	—	k	—	—	—	—	—	—	—	—
−2	—	—	—	k	—	k	—	—	—	—	—	—	—	—
−3	—	k	—	k	—	k	—	—	—	—	—	—	—	—
$t-s$		$-6p+5$		$-4p+3$		$-2p+1$			0		$2p-2$		$4p-4$	

Table 3·10. $S\hom_{A_1}(I^s, \Sigma^t)$

s \ $t-s$	−15	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11	12
6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	k	—	—	—	k	—	—	—	k	—	k	—	—
5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	k	—	—	—	k	—	—	—	k	k	—	—	—
4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	k	—	—	—	k	—	—	—	k	—	—	—	—
3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	k	—	—	—	k	—	—	—	—	—	—	—	—
2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	k	—	k	—	—	—	—	—	—	—	—	—	—
1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	k	k	—	—	—	—	—	—	—	—	—	—	—
0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	k	—	—	—	—	—	—	—	—	—	—	—	—
−1	—	—	—	—	—	—	—	—	—	—	k	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
−2	—	—	—	—	—	—	—	—	—	k	k	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
−3	—	—	—	—	—	—	—	—	k	—	k	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
−4	—	—	—	—	—	—	k	—	—	—	k	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
−5	—	—	k	—	—	—	k	—	—	—	k	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
−6	—	k	k	—	—	—	k	—	—	—	k	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—

Table 3·11. $S\hom_{A_1}(I^sJ, \Sigma^t)$

s \ $t-s$	−15	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11	12
6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	k	—	—	—	k	—	—	—	k	—	—	—	k
5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	k	—	—	—	k	—	—	—	k	—	—	—	—
4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	k	—	—	—	k	—	k	—	—	—	—	—	—
3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	k	—	—	—	k	k	—	—	—	—	—	—	—
2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	k	—	—	—	k	—	—	—	—	—	—	—	—
1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	k	—	—	—	—	—	—	—	—	—	—	—	—
0	—	—	—	—	—	—	—	—	—	—	—	—	—	k	—	—	—	—	—	—	—	—	—	—	—	—	—	—
−1	—	—	—	—	—	—	—	—	—	—	—	—	k	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
−2	—	—	—	—	—	—	—	—	—	—	k	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
−3	—	—	—	—	—	—	k	—	—	—	k	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
−4	—	—	—	—	—	k	k	—	—	—	k	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
−5	—	—	—	—	k	—	k	—	—	—	k	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
−6	—	—	k	—	—	—	k	—	—	—	k	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—

In each table, groups not indicated are zero; in Tables 3·10 and 3·11, k means $\mathbf{F}_2$. In each quadrant of each table, the obvious periodicity continues.

These tables are the result of simple and obvious calculations.

The observations above become applicable because certain modules which arise in the applications can be written as sums of invertible modules. At this point we have to consider modules which are not finitely generated over B, so we relax that assumption. We can now form infinite sums of modules; infinite sums pass to stable classes, because an infinite sum of free modules is free.

PROPOSITION 3·12. *Let $B = E[Q_0, Q_1]$, as in* (3·1). *Then the stable class of*

$$A/(AQ_0+AQ_1) \cong A \otimes_B \mathbf{F}_p$$

is

$$\prod_{r=0}^{\infty} (1+K_r+K_r^2+\ldots+K_r^{p-1}),$$

where K_r is the invertible class $\Sigma^{a(r)}I^{b(r)}$ with

$$a(r)+b(r) = 2(p-1)p^r,$$

$$b(r) = \frac{p^r-1}{p-1}.$$

PROPOSITION 3·13. *Take $p = 2$, and let $B = A_1$, as in* (3·2). *Then the stable class of*

$$A/(ASq^1+ASq^2) \cong A \otimes_B \mathbf{F}_2$$

is

$$(1+\Sigma^3IJ)(1+\Sigma^5I^3)(1+\Sigma^9I^7)\ldots(1+\Sigma^{2^r+1}I^{2^r-1})\ldots.$$

In each proposition the infinite product is to be interpreted by expanding it as an infinite sum; for example, in Proposition 3·13 it means

$$1+\Sigma^3IJ+\Sigma^5I^3+\Sigma^8I^4J+\Sigma^9I^7+\ldots.$$

Proposition 3·13 is a reformulation by the first author of a lemma of Mahowald.

Proof of Proposition 3·12. We propose to proceed in the dual, and calculate the stable class of $(A/(AQ_0+AQ_1))^*$. According to our principles, we should give this dual space the structure of a left B-module, letting b act by the dual of the map $a \mapsto (cb)a$ on $A/(AQ_0+AQ_1)$. However, by using the conjugation c of A we can throw $A/(AQ_0+AQ_1)$ onto $A/(Q_0A+Q_1A)$, and use the map $a' \mapsto a'b$ on $A/(Q_0A+Q_1A)$. This is convenient for purposes of calculation. In fact, assuming p odd, the dual A^* of A is the tensor product of an exterior algebra $E[\tau_0, \tau_1, \tau_2, \ldots]$ and a polynomial algebra $\mathbf{F}_p[\xi_1, \xi_2, \xi_3, \ldots]$. The dual of the quotient $A/(Q_0A+Q_1A)$ is the subalgebra

$$E[\tau_2, \tau_3, \ldots] \oplus \mathbf{F}_p[\xi_1, \xi_2, \xi_3, \ldots].$$

We must calculate its homology for the boundaries obtained by dualizing the maps $a \mapsto aQ_0$ and $a \mapsto aQ_1$ of $A/(Q_0A+Q_1A)$.

Under the first boundary our complex may be expressed as a tensor product of chain complexes. Here the first factor is the polynomial algebra $\mathbf{F}_p[\xi_1]$, with the zero boundary; while the rth factor for $r \geqslant 2$ has a base of monomials

$$\tau_r\xi_r^i \quad \text{and} \quad \xi_r^i \quad (i = 0, 1, 2, \ldots)$$

with the boundary

$$\tau_r\xi_r^i \to \xi_r^{i+1}, \quad \xi_r^i \mapsto 0.$$

Thus for $r \geqslant 2$ the homology of the rth factor is $\mathbf{F}_p$, generated by 1 in degree 0. We conclude that the homology of the tensor product is a polynomial algebra on one generator ξ_1.

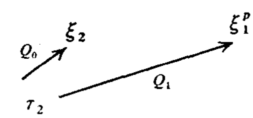
Under the second boundary our complex may also be expressed as a tensor product of chain complexes. This time the rth factor has a base of monomials $\tau_{r+1}\xi_r^i$ and ξ_r^i ($i = 0, 1, 2, \ldots$) with the boundary

$$\tau_{r+1}\xi_r^i \mapsto \xi_r^{i+p}, \quad \xi_r^i \mapsto 0.$$

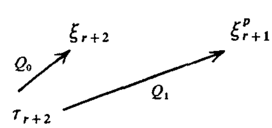
We conclude that the homology of the tensor product is a truncated algebra, given by generators $\xi_1, \xi_2, \ldots$ and relations $\xi_1^p = 0$, $\xi_2^p = 0, \ldots$.

We now seek appropriate B-modules to map into $A/(Q_0A + Q_1A)^*$, so that eventually we may obtain a map inducing isomorphisms of $H(\ ; Q_0)$ and $H(\ ; Q_1)$, and so apply the theorem of Adams and Margolis to deduce that we have a stable equivalence of B-modules.

We take L_0 to be the submodule generated by ξ_1. We take L_1 to be the following submodule.



Suppose now, as an inductive hypothesis, that we have constructed L_r and mapped it to $(A/(Q_0A + Q_1A))^*$, so that L_r contains a submodule L_r' which maps isomorphically to the B-submodule which has the k-base ξ_{r+1}. Then using the algebra structure of $(A/(Q_0A + Q_1A))^*$, we can map the tensor power $(L_r)^p$ to $(A/(Q_0A + Q_1A))^*$ so that the submodule $(L_r')^p$ maps isomorphically to the B-submodule which has the k-base ξ_{r+1}^p. We now construct L_{r+1} by adjoining to $(L_r)^p$ the following submodule.



(Of course we identify $(L_r')^p$ with the part of this new submodule which has the k-base ξ_{r+1}^p.) We take L_{r+1}' to be the part of the new submodule which has the k-base ξ_{r+2}. This completes the induction.

The result of this induction is that $H(L_r; Q_0)$ has dimension 1 over $\mathbf{F}_p$, and its generator maps to the homology class of $\xi_1^{p^r}$, while $H(L_r; Q_1)$ has dimension 1 over $\mathbf{F}_p$, and its generator maps to the homology class of ξ_{r+1}.

By using the algebra structure of $(A/(Q_0A + Q_1A))^*$, we can now construct a map

$$\prod_{r=0}^{\infty} (1 + L_r + L_r^2 + \ldots + L_r^{p-1}) \to (A/(Q_0A + Q_1A))^*;$$

this map induces an isomorphism of $H(\ ; Q_0)$ and $H(\ ; Q_1)$, and is therefore a stable equivalence by the theorem of Adams and Margolis. Moreover, as we have said, $(A/(Q_0A + Q_1A))^*$ is isomorphic to $(A/(AQ_0 + AQ_1))^*$ with its correct structure as a B-module.

Now, the infinite product

$$\prod_{r=0}^{\infty}(1+L_r+L_r^2+\ldots+L_r^{p-1})$$

represents an infinite sum which is locally finite, and duality commutes with locally finite sums. Moreover, the modules L_r are invertible by Lemma 3·5; so all the summands in the infinite sum are invertible; but for an invertible module M we have $M^* \simeq M^{-1}$. Therefore we can write the dual formula in the form

$$A/AQ_0+AQ_1 \simeq \prod_{r=0}^{\infty}(1+L_r^{-1}+L_r^{-2}+\ldots+L_r^{-p+1}),$$

or as

$$A/AQ_0+AQ_1 \simeq \prod_{r=0}^{\infty}(1+K_r+K_r^2+\ldots+K_r^{p-1})$$

if we set $K_r = L_r^{-1}$. Finally, by calculating the invariant given in the proof of Theorem 3·6, we see that $K_r = L_r^{-1} \simeq \Sigma^{a(r)}I^{b(r)}$ where

$$a(r)+b(r) = 2(p-1)p^r,$$
$$b(r) = \frac{p^r-1}{p-1}.$$

The proof is valid for $p = 2$ if suitably interpreted (interpret τ_i as ζ_{i+1} and ξ_i as ζ_i^2). Alternatively, the result for $p = 2$ is given in (2), p. 335, line 1. This completes the proof of Proposition 3·12.

Proof of Proposition 3·13. This is wholly parallel to the proof of Proposition 3·12. Again, we proceed in the dual and calculate the stable class of $(A/(ASq^1+ASq^2))^*$. We use the conjugation map c to throw $A/(ASq^1+ASq^2)$ on $A/(Sq^1A+Sq^2A)$. The dual A^* of A is the polynomial algebra $\mathbf{F}_2[\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots]$, and the dual of the quotient $A/(Sq^1A+Sq^2A)$ is the subalgebra $\mathbf{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4, \ldots, \zeta_n, \ldots]$. We must calculate its homology for the boundary obtained by dualizing the maps $a \mapsto aSq^1$ and $a \mapsto aSq^{01}$ of $A/(Sq^1A+Sq^2A)$.

Under the first boundary we get a tensor product of chain complexes, where the first factor is the polynomial algebra $\mathbf{F}_2[\zeta_1^4]$ with the zero boundary, and the rth factor for $r \geq 2$ has a base of monomials ζ_r^{2i} and $\zeta_r^{2i}\zeta_{r+1}$ $(i = 0, 1, 2, \ldots)$ with the boundary

$$\zeta_r^{2i}\zeta_{r+1} \mapsto \zeta_r^{2i+2}, \quad \zeta_r^{2i} \mapsto 0.$$

We conclude that the homology of the tensor product is a polynomial algebra on one generator ζ_1^4.

Under the second boundary our complex may also be expressed as a tensor product of chain complexes. This time the first factor has a base of monomials ζ_1^{4i} and $\zeta_1^{4i}\zeta_3$ $(i = 0, 1, 2, \ldots)$ with boundary

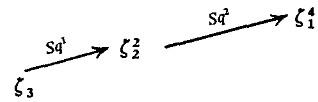
$$\zeta_1^{4i}\zeta_3 \mapsto \zeta_1^{4i+4}, \quad \zeta_1^{4i} \mapsto 0;$$

the rth factor for $r \geq 2$ has a base of monomials ζ_r^{2i} and $\zeta_r^{2i}\zeta_{r+2}$ with boundary

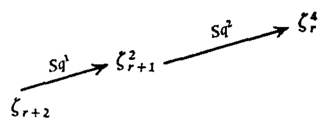
$$\zeta_r^{2i}\zeta_{r+2} \mapsto \zeta_r^{2i+4}, \quad \zeta_r^{2i} \mapsto 0.$$

We conclude that the homology of our tensor product is an exterior algebra on generators $\zeta_2^2, \zeta_3^2, \zeta_4^2, \ldots$.

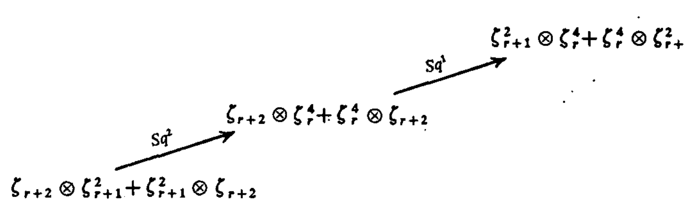
We now (again) seek appropriate B-modules to map into $(A/(Sq^1A+Sq^2A))^*$. We take L_1 to be the following submodule.



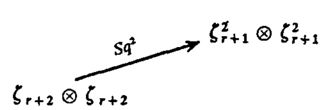
Suppose, as an inductive hypothesis, that we have constructed L_r and mapped it to $(A/(Sq^1A+Sq^2A))^*$ so that L_r contains a submodule L'_r which maps isomorphically to the following submodule.



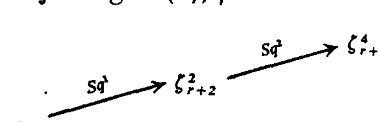
Then using the algebra structure of $(A/(Sq^1A+Sq^2(A))^*$, we can map the tensor square $(L_r)^2$ to $(A/(Sq^1A+Sq^2A))^*$. Now $(L'_r)^2$ does not map isomorphically to its image; the kernel K is a submodule which (with an obvious notation) we may display as follows.



Let us form the quotient $(L_r)^2/K$; this quotient maps to $(A/(Sq^1A+Sq^2A))^*$, and has the following submodule M.



We now construct L_{r+1} by adjoining to $(L_r)^2/K$ the following submodule.



(Of course we identify M with the part of the new submodule which has the k-base $\zeta_{r+2}^2, \zeta_{r+1}^4$.) We take L'_{r+1} to be the new submodule we adjoined. This completes the induction.

The result of this induction is that $H(L_r; Sq^1)$ has dimension 1 over $\mathbf{F}_2$, and its generator maps to the homology class of $\zeta_1^{2^{r+1}}$, while $H(L_r; Sq^{01})$ has dimension 1 over $\mathbf{F}_2$, and its generator maps to the homology class of ζ_{r+1}^2.

By using the algebra structure of $(A/(Sq^1A+Sq^2A))^*$, as before, we can construct a map

$$(1+L_1)(1+L_2)(1+L_3)\ldots \to (A/(Sq^1A+Sq^2A))^*;$$

this map induces an isomorphism of $H(\ ;Sq^1)$ and $H(\ ;Sq^{01})$, and is therefore a stable equivalence by the theorem of Adams and Margolis. Dualizing exactly as in the proof of Theorem 3·12, we find

$$A/(ASq^1+ASq^2) \simeq (1+K_1)(1+K_2)(1+K_3)\ldots,$$

where $K_r = L_r^{-1}$. Finally, by calculating the invariants given in the proof of Theorem 3·7, we see that

$$K_1 = L_1^{-1} \simeq \Sigma^3 IJ$$

while

$$K_r = L_r^{-1} \simeq \Sigma^{2^r+1} I^{2^r-1} \quad \text{for} \quad r \geqslant 2.$$

This completes the proof of Proposition 3·13.

4. *Calculation of* Ext. In this section we will prove the following two results.

PROPOSITION 4·1. $\mathrm{Ext}_A^{s,t}(A/AQ_0+AQ_1,\ A/AQ_0+AQ_1) = 0$ *provided* $s > 0$, $t-s$ *is odd and* $t-s > -2p+1$.

PROPOSITION 4·2. *Assume that* $p = 2$, *and* M *is one of the following four* A*-modules:*

$$A/(ASq^1+ASq^2),\quad A/ASq^2,\quad A/ASq^3,\quad A/(ASq^1+ASq^2Sq^3).$$

Then

$$\mathrm{Ext}_A^{s,t}(M,M) = 0 \quad \textit{for} \quad s>0, \quad t-s = -1.$$

Proof of Proposition 4·1. Let $B = E[Q_0, Q_1]$, as in (3·1). Then A is free as a right B-module, and we have

$$A/(AQ_0+AQ_1) \cong A \otimes_B \mathbf{F}_p.$$

By a standard change-of-rings theorem, we have

$$\mathrm{Ext}_A^{s,t}(A\otimes_B \mathbf{F}_p, M) \cong \mathrm{Ext}_B^{s,t}(\mathbf{F}_p, M).$$

If we take $M = A\otimes_B \mathbf{F}_p$, then by Proposition 3·12 it is stably equivalent to a sum of modules $\Sigma^a I^b$ with $a+b$ even and $a+b \geqslant 0$. By Lemma 3·8, we have

$$\mathrm{Ext}_B^{s,t}(\mathbf{F}_p, \Sigma^a I^b) \simeq S\,\mathrm{hom}_B(I^{s-b}, \Sigma^{t+a}).$$

By Table 3·9 (or by the calculations implicit in it), this is zero if $t+a-s+b$ is odd and greater than $-2p+1$. This proves Proposition 4·1.

The reader may notice how little use we have made of the precise details in Proposition 3·12, and may perhaps wonder if we could not rearrange the proof so as to omit much of section 3. So far as Proposition 4·1 goes this would be possible; in order to prove Proposition 4·1, it is sufficient to know merely the following about

$$M = A/AQ_0+AQ_1;$$

M is bounded below, $H(M;Q_0) = 0$ in even negative degrees and $H(M;Q_1) = 0$ in odd degrees. Unfortunately, similar remarks do not apply to Proposition 4·2. It can be shown by counter-examples that to prove Proposition 4·2 we need to know, at least, that M is stably equivalent to a sum of modules each finitely generated over B. For this we have to use the proof of Proposition 3·13; and if we have to use the proof, it seems unnecessarily obscure not to state what the proof proves.

We next prove Proposition 4·2, so far as it concerns the module

$$M = A/(ASq^1+ASq^2).$$

This is parallel to the proof of Proposition 4·1. Let B be A_1, as in (3·2). Then A is free as a right module over B, and we have

$$A/(ASq^1+ASq^2) \cong A \otimes_B \mathbf{F}_2.$$

By the same change-of-rings theorem, we have

$$\mathrm{Ext}_A^{s,t}(A \otimes_B \mathbf{F}_2, M) \cong \mathrm{Ext}_B^{s,t}(\mathbf{F}_2, M).$$

If we take $M = A \otimes_B \mathbf{F}_2$, then by Proposition 3·13 it is stably equivalent to a sum of modules $\Sigma^a I^b J^c$ with $a+b \equiv 0 \bmod 4$ and $a+b \geqslant 0$. By Lemma 3·8, we have

$$\mathrm{Ext}_B^{s,t}(\mathbf{F}_2, \Sigma^a I^b J^c) \cong S\,\mathrm{hom}_B(I^{s-b}J^{-c}, \Sigma^{t+a}).$$

By Tables 3·10, 3·11 (or by the calculations implicit in them) this is zero if

$$t+a-s+b \equiv -1 \bmod 4 \quad \text{and} \quad t+a-s+b \geqslant -1.$$

This proves that

$$\mathrm{Ext}_A^{s,t}(A/ASq^1+ASq^2, A/ASq^1+ASq^2) = 0$$

for $s > 0$, $t-s = -1$.

We next wish to deduce from this result the other three cases of Proposition 4·2. For this we use algebraic arguments analogous to the topological arguments one would use if one wished to prove that (with the notation of section 1) $[\mathbf{Y}, \mathbf{Y}]$ is essentially independent of d. First we need a subsidiary result.

LEMMA 4·3.

(i) $\mathrm{Ext}_A^{s,t}(A/ASq^1+ASq^2, A/ASq^1) = 0$ *for* $s > 0, t-s > -5$.

(ii) $\mathrm{Ext}_A^{s,t}(A/ASq^1, \Sigma^2 A/ASq^2) = 0$ *for* $s > 0, t-s > -5$.

Proof. (i) As before, the usual change-of-rings theorem gives

$$\mathrm{Ext}_A^{s,t}(A/ASq^1+ASq^2, A/ASq^1) \cong \mathrm{Ext}_B^{s,t}(\mathbf{F}_2, A/ASq^1).$$

Now of course we have an exact sequence

$$0 \to A/ASq^1 \to \Sigma^{-1}A \to \Sigma^{-1}A/ASq^1 \to 0$$

in which $\Sigma^{-1}A$ is free over B, and therefore injective. Proceeding dually to the proof of Lemma 3·8, and in particular applying Lemma 3·3 (c), we get

$$\mathrm{Ext}_B^{s,t}(\mathbf{F}_2, A/ASq^1) \cong S\,\mathrm{hom}_B(\mathbf{F}_2, \Sigma^{t-s}A/ASq^1).$$

But by direct calculation,

$$S\,\mathrm{hom}_B(\Sigma^u, A/ASq^1) = 0$$

for $u < 5$. This proves part (i).

(ii) Let A_0 be the subalgebra of A generated by Sq^1; its augmentation ideal I_0 is Σ. The usual change-of-rings theorem gives

$$\mathrm{Ext}_A^{s,t}(A/ASq^1, \Sigma^2 A/ASq^2) \cong \mathrm{Ext}_{A_0}^{s,t}(\mathbf{F}_2, \Sigma^2 A/ASq^2).$$

Now a trivial analogue of Lemma 3·8 shows that

$$\mathrm{Ext}_A^{s,t}(\mathbf{F}_2, \Sigma^2 A/ASq^2) \cong S\hom_{A_0}(\mathbf{F}_2, \Sigma^{t-s+2} A/ASq^2).$$

But by direct calculation,

$$S\hom_{A_0}(\Sigma^u, A/ASq^2) = 0$$

for $u < 3$. This proves the lemma.

We now use the following exact sequence, which arises in the mod 2 cohomology of the Postnikov system for **bo**, as in section 2.

$$0 \to \Sigma^2 A/ASq^2 \to A/ASq^1 \to A/ASq^1 + ASq^2 \to 0.$$

From this, we get the following two exact sequences.

$$\mathrm{Ext}_A^{s,t}\left(\frac{A}{ASq^1+ASq^2}, \frac{A}{ASq^1+ASq^2}\right) \to \mathrm{Ext}_A^{s+1,t}\left(\frac{A}{ASq^1+ASq^2}, \Sigma^2\frac{A}{ASq^2}\right)$$
$$\to \mathrm{Ext}_A^{s+1,t}\left(\frac{A}{ASq^1+ASq^2}, \frac{A}{ASq^1}\right),$$

$$\mathrm{Ext}_A^{s,t}\left(\frac{A}{ASq^1}, \Sigma^2\frac{A}{ASq^2}\right) \to \mathrm{Ext}_A^{s,t}\left(\Sigma^2\frac{A}{ASq^2}, \Sigma^2\frac{A}{ASq^2}\right)$$
$$\to \mathrm{Ext}_A^{s+1,t}\left(\frac{A}{ASq^1+ASq^2}, \Sigma^2\frac{A}{ASq^2}\right).$$

In the first exact sequence, using our previous result on the left-hand group and Lemma 4·3 (i) on the right-hand group, we see that the middle group is zero for $s > 0, t-s = -1$. Now the second exact sequence, using Lemma 4·3 (ii) on the left-hand group, shows that

$$\mathrm{Ext}_A^{s,t}(\Sigma^2 A/ASq^2, \Sigma^2 A/ASq^2) = 0$$

for $s > 0$, $t-s = -1$.

Now we use the following exact sequence.

$$0 \to \Sigma^4 A/ASq^3 \to \Sigma^2 A \to \Sigma^2 A/ASq^2 \to 0.$$

Since $\Sigma^2 A$ is both projective and injective, this shows that for $s > 0$ we have

$$\mathrm{Ext}_A^{s,t}\left(\Sigma^2\frac{A}{ASq^2}, \Sigma^2\frac{A}{ASq^2}\right) \cong \mathrm{Ext}_A^{s+1,t}\left(\Sigma^2\frac{A}{ASq^2}, \Sigma^4\frac{A}{ASq^3}\right)$$
$$\cong \mathrm{Ext}_A^{s,t}\left(\Sigma^4\frac{A}{ASq^3}, \Sigma^4\frac{A}{ASq^3}\right).$$

This group is therefore zero for $s > 0, t-s = -1$.

Finally we use the exact sequence

$$0 \to \Sigma^7\frac{A}{ASq^1+ASq^2Sq^3} \to \Sigma^4 A \to \Sigma^4\frac{A}{ASq^3} \to 0,$$

and this shows similarly that

$$\mathrm{Ext}_A^{s,t}\left(\Sigma^4 \frac{A}{ASq^3}, \Sigma^4 \frac{A}{ASq^3}\right) \cong \mathrm{Ext}_A^{s,t}\left(\Sigma^7 \frac{A}{ASq^1+ASq^2Sq^3}, \Sigma^7 \frac{A}{ASq^1+ASq^2Sq^3}\right)$$

for $s > 0$. This completes the proof of Proposition 4·2.

Similar arguments work for the exact sequence

$$0 \to \Sigma^{12} \frac{A}{ASq^1+ASq^2} \to \Sigma^7 \frac{A}{ASq^1} \to \Sigma^7 \frac{A}{ASq^1+ASq^2Sq^3} \to 0,$$

but for our purposes this is not necessary.

5. *Proof of Theorem* 1·1. In this section we will prove Theorem 1·1; so we assume that **X** and **Y** are as in Theorem 1·1. To deal with the difficulties over the convergence of the Adams spectral sequence, our plan is to approximate **X** by finite spectra $\mathbf{W}^n$, and construct maps $f^n\colon \mathbf{W}^n \to \mathbf{Y}$. In order to make the maps f^n compatible as we vary n, we plan to use the known endomorphisms of the fixed spectrum **Y**. We take these points in order.

PROPOSITION 5·1. *Let* **X** *be a spectrum whose homotopy groups are finitely generated modules over* Λ, *and bounded below. Then* X *is equivalent to a smash-product* $\mathbf{M}\Lambda \wedge \mathbf{W}$, *where* $\mathbf{M}\Lambda$ *is a Moore spectrum for the group* Λ, *and* **W** *is a spectrum whose skeletons* $\mathbf{W}^n$ *are all finite.*

Broadly speaking, this result says that any p-local spectrum **X** is the localization of a global spectrum **W**, and any p-complete spectrum **X** is the completion of a global spectrum **W**. We should perhaps only sketch the proof, since the result should appear in any complete treatment of localization and completion (compare (9, 13)). But for completeness we give at least a sketch.

It clearly follows from Proposition 5·1 that if **X** is as assumed there, then **X** is a module spectrum over the ring spectrum $\mathbf{M}\Lambda$. It is convenient to begin by proving at least part of this.

As in section 1, we define the map $\mathbf{X} \to \mathbf{M}\Lambda \wedge \mathbf{X}$ to be the composite

$$\mathbf{X} \simeq \mathbf{S}^0 \wedge \mathbf{X} \simeq \mathbf{MZ} \wedge \mathbf{X} \to \mathbf{M}\Lambda \wedge \mathbf{X},$$

where the last map is induced by the injection $\mathbf{Z} \to \Lambda$.

LEMMA 5·2. (i) *Let* **X** *be a spectrum whose homotopy groups are finitely generated modules over* Λ, *and bounded below. Then there is a map* $\nu\colon \mathbf{M}\Lambda \wedge \mathbf{X} \to \mathbf{X}$ *such that the composite*

$$\mathbf{X} \to \mathbf{M}\Lambda \wedge \mathbf{X} \xrightarrow{\nu} \mathbf{X}$$

is homotopic to the identity.

(ii) *Moreover, the induced map of homotopy groups*

$$\nu_*\colon \Lambda \otimes \pi_r(\mathbf{X}) \to \pi_r(\mathbf{X})$$

is the Λ*-module action map.*

Much more is true, but we state just what is needed for the proof of Proposition 5·1.

It is convenient to continue with subsidiary results. For example, part (ii) of Lemma 5·2 will follow immediately from part (i) by using the following result.

LEMMA 5·3. *Let A be a Λ-module, and B a finitely generated Λ-module; then any homomorphism $\theta: A \to B$ of abelian groups is a homomorphism of Λ-modules.*

Proof. If $\Lambda = \mathbf{Z}_{(p)}$ this follows by trivial algebra; if $\Lambda = \mathbf{Z}_p$ it follows because θ is continuous for the p-adic topology.

LEMMA 5·4. *Let V be a vector-space over the field $\mathbf{Q}$ of rational numbers; and let A be an abelian group which is complete and Hausdorff for the p-adic topology (e.g. a finitely generated module over $\mathbf{Z}_p$). Then*

$$\mathrm{Hom}_{\mathbf{Z}}(V, A) = 0, \quad \mathrm{Ext}_{\mathbf{Z}}(V, A) = 0.$$

Proof. The assertion about Hom is trivial; and since V is a direct sum of copies of $\mathbf{Q}$, it is sufficient to prove $\mathrm{Ext}_{\mathbf{Z}}(\mathbf{Q}, A) = 0$. We construct a $\mathbf{Z}$-free resolution of $\mathbf{Q}$

$$0 \to C_1 \xrightarrow{d} C_0 \xrightarrow{\epsilon} \mathbf{Q} \to 0$$

as follows. Let C_1, C_0 be $\mathbf{Z}$-free on bases $\{b_i\}, \{c_i\}$, $i \geqslant 1$; define d, ϵ by

$$d(b_i) = c_i - (i+1)\, c_{i+1},$$
$$\epsilon(c_i) = 1/i\,!.$$

Applying $\mathrm{Hom}_{\mathbf{Z}}(\ , A)$ we get an exact sequence

$$0 \leftarrow \mathrm{Ext}_{\mathbf{Z}}(\mathbf{Q}, A) \leftarrow \prod_{i=1}^{\infty} A \xleftarrow{\delta} \prod_{i=1}^{\infty} A,$$

where
$$\delta\{a_i\} = \{a_i - (i+1)\, a_{i+1}\}.$$

Our problem, then, is to take a vector $\{\alpha_i\} \in \prod_{i=1}^{\infty} A$ and solve the equations

$$\alpha_i = a_i - (i+1)\, a_{i+1} \quad (i = 1, 2, 3, \ldots).$$

The solution is
$$a_i = \sum_{j \geqslant i} \frac{j\,!}{i\,!} \alpha_j;$$

the series converges to a unique limit in A because we assume the p-adic topology on A is complete and Hausdorff. This proves the lemma.

Proof of Lemma 5·2. The homotopy groups of $\mathbf{X}$ are in any case modules over $\mathbf{Z}_{(p)}$; so the map $\mathbf{X} \to \mathbf{MZ}_{(p)} \wedge \mathbf{X}$ is an equivalence, because the induced map of homotopy maps

$$\pi_r(\mathbf{X}) \to \mathbf{Z}_{(p)} \otimes \pi_r(\mathbf{X})$$

is iso. Thus the map $\mathbf{X} \to \mathbf{MZ}_{(p)} \wedge \mathbf{X}$ has an inverse $\mathbf{MZ}_{(p)} \wedge \mathbf{X} \to \mathbf{X}$, unique up to homotopy.

In the p-complete case $\Lambda = \mathbf{Z}_p$, we note that the quotient $\mathbf{Z}_p/\mathbf{Z}_{(p)}$ is a vector-space $\mathbf{Q}$; therefore $H_*(\mathbf{MZ}_p \wedge \mathbf{X}, \mathbf{MZ}_{(p)} \wedge \mathbf{X})$ is a vector-space over $\mathbf{Q}$. In view of Lemma 5·4, the universal coefficient theorem yields

$$H^*(\mathbf{MZ}_p \wedge \mathbf{X}, \mathbf{MZ}_{(p)} \wedge \mathbf{X}; \pi_*(\mathbf{X})) = 0.$$

Now obstruction theory shows that there is a unique map $MZ_p \wedge X \to X$ extending the map $MZ_{(p)} \wedge X \to X$ obtained above.

This proves part (i) of Lemma 5·2, and part (ii) follows as remarked above.

Proof of Proposition 5·1. Alas, we have to divide cases. First we consider the p-local case $\Lambda = Z_{(p)}$. We proceed by induction over n. Suppose constructed a finite n-skeleton W^n and a map $e^n\colon W^n \to X$ such that the homomorphism

$$\Lambda \otimes \pi_r(W^n) \xrightarrow{1\otimes e^n_*} \Lambda \otimes \pi_r(X) \to \pi_r(X)$$

is iso for $r < n$ and epi for $r = n$. Consider the homomorphism

$$e^n_*\colon \pi_n(W^n) \to \pi_n(X).$$

Its kernel is a finitely generated module over Z, and therefore we can find a finite number of maps $f_i\colon S^n \to W^n$ which generate it. Construct V by attaching to W^n stable $(n+1)$-cells E^{n+1}_i, using the maps f_i as attaching maps; since the composites $e^n f_i\colon S^n \to X$ are null homotopic, we can extend the map $e^n\colon W^n \to X$ over V. The induced homomorphism

$$\pi_n(V) \to \pi_n(X)$$

is now monomorphic. Since localization preserves exactness, we infer that

$$\Lambda \otimes \pi_n(V) \to \Lambda \otimes \pi_n(X) \xrightarrow{\simeq} \pi_n(X)$$

is also mono. (At this point there seems to be no such simple argument for the p-complete case.)

We now take a finite number of maps $g_j\colon S^{n+1} \to X$ which generate $\pi_{n+1}(X)$ as a Λ-module. We construct

$$W^{n+1} = V \vee \bigvee_j S^{n+1}_j,$$

and we extend the map $V \to X$ over W^{n+1} by using the map g_j on S^{n+1}_j. This ensures that the homomorphism

$$\Lambda \otimes \pi_{n+1}(W^{n+1}) \to \pi_{n+1}(X)$$

is epi. This completes the induction, and constructs a spectrum W with finite skeletons W^n and a map $e\colon W \to X$ such that

$$\Lambda \otimes \pi_r(W) \xrightarrow{1\otimes e_*} \Lambda \otimes \pi_r(X) \to \pi_r(X)$$

is iso for all r. Let $\nu\colon M\Lambda \wedge X \to X$ be as in Lemma 5·2; then the composite

$$M\Lambda \wedge W \xrightarrow{1\wedge e} M\Lambda \wedge X \xrightarrow{\nu} X$$

induces on homology groups the isomorphism just mentioned, and so is an equivalence. This completes the p-local case.

Secondly we consider the p-complete case $\Lambda = Z_p$. The homotopy groups $\pi_r(X)$ are finitely generated modules over Z_p. We can find subgroups $\sigma_r \subset \pi_r(X)$ which are finitely generated modules over $Z_{(p)}$ and such that the composite

$$Z_p \otimes \sigma_r \xrightarrow{1\otimes i} Z_p \otimes \pi_r(X) \to \pi_r(X)$$

is iso for each r. (Take the whole of the p-torsion subgroup of $\pi_r(\mathbf{X})$, and one summand $\mathbf{Z}_{(p)}$ for each summand $\mathbf{Z}_p$ in $\pi_r(\mathbf{X})$.) The quotient $\pi_r(\mathbf{X})/\sigma_r$ is a direct sum of copies of $\mathbf{Z}_p/\mathbf{Z}_{(p)}$, and is a vector space over $\mathbf{Q}$. We can find a generalized Eilenberg–MacLane spectrum $\mathbf{E}$ such that $\pi_r(\mathbf{E}) \cong \pi_r(\mathbf{X})/\sigma_r$ for each r, and we can find a map $f: \mathbf{X} \to \mathbf{E}$ such that f induces on homotopy groups the projection $\pi_r(\mathbf{X}) \to \pi_r(\mathbf{X})/\sigma_r$. Let $\mathbf{F}$ be the fibre of f; then the injection $i: \mathbf{F} \to \mathbf{X}$ induces on homotopy groups the injection $\sigma_r \to \pi_r(\mathbf{X})$.

Alternatively, instead of using a fibering to realize the exact sequence

$$0 \to \sigma_r \to \pi_r(\mathbf{X}) \to \pi_r(\mathbf{X})/\sigma_r \to 0,$$

it is slightly more in line with general theory to use a pullback diagram to realize the following Cartesian square.

$$\begin{array}{ccc} \sigma_r = \mathbf{Z}_{(p)} \otimes \sigma_r & \to & \mathbf{Z}_p \otimes \sigma_r \cong \pi_r(\mathbf{X}) \\ \downarrow & & \downarrow \\ \mathbf{Q} \otimes \sigma_r & \to & \mathbf{Q}_p \otimes \sigma_r \end{array}$$

(Here $\mathbf{Q}_p$ is the field of p-adic numbers.) But really it makes no difference.

In any case, let $\nu: \mathbf{MZ}_p \wedge \mathbf{X} \to \mathbf{X}$ be as in Lemma 5·2; then the composite

$$\mathbf{MZ}_p \wedge \mathbf{F} \xrightarrow{1 \wedge i} \mathbf{MZ}_p \wedge \mathbf{X} \xrightarrow{\nu} \mathbf{X}$$

induces an isomorphism of homotopy groups, and so is an equivalence.

By the p-local case, which we have already established, we can find a spectrum $\mathbf{W}$ with finite skeletons such that $\mathbf{F} \simeq \mathbf{MZ}_{(p)} \wedge \mathbf{W}$. Then we have

$$\mathbf{X} = \mathbf{MZ}_p \wedge \mathbf{F} \simeq \mathbf{MZ}_p \wedge \mathbf{MZ}_{(p)} \wedge \mathbf{W}.$$

But since $\mathbf{Z}_p \otimes \mathbf{Z}_{(p)} \cong \mathbf{Z}_p$, $\mathbf{MZ}_p \wedge \mathbf{MZ}_{(p)}$ is again a Moore spectrum $\mathbf{MZ}_p$. This completes the proof.

One benefit which we obtain from having $\mathbf{W}^n$ finite is the following result, which is more or less standard.

LEMMA 5·5. *Let $\mathbf{W}^n$ be a finite spectrum and $\mathbf{Y}$ a spectrum whose homotopy groups $\pi_r(\mathbf{Y})$ are finitely generated modules over Λ. Consider maps $f: \mathbf{W}^n \to \mathbf{Y}$ such that*

$$f_*: \pi_r(\mathbf{W}^n) \otimes \mathbf{Q} \to \pi_r(\mathbf{Y}) \otimes \mathbf{Q}$$

is zero for all r. Then such maps fall into finitely many homotopy classes.

Proof. Let $\mathbf{Y}_\mathbf{Q}$ be the rationalization of $\mathbf{Y}$. Since $\mathbf{Y}_\mathbf{Q}$ is a generalized Eilenberg–MacLane spectrum, the hypothesis

$$f_* = 0: \pi_r(\mathbf{W}^n) \otimes \mathbf{Q} \to \pi_r(\mathbf{Y}) \otimes \mathbf{Q} \quad \text{for all } r$$

implies that the composite $\mathbf{W}^n \xrightarrow{f} \mathbf{Y} \to \mathbf{Y}_\mathbf{Q}$ is nullhomotopic. Now the obvious map

$$[\mathbf{W}^n, \mathbf{Y}] \otimes \mathbf{Q} \to [\mathbf{W}^n, \mathbf{Y}_\mathbf{Q}]$$

is iso when $\mathbf{W}^n$ is finite ((2), Proposition 6·7, p. 202). Therefore such maps f lie in the torsion subgroup of $[\mathbf{W}^n, \mathbf{Y}]$. But under the given hypotheses $[\mathbf{W}^n, \mathbf{Y}]$ is a finitely generated Λ-module, so its torsion subgroup is finite. This proves the lemma.

We will now discuss the endomorphisms of the fixed spectrum $\mathbf{Y}$.

LEMMA 5·6. *Let m be an integer for which $\pi_m(\mathbf{Y}) \simeq \Lambda$. Then there is a map $\phi(m)\colon \mathbf{Y} \to \mathbf{Y}$ with the following properties*:

(i) $\phi(m)^*\colon H^*(\mathbf{Y}; \mathbf{F}_p) \leftarrow H^*(\mathbf{Y}; \mathbf{F}_p)$ *is zero.*

(ii) $\phi(m)_*\colon \pi_t(\mathbf{Y}) \to \pi_t(\mathbf{Y})$ *is a multiple of p for all t.*

(iii) $\phi(m)_*\colon \pi_{2r}(\mathbf{Y}) \to \pi_{2r}(\mathbf{Y})$ *is zero for $2r < m$, while for $2r = m$ it is multiplication by a non-zero scalar $\delta_m \in \Lambda$.*

The proof we give is not intended to lead to the best possible value for δ_m. Such a value would be relevant if we wished to consider $[\mathbf{Y}, \mathbf{Y}]$, and we do know the right numbers, but they are not relevant for our present purposes. This modest aim allows us to use a shorter proof. In particular, the reader will see that while the proof below may appear to be constructed with the complex case in mind, it is true word for word in the real case also, though in a rather wasteful way.

Proof of Lemma 5·6. The spectrum $\mathbf{Y}$ admits a map

$$\psi^k\colon \mathbf{Y} \to \mathbf{Y}$$

for each integer k prime to p (or even for each unit k in $\mathbf{Z}_{(p)}$). The induced map

$$(\psi^k)_*\colon \pi_{2r}(\mathbf{Y}) \to \pi_{2r}(\mathbf{Y})$$

is multiplication by k^r. If we take a Λ-linear combination $\sum_i \lambda_i \psi^{k_i}$, the induced map

$$(\sum_i \lambda_i \psi^{k_i})_*\colon \pi_{2r}(\mathbf{Y}) \to \pi_{2r}(\mathbf{Y})$$

is multiplication by $\sum_i \lambda_i (k_i)^r$. Let r run over the range $d \leqslant 2r \leqslant m$, where d is as in section 1. By taking as many i's as there are r's, and suitable coefficients $\lambda_i \in \mathbf{Z}_{(p)}$, we can ensure that

$$\sum_i \lambda_i (k_i)^r = 0 \quad \text{for} \quad d \leqslant 2r < m,$$

while for $2r = m$ we have

$$\sum_i \lambda_i (k_i)^r = \Delta,$$

where Δ is the determinant of the matrix whose (i, r)th entry is $(k_i)^r$. After removing from Δ a power of $\prod_i k_i$ which is a unit in $\mathbf{Z}_{(p)}$, we obtain a Vandermonde determinant, and this can be made non-zero by choosing the k_i distinct.

We can now satisfy all the conditions by taking

$$\phi(m) = p \sum_i \lambda_i \psi^{k_i}.$$

This proves the lemma.

Let m run over the integers for which $\pi_m(\mathbf{Y}) \simeq \Lambda$. By Lemma 5·3 we have

$$\operatorname{Hom}_{\mathbf{Z}}(\pi_m(\mathbf{X}), \pi_m(\mathbf{Y})) = \operatorname{Hom}_{\Lambda}(\pi_m(\mathbf{X}), \pi_m(\mathbf{Y})),$$

which, by our standing assumptions, is a finitely generated Λ-module. If we take it mod δ_m (where δ_m is as in Lemma 5·6) we get a finite group; let

$$\alpha_{m1}, \alpha_{m2}, \ldots\colon \pi_m(\mathbf{X}) \to \pi_m(\mathbf{Y})$$

be a finite set of homomorphisms containing one representative from each residue class mod δ_m.

We suppose given an isomorphism

$$\theta: H^*(\mathbf{X}; \mathbf{F}_p) \leftarrow H^*(\mathbf{Y}; \mathbf{F}_p)$$

as in Theorem 1·1.

LEMMA 5·7. *For each n there is a map* $f = f^n: \mathbf{W}^n \to \mathbf{Y}$ *with the following properties.*

(i) *The induced map* f^* *of mod p cohomology is the composite*

$$H^*(\mathbf{W}^n; \mathbf{F}_p) \leftarrow H^*(\mathbf{X}; \mathbf{F}_p) \xleftarrow{\theta} H^*(\mathbf{Y}; \mathbf{F}_p).$$

(ii) *The map* $\Lambda \otimes \pi_r(\mathbf{W}^n) \xrightarrow{1 \otimes f_*} \Lambda \otimes \pi_r(\mathbf{Y}) \to \pi_r(\mathbf{Y})$

is iso for $r < n$, *epi for* $r = n$.

(iii) *For each m such that* $\pi_m(\mathbf{Y}) \simeq \Lambda$ *and* $m < n$ *the isomorphism*

$$\pi_m(\mathbf{X}) \xleftarrow{\cong} \Lambda \otimes \pi_m(\mathbf{W}^n) \xrightarrow{1 \otimes f_*} \Lambda \otimes \pi_m(\mathbf{Y}) \to \pi_m(\mathbf{Y})$$

is one of the representatives α_{mi} *chosen above.*

Proof. First recall that $\mathbf{Y}$ is the spectrum $\mathbf{bg}_\Lambda$ obtained by introducing coefficients Λ into a spectrum $\mathbf{bg}$, and that the map $\mathbf{bg} \to \mathbf{bg}_\Lambda$ induces an isomorphism

$$H^*(\mathbf{bg}; \mathbf{F}_p) \leftarrow H^*(\mathbf{bg}_\Lambda; \mathbf{F}_p).$$

We define the A-module map ϕ so that the following diagram is commutative.

$$\begin{array}{ccc} H^*(\mathbf{X}; \mathbf{F}_p) & \xleftarrow{\theta} & H^*(\mathbf{Y}; \mathbf{F}_p) \\ \downarrow & & \downarrow \cong \\ H^*(\mathbf{W}^n; \mathbf{F}_p) & \xleftarrow{\phi} & H^*(\mathbf{bg}; \mathbf{F}_p) \end{array}$$

We begin by showing that there is a map $\mathbf{W}^n \to \mathbf{bg}$ whose induced map of mod p cohomology is ϕ.

In fact, the spectrum $\mathbf{bg}$ admits an Adams resolution of the conventional sort. By mapping any spectrum $\mathbf{X}$ into this resolution we obtain an 'Adams spectral sequence'; of course we assert nothing about the convergence of this spectral sequence. However, the spectral sequence is functorial for maps of $\mathbf{X}$; and the usual theorem for identifying its E_2 term is valid, for this depends only on hypotheses of finite generation in the Adams resolution, and the Adams resolution of $\mathbf{bg}$ is conventional. The isomorphism

$$H^*(\mathbf{X}; \mathbf{F}_p) \xleftarrow{\theta} H^*(\mathbf{Y}; \mathbf{F}_p) \xrightarrow{\cong} H^*(\mathbf{bg}; \mathbf{F}_p)$$

gives an element of the term E_2^{00} of the spectral sequence for $\mathbf{X}$. All differentials d_r are zero on this element, by Proposition 4·1 or 4·2 according to the case. By functoriality using the map $\mathbf{W}^n \to \mathbf{X}$, all differentials d_r are zero on the element ϕ in the term E_2^{00} of the spectral sequence for $\mathbf{W}^n$. But the spectral sequence for $\mathbf{W}^n$ is convergent in the usual way, because $\mathbf{W}^n$ is a finite complex. So there is a map $\mathbf{W}^n \to \mathbf{bg}$ which induces the map ϕ of cohomology. By taking the composite

$$\mathbf{W}^n \to \mathbf{bg} \to \mathbf{bg}_\Lambda = \mathbf{Y},$$

we get a map $f: \mathbf{W}^n \to \mathbf{Y}$ which satisfies clause (i) of the conclusion.

Next we show that any such map f satisfies clause (ii) of the conclusion. The map

$$\mathbf{W} \to \mathbf{M}\Lambda \wedge \mathbf{W} \xrightarrow{\simeq} \mathbf{X}$$

induces isomorphisms of mod p cohomology; the map

$$H^r(\mathbf{W}^n; \mathbf{F}_p) \leftarrow H^r(\mathbf{W}; \mathbf{F}_p)$$

is iso for $r < n$, mono for $r = n$. It follows that the induced map

$$f^*: H^r(\mathbf{W}^n; \mathbf{F}_p) \leftarrow H^r(\mathbf{Y}; \mathbf{F}_p)$$

is iso for $r < n$, mono for $r = n$. Let $\bar{f}$ be the composite

$$\mathbf{M}\Lambda \wedge \mathbf{W}^n \xrightarrow{1 \wedge f} \mathbf{M}\Lambda \wedge \mathbf{Y} \xrightarrow{\nu} \mathbf{Y}$$

where ν is as in Lemma 5·2; then the restriction of $\bar{f}$ to $\mathbf{W}^n$ is homotopic to f. It follows that

$$\bar{f}^*: H^r(\mathbf{M}\Lambda \wedge \mathbf{W}^n; \mathbf{F}_p) \leftarrow H^r(\mathbf{Y}; \mathbf{F}_p)$$

is iso for $r < n$, mono for $r = n$. By the usual device of a mapping-cylinder, we can assume that $\bar{f}$ is an embedding; actually we do this only in order to write relative groups, and we could equally well use the corresponding groups of the map $\bar{f}$. In any case, we have

$$H^r(\mathbf{Y}, \mathbf{M}\Lambda \wedge \mathbf{W}^n; \mathbf{F}_p) = 0 \quad \text{for} \quad r \leqslant n.$$

By duality, we have

$$H_r(\mathbf{Y}, \mathbf{M}\Lambda \wedge \mathbf{W}^n; \mathbf{F}_p) = 0 \quad \text{for} \quad r \leqslant n.$$

We now argue by induction over r; suppose $\pi_s(\mathbf{Y}, \mathbf{M}\Lambda \wedge \mathbf{W}^n) = 0$ for $s < r$, where $r \leqslant n$. Then the Hurewicz theorem gives

$$\pi_r(\mathbf{Y}, \mathbf{M}\Lambda \wedge \mathbf{W}^n) \otimes \mathbf{F}_p = 0.$$

The homotopy sequence of the pair $(\mathbf{Y}, \mathbf{M}\Lambda \wedge \mathbf{W}^n)$ gives a short exact sequence of groups

$$0 \to A \to \pi_r(\mathbf{Y}, \mathbf{M}\Lambda \wedge \mathbf{W}^n) \to B \to 0$$

in which A and B are finitely generated modules over Λ. We have $B \otimes \mathbf{F}_p = 0$, so $B = 0$ (either by Nakayama's Lemma or by the structure theory for finitely generated modules). Hence the map $A \to \pi_r(\mathbf{Y}, \mathbf{M}\Lambda \wedge \mathbf{W}^n)$ is iso, and the argument of the last sentence shows that $\pi_r(\mathbf{Y}, \mathbf{M}\Lambda \wedge \mathbf{W}^n) = 0$. This completes the induction, which shows that $\pi_r(\mathbf{Y}, \mathbf{M}\Lambda \wedge \mathbf{W}^n) = 0$ for $r \leqslant n$, and establishes clause (ii) of the conclusion.

It remains to take our map f: $\mathbf{W}^n \to \mathbf{Y}$ and modify it so as to satisfy clause (iii). We do this by induction over m. Suppose that f_0: $\mathbf{W}^n \to \mathbf{Y}$ satisfies clauses (i) and (ii), and also satisfies clause (iii) in degrees $m' < m$. Let us replace f_0 by

$$f_0 + \lambda \phi(m) f_0,$$

where $\lambda \in \Lambda$ and $\phi(m)$ is as in Lemma 5·6. This process does not affect

$$f_0^*: H^*(\mathbf{W}^n; \mathbf{F}_p) \leftarrow H^*(\mathbf{Y}; \mathbf{F}_p).$$

It does not affect clause (ii), either because clause (ii) follows from clause (i), or because we alter

$$f_{0*}: \pi_r(\mathbf{W}^n) \to \pi_r(\mathbf{Y})$$

by a multiple of p. It does not affect the induced isomorphisms $\pi_{m'}(\mathbf{X}) \to \pi_{m'}(\mathbf{Y})$ for $m' < m$. However by varying λ we can make the induced isomorphism

$$\pi_m(\mathbf{X}) \to \pi_m(\mathbf{Y})$$

run over a residue class mod δ_m, so by a suitable choice of λ we can make it one of the representatives α_{mi}. This completes the induction and proves the lemma.

For the next lemma, we suppose that $\pi_n(\mathbf{Y}) \otimes \mathbf{Q} = 0$ (as happens, for example, when n is odd).

LEMMA 5·8. *The maps f: $\mathbf{W}^n \to \mathbf{Y}$ with the properties stated in Lemma* 5·7 *fall into finitely many homotopy classes.*

Proof. If f is as in Lemma 5·7, then the induced homomorphism

$$f_*: \pi_*(\mathbf{W}^n) \otimes \mathbf{Q} \to \pi_*(\mathbf{Y}) \otimes \mathbf{Q}$$

is wholly determined by the α_{mi} in Lemma 5·7. (In fact, $\pi_r(\mathbf{W}^n) \otimes \mathbf{Q} = 0$ for $r > n$, while $\pi_n(\mathbf{Y}) \otimes \mathbf{Q} = 0$ by assumption.) There are only a finite number of m's to be considered, and for each m only a finite number of α_{mi}, so there are only a finite number of possibilities for the induced homomorphism

$$f_*: \pi_*(\mathbf{W}^n) \otimes \mathbf{Q} \to \pi_*(\mathbf{Y}) \otimes \mathbf{Q}.$$

But for each induced homomorphism there are only finitely many homotopy classes of maps f, by Lemma 5·5. This proves the lemma.

LEMMA 5·9. *We can choose a sequence of maps f^{2n+1}: $\mathbf{W}^{2n+1} \to \mathbf{Y}$ with the properties stated in Lemma* 5·7 *so that $f^{2n+1}|\mathbf{W}^{2n-1} \simeq f^{2n-1}$ for each n.*

Proof. If we take a map f: $\mathbf{W}^N \to \mathbf{Y}$ with the properties stated in Lemma 5·7, then for any $n < N$ the restriction $f \mid \mathbf{W}^n$ also has these properties.

Suppose, as an inductive hypothesis, that we have chosen f^{2n-1}: $\mathbf{W}^{2n-1} \to \mathbf{Y}$ so that for an infinity of N, f^{2n-1} extends to some map g^N: $\mathbf{W}^N \to \mathbf{Y}$ with the properties stated in Lemma 5·7. (The induction starts when $2n-1 < d$; then there are maps g^N for all N by Lemma 5·7, and their restrictions to $\mathbf{W}^{2n-1}$ are all necessarily null homotopic.) Consider the restrictions $g^N|\mathbf{W}^{2n+1}$ of these maps g^N. They lie in a finite set of homotopy classes by Lemma 5·8. So at least one homotopy class arises for an infinity of g^N. Choose f^{2n+1} in such a homotopy class. This completes the induction and proves the lemma.

Proof of Theorem 1·1. The map

$$[\mathbf{W}, \mathbf{Y}] \to \varprojlim_n [\mathbf{W}^{2n+1}, \mathbf{Y}]$$

is epi; therefore the sequence of maps f^{2n+1} of Lemma 5·9 arises from a map f^∞: $\mathbf{W} \to \mathbf{Y}$. From clauses (i) and (ii) of Lemma 5·7 we see that the map of cohomology induced by f^∞ is the composite

$$H^*(\mathbf{W}; \mathbf{F}_p) \leftarrow H^*(\mathbf{X}; \mathbf{F}_p) \xleftarrow{\theta} H^*(\mathbf{Y}; \mathbf{F}_p),$$

and that (if ν is as in Lemma 5·2) the composite

$$\mathbf{M\Lambda \wedge W} \xrightarrow{1\wedge f^{\infty}} \mathbf{M\Lambda \wedge Y} \xrightarrow{\nu} \mathbf{Y}$$

is an equivalence. It only remains to take our map f: $\mathbf{X} \rightarrow \mathbf{Y}$ to be the composite

$$\mathbf{X} \xleftarrow{\simeq} \mathbf{M\Lambda \wedge W} \xrightarrow{1\wedge f^{\infty}} \mathbf{M\Lambda \wedge Y} \xrightarrow{\nu} \mathbf{Y}.$$

This completes the proof.

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