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An ∞-categorical approach to R-line bundles, R-module Thom spectra, and twisted R-homology

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Abstract

We develop a generalization of the theory of Thom spectra using the language of ∞-categories. This treatment exposes the conceptual underpinnings of the Thom spectrum functor: we use a new model of parameterized spectra, and our definition is motivated by the geometric definition of Thom spectra of May–Sigurdsson. For an A_∞-ring spectrum R, we associate a Thom spectrum to a map of ∞-categories from the ∞-groupoid of a space X to the ∞-category of free rank one R-modules, which we show is a model for BGL_1R; we show that BGL_1R classifies homotopy sheaves of rank one R-modules, which we call R-line bundles. We use our R-module Thom spectrum to define the twisted R-homology and cohomology of R-line bundles over a space classified by a map $X \to BGL_1R$, and we recover the generalized theory of orientations in this context. In order to compare this approach to the classical theory, we characterize the Thom spectrum functor axiomatically, from the perspective of Morita theory.

1. *Introduction*

In the companion to this paper [**2**], we review and extend the work of [**14**] on Thom spectra and orientations, using the theory of structured ring spectra. To an A_∞-ring spectrum R, we associate a space BGL_1R, and to a map of spaces $f : X \to BGL_1R$ we associate an R-module Thom spectrum Mf such that R-module orientations $Mf \to R$ correspond to null-homotopies of f.

Letting S denote the sphere spectrum, one finds that BGL_1S is the classifying space for stable spherical fibrations, and if f factors as

$$f : X \xrightarrow{g} BGL_1S \longrightarrow BGL_1R,$$

then Mg is equivalent to the usual Thom spectrum of the spherical fibration classified by g, $Mf \simeq Mg \wedge R$, and R-module orientations $Mf \to R$ correspond to classical orientations $Mg \to R$.

Rich as it is, the classical theory has a number of shortcomings. For example, one expects Thom spectra as above to arise from bundles of R-modules. However, in the approaches of [**2**] as well as [**9**, **14**], such a bundle theory is more a source of inspiration than of concrete constructions or proofs. A related problem is that, with the constructions in [**2**, **9**, **14**], it is difficult to identify the functor represented by the homotopy type BGL_1R. The parameterized homotopy theory of [**15**] is one approach to the bundles of R-modules we have in mind, but the material on Thom spectra in that work focuses on spherical fibrations, and the discussion

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of twisted generalized cohomology in § 22 of that book requires a model for GL_1R which is a genuine topological monoid, a situation which may not arise from the ambient geometry.

In this paper, we introduce a new approach to parameterized spaces and spectra, Thom spectra, and orientations, based on the theory of ∞-categories. Our treatment has a number of attractive features. We use a simple theory of parameterized spectra as homotopy local systems of spectra. We give a model for BGL_1R which, by construction, evidently classifies homotopy local systems of free rank one R-modules. Using this model, we are able to give an intuitive and effective construction of the Thom spectrum. Our Thom spectrum functor is an ∞-categorical left adjoint, and so clearly commutes with homotopy colimits, and comes with an obstruction theory for orientations. We also discuss an axiomatic approach to the theory of generalized Thom spectra which allows us easily to check that our construction specializes to the other existing constructions, such as [**9**].

To begin, let us consider spaces over a space X. Since the singular complex functor from spaces to simplicial sets induces an equivalence of ∞-categories (where the latter is equipped with the Kan model structure), and both are equivalent to the ∞-category of ∞-groupoids, we will typically not distinguish between a space X and its singular complex $\mathrm{Sing}(X)$. We will also use the term 'fundamental ∞-groupoid of X' for any ∞-category equivalent to $\mathrm{Sing}(X)$. In particular, we may view spaces as ∞-groupoids, and hence as ∞-categories. Moreover, the ∞-category $\mathcal{T}$ of spaces is the prototypical example of an ∞-topos, so that, for any space X, there is a canonical equivalence

$$\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{T}) \longrightarrow \mathcal{T}_{/X},$$

between the ∞-categories of presheaves of spaces on X and spaces over X (see [**11**, 2.2.1.2]). Thus, the ∞-category $\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{T})$ is a model for the ∞-category of spaces over X.

Note that, since X is an ∞-groupoid, there is a canonical contractible space of equivalences $X \simeq X^{\mathrm{op}}$, and so of equivalences

$$\mathrm{Fun}(X, \mathcal{T}) \longrightarrow \mathrm{Fun}(X^{\mathrm{op}}, \mathcal{T}).$$

We prefer to use $\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{T})$ to emphasize the analogy with presheaves.

Now let R be a ring spectrum. We say that an R-module M is free of rank one if there is an equivalence of R-modules $M \to R$, and we write R-line $\subset$ R-mod for the subcategory consisting of the free rank one R-modules and the equivalences thereof. By construction, R-line is an ∞-groupoid, that is, a Kan complex. In a precise sense which we now explain, R-mod classifies bundles of R-modules, and R-line classifies bundles of free rank one R-modules whose fibers are glued together by R-linear equivalences.

Given a space X, a functor (that is, a map of simplicial sets)

$$L : X^{\mathrm{op}} \longrightarrow R\text{-mod}$$

is a sort of local coefficient system: for each point $p \in X$, we have an R-module L_p. To a path $\gamma : p \to q$ in X, L associates an equivalence of R-modules

$$L_\gamma : L_q \simeq L_p. \tag{1.1}$$

From a homotopy of paths $h : \gamma \to \gamma'$, we get a path

$$L_h : L_{\gamma'} \longrightarrow L_\gamma, \tag{1.2}$$

in the space of R-module equivalences $L_p \to L_q$, and so forth for higher homotopies. More precisely, L is a 'homotopy local system' of R-modules. The fact that the data of a functor from X^{op} to R-mod are precisely the higher coherence conditions for a homotopy local system of R-modules is what makes the theory of ∞-categories so effective in this context. With this in mind, we make the following definition.

DEFINITION 1.3 (§ 2.4). Let X be a space. A *bundle of R-modules over X* is a functor

$$f : X^{\mathrm{op}} \longrightarrow R\text{-mod}.$$

A *bundle of R-lines over X* is a functor

$$f : X^{\mathrm{op}} \longrightarrow R\text{-line}.$$

We write $\mathrm{Fun}(X^{\mathrm{op}}, R\text{-mod})$ and $\mathrm{Fun}(X^{\mathrm{op}}, R\text{-line})$ for the ∞-categories of bundles of R-modules and R-lines over X.

Our definition of the Thom spectrum is motivated by the May–Sigurdsson description of the 'neo-classical' Thom spectrum as the composite of the pullback of a universal parameterized spectrum followed by the base change along the map to a point [**15**, 23.7.1, 23.7.4].

DEFINITION 1.4 (§ 2.5). Let X be a space. The Thom R-module spectrum Mf of a bundle of R-lines over X

$$f : X^{\mathrm{op}} \longrightarrow R\text{-line}$$

is the colimit of the functor

$$X^{\mathrm{op}} \xrightarrow{f} R\text{-line} \longrightarrow R\text{-mod},$$

obtained by composing with the inclusion $R\text{-line} \subset R\text{-mod}$.

It is very easy to describe the obstruction theory for orientations in this setting. The colimit in Definition 1.4 means that the space of R-module maps

$$Mf \longrightarrow R$$

is equivalent to the space of maps of bundles of R-modules

$$f \longrightarrow R_X,$$

where R_X denotes the trivial bundle of R-lines over X, that is, the constant functor $X^{\mathrm{op}} \to R\text{-line}$ with value $R \in R\text{-line}$.

DEFINITION 1.5 (Definition 2.22). The space of *orientations* $Mf \to R$ is the pullback in the diagram

$$\begin{tikzcd}
\mathrm{Orientations}(Mf, R) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{map}_{R\text{-mod}}(Mf, R) \arrow[d, "\simeq"] \\
\mathrm{map}_{R_X\text{-line}}(f, R_X) \arrow[r] & \mathrm{map}_{R_X\text{-mod}}(f, R_X).
\end{tikzcd}$$

That is, orientations $Mf \to R$ are those R-module maps that correspond to trivializations $f \simeq R_X$ of the bundle of R-lines f.

Put another way, let R-triv be the ∞-category of trivialized R-lines: R-lines L equipped with a specific equivalence $L \xrightarrow{\simeq} R$. R-triv is a contractible Kan complex, and the natural map

$$R\text{-triv} \longrightarrow R\text{-line} \tag{1.6}$$

is a Kan fibration. We then have the following.

THEOREM 1.7 (Theorem 2.24). *If $f : X^{\mathrm{op}} \to R\text{-line}$ is a bundle of R-lines over X, then the space of orientations $Mf \to R$ is equivalent to the space of lifts in the diagram*

$$\begin{tikzcd} & R\text{-triv} \arrow[d] \\ X \arrow[ur, dashed] \arrow[r, "f"'] & R\text{-line}. \end{tikzcd} \tag{1.8}$$

Analogous considerations lead to a version of the Thom isomorphism in this setting.

Finally, using this notion of R-module Thom spectrum, we can define the twisted R-homology and R-cohomology of a space $f : X \to R\text{-line}$ equipped with an R-line bundle by the formulas

$$R_n^f(X) = \pi_0 \operatorname{map}_R(\Sigma^n R, Mf) \cong \pi_n Mf, \tag{1.9}$$

$$R_f^n(X) = \pi_0 \operatorname{map}_R(Mf, \Sigma^n R). \tag{1.10}$$

In the presence of an orientation, we have the following untwisting result.

COROLLARY 1.11. *If $f : X^{\mathrm{op}} \to R\text{-line}$ admits an orientation, then $Mf \simeq R \wedge \Sigma_+^\infty X$, and the twisted R-homology and R-cohomology spectra*

$$R^f(X) \simeq R \wedge \Sigma_+^\infty X, \tag{1.12}$$

$$R_f(X) \simeq \operatorname{Map}(\Sigma_+^\infty X, R), \tag{1.13}$$

reduce to the ordinary R-homology and R-cohomology spectra of X (here Map *denotes the function spectrum).*

In § 3, we relate the theory developed in this paper to other approaches, such as [**2**, **9**, **14**]. We rely on the fact that R-line is a model for BGL_1R. Indeed, the fiber of (1.6) at R is $\operatorname{Aut}_R(R)$, by which we mean the derived space of R-linear self-homotopy equivalences of R (for example, see [**2**, § 2]). More precisely, we have the following.

COROLLARY 1.14 (Corollary 2.14). *The Kan fibration*

$$\operatorname{Aut}_R(R) \longrightarrow R\text{-triv} \longrightarrow R\text{-line} \tag{1.15}$$

is a model in simplicial sets for the quasifibration $GL_1R \to EGL_1R \to BGL_1R$.

REMARK 1.16. In fact, since geometric realization carries Kan fibrations to Serre fibrations [**16**], upon geometric realization we obtain a Serre fibration that models

$$GL_1R \longrightarrow EGL_1R \longrightarrow BGL_1R,$$

in topological spaces. The approach taken in [**2**, **14**] is only known to produce a quasifibration.

The equivalence $B\operatorname{Aut}(R) \simeq R\text{-line}$ implies the following description of the Thom spectrum functor of Definition 1.4, which plays a role in § 3 when we compare our approaches to Thom spectra. Recall that if x is an object in an ∞-category $\mathcal{C}$, then $\operatorname{Aut}_{\mathcal{C}}(x)$ is a group-like monoidal ∞-groupoid, that is, a group-like A_∞-space; conversely if G is a group-like monoidal ∞-groupoid, then we can form the ∞-category BG with a single object $*$ and G as automorphisms. Moreover, an action of G on x is just a functor $BG \to \mathcal{C}$.

THEOREM 1.17. *Let G be a group-like monoidal ∞-groupoid. A map $BG \to R$-line specifies an R-linear action of G on R, and then the Thom spectrum is equivalent to the (homotopy) quotient R/G.*

The preceding theorem follows immediately from the construction of the Thom spectrum, since by definition the quotient in the statement is the colimit of the map of ∞-categories $BG \to R$-line $\to R$-mod.

Turning to the comparisons, the definitions of [**2**] and this paper give two constructions of an R-module Thom spectrum from a map $f : X \to BGL_1R$. Roughly speaking, the 'algebraic' model studied in [**2**] takes the pullback P in the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & EGL_1R \arrow[d] \\ X \arrow[r, "f"] & BGL_1R \end{tikzcd}$$

and sets

$$M_{\mathrm{alg}} f = \Sigma^\infty_+ P \wedge_{\Sigma^\infty_+ GL_1R} R. \tag{1.18}$$

The 'geometric' model in this paper sets

$$M_{\mathrm{geo}} f = \mathrm{colim}(X^{\mathrm{op}} \xrightarrow{f} BGL_1R \simeq R\text{-line} \longrightarrow R\text{-mod}).$$

It is possible to show by a direct calculation that these two constructions are equivalent; we do this in § 3.7. The bulk of § 3 is concerned with a more general characterization of the Thom spectrum functor from the point of view of Morita theory. Here, we also show that our Thom spectrum recovers the Thom spectrum of [**9**] in the special case of a map $f : X \to BGL_1S$.

In (1.18), the Thom spectrum M_{alg} is a derived smash product with R, regarded as an $\Sigma^\infty_+ GL_1R$-R bimodule, specified by the canonical action of $\Sigma^\infty_+ GL_1R$ on R. Recalling that the target category of R-modules is stable, we can regard this Thom spectrum as given by a functor from (right) $\Sigma^\infty_+ GL_1R$-modules to R-modules. Now, roughly speaking, Morita theory (more precisely, the Eilenberg–Watts theorem) implies that any continuous functor from (right) $\Sigma^\infty_+ GL_1R$-modules to R-modules that preserves homotopy colimits and takes GL_1R to R can be realized as tensoring with an appropriate $\Sigma^\infty_+ GL_1R$-R bimodule. In particular, this tells us that the Thom spectrum functor is characterized among such functors by the additional data of the action of GL_1R on R.

We develop these ideas in the setting of ∞-categories. Let $\mathcal{T}$ be the ∞-category of spaces. Given a colimit-preserving functor $F : \mathcal{T}_{/B\,\mathrm{Aut}(R)} \to R$-mod which sends $*/B\,\mathrm{Aut}(R)$ to R, we can restrict along the Yoneda embedding (3.2)

$$B\,\mathrm{Aut}(R) \longrightarrow \mathcal{T}_{/B\,\mathrm{Aut}(R)} \xrightarrow{F} R\text{-mod};$$

since it takes the object of $B\,\mathrm{Aut}(R)$ to R, we may view this as a functor

$$k : B\,\mathrm{Aut}(R) \longrightarrow B\,\mathrm{Aut}(R).$$

Conversely, given an endomorphism k of $B\,\mathrm{Aut}(R)$, we get a colimit-preserving functor

$$F : \mathcal{T}_{/B\,\mathrm{Aut}(R)} \longrightarrow R\text{-mod},$$

whose value on $B^{\mathrm{op}} \to B\,\mathrm{Aut}(R)$ is

$$\mathrm{colim}(B^{\mathrm{op}} \longrightarrow B\,\mathrm{Aut}(R) \xrightarrow{k} B\,\mathrm{Aut}(R) \hookrightarrow R\text{-mod}).$$

About this correspondence, we prove the following.

Proposition 1.19 (Corollary 3.13). *A functor F from the ∞-category $\mathcal{T}_{/B\,\mathrm{Aut}(R)}$ to the ∞-category of R-modules is equivalent to the Thom spectrum functor if and only if it preserves colimits and its restriction along the Yoneda embedding*

$$B\,\mathrm{Aut}(R) \longrightarrow \mathcal{T}_{/B\,\mathrm{Aut}(R)} \xrightarrow{F} R\text{-mod}$$

is equivalent to the canonical inclusion

$$B\,\mathrm{Aut}(R) \xrightarrow{\simeq} R\text{-line} \longrightarrow R\text{-mod}.$$

It follows easily (Proposition 3.20) that the Thom spectrum functors M_{geo} and M_{alg} are equivalent. It also follows that, as in Proposition 1.17, the Thom spectrum of a group-like A_∞-map $\varphi : G \to GL_1S$ is the (homotopy) quotient

$$\mathrm{colim}(BG^{\mathrm{op}} \longrightarrow R\text{-mod}) \simeq R/G.$$

This observation is the basis for our comparison with the Thom spectrum of Lewis and May. In § 3.6, we show that the Lewis–May Thom spectrum associated to the map $B\varphi : BG \to BGL_1S$ is a model for the (homotopy) quotient S/G.

Proposition 1.20 (Corollary 3.24). *The Lewis–May Thom spectrum associated to a map*

$$f : B \longrightarrow BGL_1S$$

is equivalent to the Thom spectrum associated by Definition 1.4 to the map of ∞-categories

$$B^{\mathrm{op}} \xrightarrow{f} BGL_1S \simeq S\text{-line}.$$

2. *Parameterized spectra and Thom spectra*

In this section, we show that the theory of ∞-categories provides a powerful technical and conceptual framework for the study of Thom spectra and orientations. We chose to use the theory of quasicategories as developed by Joyal [**8**] and Lurie [**11**], but for the theory of R-module Thom spectra and orientations, all that is really required is a good ∞-category of R-modules.

2.1. ∞-*Categories and* ∞-*groupoids*

For the purposes of this paper, an ∞-category will always mean a quasicategory in the sense of Joyal [**8**]. This is the same as a weak Kan complex in the sense of Boardman and Vogt [**5**]; the different terminology reflects the fact that these objects simultaneously generalize the homotopy theories of categories and of spaces. There is nothing essential in our use of quasicategories, and any other sufficiently well-developed theory of ∞-categories (more precisely, $(\infty, 1)$-categories) would suffice.

Given two ∞-categories $\mathcal{C}$ and $\mathcal{D}$, the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$ is simply the simplicial set of maps from $\mathcal{C}$ to $\mathcal{D}$, considered as simplicial sets. More generally, for any simplicial set X there is an ∞-category of functors from X to $\mathcal{C}$, written as $\mathrm{Fun}(X, \mathcal{C})$; by Lurie [**11**, Proposition 1.2.7.2, 1.2.7.3], the simplicial set $\mathrm{Fun}(X, \mathcal{C})$ is an ∞-category whenever $\mathcal{C}$ is, even for an arbitrary simplicial set X.

This description of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ gives rise to simplicial categories of ∞-categories and ∞-groupoids. For our purposes, it is important to have ∞-*categories* Cat_∞ and Gpd_∞ of ∞-categories and ∞-groupoids, respectively. We construct these ∞-categories by a general technique for converting a simplicial category to an ∞-category: there is a simplicial nerve functor N from simplicial categories to ∞-categories which is the right adjoint of a Quillen equivalence

$\mathfrak{C} : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Cat}_\Delta : \mathrm{N}$ (see [**11**, §§ 1.1.5.5, 1.1.5.12, 1.1.5.13]). Note that this process also gives rise to a standard passage from a simplicial model category to an ∞-category which retains the homotopical information encoded by the simplicial model structure. Specifically, given a simplicial model category $\mathcal{M}$, one restricts to the simplicial category on the cofibrant–fibrant objects, $\mathcal{M}^{\mathrm{cf}}$. Then applying the simplicial nerve yields an ∞-category $\mathrm{N}(\mathcal{M}^{\mathrm{cf}})$.

In particular, Cat_∞ is the simplicial nerve of the simplicial category of ∞-categories, in which the mapping spaces are made fibrant by restricting to maximal Kan subcomplexes, and Gpd_∞ is the full ∞-subcategory of Cat_∞ on the ∞-groupoids. We recall that the Quillen equivalence between the standard model structure on topological spaces and the Kan model structure on simplicial sets induces an equivalence on underlying ∞-categories. Thus, as all the constructions we perform in this paper are homotopy invariant, we will typically regard topological spaces as ∞-groupoids via their singular complexes.

Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ admits a maximal ∞-subgroupoid $\mathcal{C}^{\simeq}$, which is by definition the pullback (in simplicial sets) of the diagram

$$\begin{tikzcd} \mathcal{C}^{\simeq} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathrm{N}\,\mathrm{ho}(\mathcal{C})^{\simeq} \arrow[r] & \mathrm{N}\,\mathrm{ho}(\mathcal{C}) \end{tikzcd} \tag{2.1}$$

where $\mathrm{N}\,\mathrm{ho}(\mathcal{C})$ denotes the nerve of the homotopy category of $\mathcal{C}$ and $\mathrm{N}\,\mathrm{ho}(\mathcal{C})^{\simeq}$ is the maximal subgroupoid. Thus, if a and b are objects of $\mathcal{C}$, $\mathcal{C}^{\simeq}(a,b)$ is the subcategory of $\mathcal{C}(a,b)$ consisting of the *equivalences*.

2.2. *Parameterized spaces*

Let X be an ∞-groupoid, which we view as the fundamental ∞-groupoid of a topological space. There are two canonically equivalent ∞-topoi associated to X; namely, the slice ∞-category $\mathrm{Gpd}_{\infty/X}$ of ∞-groupoids over X, and the ∞-category $\mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Gpd}_\infty)$ of presheaves of ∞-groupoids on X. The equivalence

$$\mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Gpd}_\infty) \simeq \mathrm{Gpd}_{\infty/X}$$

sends a functor to its colimit, regarded as a space over X, and may be regarded as a generalization of the equivalence between (free) G-spaces and spaces over BG (see [**11**, 2.2.1.2]). In particular, a terminal object $1 \in \mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Gpd}_\infty)$ must be sent to a terminal object $\mathrm{id}_X \in \mathrm{Gpd}_{\infty/X}$, which in this special case recovers the formula

$$\operatorname*{colim}_{X^{\mathrm{op}}} 1 \simeq X. \tag{2.2}$$

Remark 2.3. As explained in § 1, the data of a functor $L : X^{\mathrm{op}} \to \mathrm{Gpd}_\infty$ encodes the data of a homotopy local system of spaces on X.

Remark 2.4. Since X is an ∞-groupoid, we have a canonical contractible space of equivalences $X \simeq X^{\mathrm{op}}$, which induces equivalences

$$\mathrm{Fun}(X, \mathrm{Gpd}_\infty) \simeq \mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Gpd}_\infty) \simeq \mathrm{Gpd}_{\infty/X}\,.$$

Although notationally a bit more complicated, we think it is slightly more natural to regard spaces over X as contravariant functors, instead of covariant functors, from X to Gpd_∞. One reason for this is that, this way, the Yoneda embedding appears naturally as a functor $X \to \mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Gpd}_\infty)$, and this will play an important role in our treatment of the Thom spectrum functor (cf. Proposition 3.12).

LEMMA 2.5. *The base-change functor $f^*: \mathrm{Gpd}_{\infty/X} \to \mathrm{Gpd}_{\infty/X'}$ admits a right adjoint. In particular, f^* commutes with colimits.*

Proof. For the proof, see [**11**, 6.1.3.14]. □

REMARK 2.6. If X is an ∞-groupoid, then via the equivalence of ∞-categories $\mathrm{Gpd}_{\infty/X} \simeq \mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Gpd}_\infty)$, the Yoneda embedding $X \to \mathrm{Gpd}_{\infty/X}$ sends the point x of X to the 'path fibration' $X_{/x} \to X$. (This follows from an analysis of the 'unstraightening' functor that provides the right adjoint in [**11**, 2.2.1.2].)

2.3. *Parameterized spectra*

An ∞-category $\mathcal{C}$ is *stable* if it has a zero object, finite limits, and the endofunctor $\Omega : \mathcal{C} \to \mathcal{C}$, defined by sending X to the limit of the diagram $* \to X \leftarrow *$, is an equivalence [**12**, 1.1.1.9, 1.4.2.27]. It follows that the left adjoint Σ of Ω is also an equivalence, that finite products and finite coproducts agree, and that square $\Delta^1 \times \Delta^1 \to \mathcal{C}$ is a pullback if and only if it is a pushout (so that $\mathcal{C}$ has all finite colimits as well). A morphism of stable ∞-categories is an exact functor, meaning a functor which preserves finite limits and colimits [**12**, 1.1.4.1].

More generally, given any ∞-category $\mathcal{C}$ with finite limits, the *stabilization* of $\mathcal{C}$ is the limit (in the ∞-category of ∞-categories) of the tower

$$\cdots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*,$$

where $\mathcal{C}_*$ denotes the pointed ∞-category associated to $\mathcal{C}$ (the full ∞-subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ on those arrows whose source is a final object $*$ of $\mathcal{C}$). Provided $\mathcal{C}$ is presentable, $\mathrm{Stab}(\mathcal{C})$ comes equipped with a stabilization functor $\Sigma^\infty_+ : \mathcal{C} \to \mathrm{Stab}(\mathcal{C})$ functor from $\mathcal{C}$ (see [**12**, 1.4.4.4]), formally analogous to the suspension spectrum functor, and left adjoint to the zero-space functor $\Omega^\infty_- : \mathrm{Stab}(\mathcal{C}) \to \mathcal{C}$ (the subscript indicates that we forget the basepoint).

If one works entirely in the world of presentable stable ∞-categories and left adjoint functors thereof, then Stab is left adjoint to the inclusion into the ∞-category of presentable ∞-categories of the full ∞-subcategory of presentable stable ∞-categories. In other words, a morphism of presentable ∞-categories $\mathcal{C} \to \mathcal{D}$ such that $\mathcal{D}$ is stable factors (uniquely up to a contractible space of choices) through the stabilization $\Sigma^\infty_+ : \mathcal{C} \to \mathrm{Stab}(\mathcal{C})$ of $\mathcal{C}$ (cf. [**12**, 1.4.4.4, 1.4.4.5]). The ∞-category $\mathrm{Gpd}_{\infty/X}$ of spaces over a fixed space X is presentable.

The discussion so far suggests two models for the ∞-category of spectra over X: one is $\mathrm{Stab}(\mathrm{Gpd}_{\infty/X})$, and the other is $\mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Stab}(\mathrm{Gpd}_\infty))$. In fact, these are equivalent: for any ∞-groupoid X, the equivalence $\mathrm{Gpd}_{\infty/X} \simeq \mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Gpd}_\infty)$ induces an equivalence of stabilizations

$$\mathrm{Stab}(\mathrm{Gpd}_{\infty/X}) \simeq \mathrm{Stab}(\mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Gpd}_\infty)).$$

Since limits in functor categories are computed pointwise, one easily checks that

$$\mathrm{Stab}(\mathrm{Gpd}_{\infty/X}) \simeq \mathrm{Stab}(\mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Gpd}_\infty)) \simeq \mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Stab}(\mathrm{Gpd}_\infty)).$$

REMARK 2.7. May and Sigurdsson [**15**] build a simplicial model category $\mathcal{S}_X$ of orthogonal spectra parameterized by a topological X. In [**1**], we prove that there is an equivalence of ∞-categories between the simplicial nerve of the May–Sigurdsson category of parameterized orthogonal spectra $\mathrm{N}(\mathcal{S}_X^{\mathrm{cf}})$ and the ∞-category $\mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Stab}(\mathrm{Gpd}_\infty))$.

2.4. *Parameterized R-modules and R-lines*

We now fix an A_∞-ring spectrum R. Recall [**12**, 4.2.1.36] that there exists a presentable stable ∞-category R-mod of (right) R-module spectra, and that this ∞-category possesses a distinguished object R.

Definition 2.8. An R-line is an R-module M which admits an R-module equivalence $M \simeq R$.

Let R-line denote the full ∞-subgroupoid of R-mod spanned by the R-lines. This is not the same as the full ∞-subcategory of R-mod on the R-lines, as a map of R-lines is by definition an equivalence. We regard R-line as a pointed ∞-groupoid via the distinguished object R.

Proposition 2.9. *There is a canonical equivalence of ∞-groupoids*

$$B\,\mathrm{Aut}_R(R) \simeq R\text{-line},$$

and $\mathrm{Aut}_R(R) \simeq GL_1 R$ *as monoidal ∞-groupoids.*

Proof. We regard $B\,\mathrm{Aut}_R(R) \subset R$-mod as the full subgroupoid of R-mod consisting of the single R-module R. Hence, $B\,\mathrm{Aut}_R(R)$ is naturally a full subgroupoid of R-line, and the fully faithful inclusion $B\,\mathrm{Aut}_R(R) \subset R$-line is also essentially surjective by definition of R-line. It is therefore an equivalence, so it only remains to show that $GL_1 R \simeq \mathrm{Aut}_R(R)$ as monoidal ∞-groupoids. This follows from the fact that $\mathrm{End}_R(R) \simeq \Omega^\infty(R)$, and $\mathrm{Aut}_R(R) \subset \mathrm{End}_R(R)$ is, by (2.1), the monoidal subspace defined by the same condition as $GL_1 R \subset \Omega^\infty(R)$; namely, as the pullback

$$\begin{tikzcd}
\mathrm{Aut}_R(R) \arrow[r] \arrow[d] & \mathrm{End}_R(R) \arrow[d] \\
\pi_0\,\mathrm{End}_R(R)^\times \arrow[r] & \pi_0\,\mathrm{End}_R(R)
\end{tikzcd}$$

where $\pi_0\,\mathrm{End}_R(R)^\times \cong \pi_0(R)^\times$ denotes the invertible homotopy classes of endomorphisms in the ordinary category $\mathrm{ho}(R\text{-mod})$. □

Definition 2.10. Let X be a space. The ∞-category of R-modules over X is the ∞-category $\mathrm{Fun}(X^{\mathrm{op}}, R\text{-mod})$ of presheaves of R-modules on X; similarly, the ∞-category of R-lines over X is the ∞-category $\mathrm{Fun}(X^{\mathrm{op}}, R\text{-line})$ of presheaves of R-lines on X.

We will denote by R_X the constant functor $X^{\mathrm{op}} \to R$-line $\to R$-mod which has value R, and sometimes write

$$R_X\text{-mod} = \mathrm{Fun}(X^{\mathrm{op}}, R\text{-mod}),$$

for the ∞-category of R-modules over X, and

$$R_X\text{-line} = \mathrm{Fun}(X^{\mathrm{op}}, R\text{-line}),$$

for the full ∞-subgroupoid spanned by those R-modules over X which factor

$$X^{\mathrm{op}} \longrightarrow R\text{-line} \longrightarrow R\text{-mod},$$

through the inclusion of the full ∞-subgroupoid R-line $\to R$-mod.

LEMMA 2.11. *The fiber over X of the projection* $\mathrm{Gpd}_{\infty/R\text{-line}} \to \mathrm{Gpd}_\infty$ *is equivalent to the ∞-groupoid* R_X-line.

Proof. $R_X\text{-line} \simeq \mathrm{Fun}(X^{\mathrm{op}}, R\text{-line}) \simeq \mathrm{map}_{\mathrm{Gpd}_\infty}(X^{\mathrm{op}}, R\text{-line})$, and, in general, the ∞-groupoid $\mathrm{map}_{\mathcal{C}}(a,b)$ of maps from a to b in the ∞-category $\mathcal{C}$ may be calculated as the fiber over a of the projection $\mathcal{C}_{/b} \to \mathcal{C}$. □

DEFINITION 2.12. A *trivialization* of an R_X-module L is an R_X-module equivalence $L \to R_X$. The ∞-category R_X-triv of trivialized R-lines is the slice category

$$R_X\text{-triv} \overset{\text{def}}{=} R_X\text{-line}_{/R_X}.$$

The objects of R_X-triv are *trivialized* R_X-lines, which is to say R_X-lines L with a trivialization $L \to R_X$; more generally, an n-simplex $\Delta^n \to R_X$-triv of R_X-triv is a map $\Delta^n \star \Delta^0 \to R_X$-line of R_X-line which sends Δ^0 to R_X. There is a canonical projection

$$\iota_X : R_X\text{-triv} \longrightarrow R_X\text{-line},$$

which sends the n-simplex $\Delta^n \star \Delta^0 \to R_X$-line to the n-simplex $\Delta^n \to \Delta^n \star \Delta^0 \to R_X$-line; according to (the dual of) [**11**, Corollary 2.1.2.4], this is a right fibration, and hence a Kan fibration as R_X-line is an ∞-groupoid [**11**, Lemma 2.1.3.2].

LEMMA 2.13. *Let X be an ∞-groupoid. Then* R_X-triv *is a contractible ∞-groupoid, and the fiber, over a given R_X-line f, of the projection*

$$\iota_X : R_X\text{-triv} \longrightarrow R_X\text{-line}$$

is the (possibly empty) ∞-groupoid $\mathrm{map}_{R_X\text{-line}}(f, R_X)$.

Proof. Once again, use the description of $\mathrm{map}_{\mathcal{C}}(a,b)$ as the fiber over a of the projection $\mathcal{C}_{/b} \to \mathcal{C}$, together with the fact that if $\mathcal{C}$ is an ∞-groupoid, then $\mathcal{C}_{/b}$, an ∞-groupoid with a final object, is contractible. □

COROLLARY 2.14. *The Kan fibration*

$$\mathrm{Aut}_R(R) \longrightarrow R\text{-triv} \longrightarrow R\text{-line}$$

is a simplicial model for the quasifibration $GL_1R \to EGL_1R \to BGL_1R$.

Proof. By the preceding discussion, R-triv is a contractible Kan complex and the projection R-triv $\to R$-line is a Kan fibration. The result follows from Proposition 2.9, where we showed that $\mathrm{Aut}_R(R) \simeq GL_1R$. □

For X the terminal Kan complex, we write R-triv in place of R_X-triv and $\iota : R\text{-triv} \to R\text{-line}$ in place of ι_X. Given $f : X \to R$-line, we refer to a factorization

$$\begin{tikzcd} & R\text{-triv} \arrow[d, "\iota"] \\ X^{\mathrm{op}} \arrow[ur, dashed] \arrow[r, "f"'] & R\text{-line} \end{tikzcd} \tag{2.15}$$

of f through ι as a *trivialization* of f.

Definition 2.16. We write $\mathrm{Triv}(f)$ for the space of trivializations of f; explicitly, it is the fiber over f in the fibration

$$\mathrm{Fun}(X^{\mathrm{op}}, R\text{-triv}) \xrightarrow{\iota} \mathrm{Fun}(X^{\mathrm{op}}, R\text{-line}).$$

Corollary 2.17. *There is a canonical equivalence of ∞-groupoids*

$$\mathrm{Fun}(X^{\mathrm{op}}, R\text{-triv}) \simeq R_X\text{-triv}.$$

Moreover, $\mathrm{Triv}(f)$ *is equivalent to* $\mathrm{map}_{R_X\text{-line}}(f, R_X)$.

Proof. For the first claim, we have

$$\mathrm{Fun}(X^{\mathrm{op}}, R\text{-line}_{/R}) \simeq \mathrm{Fun}(X^{\mathrm{op}}, R\text{-line})_{/p^*R} \simeq R_X\text{-line}_{/R_X}.$$

For the second, compare the two pullback diagrams

$$\begin{tikzcd} \mathrm{map}_{R_X\text{-line}}(f, R_X) \arrow[r] \arrow[d] & R_X\text{-line}_{/R_X} \arrow[d] \\ \{f\} \arrow[r] & R_X\text{-line} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{Triv}(f) \arrow[r] \arrow[d] & \mathrm{Fun}(X^{\mathrm{op}}, R\text{-triv}) \arrow[d] \\ \{f\} \arrow[r] & \mathrm{Fun}(X^{\mathrm{op}}, R\text{-line}) \end{tikzcd}$$

in which the two right-hand fibrations are equivalent. □

A map of spaces $f : X \to Y$ gives rise to a restriction functor

$$f^* : R_Y\text{-mod} \longrightarrow R_X\text{-mod},$$

which admits a right adjoint f_* as well as a left adjoint $f_!$. This means that, given an R_X-module L and an R_Y-module M, there are natural equivalences of ∞-groupoids

$$\mathrm{map}(f_!L, M) \simeq \mathrm{map}(L, f^*M)$$

and

$$\mathrm{map}(f^*M, L) \simeq \mathrm{map}(M, f_*L).$$

An important point about these functors is the following.

Proposition 2.18. *Let $\pi : X \to *$ be the projection to a point and let $\pi^* : R\text{-mod} \to R_X\text{-mod}$ be the resulting functor. If M is an R-module, then*

$$\pi_!\pi^*M \simeq \Sigma^\infty_+ X \wedge M. \tag{2.19}$$

Proof. We use the equivalence $R_X\text{-mod} \simeq \mathrm{Fun}(X^{\mathrm{op}}, R\text{-mod})$, and compute in $\mathrm{Fun}(X^{\mathrm{op}}, R\text{-mod})$. In that case the left-hand side in (2.19) is the colimit of the constant map of ∞-categories

$$X^{\mathrm{op}} \xrightarrow{M} R\text{-mod}.$$

This map is equivalent to the composition

$$X^{\mathrm{op}} \xrightarrow{1} \mathrm{Gpd}_\infty \xrightarrow{\Sigma^\infty_+} \mathrm{Stab}(\mathrm{Gpd}_\infty) \xrightarrow{(-)\wedge M} R\text{-mod}.$$

The second two functors in this composition commute with colimits, and equation (2.2) says that $X \simeq \operatorname{colim}(1 : X^{\mathrm{op}} \to \mathrm{Gpd}_\infty)$. □

2.5. *Thom spectra*

We continue to fix an A_∞-ring spectrum R.

DEFINITION 2.20. The Thom R-module spectrum is the functor

$$M : \mathrm{Gpd}_{\infty/R\text{-line}} \longrightarrow R\text{-mod},$$

which sends $f : X^{\mathrm{op}} \to R$-line to the colimit of the composite

$$X^{\mathrm{op}} \xrightarrow{f} R\text{-line} \xrightarrow{i} R\text{-mod}.$$

Equivalently Mf is the left Kan extension

$$Mf \overset{\mathrm{def}}{=} p_!(i \circ f)$$

along the map $p : X^{\mathrm{op}} \to *$.

PROPOSITION 2.21. *Let G be an ∞-group (a group-like monoidal ∞-groupoid) with classifying space BG and suppose given a map $f : BG \to R$-line. Then*

$$Mf \simeq R/G,$$

where G acts on R via the map $\Omega f : G \simeq \Omega BG \to \Omega(R\text{-line}) \simeq \mathrm{Aut}_R(R)$.

Proof. Both Mf and R/G are equivalent to the colimit of the composite functor $BG^{\mathrm{op}} \to B\,\mathrm{Aut}_R(R) \simeq R\text{-line} \to R\text{-mod}$. □

2.6. *Orientations*

With these in place, one can analyze the space of orientations in a straightforward manner, as follows. First of all observe that, by definition, we have an equivalence

$$\mathrm{map}_{R\text{-mod}}(Mf, R) \simeq \mathrm{map}_{R_X\text{-mod}}(f, p^*R).$$

DEFINITION 2.22. The space of *orientations* of Mf is the pullback

$$\begin{tikzcd} \mathrm{Orientations}_R(Mf, R) \arrow[r] \arrow[d, "\simeq"] & \mathrm{map}_{R\text{-mod}}(Mf, R) \arrow[d, "\simeq"] \\ \mathrm{map}_{R_X\text{-line}}(f, p^*R) \arrow[r] & \mathrm{map}_{R_X\text{-mod}}(f, p^*R). \end{tikzcd} \tag{2.23}$$

The ∞-groupoid $\mathrm{Orientations}_R(Mf, R)$ enjoys an obstruction theory analogous to that of the space of orientations described in [**2**]. The following theorem is the analog in this context of [**2**, 3.20].

THEOREM 2.24. *Let $f : X^{\mathrm{op}} \to R$-line be a map, with associated Thom R-module Mf. Then the space of orientations $Mf \to R$ is equivalent to the space of lifts in the diagram*

$$\begin{tikzcd} & R\text{-triv} \arrow[d, "\iota"] \\ X^{\mathrm{op}} \arrow[ur, dashed] \arrow[r, "f"'] & R\text{-line}. \end{tikzcd} \tag{2.25}$$

Proof. Corollary 2.17 says that the space $\mathrm{Triv}(f)$ of factorizations of f through ι is equivalent to the mapping space $\mathrm{map}_{R_X\text{-line}}(f, p^*R)$. $\square$

COROLLARY 2.26. *An orientation of Mf determines an equivalence of R-modules*

$$Mf \simeq \Sigma^\infty_+ X \wedge R.$$

Proof. If $\mathcal{C}$ is an ∞-category, write $\mathrm{Iso}(\mathcal{C})(a,b)$ for the subspace $\mathrm{map}_{\mathcal{C}^\simeq}(a,b) \subseteq \mathrm{map}_{\mathcal{C}}(a,b)$ consisting of equivalences (see (2.1)). By definition, $\mathrm{map}_{R_X\text{-line}}(f, R_X) = \mathrm{Iso}(R_X\text{-mod})(f, p^*R)$, and so (2.23) gives an equivalence $\mathrm{Orientations}_R(Mf, R) \longrightarrow \mathrm{Iso}(R_X\text{-mod})(f, p^*R)$. The desired map is the composite

$$\mathrm{Iso}(R_X\text{-mod})(f, p^*R) \longrightarrow \mathrm{Iso}(R\text{-mod})(p_!f, p_!p^*R) \longrightarrow \mathrm{Iso}(R\text{-mod})(p_!f, \Sigma^\infty_+ X \wedge R).$$

Here the second map applies $p_!$ and the last map composes with the equivalence $p_!p^*R \to \Sigma^\infty_+ X \wedge R$ of Proposition 2.18. $\square$

2.7. *Twisted homology and cohomology*

Recall that the R-module Thom spectrum Mf of the map $f : X^{\mathrm{op}} \to R$-line, which we think of as classifying an R-line bundle on X, is the pushforward $Mf \simeq p_!f$ of the composite

$$X^{\mathrm{op}} \xrightarrow{f} R\text{-line} \longrightarrow R\text{-mod}.$$

The homotopy groups $\pi_n Mf$ can be computed as homotopy classes of R-module maps from $\Sigma^n R$ to Mf, which is a convenient formulation because the twisted R-cohomology groups are dually homotopy classes of R-module maps from Mf to $\Sigma^n R$.

DEFINITION 2.27. Let R be an A_∞-ring spectrum, let X be a space with projection $p : X \to *$ to the point, and let $f : X \to R$-line be an R-line bundle on X. Then the f-twisted R-homology and R-cohomology of X are the mapping spectra

$$R^f(X) = \mathrm{Map}_R(R, Mf) \simeq Mf,$$
$$R_f(X) = \mathrm{Map}_R(Mf, R) \simeq \mathrm{Map}_{R_X}(f, R_X),$$

formed in the stable ∞-category R-mod of R-modules (or $\mathrm{Fun}(X^{\mathrm{op}}, R\text{-mod})$ of R_X-modules).

Here recall that $R_X \simeq p^*R$ is the constant bundle of R-modules $X^{\mathrm{op}} \to R\text{-line} \to R\text{-mod}$, and the equivalence of mapping spectra $\mathrm{Map}_R(Mf, R) \simeq \mathrm{Map}_{R_X}(f, R_X)$ follows from the equivalence, for each integer n, of mapping spaces

$$\mathrm{map}_R(p_!f, \Sigma^n R) \simeq \mathrm{map}_{R_X}(f, p^*\Sigma^n R)$$

that results from the fact that p^* is right adjoint to $p_!$.

Note that, since R is only assumed to be an A_∞-ring spectrum, the homotopy category of R-mod does not usually admit a closed monoidal structure with unit R; nevertheless, we still regard $R_f(X)$ as the 'R-dual' spectrum $\mathrm{Map}_R(Mf, R)$ of $Mf \simeq R^f(X)$, or as the 'spectrum of (global) sections' $\mathrm{Map}_{R_X}(f, R_X)$ of the R-line bundle f. Also, the notation $R^f(X)$ and $R_f(X)$ is designed so that, for an integer n, we have the f-twisted R-homology and R-cohomology *groups*

$$\begin{aligned} R^f_n(X) &= \pi_0 \operatorname{map}_R(\Sigma^n R, Mf) \cong \pi_n Mf, \\ R^n_f(X) &= \pi_0 \operatorname{map}_R(Mf, \Sigma^n R) \cong \pi_0 \operatorname{map}_{R_X}(f, p^* \Sigma^n R). \end{aligned}$$

A consequence of our work with orientations is the following untwisting result.

COROLLARY 2.28. *If $f : X^{\mathrm{op}} \to R\text{-line}$ admits an orientation, then $Mf \simeq R \wedge \Sigma^\infty_+ X$, and the twisted R-homology and R-cohomology spectra*

$$\begin{aligned} R^f(X) &\simeq R \wedge \Sigma^\infty_+ X, \\ R_f(X) &\simeq \mathrm{Map}(\Sigma^\infty_+ X, R), \end{aligned}$$

reduce to the ordinary R-homology and R-cohomology spectra of X.

Proof. Indeed, Corollary 2.26 gives equivalences $\mathrm{Map}_R(R, Mf) \simeq Mf \simeq R \wedge \Sigma^\infty_+ X$ and $\mathrm{Map}_R(R \wedge \Sigma^\infty_+ X, R) \simeq \mathrm{Map}(\Sigma^\infty_+ X, R)$. □

3. *Morita theory and Thom spectra*

In this section, we interpret the construction of the Thom spectrum from the perspective of Morita theory. This viewpoint is implicit in the 'algebraic' definition of the Thom spectrum of $f : X \to BGL_1 R$ in [2] as the derived smash product

$$M_{\mathrm{alg}} f \overset{\mathrm{def}}{=} \Sigma^\infty_+ P \wedge^{\mathrm{L}}_{\Sigma^\infty_+ GL_1 R} R,$$

where P is the pullback of the diagram

$$X \longrightarrow BGL_1 R \longleftarrow EGL_1 R.$$

As passage to the pullback induces an equivalence between spaces over $BGL_1 R$ and $GL_1 R$ spaces, and the target category of R-modules is stable, we can regard the Thom spectrum as essentially given by a functor from (right) $\Sigma^\infty_+ GL_1 R$-modules to R-modules.

Roughly speaking, Morita theory (more precisely, the Eilenberg–Watts theorem) implies that any continuous functor from (right) $\Sigma^\infty_+ GL_1 R$-modules to (right) R-modules which preserves homotopy colimits and takes $GL_1 R$ to R can be realized as tensoring with an appropriate $(\Sigma^\infty_+ GL_1 R)$-R bimodule. In particular, this tells us that the Thom spectrum functor is characterized amongst such functors by the additional data of the action of $GL_1 R$ on R, equivalently a map $BGL_1 R \to BGL_1 R$.

Beyond its conceptual appeal, this viewpoint on the Thom spectrum functor provides the basic framework for comparing the construction which we have discussed in this paper with M_{alg} and also with the 'neo-classical' construction of Lewis and May and the parameterized construction of May and Sigurdsson.

After discussing the analog of the classical Eilenberg–Watts theorem in the context of ring spectra in § 3.1, in § 3.2 we classify colimit-preserving functors between ∞-categories. Our classification leads in § 3.3 to a characterization of the 'geometric' Thom spectrum functor $M = M_{\mathrm{geo}}$ of this paper, which serves as the basis for comparison with the 'algebraic' Thom spectrum M_{alg} from [2].

In § 3.4, we briefly review the construction of M_{alg}, and characterize it using Morita theory. In § 3.5, we prove the equivalence of M_{geo} and M_{alg}. The close relationship between our ∞-categorical construction of the Thom spectrum and the definition of May and Sigurdsson [**15**, 23.7.1, 23.7.4] allows us (in § 3.6) to compare May and Sigurdsson's construction of the Thom spectrum (and by extension the 'neo-classical' Lewis–May construction) to the ones in this paper.

In § 3.7, we also sketch a direct comparison between M_{geo} and M_{alg}; although the argument does not characterize the functor among all functors from GL_1R-modules to R-modules, we believe it provides a useful concrete depiction of the situation.

3.1. *The Eilenberg–Watts theorem for categories of module spectra*

The key underpinning of classical Morita theory is the Eilenberg–Watts theorem, which for rings A and B establishes an equivalence between the category of colimit-preserving functors A-mod $\to B$-mod and the category of (A, B)-bimodules. The proof of the theorem proceeds by observing that any functor $T : A\text{-mod} \to B\text{-mod}$ specifies a bimodule structure on TA with the A-action given by the composite

$$A \longrightarrow F_A(A, A) \longrightarrow F_B(TA, TA).$$

It is then straightforward to check that the functor $- \otimes_A TA$ is isomorphic to the functor T, using the fact that both of these functors preserve colimits.

In this section, we discuss the generalization of this result to the setting of categories of module spectra. The situation here is more complicated than in the discrete case; for instance, it is well known that there are equivalences between categories of module spectra which are not given by tensoring with bimodules, and there are similar difficulties with the most general possible formulation of the Eilenberg–Watts theorem. However, much of the subtlety here comes from the fact that unlike in the classical situation, compatibility with the enrichment in spectra is not automatic (see, for example, the excellent recent paper of Johnson [**7**] for a comprehensive discussion of the situation). By assuming our functors are enriched, we can recover a close analog of the classical result.

Let A and B be (cofibrant) S-algebras and let T be an enriched functor

$$T : A\text{-mod} \longrightarrow B\text{-mod}.$$

Specifically, we assume that T induces a map of function spectra $F_A(X, Y) \to F_B(TX, TY)$, and furthermore that T preserves tensors (in particular, homotopies) and homotopy colimits. For instance, these conditions are satisfied if T is a Quillen left-adjoint. The assumption that T is homotopy-preserving implies that T preserves weak equivalences between cofibrant objects and so admits a total left-derived functor $T^{\mathrm{L}} : \mathrm{ho}\, A\text{-mod} \to \mathrm{ho}\, B\text{-mod}$. Furthermore, $T(A)$ is an A-B bimodule with the bimodule structure induced just as above.

Using an elaboration of the arguments of [**18**, 4.1.2] (see also [**17**, 4.20]), we now can prove the following Eilenberg–Watts theorem in this setting. We will work in the EKMM categories of S-modules [**6**], so we can assume that all objects are fibrant.

Proposition 3.1. *Given the hypotheses of the preceding discussion, there is a natural isomorphism in* $\mathrm{ho}\, B$-mod *between the total left-derived functor* $T^{\mathrm{L}}(-)$ *and the derived smash product* $(-) \wedge^{\mathrm{L}} T(A)$, *regarding* $T(A)$ *as a bimodule as above.*

Proof. By continuity, there is a natural map of B-modules

$$(-) \wedge_A T(A) \longrightarrow T(-).$$

Let T' denote a cofibrant replacement of $T(A)$ as an A-B bimodule. Since the functor $(-) \wedge_A T'$ preserves weak equivalences between cofibrant A-modules, there is a total left-derived functor $(-) \wedge_A^{\mathrm{L}} T'$ which models $(-) \wedge_A^{\mathrm{L}} T(A)$. Thus, the composite

$$(-) \wedge_A T' \longrightarrow (-) \wedge_A T(A) \longrightarrow T(-)$$

descends to the homotopy category to produce a natural map

$$(-) \wedge_A^{\mathrm{L}} T(A) \longrightarrow T^{\mathrm{L}}(-).$$

The map is clearly an equivalence for the free A-module of rank 1, that is, A. Since both sides commute with homotopy colimits, we can inductively deduce that the first map is an equivalence for all cofibrant A-modules, and this implies that the map of derived functors is an isomorphism. □

To characterize the Thom spectrum functor amongst functors from spaces over BGL_1R to R-modules, it is useful to formulate Proposition 3.1 in terms of ∞-categories. One reason is that (as we recall in § 3.4) the 'algebraic' Thom spectrum of [**2**] is the composition of a right derived functor (which is an equivalence) and a left derived functor. We remark that much of the technical difficulty in the neo-classical theory of the Thom spectrum functor arises from the difficulties involved in dealing with point-set models of such composites. This is the kind of formal situation that the ∞-category framework handles well.

3.2. *Colimit-preserving functors*

In this section, we study functors between ∞-categories which preserve colimits. Specializing to module categories, we obtain a version of the Eilenberg–Watts theorem which applies to both the algebraic and the geometric Thom spectrum.

We begin by considering cocomplete ∞-categories. Let $\mathcal{C}$ be a small ∞-category, and consider the ∞-topos $\mathrm{Pre}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{T})$ of presheaves of ∞-groupoids on $\mathcal{C}$. Recall that $\mathrm{Pre}(\mathcal{C})$ comes equipped with a fully faithful Yoneda embedding

$$\mathcal{C} \longrightarrow \mathrm{Pre}(\mathcal{C}), \tag{3.2}$$

which exhibits $\mathrm{Pre}(\mathcal{C})$ as the 'free cocompletion' [**11**, 5.1.5.8] of $\mathcal{C}$. More precisely, writing $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ consisting of the colimit-preserving functors, we have the following lemma.

LEMMA 3.3 [**11**, 5.1.5.6]. *For any cocomplete ∞-category $\mathcal{D}$, precomposition with the Yoneda embedding induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Pre}(\mathcal{C}), \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D}). \tag{3.4}$$

We shall be particularly interested in the case where $\mathcal{C}$ is an ∞-groupoid, so that

$$\mathrm{Pre}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Gpd}_\infty) \simeq \mathrm{Gpd}_{\infty/\mathcal{C}}, \tag{3.5}$$

as in Remark 2.4. In particular, given a functor $f : \mathcal{C} \to \mathcal{D}$, we may extend by colimits to a colimit-preserving functor $\tilde{f}\colon\ \mathrm{Gpd}_{\infty/\mathcal{C}} \to \mathcal{D}$.

COROLLARY 3.6. *If $g : \mathrm{Gpd}_{\infty/\mathcal{C}} \to \mathcal{D}$ is any colimit-preserving functor whose restriction along the Yoneda embedding $\mathcal{C} \to \mathrm{Gpd}_{\infty/\mathcal{C}}$ is equivalent to f, then g is equivalent to $\tilde{f}$.*

LEMMA 3.7 [12, 1.4.4.4, 1.4.4.5]. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories such that $\mathcal{D}$ is stable. Then*

$$\Omega^\infty_- : \mathrm{Stab}(\mathcal{C}) \longrightarrow \mathcal{C}$$

admits a left adjoint

$$\Sigma^\infty_+ : \mathcal{C} \longrightarrow \mathrm{Stab}(\mathcal{C}),$$

and precomposition with the Σ^∞_+ induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Stab}(\mathcal{C}), \mathcal{D}) \longrightarrow \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}).$$

Combining the universal properties of stabilization and the Yoneda embedding, we obtain the following equivalence of ∞-categories.

COROLLARY 3.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories such that $\mathcal{D}$ is stable and presentable. Then there are equivalences of ∞-categories*

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Stab}(\mathrm{Pre}(\mathcal{C})), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathrm{Pre}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Proof. This follows from the last two lemmas. □

Now suppose that $\mathcal{C}$ and $\mathcal{D}$ have distinguished objects, given by maps $* \to \mathcal{C}$ and $* \to \mathcal{D}$ from the trivial ∞-category $*$. Then $\mathrm{Pre}(\mathcal{C})$ and $\mathrm{Stab}(\mathrm{Pre}(\mathcal{C}))$ inherit distinguished objects via the composite

$$* \longrightarrow \mathcal{C} \xrightarrow{i} \mathrm{Pre}(\mathcal{C}) \xrightarrow{\Sigma^\infty_+} \mathrm{Stab}(\mathrm{Pre}(\mathcal{C})),$$

where i denotes the Yoneda embedding. Note that the fiber sequence

$$\mathrm{Fun}_{*/}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}(*, \mathcal{D}) \simeq \mathcal{D}$$

shows that the ∞-category of pointed functors is equivalent to the fiber of the evaluation map $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$ over the distinguished object of $\mathcal{D}$.

PROPOSITION 3.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories with distinguished objects such that $\mathcal{D}$ is stable and cocomplete. Then there are equivalences of ∞-categories*

$$\mathrm{Fun}^{\mathrm{L}}_{*/}(\mathrm{Stab}(\mathrm{Pre}(\mathcal{C})), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{L}}_{*/}(\mathrm{Pre}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}_{*/}(\mathcal{C}, \mathcal{D}).$$

Proof. Take the fiber of $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Stab}(\mathrm{Pre}(\mathcal{C})), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathrm{Pre}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ over $* \to \mathcal{D}$. □

COROLLARY 3.10. *Let G be a group-like monoidal ∞-groupoid G, let BG be a one-object ∞-groupoid with $G \simeq \mathrm{Aut}_{BG}(*)$, and let $\mathcal{D}$ be a stable and cocomplete ∞-category with a distinguished object $*$. Then*

$$\begin{aligned}\mathrm{Fun}^{\mathrm{L}}_{*/}(\mathrm{Stab}(\mathrm{Pre}(BG)), \mathcal{D}) &\simeq \mathrm{Fun}^{\mathrm{L}}_{*/}(\mathrm{Pre}(BG), \mathcal{D}) \\ &\simeq \mathrm{Fun}_{*/}(BG, \mathcal{D}) \simeq \mathrm{Fun}(BG, B\,\mathrm{Aut}_{\mathcal{D}}(*));\end{aligned}$$

that is, specifying an action of G on the distinguished object $*$ of $\mathcal{D}$ is equivalent to specifying a pointed colimit-preserving functor from $\mathrm{Pre}(BG)$ (or its stabilization) to $\mathcal{D}$.

Proof. A basepoint-preserving functor $BG \to \mathcal{D}$ necessarily factors through the full subgroupoid $B\,\mathrm{Aut}_{\mathcal{D}}(*)$. □

Note that the ∞-category $\mathrm{Fun}(BG, B\,\mathrm{Aut}_{\mathcal{D}}(*))$ is actually an ∞-groupoid, as $B\,\mathrm{Aut}_{\mathcal{D}}(*)$ is an ∞-groupoid.

Putting this all together, consider the case in which the target ∞-category $\mathcal{D}$ is the ∞-category of right R-modules for an associative S-algebra R, pointed by the free rank one R-module R. Then $\mathrm{Aut}_{\mathcal{D}}(*) \simeq GL_1 R$, and we have an ∞-categorical version of the Eilenberg–Watts theorem.

COROLLARY 3.11. *The space of pointed colimit-preserving maps from the ∞-category of spaces over BG to the ∞-category of R-modules is equivalent to the space of monoidal maps from G to $GL_1 R$, or equivalently the space of maps from BG to $BGL_1 R$.*

3.3. *∞-Categorical Thom spectra, revisited*

We now return to the definition of Thom spectra from § 2 and interpret that construction in light of the work of the previous subsections. To avoid confusion with the Thom spectrum constructed in [**2**], in this section we write M_{geo} for the Thom spectrum of § 2.

Let R be an algebra in $\mathrm{Stab}(\mathrm{Gpd}_\infty)$, and form the ∞-categories R-mod and R-line. Given a map of ∞-groupoids

$$f : X \longrightarrow R\text{-line},$$

the 'geometric' Thom spectrum we constructed in § 2 is the pushforward of the restriction to X of the tautological R-line bundle $\mathrm{id}_{R\text{-line}}$, the identity of R-line. More precisely, $M_{\mathrm{geo}} f \simeq \mathrm{colim}(f : X \to R\text{-line} \to R\text{-mod})$, and in particular, M_{geo} preserves (∞-categorical) colimits.

PROPOSITION 3.12. *The restriction of $M_{\mathrm{geo}} : \mathrm{Gpd}_{\infty/R\text{-line}} \to R\text{-mod}$ along the Yoneda embedding*

$$R\text{-line} \longrightarrow \mathrm{Fun}(R\text{-line}^{\mathrm{op}}, \mathrm{Gpd}_\infty) \simeq \mathrm{Gpd}_{\infty/R\text{-line}}$$

is equivalent to the inclusion R-line $\longrightarrow$ R-mod of the full ∞-subgroupoid on R.

Proof. Consider the colimit-preserving functor $\mathrm{Gpd}_{\infty/R\text{-line}} \to R\text{-mod}$ induced by the canonical inclusion $R\text{-line} \to R\text{-mod}$. As we explain in Corollary 3.6, it sends $X \to R\text{-line}$ to the colimit of the composite $X \to R\text{-line} \to R\text{-mod}$. □

Together with Corollary 3.6, the proposition implies the following.

COROLLARY 3.13 (Proposition 1.19). *A functor $\mathrm{Gpd}_{\infty/R\text{-line}} \to R\text{-mod}$ is equivalent to M_{geo} if and only if it preserves colimits and its restriction along the Yoneda embedding $R\text{-line} \to \mathrm{Fun}(R\text{-line}^{\mathrm{op}}, \mathrm{Gpd}_\infty) \simeq \mathrm{Gpd}_{\infty/R\text{-line}}$ is equivalent to the inclusion of R-line into R-mod.*

3.4. *A review of the algebraic Thom spectrum functor*

We briefly recall the 'algebraic' construction of the Thom spectrum from [**2**]. For an A_∞-ring spectrum R, the classical construction yields $GL_1 R$ as an A_∞-space. This means we expect to be able to form constructions $BGL_1 R$ and $EGL_1 R$, and so, given a classifying map

$f: X \to BGL_1R$, obtain a GL_1R-space P as the pullback of the diagram

$$X \longrightarrow BGL_1R \longleftarrow EGL_1R.$$

We then define the Thom spectrum associated to f as the derived smash product

$$M_{\mathrm{alg}}f \overset{\mathrm{def}}{=} \Sigma^\infty_+ P \wedge^{\mathrm{L}}_{\Sigma^\infty_+ GL_1R} R, \tag{3.14}$$

where R is the $\Sigma^\infty_+ GL_1R$-R bimodule specified by the canonical action of $\Sigma^\infty_+ GL_1R$ on R.

In order to make this outline precise, the companion paper used the technology of $*$-modules [**3**, **4**], which are a symmetric monoidal model for the category of spaces such that monoids are precisely A_∞-spaces and commutative monoids are precisely E_∞-spaces. Denote the category of $*$-modules by $\mathcal{M}_*$. As an A_∞ (or E_∞) space, GL_1R gives rise to a monoid in the category of $*$-modules. We will abusively continue to use the notation GL_1R to denote a model of GL_1R which is cofibrant as a monoid in $*$-modules. We can compute BGL_1R and EGL_1R as two-sided bar constructions with respect to the symmetric monoidal product $\boxtimes$:

$$E_\boxtimes GL_1R = B_\boxtimes(*, GL_1R, GL_1R) \quad \text{and} \quad B_\boxtimes GL_1R = B_\boxtimes(*, GL_1R, *).$$

The map $E_\boxtimes GL_1R \to B_\boxtimes GL_1R$ models the universal quasifibration [**2**, 3.8]. Furthermore, there is a homotopically well-behaved category $\mathcal{M}_{GL_1R}$ of GL_1R-modules in $\mathcal{M}_*$ (see [**2**, 3.6]).

Now, given a fibration of $*$-modules $f: X \to B_\boxtimes GL_1R$, we take the pullback of the diagram

$$X \longrightarrow B_\boxtimes GL_1R \longleftarrow E_\boxtimes GL_1R$$

to obtain a GL_1R-module P. This procedure defines a functor from $*$-modules over $B_\boxtimes GL_1R$ to GL_1R-modules; since we are assuming f is a fibration, we are computing the derived functor. Applying $\Sigma^\infty_{\mathbb{L}+}$, we obtain a right $\Sigma^\infty_{\mathbb{L}+} GL_1R$-module $\Sigma^\infty_{\mathbb{L}+}P$, and so we can define $M_{\mathrm{alg}}f$ as above. (Here $\Sigma^\infty_{\mathbb{L}+}$ is the appropriate model of Σ^∞_+ in this setting.)

The functor which sends f to P induces an equivalence of ∞-categories

$$\mathrm{N}((\mathcal{M}_{*/B_\boxtimes GL_1R})^{\mathrm{cf}}) \simeq \mathrm{N}((\mathcal{M}_{GL_1R})^{\mathrm{cf}}),$$

as a consequence of [**2**, 3.19]. Together with Proposition 3.1, this gives a characterization of the algebraic Thom spectrum functor.

Proposition 3.15. *Let*

$$T: \mathcal{M}_{GL_1R} \longrightarrow \mathcal{M}_R$$

be a continuous, colimit-preserving functor which sends GL_1R to an R-module R' homotopy equivalent to R in such a way that

$$GL_1R \simeq \mathrm{End}_{\mathcal{M}_{GL_1R}}(GL_1R) \longrightarrow \mathrm{End}_{\mathcal{M}_R}(R') \simeq \mathrm{End}_{\mathcal{M}_R}(R)$$

is homotopy equivalent to the inclusion $GL_1R \simeq \mathrm{Aut}(R) \to \mathrm{End}(R)$. (Here Aut *and* End *refer to the derived automorphism and endomorphism spaces, respectively.) Then T^{L}, the left-derived functor of T, is homotopy equivalent to*

$$\Sigma^\infty_{\mathbb{L}+}(-) \wedge^{\mathrm{L}}_{\Sigma^\infty_{\mathbb{L}+} GL_1R} R : \mathcal{M}_{GL_1R} \to \mathcal{M}_R.$$

Proof. The stability of R-mod and Proposition 3.1 together imply that T^{L} is homotopy equivalent to $\Sigma^\infty_{\mathbb{L}+}(-) \wedge^{\mathrm{L}}_{\Sigma^\infty_{\mathbb{L}+} GL_1R} B$ for some $(\Sigma^\infty_{\mathbb{L}+} GL_1R, R)$-bimodule B. Since $T(\Sigma^\infty_{\mathbb{L}+} GL_1R) \simeq R$, we must have $B \simeq R$; since the left action of GL_1R on itself induces (via the equivalence $R' \simeq R$) the canonical action of $\Sigma^\infty_{\mathbb{L}+} GL_1R$ on R, we conclude that $B \simeq R$ as $(\Sigma^\infty_{\mathbb{L}+} GL_1R, R)$-bimodules. □

3.5. *Comparing notions of Thom spectrum*

In this section, we show that, on underlying ∞-categories, the algebraic Thom R-module functor is equivalent to the geometric Thom spectrum functor via the characterization of Corollary 3.13.

Let $\mathcal{M}_S$ be the category of EKMM S-modules [**6**]. According to the discussion in [**12**, § 1.4.3] (and using the comparisons of [**13**]), there is an equivalence of ∞-categories

$$N\mathcal{M}_S^{\mathrm{cf}} \simeq \mathrm{Stab}(\mathrm{Gpd}_\infty), \tag{3.16}$$

which induces equivalences of ∞-categories of algebras and commutative algebras

$$\mathrm{N}\,\mathrm{Alg}(\mathcal{M}_S)^{\mathrm{cf}} \simeq \mathrm{Alg}(\mathrm{Stab}(\mathrm{Gpd}_\infty)), \quad \mathrm{N}\,\mathrm{CAlg}(\mathcal{M}_S)^{\mathrm{cf}} \simeq \mathrm{CAlg}(\mathrm{Stab}(\mathrm{Gpd}_\infty)). \tag{3.17}$$

Let R be a cofibrant–fibrant EKMM S-algebra and let R' be the corresponding algebra in $\mathrm{Alg}(\mathrm{Stab}(\mathrm{Gpd}_\infty))$. The equivalence (3.16) induces an equivalence of ∞-categories

$$\mathrm{N}(\mathcal{M}_R^{\mathrm{cf}}) \simeq R'\text{-mod}. \tag{3.18}$$

Proposition 2.14 gives an equivalence of ∞-groupoids

$$BGL_1R \simeq \mathrm{N}((R\text{-line})^{\mathrm{cf}}), \tag{3.19}$$

and so, putting (3.18) and (3.19) together with the comparisons of [**2**, 3.7], we have equivalences of ∞-categories

$$\mathrm{N}((\mathcal{M}_{*/\mathcal{B}_\boxtimes GL_1R})^{\mathrm{cf}}) \simeq \mathrm{N}((\mathrm{Top}_{/BGL_1R})^{\mathrm{cf}}) \simeq \mathrm{Gpd}_{\infty/R'\text{-line}}.$$

Proposition 3.20. *The functor*

$$\mathrm{Gpd}_{\infty/R'\text{-mod}} \simeq \mathrm{N}((\mathrm{Top}_{/BGL_1R})^{\mathrm{cf}}) \xrightarrow{NM_{\mathrm{alg}}} \mathrm{N}(\mathcal{M}_R^{\mathrm{cf}}) \simeq R'\text{-mod},$$

obtained by passing the Thom R-module functor M_{alg} of [**2**] *though the indicated equivalences is equivalent to the Thom R'-module functor of* § 2.

Proof. Let $\mathcal{C}$ denote the topological category with a single object $*$ and

$$\mathrm{map}_{\mathcal{C}}(*,*) = GL_1R = \mathrm{Aut}_R(R^{\mathrm{cf}}) \simeq \mathrm{Aut}_{R'}(R').$$

Note that $\mathcal{C}$ is naturally a topological subcategory of $\mathcal{M}_{GL_1R}$ (the full topological subcategory on GL_1R) and by definition a topological subcategory of $\mathcal{M}_R$. Note also that

$$N\mathcal{C} \simeq B\,\mathrm{Aut}(R') \simeq R'\text{-line}.$$

As in Proposition 3.15, the continuous functor

$$T^{\mathrm{L}} : \mathcal{M}_{GL_1R} \longrightarrow \mathcal{M}_R$$

determined by M_{alg} has the property that its restriction to $\mathcal{C}$ is equivalent to the inclusion of the topological subcategory $\mathcal{C} \to \mathcal{M}_R$. Taking simplicial nerves, and recalling that

$$\mathrm{N}(\mathcal{M}_{GL_1R}^{\mathrm{cf}}) \simeq \mathrm{N}((\mathrm{Top}_{/BGL_1R})^{\mathrm{cf}}) \simeq \mathrm{Fun}(N\mathcal{C}^{\mathrm{op}}, \mathrm{Gpd}_\infty),$$

we see that

$$\mathrm{N}(T^{\mathrm{L}}) : \mathrm{Fun}(N\mathcal{C}^{\mathrm{op}}, \mathrm{Gpd}_\infty) \simeq \mathrm{N}(\mathcal{M}_{GL_1R}^{\mathrm{cf}}) \longrightarrow \mathrm{N}(\mathcal{M}_R^{\mathrm{cf}}) \simeq R'\text{-mod}$$

is a colimit-preserving functor whose restriction along the Yoneda embedding

$$\mathrm{N}\mathcal{C} \to \mathrm{Fun}(N\mathcal{C}^{\mathrm{op}}, \mathrm{Gpd}_\infty) \simeq \mathrm{Gpd}_{\infty/R'\text{-line}}$$

is equivalent to the inclusion of the ∞-subcategory $\mathrm{N}\mathcal{C} \simeq R'\text{-line} \to R'\text{-mod}$. It follows from Corollary 3.13 that $\mathrm{N}(T^{\mathrm{L}})$ is equivalent to the 'geometric' Thom spectrum functor of § 2. □

Remark 3.21. The argument also implies the following apparently more general result. Recall from § 3.2 that any map $k : BGL_1R \to BGL_1R$ defines a functor from the ∞-category of spaces over BGL_1R to the ∞-category of R-modules, defined by sending $f : X \to BGL_1R$ to the colimit of the composite

$$X^{\mathrm{op}} \xrightarrow{f} BGL_1R \xrightarrow{k} BGL_1R \to R\text{-mod}. \tag{3.22}$$

On the other hand, according to Proposition 3.26, we can describe the derived smash product from § 3.1 associated to k as the colimit of the composite

$$X^{\mathrm{op}} \xrightarrow{f} BGL_1R \xrightarrow{k} BGL_1R \xrightarrow{\Sigma^\infty_+} \Sigma^\infty_+ GL_1R\text{-mod} \xrightarrow{(-)\wedge_{\Sigma^\infty_+ GL_1R} R} R\text{-mod}.$$

Since both functors are given by the formula $M(k \circ f)$, the Thom R-module of f composed with k, we conclude that these two procedures are equivalent for any k, not just the identity.

3.6. *The 'neo-classical' Thom spectrum functor*

In this section, we compare the Lewis–May operadic Thom spectrum functor to the Thom spectrum functors discussed in this paper. Since the May–Sigurdsson construction of the Thom spectrum in terms of a parameterized universal spectrum over BGL_1S (see [**15**, 23.7.4]) is easily seen to be equivalent to the space-level Lewis–May description, this will imply that all of the known descriptions of the Thom spectrum functor agree up to homotopy. Our comparison proceeds by relating the Lewis–May model to the quotient description of Proposition 2.21.

We begin by briefly reviewing the Lewis–May construction of the Thom spectrum functor; the interested reader is referred to Lewis' thesis, published as [**9**, Chapter IX], and the excellent discussion in [**15**, Chapter 22] for more details and proofs of the foundational results below. Nonetheless, we have tried to make our discussion relatively self-contained.

The starting point for the Lewis–May construction is an explicit construction of GL_1S in terms of a diagrammatic model of infinite loop spaces. Let $\mathscr{I}_c$ be the symmetric monoidal category of finite or countably infinite-dimensional real inner product spaces and linear isometries. Define an $\mathscr{I}_c$-space to be a continuous functor from $\mathscr{I}_c$ to spaces. The usual left Kan extension construction (i.e., Day convolution) gives the diagram category of $\mathscr{I}_c$-spaces a symmetric monoidal structure. It turns out that monoids and commutative monoids for this category model, respectively, A_∞ and E_∞ spaces; for technical felicity, we focus attention on the commutative monoids which satisfy two additional properties.

(1) The map $T(V) \to T(W)$ associated to a linear isometry $V \to W$ is a homeomorphism onto a closed subspace.

(2) Each $T(W)$ is the colimit of the $T(V)$, where V runs over the finite-dimensional subspaces of W and the maps in the colimit system are restricted to the inclusions.

Denote such a functor as an $\mathscr{I}_c$-FCP (functor with cartesian product) [**15**, 23.6.1]; the requirement that T be a diagrammatic commutative monoid implies the existence of a 'Whitney sum' natural transformation $T(U) \times T(V) \to T(U \oplus V)$. This terminology is of course deliberately evocative of the notion of FSP (functor with smash product), which is essentially an orthogonal ring spectrum [**13**].

An $\mathscr{I}_c$-FCP gives rise to an E_∞-space structured by the linear isometries operad $\mathfrak{L}$; specifically, $T(\mathbb{R}_\infty) = \mathrm{colim}_V\, T(V)$ is an $\mathfrak{L}$-space with the operad maps induced by the Whitney sum [**14**, 1.9; **15**, 23.6.3]. In fact, as alluded to above, one can set up a Quillen equivalence between the category of $\mathscr{I}_c$-FCPs and the category of E_∞-spaces, although we do not discuss this matter herein (see [**10**] for a nice treatment of this comparison).

Moving on, we now focus attention on the $\mathscr{I}_c$-FCP specified by taking $V \subset \mathbb{R}^\infty$ to the space of based homotopy self-equivalences of S^V; this is classically denoted by $F(V)$. Passing to the colimit over inclusions, $F(\mathbb{R}^\infty) = \mathrm{colim}_V\, F(V)$ becomes an $\mathfrak{L}$-space which models GL_1S, this is essentially one of the original descriptions from [**14**]. Furthermore, since each $F(V)$ is a monoid, applying the two-side bar construction levelwise yields an FCP specified by $V \mapsto BF(V)$; here $BF(V)$ denotes the bar construction $B(*, F(V), *)$, and the Whitney sum transformation is defined using the homeomorphism $B(*, F(V), *) \times B(*, F(W), *) \cong B(*, F(V) \times F(W), *)$. The colimit $BF(\mathbb{R}^\infty)$ provides a model for BGL_1S.

Now, since $F(V)$ acts on S^V, we can also form the two-sided bar construction $B(*, F(V), S^V)$, abbreviated $EF(V)$, and there is a universal quasifibration

$$\pi_V : EF(V) = B(*, F(V), S^V) \longrightarrow B(*, F(V), *) = BF(V),$$

which classifies spherical fibrations with fiber S^V. Given a map $X \to BF(\mathbb{R}^\infty)$, by pulling back subspaces $BF(V) \subset BF(\mathbb{R}^\infty)$, we get an induced filtration on X; denote the space corresponding to pulling back along the inclusion of $V \in \mathbb{R}^\infty$ by $X(V)$ (see [**9**, IX.3.1]).

Denote by $Z(V)$ the pullback

$$X(V) \longrightarrow BF(V) \longleftarrow EF(V).$$

The Vth space of the Thom prespectrum is then obtained by taking the Thom space of $Z(V) \to X(V)$, that is, by collapsing out the section induced from the basepoint inclusion $* \to S^V$; denote the resulting prespectrum by TF (see [**9**, IX.3.2], and note that some work is involved in checking that these spaces in fact assemble into a prespectrum).

Next, we will verify that the prespectrum TF associated to the identity map on $BF(\mathbb{R}^\infty)$ is stably equivalent to the homotopy quotient $S/GL_1S \simeq S/F(\mathbb{R}^\infty)$. For a point-set description of this homotopy quotient, it follows from [**2**, 3.9] that the category of EKMM (commutative) S-algebras is tensored over (commutative) monoids in $*$-modules: the tensor of a monoid in $*$-modules M and an S-algebra A is $\Sigma^\infty_{\mathbb{L}+} M \wedge A$, with multiplication

$$\begin{aligned}(\Sigma^\infty_{\mathbb{L}+} M \wedge A) \wedge (\Sigma^\infty_{\mathbb{L}+} M \wedge A) &\cong (\Sigma^\infty_{\mathbb{L}+} M \wedge \Sigma^\infty_{\mathbb{L}+} M) \wedge (A \wedge A)\\ &\cong (\Sigma^\infty_{\mathbb{L}+} (M \boxtimes M)) \wedge (A \wedge A) \longrightarrow (\Sigma^\infty_{\mathbb{L}+} M) \wedge A.\end{aligned}$$

Thus, we can model the homotopy quotient as a bar construction in the category of (commutative) S-algebras. However, we can also describe the homotopy quotient as $\operatorname{colim}_V S/F(V)$, where here we use the structure of $F(V)$ as a monoid acting on S^V. It is this 'space-level' description that we will employ in the comparison below.

We find it most convenient to reinterpret the Lewis–May construction in this situation, as follows: The Thom space in this case is by definition the cofiber $(EF(V), BF(V))$ of the inclusion $BF(V) \to EF(V)$ induced from the basepoint inclusion $* \to S^V$. Now,

$$BF(V) \simeq */F(V),$$

and similarly

$$EF(V) \simeq S^V/F(V).$$

Hence, the Thom space is likewise the cofiber $(S^V, *)/F(V)$ of the inclusion $* \to S^V$, viewed as a *pointed* space.

More generally, we can regard the prespectrum $\{MF(V)\}$ as equivalently described as

$$MF(V) \stackrel{\text{def}}{=} S^V/F(V),$$

the homotopy quotient of the *pointed* space S^V by $F(V)$ via the canonical action, with structure maps induced from the quotient maps $S^V \to S^V/F(V)$ together with the pairings

$$\begin{aligned}MF(V) \wedge MF(W) \simeq S^V/F(V) \wedge S^W/F(W) &\longrightarrow S^{V\oplus W}/F(V) \times F(W)\\ &\longrightarrow S^{V\oplus W}/F(V \oplus W),\end{aligned}$$

where $F(V) \times F(W) \to F(V \oplus W)$ is the Whitney sum map of F. It is straightforward to check that the structure maps in terms of the bar construction described in [**9**, IX.3.2] realize these structure maps.

The associated spectrum MF is then the colimit $\operatorname{colim}_V S/F(V) \simeq S/F(\mathbb{R}^\infty)$. A key point is that the Thom spectrum functor can be described as the colimit over shifts of the Thom spaces [**9**, IX.3.7, IX.4.4]:

$$MF = \operatorname*{colim}_V \Sigma^{-V}\Sigma^\infty MF(V).$$

Furthermore, using the bar construction, we can see that the spectrum quotient $(\Sigma^V S)/F(V)$ is equivalent to $\Sigma^\infty S^V/F(V)$. Putting these facts together, we have the following chain of equivalences:

$$\begin{aligned} MF &= \operatorname*{colim}_V \Sigma^{-V}\Sigma^\infty MF(V) = \operatorname*{colim}_V \Sigma^{-V}\Sigma^\infty S^V/F(V) \\ &\simeq \operatorname*{colim}_V \Sigma^{-V}(\Sigma^V S)/F(V) \simeq \operatorname*{colim}_V (\Sigma^{-V}\Sigma^V S)/F(V) \simeq S/F(\mathbb{R}^\infty). \end{aligned}$$

More generally, a slight elaboration of this argument implies the following proposition.

Proposition 3.23. *The Lewis–May Thom spectrum MG associated to a group-like A_∞-map $\varphi : G \to GL_1 S$ modeled by the map of $\mathscr{I}_c$-FCPs $G \to F$ is equivalent to the spectrum S/G, the homotopy quotient of the sphere by the action of φ.*

Note that any A_∞-map $X \to F(\mathbb{R}^\infty)$ can be rectified to a map of $\mathscr{I}_c$-FCPs $X' \to F$ (see [**10**]).

Corollary 3.24. *Given a map of spaces $f : X \to BGL_1 S$, write $M_{\mathrm{LM}} f$ for the spectrum associated to the Lewis–May Thom spectrum of f. Then $M_{\mathrm{LM}} f \simeq M_{\mathrm{geo}} f$ as objects of the ∞-category of spectra.*

Proof. A basic property of the Thom spectrum functor M_{LM} is that it preserves colimits [**9**, IX.4.3]. Thus, we can assume that X is connected. In this case, $X \simeq BG$ for some group-like A_∞-space G, and $f : BG \to BGL_1 S$ is the delooping of an A_∞-map $G \to GL_1 S$. Hence, $M_{\mathrm{geo}} f \simeq S/G$ by Proposition 3.23 and $M_{\mathrm{LM}} f \simeq M_{\mathrm{geo}} f$ by Proposition 2.21. □

3.7. *The algebraic Thom spectrum functor as a colimit*

We sketch another approach to the comparison of the 'geometric' and 'algebraic' Thom spectrum functors. This approach has the advantage of giving a direct comparison of the two functors. It has the disadvantage that it does not characterize the Thom spectrum functor among functors

$$\mathcal{T}_{/BGL_1 R} \longrightarrow R\text{-mod},$$

and it does not exhibit the conceptual role played by Morita theory. Instead, it identifies both functors as colimits.

Suppose that R is an S-algebra. Let R-mod be the associated ∞-category of R-modules, let R-line be the sub-∞-groupoid of R-lines, and let $j : R\text{-line} \to R\text{-mod}$ denote the inclusion. For a space X, the 'geometric' Thom spectrum functor sends a map $f : X^{\mathrm{op}} \to R\text{-line}$ to

$$\operatorname{colim}(X^{\mathrm{op}} \xrightarrow{f} R\text{-line} \xrightarrow{j} R\text{-mod}).$$

As in [**2**, § 3], let G be a cofibrant replacement of $GL_1 R$ as a monoid in $*$-modules. By definition of R-line, we have an equivalence $B_{\boxtimes} G \simeq R\text{-line}$. But observe that we also have a natural equivalence

$$B_{\boxtimes}G \simeq G\text{-line}.$$

That is, let G-mod $= N(\mathcal{M}_G{}^{\mathrm{cf}})$ be the ∞-category of G-modules and let G-line be the maximal ∞-groupoid generated by the G-lines, that is, G-modules which admit a weak equivalence to G. By construction, G-line is connected, and so equivalent to $B\,\mathrm{Aut}(G) \simeq B_{\boxtimes}G$.

Recall that we have an equivalence of ∞-categories

$$\mathrm{Gpd}_{\infty/G\text{-line}} \simeq G\text{-mod}. \tag{3.25}$$

The key observation is the following. Let $k : G\text{-line} \to G\text{-mod}$ denote the tautological inclusion. To a map of ∞-groupoids

$$f : X^{\mathrm{op}} \longrightarrow G\text{-line},$$

we can associate the G-module

$$P_f = \mathrm{colim}(X^{\mathrm{op}} \xrightarrow{f} G\text{-line} \xrightarrow{k} G\text{-mod}).$$

Inspecting the proof of [**11**, 2.2.1.2] implies that the functor $P : \mathrm{Gpd}_{\infty/G\text{-line}} \to G\text{-mod}$ gives the equivalence (3.25).

In other words, if $f : X \to B_{\boxtimes}G$ is a fibration of $*$-modules, then we can form P as in the pullback along $E_{\boxtimes}G \to B_{\boxtimes}G$. Alternatively, we can form

$$f : X \longrightarrow B_{\boxtimes}G \simeq G\text{-line},$$

and then form $P_f = \mathrm{colim}(kf)$, and obtain an equivalence of G-modules

$$P_f \simeq P.$$

PROPOSITION 3.26. *Let $f : X \to B_{\boxtimes}G$ be a fibration of $*$-modules. The 'algebraic' Thom spectrum functor sends f to*

$$\mathrm{colim}(X^{\mathrm{op}} \xrightarrow{f} B_{\boxtimes}G \simeq G\text{-line} \xrightarrow{k} G\text{-mod} \xrightarrow{\Sigma^{\infty}_{+}} \Sigma^{\infty}_{+}G\text{-mod} \xrightarrow{\wedge_{\Sigma^{\infty}_{+}G}R} R\text{-mod}).$$

Proof. We have

$$P \simeq \mathrm{colim}(X^{\mathrm{op}} \xrightarrow{f} B_{\boxtimes}G \simeq G\text{-line} \xrightarrow{k} G\text{-mod}), \tag{3.27}$$

and so

$$\begin{aligned} Mf &= \Sigma^{\infty}_{+}P \wedge_{\Sigma^{\infty}_{+}G} R \\ &\simeq \Sigma^{\infty}_{+}\,\mathrm{colim}(kf) \wedge_{\Sigma^{\infty}_{+}G} R \\ &\simeq \mathrm{colim}(\Sigma^{\infty}_{+}kf) \wedge_{\Sigma^{\infty}_{+}G} R \\ &\simeq \mathrm{colim}(\Sigma^{\infty}_{+}kf \wedge_{\Sigma^{\infty}_{+}G} R). \end{aligned}$$

□

From this point of view, the coincidence of the two Thom spectrum functors amounts to the fact that diagram

$$\begin{tikzcd} X^{\mathrm{op}} \arrow[r, "f"] \arrow[dr, "f"'] & G\text{-line} \arrow[r, "k"] \arrow[d, "\simeq\ \Sigma^{\infty}_{+}(-)\wedge_{\Sigma^{\infty}_{+}G}R"] & G\text{-mod} \arrow[d, "\Sigma^{\infty}_{+}(-)\wedge_{\Sigma^{\infty}_{+}G}R"] \\ & R\text{-line} \arrow[r, "j"'] & R\text{-mod} \end{tikzcd}$$

evidently commutes.

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