### CHROMATIC COMPLEXITY OF ALGEBRAIC K-THEORY OF y(n)

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ABSTRACT. The family of Thom spectra y(n) interpolate between the sphere spectrum and the mod two Eilenberg-MacLane spectrum. Computations of Mahowald, Ravenel, and Shick and the authors show that the  $E_1$  ring spectrum y(n) has chromatic complexity n. We show that topological periodic cyclic homology of y(n) has chromatic complexity n + 1. This gives evidence that topological periodic cyclic homology shifts chromatic height at all chromatic heights, supporting a variant of the Ausoni–Rognes red-shift conjecture. We also show that relative algebraic K-theory, topological cyclic homology, and topological negative cyclic homology of y(n) at least preserve chromatic complexity.

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### 1. INTRODUCTION

The Lichtenbaum-Quillen conjectures [26][39] describe the relationship between étale cohomology and algebraic K-theory and consequently the relationship between orders of algebraic Kgroups and special values of Dedekind zeta functions. Following Waldhausen [46], the Lichtenbaum-Quillen conjectures can be phrased homotopy theoretically, for odd primes p, as the question of whether the map

$$S/p \wedge K(A) \rightarrow v_1^{-1}S/p \wedge K(A)$$

is an isomorphism in homotopy in sufficiently high degrees, where A is a nice enough ring with  $1/p \in A$  (see [36] for a nice survey). Here S/p denotes the mod p Moore spectrum for an odd prime p and  $v_1^{-1}S/p$  is the telescope of the periodic self-map  $v_1: \Sigma^{2p-2}S/p \to S/p$ . These conjectures have recently been proven as a consequence of the Norm Residue Theorem of Rost-Voevodsky (see [21] for a self-contained proof).

The chromatic red-shift conjectures of Ausoni and Rognes [10] generalize the Lichtenbaum-Quillen conjectures to higher chromatic complexities. To discuss the red-shift conjectures, we need to introduce precise notions of chromatic complexity. We say that a *p*-local finite spectrum V has type *n* if

$$n = \min\{m : K(m)_*(V) \neq 0\},\$$

where K(n) is the *n*-th Morava K-theory. By the periodicity theorem [24, Thm. 9] any type *n* spectrum V has a periodic  $v_n$ -self map

$$v_n^d \colon \Sigma^{(2p^n-2)d} V \to V$$

so we can define the telescope  $v_n^{-1}V$ . The chromatic red-shift problem of Rognes [41] (predating [10]), states that if A is an  $E_1$  ring spectrum with pure fp type n then the p-completion of K(A) has pure fp type n + 1.<sup>1</sup> This generalizes the Lichtenbaum-Quillen conjectures since if K(A) has pure fp type n + 1 then

$$V \wedge K(A) \rightarrow v^{-1}V \wedge K(A)$$

is an isomorphism in homotopy in sufficiently high degrees for some finite spectrum V of type n + 1.

The spectrum  $H\mathbb{F}_p$  has pure fp type -1 by convention, so Quillen's computation [38] that  $K(\mathbb{F}_p)_p \simeq H\mathbb{Z}_p$  shows that  $K(\mathbb{F}_p)_p$  has pure fp type 0. If A is the ring of integers in a finite extension of  $\mathbb{Q}_p$  then the Eilenberg-MacLane spectrum HA has pure fp type 0 and a special case of the chromatic red-shift problem is the Lichtenbaum-Quillen conjecture. Ausoni and Rognes computed the algebraic K-theory of the connective covers of the Adams summand and complex K-theory mod  $(p, v_1)$  and showed that each have pure fp type 2 [9][8]. Since the Adams summand and complex K-theory have pure fp type 1, this gives further evidence for the chromatic red shift problem. The red-shift conjectures have been further investigated in recent work of Clausen-Mathew-Naumann-Noel [15], Land-Meier-Tamme [25], and the first author [4].

In this paper, we make the case for formulating the red-shift conjecture using vanishing of Morava K-theory as our notion of chromatic complexity when the input R is an  $E_1$  ring spectrum that does not have finitely presented mod p homology. In this case, R will not have pure fp type mfor any finite m by [31] (cf. Corollary 2.4) so shifts in chromatic complexity would not be visible using the notion of pure fp height. In particular, we consider a family of  $E_1$  ring spectra

$$S = y(0) \rightarrow y(1) \rightarrow \dots y(\infty) = H\mathbb{F}_2$$

defined by Mahowald [29]. It is known by Proposition 2.22 at p = 2 and [30] at odd primes that

$$n = \min\{m : K(m)_*(y(n)) \neq 0\}$$

so y(n) may be considered type n even though it is not a finite spectrum.

We also make the case for studying the red-shift conjecture, not just for algebraic K-theory and topological cyclic homology, but also for topological periodic cyclic homology and topological negative cyclic homology. Indeed, a shift in chromatic height is already detected in these approximations to algebraic K-theory in many examples where red-shift phenomena has been detected (eg. [9, 4]), so this idea is certainly not new. One of our goals is to give evidence at all chromatic heights for the following red-shift question for topological periodic cyclic homology.

**Question 1.1.** If A is an  $E_1$  ring spectrum such that  $K(m)_*(A) = 0$  for  $0 \le m \le n-1$  then

$$K(m)_*(TP(A)) \cong 0$$
 for  $1 \le m \le n$ .

The main result of this paper answers this question affirmatively at all chromatic heights for the family of  $E_1$  ring spectra y(n).

**Theorem 1.2.** For  $1 \le m \le n$ , there are isomorphisms

$$K(m)_*(TP(y(n))) \cong 0.$$

The key case here is the case m = n, which demonstrates an increase in chromatic complexity. We also prove that relative algebraic K-theory at least preserves vanishing of Morava K-theory. Note that the map  $y(n) \to H\mathbb{F}_2$  is a K(m)-equivalence for  $0 \le m \le n-1$ , since y(n) has type nand  $H\mathbb{F}_2$  has type  $\infty$ . Given a map of  $E_1$  ring spectra  $f: A \to B$ , we write  $K(A, B) := \operatorname{fib}(K(f))$ .

<sup>&</sup>lt;sup>1</sup>Following [41], X has pure fp height n if there is a finite spectrum V of type n with  $v_n$ -self map  $v_n^d$  such that  $\pi_*(V \wedge X)$  is a finitely generated free  $P(v_n^d)$ -module. In [41], the red-shift conjecture is phrased for p-typical topological cyclic homology, but we phrase it this way to ease exposition. This does not pose a problem in most cases of interest by [16, Thm 2.2.1].

**Theorem 1.3.** There are isomorphisms

$$K(m)_*(K(y(n);H\mathbb{F}_2)) \cong 0$$

for  $0 \le m \le n - 1$ .

Since the first draft of this paper appeared, Land-Meier-Tamme [25] have also proven the theorem above by different methods.

The results above follow from analyzing the homological Tate and homotopy fixed point spectral sequences [14] arising from the Greenlees filtration of TP(y(n)), denoted  $\{TP(y(n))[i]\}_{i\in\mathbb{Z}}$ , and the skeletal filtration of  $TC^{-}(y(n))$ , denoted  $\{TC^{-}(y(n))[i]\}_{i\in\mathbb{N}}$  (Sections 4.2 and 5.1). We define continuous Morava K-theory as

$$K(m)^{c}_{*}(TP(y(n))) := \lim_{i} K(m)_{*}(TP(y(n))[i]), \text{ and}$$
$$K(m)^{c}_{*}(TC^{-}(y(n))) := \lim_{i} K(m)_{*}(TC^{-}(y(n))[i]).$$

We first prove vanishing results for continuous Morava K-theory of topological periodic cyclic homology and topological negative cyclic homology of y(n).

**Theorem 1.4** (Theorems 4.17 and 5.8). For all  $1 \le n \le \infty$ , there are isomorphisms

$$K(m)^c_*(TP(y(n))) \cong 0$$

for  $1 \leq m \leq n$ . Further, there are isomorphisms

$$K(\ell)^c_*(TC^-(y(n))) \cong 0$$

for  $1 \leq \ell \leq n-1$ .

We then show that in Example 5.10 that

$$K(1)^{c}_{*}(TC^{-}(y(1))) \neq 0.$$

Along with the previous theorem, this suggests that topological periodic cyclic homology may be better suited for studying shifts in chromatic complexity using vanishing of Morava K-theory than topological negative cyclic homology, topological cyclic homology, and algebraic K-theory.

1.1. **Outline.** In Section 2, we recall the construction and basic properties of the spectra y(n). We compute vanishing of Morava K-theory of y(n) using Margolis homology and the localized Adams spectral sequence. We also construct Thom spectra z(n) which are integral analogs of y(n), i.e. they are spectra which interpolate between S and  $H\mathbb{Z}$ . We show that the spectra z(n) have a self map  $v_n$  and, again using Margolis homology, we compute vanishing of Morava K-theory of z(n) and the cofiber  $z(n)/v_n$  of this self map. We believe that the spectra z(n) are of independent interest. In Section 3, we analyze the Bökstedt spectral sequence converging to the mod two homology of THH(y(n)). We also prove a key technical proposition (Proposition 3.8) about the map  $H_*(THH(y(n))) \to H_*(THH(H\mathbb{F}_2))$  which we use in subsequent sections. In Section 4, we analyze the topological periodic cyclic homology  $TP(y(n)) := THH(y(n))^{t^{\mathrm{T}}}$ . The key tool is the homological Tate spectral sequence of Bruner-Rognes [14]. In Section 5, we carry out a similar analysis for topological negative cyclic homology  $TC^-(y(n)) := THH(y(n))^{h^{\mathrm{T}}}$ . In Section 6, we study the topological cyclic homology TC(y(n)) and algebraic K-theory K(y(n)).

1.2. **Conventions.** Throughout, we write  $H_*(X)$  (resp.  $H^*(X)$ ) for homology (resp. cohomology) of a space or spectrum X with coefficients in  $\mathbb{F}_p$ . We write  $\mathcal{A} := H^*(H\mathbb{F}_2)$  for the Steenrod algebra, which is a Hopf algebra with generators  $Sq^{2^i}$  and relations given by the Adem relations. The dual of the Steenrod algebra will be denoted  $\mathcal{A}_* := H_*(H\mathbb{F}_2)$  and it is isomorphic to  $P(\bar{\xi}_i \mid i \geq 1)$  where  $\bar{\xi} := \chi(\xi_i)$  is the image of the usual Milnor generators under the antipode  $\chi$  of the Hopf algebra  $\mathcal{A}_*$ . The coproduct  $\psi: \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$  is given by the formula

(1) 
$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{2^i}.$$

We will let  $E(n) := E(Q_0, \ldots, Q_n)$  denote the subalgebra of the Steenrod algebra generated by the first n + 1 Milnor primitives and we denote by

$$\mathcal{E} := E(\infty) = E(Q_0, Q_1, \ldots)$$

the subalgebra generated by all of the Milnor primitives. As usual,  $E(n)_*$  and  $\mathcal{E}_*$  will be the  $\mathbb{F}_p$ -linear duals of these subalgebras, respectively. As is customary, there is often an implicit prime p in our notation (for example the notation  $H_*(X)$ , y(n), and K(m)) and this implicit prime will be p = 2unless otherwise stated. Given a functor from the category of  $E_1$  ring spectra to the category of spectra E and a map  $f: A \to B$  of  $E_1$  ring spectra, we will write  $E(A, B) := \operatorname{fib}(E(f))$ .

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### 2. Families of Thom Spectra

In this section, we recall the Thom spectra y(n) which interpolate between the sphere spectrum and the mod two Eilenberg-MacLane spectrum. We also introduce a family of spectra z(n) which interpolate between the sphere spectrum and the integral Eilenberg-MacLane spectrum. Basic properties of both families, such as their homology and multiplicative structure, are discussed in Section 2.1. In Section 2.2, we recall Margolis homology and the localized Adams spectral sequence, and in Section 2.3, we apply them to compute the chromatic complexity of the spectra y(n), z(n), and  $z(n)/v_n$  (Proposition 2.22).

2.1. Construction of the Thom spectra y(n) and z(n). We begin with Mahowald's construction of  $H\mathbb{F}_2$  and  $H\mathbb{Z}$  as Thom spectra [29]. Let  $f = \Omega^2 w: \Omega^2 S^3 \to \Omega^2 B^3 O \simeq BO$  be the two-fold looping of the generator  $w: S^3 \to B^3 O$  of  $\pi_3(B^3 O) \cong \pi_0(O) \cong \mathbb{Z}/2$ . Recall that for an  $E_{\infty}$  ring spectrum R, one can construct the group-like  $E_{\infty}$  space  $GL_1R$  [3][33]. When R = S is the sphere spectrum, its delooping  $BGL_1S$  is a model for the classifying space of stable spherical fibrations. The classical J homomorphism then gives a map of group-like  $E_{\infty}$ -spaces  $J: O \to GL_1S$ . In [29, Sec. 2.6], Mahowald showed that

(2) 
$$H\mathbb{F}_2 \simeq Th(\Omega^2 S^3 \xrightarrow{f} BO \xrightarrow{BJ} BGL_1S)$$

where Th(-) is the Thom spectrum construction. We refer the reader to [2] for a modern treatment of the Thom spectrum construction.

Similarly, Mahowald [29, Prop. 2.8] proved that

$$H\mathbb{Z} \simeq Th(\Omega^2(S^3\langle 3\rangle) \to \Omega^2 S^3 \xrightarrow{f} BO \xrightarrow{BJ} BGL_1S)$$

where  $S^3\langle 3 \rangle$  is the fiber of the map  $S^3 \to K(\mathbb{Z},3)$  and  $\iota \colon S^3\langle 3 \rangle \to S^3$  is the inclusion of the fiber.

We now produce the spectra y(n) following [29, Sec. 4.5]. The James splitting gives an equivalence  $\Omega \Sigma S^2 \simeq J_{\infty} S^2$  where  $J_{\infty} X$  is the James construction of the space X, so we can rewrite (2) as  $H\mathbb{F}_2 \simeq Th(\Omega J_{\infty}S^2 \to BGL_1S)$ . By truncating the James construction, one can define spectra  $J_k S^2$ , and there is an obvious inclusion  $i_k: J_k S^2 \hookrightarrow J_{\infty} S^2$ . Taking  $k = 2^n - 1$ , one defines

$$y(n) := Th(\Omega J_{2^n - 1}S^2 \xrightarrow{f_n} BGL_1S)$$

where  $f_n = BJ \circ f \circ \Omega i_{2^n-1}$ . (Note that one needs to *p*-localize in order to construct y(n) at odd primes, but this is not necessary at the prime 2.)

The fiber sequence  $J_{2^n-1}S^2 \to \Omega S^3 \to \Omega S^{2^{n+1}+1}$  implies that the map  $J_{2^n-1}S^2 \to \Omega S^3$ is  $(2^{n+1}-1)$ -connected. Thus there is a map  $J_{2^n-1}S^2 \to K(\mathbb{Z},2)$  given by truncating homotopy groups which is compatible with the map  $J_{\infty}S^2 \to K(\mathbb{Z},2)$ . **Construction 2.1.** Let  $n \ge 1$ . Write  $J_{2^n-1}S^2\langle 2 \rangle$  for the fiber of the map  $J_{2^n-1}S^2 \to K(\mathbb{Z},2)$  given by truncating homotopy groups. Define

$$z(n) := Th(\Omega(J_{2^n-1}S^2\langle 2\rangle) \xrightarrow{g_n} BGL_1S)$$

where  $g_n = f_n \circ \Omega i_{2^n-1} \circ \Omega \iota_{2^n-1}$  with  $\iota_k \colon J_k S^2 \langle 2 \rangle \to J_k S^2$  is the inclusion of the fiber. There is a commutative diagram

$$\begin{array}{cccc} \Omega(J_{2^{n}-1}S^{2}\langle 2\rangle) &\longrightarrow \Omega^{2}S^{3}\langle 3\rangle \\ & & & & \downarrow & & \downarrow \\ \Omega(J_{2^{n}-1}S^{2}) \xrightarrow{\Omega_{i_{2^{n}-1}}} & \Omega^{2}S^{3} & \xrightarrow{f_{n}} BGL_{1}S \\ & & & \downarrow & & \downarrow \\ & & & & \downarrow & \\ S^{1} & \xrightarrow{=} & S^{1} \end{array}$$

where the left two columns are fiber sequences, all the maps in the upper right triangle are 2-fold loop maps, and all the maps in the upper left square are 1-fold loop maps.

**Lemma 2.2.** The spectra z(n) are  $E_1$  ring spectra and the diagram

$$\begin{array}{ccc} z(n) \longrightarrow H\mathbb{Z} \\ & & \downarrow \\ & & \downarrow \\ y(n) \longrightarrow H\mathbb{F}_2 \end{array}$$

is a commutative diagram in the category of  $E_1$  ring spectra for  $n \geq 1$ .

*Proof.* This immediately follows from Lewis's theorem [18, Ch. IX] and Construction 2.1.  $\Box$ 

The homology of the spectra y(n) interpolate between the homology of the sphere spectrum  $S \simeq y(0)$  and the homology of the mod 2 Eilenberg-MacLane spectrum  $H\mathbb{F}_2 \simeq y(\infty)$ . Similarly, the homology of the spectra z(n) interpolate between the homology of z(1) and the homology of the integral Eilenberg-MacLane spectrum  $z(\infty) = H\mathbb{Z}$ . More generally, we have the following ladder of interpolations between the sphere spectrum and the mod 2 and integral Eilenberg-MacLane spectra:

Note that the structure of  $\mathcal{A}_*$  as an  $\mathcal{A}_*$ -comodule is given by the coproduct  $\psi$ , and more generally, the coaction on any sub-Hopf algebra of  $\mathcal{A}_*$  is defined to be the restriction of the coproduct.

Lemma 2.3. There are isomorphisms

$$H_*(y(n)) \cong P(\bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_n),$$
  
$$H_*(z(n)) \cong P(\bar{\xi}_1^2, \bar{\xi}_2, \dots, \bar{\xi}_n)$$

of sub- $\mathcal{A}_*$ -comodule algebras of  $H_*(H\mathbb{F}_2)$  and  $H_*(H\mathbb{Z})$ , respectively, for  $n \ge 1$ . The map  $z(n) \to y(n)$  induces the evident inclusion in homology.

*Proof.* By the Thom isomorphism, there are isomorphisms  $H_*(y(n)) \cong H_*(\Omega J_{2^n-1}S^2)$  and  $H_*(z(n)) \cong H_*(\Omega (J_{2^n-1}S^2\langle 2 \rangle))$ . The Serre spectral sequence arising from the path-loops fibration has the form

$$E_2 = H_*(J_{2^n - 1}S^2; H_*(\Omega J_{2^n - 1}S^2)) \Rightarrow H_*(PJ_{2^n - 1}S^2) = \mathbb{F}_2\{1\}$$

The homology  $H_*(J_{2^n-1}S^2; \mathbb{F}_2)$  can be computed by induction on k from the long exact sequence associated to the cofiber sequence

$$J_{k-1}S^2 \to J_k S^2 \to J_k S^2 / J_{k-1}S^2 \simeq (S^2)^{\wedge k}$$

as in [23, Lect. 3]. We have an additive isomorphism

$$H_*(J_{2^n-1}S^2; \mathbb{F}_2) \cong P_{2^n}(x)$$

where |x| = 2. In order for the relevant classes to die in the spectral sequence, we must have

$$H_*(\Omega J_{2^n-1}S^2) \cong P(\bar{\xi}_1, \dots, \bar{\xi}_n).$$

Thus  $H_*(y(n)) \cong P(\bar{\xi}_1, \ldots, \bar{\xi}_n)$ . In the analogous Serre spectral sequence

$$E_2 = H_*(J_{2^n - 1}S^2 \langle 2 \rangle; H_*(\Omega(J_{2^n - 1}S^2 \langle 2 \rangle))) \Rightarrow H_*(PJ_{2^n - 1}S^2) = \mathbb{F}_2\{1\},$$

only difference is that  $H_*(J_{2^n-1}S^2\langle 2\rangle) \cong \mathbb{F}_2\{1, x^2, \ldots, x^{2^{n+1}-1}\}$  so the computation is the same except instead of  $\bar{\xi}_1$  and its powers we only observe the powers of  $\bar{\xi}_1^2$ . Thus, we have  $H_*(z(n)) \cong P(\bar{\xi}_1^2, \bar{\xi}_2, \ldots)$ .

Finally, we claim that the maps  $H_*(y(n)) \to H_*(H\mathbb{F}_2)$  and  $H_*(z(n)) \to H_*(H\mathbb{Z})$  are monomorphisms of  $\mathcal{A}_*$ -comodule algebras and this implies the desired  $\mathcal{A}_*$ -coaction. This claim may easily be deduced by the map of Serre spectral sequences induced by the map of fiber sequences in Construction 2.1.

## **Corollary 2.4.** The spectra y(n) and z(n) are not fp type m for any finite m.

*Proof.* The cokernels of the inclusions  $H_*(y(n)) \to \mathcal{A}_*$  and  $H_*(z(n)) \to \mathcal{A}_*$  are not finitely generated as  $\mathcal{A}_*$ -comodules, so y(n) and z(n) are not finitely presented as comodules over the dual Steenrod algebra. The result then follows by [31, Prop. 3.2].

From the homology calculation above, we can also determine that y(n) and z(n) are not highly structured ring spectra in the following sense.

**Corollary 2.5.** The  $E_1$  algebra structure on y(n) and z(n) cannot be extended to an  $E_2$  algebra structure.

Proof. We give the proof for y(n); the proof for z(n) is similar. An extension of the  $E_1$  algebra structure to an  $E_2$  algebra structure implies an extension of the  $H_1$  algebra structure to an  $H_2$  algebra structure. If this is the case, then the top operation may be identified with the Dyer-Lashof operation  $Q^{|x|+1}(x)$  [13, p. 65].<sup>2</sup> Suppose the  $E_1$  algebra structure of y(n) extends to an  $E_2$  algebra structure. Then we may form a Postnikov truncation in the category of  $E_2$  algebras to produce a map  $y(n) \to H\pi_0(y(n)) = H\mathbb{F}_2$ . This map induces an inclusion  $H_*(y(n)) \hookrightarrow \mathcal{A}$  and therefore sends  $\overline{\xi}_n \in H_*(y(n))$  to  $\overline{\xi}_n \in \mathcal{A}$ . This inclusion must be compatible with  $Q^{|x|+1}$ . However, we know that  $Q^{|\overline{\xi}_n|+1}(\overline{\xi}_n) = \overline{\xi}_{n+1}$  in  $H_*(H\mathbb{F}_2)$  by [13, Thm. 2.2], but  $\overline{\xi}_{n+1}$  is not in the image of the inclusion. We can conclude that the  $E_1$  ring structure of y(n) cannot be extended to an  $E_3$  ring structure compatible with the  $E_3$  ring structure on  $H\mathbb{F}_2$ . For z(n), the argument is the same except that we consider the Postnikov truncation in  $E_2$  algebras  $z(n) \to H\pi_0(z(n)) = H\mathbb{Z}$ , which also induces an inclusion in homology  $H_*(z(n)) \hookrightarrow (\mathcal{A}//E(Q_0))_*$  and the rest of the argument is the same.

The fact that y(n) is not an  $E_2$  algebra will play a key role in Section 3.

**Convention 2.6.** We will often use the unit map  $S \to y(n)$  as well as the map  $y(n) \to H\mathbb{F}_2$ induced by the inclusion  $J_{2^n-1}S^2 \hookrightarrow J_{\infty}S^2 \simeq \Omega S^3$ . From this point on, any map  $y(n) \to H\mathbb{F}_2$ without decoration refers to the latter.

**Lemma 2.7.** The map  $y(n) \to H\mathbb{F}_2$  is  $(2^{n+1}-2)$ -connected.

*Proof.* The Adams spectral sequence converging to  $y(n)_*$  has the form

$$\operatorname{Ext}_{A_*}^{**}(\mathbb{F}_2, P(\overline{\xi}_1, \overline{\xi}_2, \dots, \overline{\xi}_n)) \Rightarrow y(n)_*$$

<sup>&</sup>lt;sup>2</sup>There is also a typo in line (1) of [13, p. 65] and  $H_{n+1}$  should read  $H_n$ . This is evident by comparing [13, Thm. 3.3] to [13, Thm. 3.1]. See [43, Thm. 1.8] where this is stated correctly in the literature at odd primes.

and the Adams spectral sequence converging to  $(H\mathbb{F}_2)_*$  has the form

$$\operatorname{Ext}_{\mathcal{A}_*}^{**}(\mathbb{F}_2, P(\bar{\xi}_1, \bar{\xi}_2, \ldots)) \cong \mathbb{F}_2 \Rightarrow (H\mathbb{F}_2)_* = \mathbb{F}_2.$$

Let  $C_{\bullet}(n)$  denote the  $E_1$  page of the Adams spectral sequence for y(n) and let  $C_{\bullet}(\infty)$  denote the  $E_1$  page for the Adams spectral sequence for  $H\mathbb{F}_2$ . These two  $E_1$  pages differ only in stems above the degree of  $\bar{\xi}_{n+1}$ . Since  $|\bar{\xi}_{n+1}| = 2^{n+1} - 1$ , the resulting  $E_2$ -terms agree up to stem  $2^{n+1} - 2$ . Since the second spectral sequence collapses at  $E_2$ , we conclude that  $\pi_i(y(n)) = 0$  for  $i \leq 2^{n+1} - 2$ .

2.2. The localized Adams spectral sequence. In this section, we recall the localized Adams spectral sequence [30][35] and discuss its applications to the study of chromatic complexity. Recall that the *m*-th connective Morava K-theory k(m) has homology  $H_*(k(m)) \cong \mathcal{A}_* \square_{E(\bar{\xi}_{m+1})} \mathbb{F}_2$  where  $\bar{\xi}_{m+1}$  is dual to the *m*-th Milnor primitive  $Q_m$ . The Adams spectral sequence converging to  $k(m)_*(X)$  has the form

$$E_2 = \operatorname{Ext}_{E(\bar{\mathcal{E}}_{m+1})}^{**}(\mathbb{F}_2, H_*(X)) \Rightarrow k(m)_*(X)$$

by the Künneth isomorphism and the change-of-rings isomorphism. The spectral sequence collapses for degree reasons when X is the sphere spectrum to show that  $k(m)_* \cong \mathbb{F}_2[v_m]$  with  $|v_m| = 2^{n+1} - 2$ .

Morava K-theory K(m) with  $K(m)_* \cong \mathbb{F}_2[v_m^{\pm 1}]$  can then be constructed as the telescope

$$K(m) = \widehat{k(m)} = \text{hocolim}\left(k(m) \xrightarrow{v_m} \Sigma^{-(2^{m+1}-2)}k(m) \xrightarrow{v_m} \cdots\right).$$

Since smash product commutes with filtered colimits,  $K(m) \wedge X$  is the telescope of the self-map  $v_m \wedge id_X$  on  $k(m) \wedge X$ .

The homotopy groups of a telescope can sometimes be computed using the localized Adams spectral sequence introduced in [35]. Our recollection follows [30].

**Construction 2.8.** [35][30] Let Y be a spectrum with with a  $v_n$ -self map  $f: Y \to \Sigma^{-d}Y$ , let  $\widehat{Y}$  be the telescope of f, and let

$$Y = Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots$$

be an Adams resolution of Y. Suppose there is a lifting  $\tilde{f}: Y \to \Sigma^{-d} Y_{s_0}$  for some  $s_0 \ge 0$ . This lifting induces maps  $\tilde{f}: Y_s \to Y_{s+s_0}$  for each  $Y_s$  in the Adams resolution above. Iterate these maps to define telescopes  $\widehat{Y_s}$  for  $s \ge 0$  and set  $Y_s = Y$  for s < 0 to produce a tower

$$\cdots \leftarrow \widehat{Y_{-1}} \leftarrow \widehat{Y_0} \leftarrow \widehat{Y_1} \leftarrow \cdots$$

The resulting conditionally convergent full plane spectral sequence

$$v_m^{-1} \operatorname{Ext}_{\mathcal{A}}^{**}(H^*(Y), \mathbb{F}_2) \Rightarrow \pi_*(Y)$$

is the localized Adams spectral sequence.

**Theorem 2.9.** [30, Thm. 2.13] For a spectrum Y equipped with maps f and  $\tilde{f}$  as above, in the localized Adams spectral sequence for  $\pi_*(\hat{Y})$  we have

- The homotopy colimit hocolim  $\widehat{Y}_{-s}$  is the telescope  $\widehat{Y}$ .
- The homotopy limit  $\operatorname{holim} \widehat{Y_s}$  is contractible if the original (unlocalized) Adams spectral sequence has a vanishing line of slope  $s_0/d$  at  $E_r$  for some finite r, i.e. if there are constants c and r such that

$$E_r^{s,t} = 0$$
 for  $s > c + (t-s)(s_0/d)$ .

(In this case, we say f has a parallel lifting f.)

• If f has a parallel lifting, this localized Adams spectral sequence converges to  $\pi_*(\hat{Y})$ .

**Remark 2.10.** Mahowald, Ravenel, and Shick compute  $v_n^{-1}E_2$  for the localized Adams spectral sequence converging to  $\pi_*(\widehat{y(n)})$  in [30, Sec. 2.3]. Their computations show that the localized Adams spectral sequence for  $\pi_*(\widehat{y(n)})$  converges.

As suggested by the notation, the  $E_2$ -page of the localized Adams spectral sequence can be computed by inverting  $v_m$  at the level of Ext-groups as in [30, Section 2.5] and [17].

We now specialize to the case  $Y = k(m) \wedge X$  and  $f = v_m \wedge id_X$  so that  $\hat{Y} \simeq K(m) \wedge X$ . By applying the Künneth isomorphism and a change-of-rings isomorphism, we see that the localized Adams spectral sequence takes the form

(3) 
$$v_m^{-1}E_2 := v_m^{-1} \operatorname{Ext}_{E(Q_m)}^{**}(H^*(X), \mathbb{F}_2) \Rightarrow K(m)_*(X).$$

Note that we have only used the  $v_m$ -map on k(m), so there is no decoration on X.

In order to compute  $v_m^{-1}E_2$ , we will use Margolis homology [32, Ch. 19] which encodes the action of the Milnor primitive  $Q_m$  on  $H^*(X)$ . For our purposes, it will be easier to work with homology and since all the examples we consider have finite type homology the distinction is minor as we note in Lemma 2.12.

**Definition 2.11.** Let M be a module over  $E(Q_m)$ . Since  $Q_m^2 = 0$ , we obtain a complex

$$\dots \xrightarrow{Q_m} M \xrightarrow{Q_m} M \xrightarrow{Q_m} M \xrightarrow{Q_m} \dots$$

The homology

$$H(M;Q_m) := \ker(Q_m|M) / \operatorname{im}(Q_m) \cap M$$

is the Margolis homology of M with respect to  $Q_m$ . If X is a spectrum, then the Margolis homology of X with respect to  $Q_m$ , denoted  $H(X; Q_m)$ , is defined by taking  $M = H_*(X)$ .

The following lemma says that we could equivalently define  $H(X;Q_m)$  by taking  $M = H^*(X)$  in Definition 2.11 and then dualizing.

**Lemma 2.12.** Let X be a spectrum with  $H^*(X)$  finite type. There is a natural isomorphism of (left)  $E(Q_m)$ -modules

$$H(H^*(X);Q_m) \cong D(H(X;Q_m))$$

where D(-) is the dual A-module.

*Proof.* The result follows from [32, Ch. 19, Prop. 12] by letting  $M = H_*(X)$  since in this case there is an isomorphism  $D(M) \cong H^*(X)$ .

The following lemma is well known, so we omit the proof.

**Lemma 2.13.** Suppose  $H_*(X)$  is bounded below and finite type. There is an isomorphism

$$\operatorname{Ext}_{E(Q_m)_*}^{*,*}(\mathbb{F}_2, H_*(X)) \cong H(X; Q_m) \otimes \mathbb{F}_2[v_m] \oplus T$$

where  $|v_m| = (1, 2^{m+1} - 1)$  and T is a simple  $v_m$ -torsion module concentrated in bidegrees (0, t) with  $t \ge \min\{i : H_i(X) \ne 0\}$ .

**Corollary 2.14.** Let  $m \ge 1$  and suppose X is a bounded below spectrum and  $H_*(X)$  is finite type. The following statements hold:

- (1) If  $H(X; Q_m) = 0$ , then  $v_m^{-1}E_2 = 0$ .
- (2) The  $E_2$  page of (3), denoted  $v_m^{-1}E_2$ , has a vanishing line of slope  $1/|v_m|$ .
- (3) The localized Adams spectral sequence associated to  $k(m) \wedge X$  with the self-map  $v_m \wedge id_X$  converges to  $K(m)_*(X)$ .

*Proof.* Parts (1) and (2) are clear from Lemma 2.13. Also see Step 2 of the proof [5, Thm. 3.5]. Statement (3) follows by applying Theorem 2.9.  $\Box$ 

When all the hypotheses for Corollary 2.14 hold including  $H(X;Q_m) = 0$  then it is a consequence that  $K(m)_*X \cong 0$  and this result also appears in [5, Thm. A.6].

2.3. Chromatic complexity of y(n), z(n), and  $z(n)/v_n$ . We now apply the localized Adams spectral sequence to determine the chromatic complexity of y(n), z(n), and a spectrum we define in this section  $z(n)/v_n$ .

The action of  $Q_m$  on the generator  $\bar{\xi}_k \in \mathcal{A}_*$  can be computed using the coproduct  $\psi \colon \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$  defined in (1). In particular, we have

(4) 
$$Q_m(\bar{\xi}_k) = \begin{cases} \bar{\xi}_{k-m-1}^{2^{m+1}} & k \ge m+1, \\ 0 & else, \end{cases}$$

where  $\bar{\xi}_0 = 1$ . This action can be extended to all of  $\mathcal{A}_*$  using the fact that  $Q_m$  acts as a derivation.

As a warm-up for later computations, we compute the Margolis homology of the dual Steenrod algebra. The chain complexes defined in the proof will be used in our computation of  $H(y(n); Q_m)$  below.

**Lemma 2.15.** The Margolis homology of the dual Steenrod algebra  $H(\mathcal{A}_*; Q_m)$ , or equivalently the Margolis homology of  $H\mathbb{F}_2$ , vanishes for all  $m \geq 0$ .

*Proof.* This is [32, Ch. 19, Prop. 1], but our proof is modeled after [1, Lem. 16.9]. We begin with  $H(\mathcal{A}_*; Q_0)$ , which is somewhat exceptional. Express  $\mathcal{A}_*$  as the tensor product of the chain complexes (with differential  $Q_0$ )

where  $r \geq 1$ . Each chain complex  $(c_r)$  has homology  $\mathbb{F}_2\{1\}$  and the chain complex  $(e_0)$  has vanishing homology, so by the Künneth isomorphism for Margolis homology [32, Ch. 19, Prop. 18], we have  $H(\mathcal{A}_*; Q_0) \cong 0.$ 

Now we compute  $H(\mathcal{A}_*; Q_1)$ . Decompose  $\mathcal{A}_*$  as the tensor product of the chain complexes (with differential  $Q_1$ )

$$(e_0) \quad \mathbb{F}_2\{1\} \leftarrow \mathbb{F}_2\{\overline{\xi}_2\},$$

- $(c_r) \quad \mathbb{F}_2\{1, \bar{\xi}_r^4\} \leftarrow \mathbb{F}_2\{\bar{\xi}_{r+2}, \bar{\xi}_r^8\} \leftarrow \mathbb{F}_2\{\bar{\xi}_r^4 \bar{\xi}_{r+2}, \bar{\xi}_r^{12}\} \leftarrow \mathbb{F}_2\{\bar{\xi}_r^8 \bar{\xi}_{r+2}, \bar{\xi}_r^{16}\} \leftarrow \cdots,$
- $(d_1) \quad \mathbb{F}_2\{1, \bar{\xi}_1, \bar{\xi}_1^2\},\$
- $(d_s) \quad \mathbb{F}_2\{1, \bar{\xi}_s^2\},$

where  $r \geq 1$  and  $s \geq 2$ . The chain complex  $(e_0)$  has vanishing homology, so by the Künneth isomorphism we have  $H(\mathcal{A}_*; Q_1) \cong 0$ .

The computation of  $H(\mathcal{A}_*; Q_m)$  for  $m \ge 2$  is similar. Decompose  $\mathcal{A}_*$  into chain complexes as above; the chain complex

$$(e_0) \quad \mathbb{F}_2\{1\} \leftarrow \mathbb{F}_2\{\xi_{m+1}\}$$

has vanishing homology, so  $H(\mathcal{A}_*; Q_m) = 0$ .

**Corollary 2.16.** The Margolis homology of  $(A//A(0))^{\vee} \cong \mathbb{F}_2[\bar{\xi}_1^2, \bar{\xi}_2, \ldots]$ , or equivalently the Margolis homology of  $H\mathbb{Z}$ , is given by

$$H(H\mathbb{Z};Q_m) \cong \begin{cases} \mathbb{F}_2 & m = 0, \\ 0 & else. \end{cases}$$

*Proof.* We begin with m = 0. The only difference between this computation and the computation for  $H_*(H\mathbb{F}_2, Q_0)$ , is that we remove the chain complex  $(e_0)$ . Since the homology of the remaining complexes  $(c_r)$  is  $\mathbb{F}_2$  in each case,  $H_*(H\mathbb{Z}, Q_0) \cong \mathbb{F}_2$ .

For m > 0, we can use the same complexes as in the previous proof after replacing  $(d_1)$  by the chain complex  $\mathbb{F}_2\{1, \bar{\xi}_1^2\}$ .

We compute the Margolis homology of y(n) and z(n) by modifying these complexes further.

**Lemma 2.17.** The Margolis homology of  $P(\bar{\xi}_1, \ldots, \bar{\xi}_n)$ , or equivalently the Margolis homology of y(n), is given by

$$H(y(n); Q_m) \cong \begin{cases} 0 & \text{if } 0 \le m \le n-1, \\ H_*(y(n)) & \text{if } m \ge n. \end{cases}$$

*Proof.* When m = 0, the first r for which the complex  $(c_r)$  cannot be defined is  $(c_n)$  since  $\bar{\xi}_{n+1} \notin H_*(y(n))$ . Therefore we replace  $(c_n)$  by the complex

$$(c'_n) \quad \mathbb{F}_2\{1\} \leftarrow \mathbb{F}_2\{\bar{\xi}_n^2\} \leftarrow \mathbb{F}_2\{\bar{\xi}_n^4\} \leftarrow \mathbb{F}_2\{\bar{\xi}_n^6\} \leftarrow \cdots$$

Then  $H_*(y(n))$  decomposes as the tensor product of the complexes  $(e_0)$ ,  $(c_r)_{1 \leq r \leq n-1}$ , and  $(c'_n)$ . The homology of  $(c'_n)$  is nontrivial but the homology of  $(e_0)$  vanishes, so  $H(y(n); Q_0) \cong 0$ .

When  $1 \le m \le n-1$ , we make a similar change. We end up with redefined chain complexes  $(c'_r)$  for  $n-m \le r \le n$ . Since we still tensor with the acyclic complex  $(e_0)$ , we still have  $H(y(n); Q_m) = 0$ .

When  $m \ge n$ , we no longer include the chain complex  $(e_0)$  since  $\overline{\xi}_{n+1} \notin H_*(y(n))$ . Since  $Q_n(\overline{\xi}_i) = 0$  for all  $1 \le i \le n$ , we see that  $H_*(y(n))$  is generated by cycles and obtain the desired isomorphism.

The same techniques adapted to the complexes used to compute  $H(H\mathbb{Z}; Q_m)$  give the following:

**Corollary 2.18.** The Margolis homology of  $P(\bar{\xi}_1^2, \bar{\xi}_2, \dots, \bar{\xi}_n)$ , or equivalently the Margolis homology of z(n), is given by

$$H(z(n); Q_m) \cong \begin{cases} P(\bar{\xi}_n^2) & \text{if } m = 0, \\ 0 & \text{if } 1 \le m \le n-1, \\ H_*(z(n)) & \text{if } m \ge n. \end{cases}$$

*Proof.* We do the same alterations to Lemma 2.17 as we did to produce the proof of Corollary 2.16 from Lemma 2.15, so we will just describe the case m = 0. We use the same chain complexes as in Lemma 2.17 except we do not include the acyclic complex  $(e_0)$ . Thus, the Margolis homology is a tensor product of copies of  $\mathbb{F}_2$  with the homology of  $(c'_n)$ . Thus,  $H_*(z(n);Q_0) \cong P(\bar{\xi}_n^2)$ .

The following lemma will be useful for further describing the chromatic complexity of z(n).

**Lemma 2.19.** The spectrum z(n) has a self map

$$v_n \colon \Sigma^{2p^n - 2} z(n) \to z(n)$$

that induces the zero map on  $K(m)_*$  for  $1 \le m < n$  and on mod p homology  $H_*$ . Moreover, we have an isomorphism of  $\mathcal{A}_*$ -comodules

$$H_*(z(n)/v_n) \cong H_*(z(n)) \otimes E(\bar{\xi}_{n+1}).$$

*Proof.* We analyze the Adams spectral sequence for z(n) in a range. First we describe the input of the Adams spectral sequence. Following [30], write  $B(n)_*$  for the Hopf algebra in the extension

$$H_*(y(n)) \to \mathcal{A}_* \to B(n)_*$$

and let  $C(n)_*$  be the Hopf algebra in the extension

$$H_*(z(n)) \to \mathcal{A}_* \to C(n)_*$$

We also have a Hopf algebra extension

$$E(\bar{\xi}_1) \to C(n)_* \to B(n)_*$$

and an associated Cartan-Eilenberg spectral sequence

$$\operatorname{Ext}_{B(n)_*}^s(\mathbb{F}_2, \operatorname{Ext}_{E(\bar{\xi}_1)_*}^t(\mathbb{F}_2, \mathbb{F}_2)) \cong \operatorname{Ext}_{B(n)_*}^s(\mathbb{F}_2, P(h_0)_t) \Rightarrow \operatorname{Ext}_{C(n)_*}^{s+t}(\mathbb{F}_2, \mathbb{F}_2).$$

The Cartan-Eilenberg spectral sequence collapses to the (s, 0)-line because  $|h_0| = |\xi_1| - 1 = 0$ . Since there are no classes in  $\operatorname{Ext}_{B(n)}^k(\mathbb{F}_2, \mathbb{F}_2)$  in adjacent degrees for  $k \leq 2^{n+1}$  and the Adams spectral sequence for y(n) collapses in this range by [30, Lem. 3.5], the Adams spectral sequence for z(n)also collapses in this range. Again, by [30, Lem. 3.5] there is an element  $v_n$  in Adams filtration one in  $\operatorname{Ext}_{C(n)_*}^{*}(\mathbb{F}_2, \mathbb{F}_2)$  and we observe that it supports a  $h_0$ -tower. Consequently, there is an element  $v_n \in \pi_{2^{n+1}-2}(z(n)_2)$  generating  $\mathbb{Z}_2$ .

Since z(n) is an  $E_1$  ring spectrum we can produce a self map as the composite

$$S^{2^{n+1}-2} \wedge z(n) \xrightarrow{v_n \wedge \mathrm{id}_{z(n)}} z(n) \wedge z(n) \to z(n).$$

This is also the map obtained by taking the adjoint to  $v_n: S^{2^{n+1}-1} \to z(n)$  in the category of (right) z(n)-modules. By Lemma 2.17 this map induces the zero map on  $K(m)_*$  for  $1 \le m < n$ . It also induces the zero map on  $H_*$  because  $v_n$  is detected by an element in Adams filtration one.

Therefore we have an extension

$$0 \to H_*(z(n)) \to H_*(z(n)/v_n) \to \Sigma^{2^{n+1}-1}H_*(z(n)) \to 0$$

of  $\mathcal{A}_*$  comodules. The group of possible  $\mathcal{A}_*$ -comodule extensions is given by

$$\operatorname{Ext}_{\mathcal{A}_{*}}^{1}(\Sigma^{2^{n+1}-1}H_{*}(z(n)),H_{*}(z(n))) \cong \operatorname{Ext}_{C(n)_{*}}^{1}(\Sigma^{2^{n+1}-1}H_{*}(z(n)),\mathbb{F}_{2}),$$

using a change of rings isomorphism and the isomorphism  $H_*(z(n)) \cong \mathcal{A}_* \square_{C(n)_*} \mathbb{F}_2$ .

We are therefore reduced to examining the possible  $C(n)_*$ -comodule extensions

$$0 \to \mathbb{F}_2 \to E \to \Sigma^{2^{n+1}-1} H_*(z(n)) \to 0,$$

but all such extensions of  $C(n)_*$ -comodules are trivial by examination of the grading preserving coaction. This implies  $H_*(z(n)/v_n) \cong H_*(z(n)) \otimes E(x)$  where  $|x| = 2^{n+1} - 1$ .

We now use the Adams spectral sequence to determine the  $\mathcal{A}_*$ -coaction on x. Note that  $z(n)/v_n$  was obtained by coning off the element in homotopy detected by the permanent cycle  $\bar{\xi}_n \otimes 1$  in the Adams spectral sequence with

$$E_2 = \operatorname{Ext}_{C(n)_*}^{*,*}(\mathbb{F}_2, E(x)).$$

The only element that can kill  $\bar{\xi}_n \otimes 1$  is x, so we have

$$d_1(x) = \bar{\xi}_{n+1} \otimes 1.$$

On the other hand,  $d_1$ -differentials in the Adams spectral sequence can be calculated using the formula for differentials in the cobar complex, so  $d_1(x) = 1 \otimes x - \psi_n(x)$  where  $\psi_n(x)$  is the coaction of x in  $H_*(z(n)/v_n)$ . We therefore see that

$$\psi_n(x) = \overline{\xi}_{n+1} \otimes 1 + 1 \otimes x + d_1$$
 boundaries.

Finally, since the composite  $\Sigma^{2^{n+1}-2}z(n) \xrightarrow{v_n} z(n) \to z(\infty) = H\mathbb{Z}$  is null-homotopic, there is a map  $z(n)/v_n \to H\mathbb{Z}$ . This map sends  $\bar{\xi}_{n+1} \otimes 1$  to the class with the same name in the cobar complex for  $H\mathbb{Z}$ . In the latter cobar complex,  $\bar{\xi}_{n+1} \otimes 1$  is killed by a differential on  $\bar{\xi}_{n+1} \in H_*(H\mathbb{Z})$ . Therefore x maps to  $\bar{\xi}_{n+1}$  under the map of spectral sequences. Since the map  $H_*(z(n)/v_n) \to H_*(H\mathbb{Z})$  is a map of  $\mathcal{A}_*$ -comodules, the coaction on x coincides with the coaction on  $\bar{\xi}_{n+1}$ . Note also that there is no room for hidden comodule extensions because |x| > |y| for all generators y of  $H_*(z(n)/v_n)$ .  $\Box$ 

**Remark 2.20.** In fact,  $z(n)/v_n$  may be constructed as the Thom spectrum of the map

$$S^{2p^n-1} \to BGL_1(z(n))$$

adjoint to  $v_n \in \pi_{2p^n-2}(GL_1(z(n))) \cong \pi_{2p^n-2}(z(n))$ . The first author would like to thank Jeremy Hahn for pointing this out.

We now determine the chromatic complexity of  $z(n)/v_n$ .

**Corollary 2.21.** The Margolis homology of  $P(\bar{\xi}_1^2, \bar{\xi}_2, \ldots, \bar{\xi}_n) \otimes E(\bar{\xi}_{n+1})$ , or equivalently the Margolis homology of  $z(n)/v_n$ , is given by

$$H(z(n)/v_n; Q_m) \cong \begin{cases} \mathbb{F}_2 & \text{if } m = 0, \\ 0 & \text{if } 1 \le m \le n \\ H_*(z(n)/v_n) & \text{if } m \ge n+1. \end{cases}$$

*Proof.* The proof in the case m = n follows by tensoring the complexes from the previous corollary with the complex

$$\mathbb{F}_2\{1\} \leftarrow \mathbb{F}_2\{\bar{\xi}_{n+1}\}.$$

Since this complex is  $Q_n$ -acyclic, we observe that  $H(z(n)/v_n; Q_n) \cong 0$ . For m = 0, we make the following adjustment. Rather than replacing  $(c_n)$  with  $(c'_n)$  as in Lemma 2.17, we keep the complex  $(c_n)$  and remove  $(c_r)$  for r > n. This has the consequence that  $H_*(z(n)/v_n, Q_0) \cong \mathbb{F}_2$ . In the case 0 < m < n, we only replace  $(c_r)$  with  $(c'_r)$  for  $n - m < r \le n$ . The Margolis homology is still trivial because we are tensoring with the acyclic complex  $\mathbb{F}_2\{1\} \leftarrow \mathbb{F}_2\{\bar{\xi}_m\}$ . The case m > n is exactly the same as in Lemma 2.17

We can assemble these Margolis homology computations to study  $K(m)_*(X)$  for X = y(n), z(n), and  $z(n)/v_n$ .

**Proposition 2.22.** The chromatic complexity of y(n), z(n) and  $z(n)/v_n$  may be described as follows:

- The spectrum y(n) is K(m)-acyclic for  $0 \le m \le n-1$ , and  $K(n)_*(y(n)) \ne 0$ .
- The spectrum z(n) is K(m)-acyclic for  $1 \le m \le n-1$ , and  $K(m)_*(z(n)) \ne 0$  for m = 0, n.
- The spectrum  $z(n)/v_n$  is K(m)-acyclic for  $1 \le m \le n$ , and  $K(m)_*(z(n)/v_n) \ne 0$  for m = 0.

*Proof.* We give the proof for y(n); the proofs for z(n) and  $z(n)/v_n$  are similar. Since y(n) is connective, the localized Adams spectral sequence converges to  $K(m)_*(y(n))$  by Lemma 2.14. By the same lemma, we have

$$v_m^{-1}E_2 = v_m^{-1}\operatorname{Ext}_{E(Q_m)}(H^*(y(n)), \mathbb{F}_2) \cong 0$$

whenever  $H(y(n); Q_m)$  vanishes. Therefore Lemma 2.17 proves the K(m)-acyclicity of y(n) for  $0 \le m \le n-1$ .

It remains to show that  $K(m)_*(y(n)) \neq 0$  and  $K(m)_*(z(n)) \neq 0$  for m = 0, n. The m = 0 cases follow from examination of the maps  $z(n) \to H\mathbb{Z}$  and  $z(n)/v_n \to H\mathbb{Z}$ , which allow us to resolve the  $h_0$ -tower in the zero stem in the Adams spectral sequence. Consequently,  $H\mathbb{Q}_0(z(n)) \cong H\mathbb{Q}_0(z(n)/v_n) \cong \mathbb{Q}$ .

The proofs that  $K(n)_*(y(n)) \neq 0$  and  $K(n)_*(z(n)) \neq 0$  are essentially the same, so we just describe the y(n) case. Since K(n) and y(n) are both  $E_1$  ring spectra, the Atiyah-Hirzebruch spectral sequence

$$H_*(y(n), K(n)) \Rightarrow K(n)_*(y(n))$$

is multiplicative, with multiplicative generators all either on the zero line or the zero column using Serre grading. The spectral sequence is a right half plane spectral sequence so the generators on the zero column cannot support differentials. The generators in the zero line are all in degrees less than or equal to  $2^n - 1$  and since  $|v_n| = 2^{n+1} - 2$  the  $E_2$ -page is isomorphic to the  $E_{2^{n+1}-1}$  page and therefore the spectral sequence collapses. Consequently, there is an isomorphism

$$K(n)_*(y(n)) \cong K(n)_* \otimes H_*(y(n)).$$

We now turn to the study of the topological Hochschild homology of y(n). We begin by computing  $H_*(THH(y(n)))$  using the Bökstedt spectral sequence [12]. We then analyze the map  $\phi_n : H_*(THH(y(n))) \to H_*(THH(H\mathbb{F}_2)).$ 

**Remark 3.1.** The calculations in this section and the sequel are complicated by two facts:

- (1) The spectrum THH(y(n)) does not admit a ring structure since y(n) is an  $E_1$  ring spectrum, but not an  $E_2$  ring spectrum. Therefore we will only prove *additive isomorphisms* throughout the remaining sections since there is no multiplicative structure *a priori* on  $H_*(THH(y(n)))$ .
- (2) There is indeterminacy in the names of many classes. For example, we only understand classes in  $H_*(THH(y(n)))$  up to lower Bökstedt filtration. This does not complicate the additive presentations in our calculations, but it does affect our understanding of deeper structure such as the coaction of the dual Steenrod algebra  $\mathcal{A}_*$ .

**Proposition 3.2.** There is an isomorphism of graded  $A_*$ -comodules

$$H_*(THH(y(n))) \cong H_*(y(n)) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \dots, \sigma\bar{\xi}_n)$$

where the coaction

$$\nu_n \colon H_*(THH(y(n))) \to H_*(THH(y(n))) \otimes \mathcal{A}_*$$

on elements  $x \in H_*(y(n))$  is determined by the restriction of the coproduct on  $\mathcal{A}_*$  to  $H_*(y(n)) \subset \mathcal{A}_*$ , the coaction on  $\sigma \xi_i$  is determined by the formula  $\nu_n(\sigma \xi_i) = (1 \otimes \sigma)\nu_n(\xi_i)$ , and the coaction on symbolic products xy is determined by  $\nu_n(xy) = \nu_n(x)\nu_n(y)$ .

*Proof.* The  $E_2$ -term of the Bökstedt spectral sequence

$$E_2^{*,*} \cong HH_*(H_*(y(n))) \cong P(\bar{\xi}_1, \dots, \bar{\xi}_n) \otimes E(\sigma\bar{\xi}_1, \dots, \sigma\bar{\xi}_n)$$

maps injectively to the  $E_2$ -term of the Bökstedt spectral sequence for  $H\mathbb{F}_2$ . The latter spectral sequence is multiplicative and all the algebra generators are concentrated in Bökstedt filtration zero and one. Consequently, the Bökstedt spectral sequence for  $H\mathbb{F}_2$  collapses and the injective map of spectral sequences implies that the Bökstedt spectral sequence for y(n) also collapses.

The Bökstedt spectral sequence is a spectral sequence of  $\mathcal{A}_*$ -comodules and the formula  $\nu_n(\sigma x) = (1 \otimes \sigma)\nu_n(x)$  holds because the operator  $\sigma$  is induced by a map of spectra  $\mathbb{T} \wedge R \to THH(R)$  (see e.g. [7, Eq. 5.11]) and this determines the  $\mathcal{A}_*$ -coaction modulo lower Bökstedt filtration.

We will use the fact that the Bökstedt spectral sequence computing  $H_*(THH(y(n)))$  agrees with the Bökstedt spectral sequence computing  $H_*(THH(H\mathbb{F}_2))$  up until degree  $2^{n+1}-2 = |\bar{\xi}_{n+1}| - 1$ . We will also frequently use the map

$$\phi_n \colon THH(y(n)) \to THH(H\mathbb{F}_2)$$

induced by the map  $y(n) \to H\mathbb{F}_2$ . The rest of this section is dedicated to studying the induced map on homology

$$(\phi_n)_* \colon H_*(THH(y(n))) \to H_*(THH(H\mathbb{F}_2)).$$

**Lemma 3.3.** Let  $x \in H_*(THH(y(n)))$ . The induced map on homology

$$(\phi_n)_* \colon H_*(THH(y(n))) \to H_*(THH(H\mathbb{F}_2))$$

has the form

$$(\phi_n)_*(x) = x + (classes of lower Bökstedt filtration).$$

*Proof.* The linearization map  $y(n) \to H\mathbb{F}_2$  induces a map of Bökstedt spectral sequences. The lemma follows from the construction of the Bökstedt spectral sequence.

The spectrum THH(y(n)) has a canonical circle action  $\mathbb{T}_+ \wedge THH(y(n)) \to THH(y(n))$ compatible with a structure map  $\sigma \colon \mathbb{T} \wedge y(n) \to THH(y(n))$ . Though THH(y(n)) may not be a ring spectrum, the map  $\sigma$  acts as though it were a derivation on  $H_*(THH(y(n)))$ , at least symbolically, as in [34, Prop. 3.2]. Indeed, the proof of [34, Prop. 3.2] only relies on R being an  $E_1$  ring spectrum. This behavior will be important for our analysis in the next section since the structure map  $\sigma$ determines the  $d^2$ -differentials in the homological  $\mathbb{T}$ -Tate spectral sequence.

If  $x \in H_*(THH(y(n)))$  satisfies  $\sigma(x) = 0$ , we will refer to x as a  $\sigma$ -cycle. If  $x = \sigma(y)$  for some  $y \in H_*(THH(y(n)))$ , we will refer to x as a  $\sigma$ -boundary. The  $\mathcal{A}_*$ -coaction on  $H_*(THH(y(n)))$ will be denoted

$$\nu_n \colon H_*(THH(y(n))) \to \mathcal{A}_* \otimes H_*(THH(y(n)))$$

for  $0 \le n \le \infty$  with the convention that  $y(\infty) = H\mathbb{F}_2$ .

**Lemma 3.4.** The induced map on homology  $(\phi_n)_*$ :  $H_*(THH(y(n))) \to H_*(THH(H\mathbb{F}_2))$  is a map of  $\mathcal{A}_*$ -comodules; i.e. the formula  $(\mathrm{id} \otimes (\phi_n)_*) \circ \nu_n = \nu_\infty \circ (\phi_n)_*$  holds.

*Proof.* This is true for any map in homology induced by a map of spectra.

We are now ready to prove the main result of this section, Proposition 3.8. Before that, we include the following example to illustrate some subtleties in understanding  $(\phi_n)_*$ .

Example 3.5. We will fully describe the map

$$P(\bar{\xi}_1) \otimes E(\sigma\bar{\xi}_1) \cong H_*(THH(y(1))) \xrightarrow{(\phi_1)_*} H_*(THH(H\mathbb{F}_2)) \cong P(\bar{\xi}_1, \bar{\xi}_2, \ldots) \otimes P(\sigma\bar{\xi}_1)$$

induced by the map  $\phi_1: THH(y(1)) \to THH(H\mathbb{F}_2)$  as a map of  $E(\sigma)$ -modules in the category of  $\mathcal{A}_*$ -comodules. We first describe this map as map of  $E(\sigma)$ -modules.

By Lemma 3.3, we have  $(\phi_1)_*(\bar{\xi}_1^i) = \bar{\xi}_1^i$  since  $\bar{\xi}_1^i \in H_i(THH(y(1)))$  has Bökstedt filtration zero. In general, the map  $(\phi_1)_*$  sends classes in  $H_*(THH(y(n)))$  in Bökstedt filtration zero to the classes with the same name in  $H_*(THH(H\mathbb{F}_2))$ .

Moving on to Bökstedt filtration one, we know that either

$$(\phi_1)_*(\sigma\bar{\xi}_1) = \sigma\bar{\xi}_1 \text{ or } (\phi_1)_*(\sigma\bar{\xi}_1) = \sigma\bar{\xi}_1 + \bar{\xi}_1^2$$

for degree reasons. If the latter formula holds, we may simply change our basis for the vector space  $H_2THH(y(1)) \cong \mathbb{F}_2\{\sigma \bar{\xi}_1, \bar{\xi}_1^2\}$  to account for this, so we may assume the former.

We now analyze the key case. Consider the class  $\bar{\xi}_1 \sigma \bar{\xi}_1 \in H_3(THH(y(1)))$ . We claim that  $(\phi_1)_*(\bar{\xi}_1\sigma \bar{\xi}_1) \neq \bar{\xi}_1\sigma \bar{\xi}_1$ . In fact, we know that  $\sigma(\bar{\xi}_1\sigma \bar{\xi}_1) = 0$  in  $H_*(THH(y(1)))$  and therefore  $\bar{\xi}_1\sigma \bar{\xi}_1$  must map to a  $\sigma$ -cycle in  $H_*(THH(H\mathbb{F}_2))$ . By Lemma 3.3, we know that  $\bar{\xi}_1\sigma \bar{\xi}_1$  maps to the class of the same name modulo classes in lower Bökstedt filtration. We also know that in  $H_*(THH(H\mathbb{F}_2))$  $\sigma(\bar{\xi}_1\sigma \bar{\xi}_1) = \sigma \bar{\xi}_2$ . Therefore,  $(\phi_1)_*(\bar{\xi}_1\sigma \bar{\xi}_1) = \bar{\xi}_1\sigma \bar{\xi}_1 + y$  where y is in Bökstedt filtration zero and  $\sigma y = \sigma \bar{\xi}_2$ . The only such element in  $H_*(THH(H\mathbb{F}_2))$  with these properties is  $\bar{\xi}_2$  itself. Thus,

$$(\phi_1)_*(\xi_1 \sigma \xi_1) = \xi_1 \sigma \xi_1 + \xi_2.$$

We then claim that  $(\phi_1)_*(\bar{\xi}_1^{2k}\sigma\bar{\xi}_1) = \bar{\xi}_1^{2k}\sigma\bar{\xi}_1$ . We know that  $\sigma(\bar{\xi}_1^{2k}\sigma\bar{\xi}_1) = 0$  in both the source and target. Therefore, the only possibility is that we add  $\sigma$ -cycles in either the source or target of the map. Since this does does not affect the map up to isomorphism of  $E(\sigma)$ -modules, we may assume  $(\phi_1)_*(\bar{\xi}_1^{2k}\sigma\bar{\xi}_1) = \bar{\xi}_1^{2k}\sigma\bar{\xi}_1$ .

We also claim that  $(\phi_1)_*(\bar{\xi}_1^{2k+1}\sigma\bar{\xi}_1) = \bar{\xi}_1^{2k+1}\sigma\bar{\xi}_1 + \bar{\xi}_1^{2k}\bar{\xi}_2$ . Again, we know  $\sigma(\bar{\xi}_1^{2k+1}\sigma\bar{\xi}_1) = 0$  in  $H_*(THH(y(1)))$  whereas  $\bar{\xi}_1^{2k+1}\sigma\bar{\xi}_1 = \bar{\xi}_1^{2k}\sigma\bar{\xi}_2$  in  $H_*(THH(H\mathbb{F}_2))$ . Therefore, we must add a term y in Bökstedt filtration zero such that  $\sigma y = \bar{\xi}_1^{2k}\sigma\bar{\xi}_2$  and the only possibility is  $\bar{\xi}_1^{2k}\sigma\bar{\xi}_2$ . This completely determines the map up to isomorphism of  $E(\sigma)$ -modules.

We now describe the map as a map of  $E(\sigma)$ -modules in the category of  $\mathcal{A}_*$ -comodules up to some indeterminacy. First, note that there are no  $\sigma$ -cycles in the degree of  $\bar{\xi}_1^{2k+1}\sigma\bar{\xi}_1$  in lower

Bökstedt filtration and thus we know the answer for  $\bar{\xi}_1^{2k+1}\sigma\bar{\xi}_1$  completely as a map of  $\mathcal{A}_*$ -comodules. This also forces the  $\mathcal{A}_*$ -comodule structure on these elements. For example, since

$$\nu_{\infty}(\bar{\xi}_{1}^{2k+1}\sigma\bar{\xi}_{1}+\bar{\xi}_{1}^{2k}\bar{\xi}_{2}) = (\bar{\xi}_{1}\otimes 1+1\otimes\bar{\xi}_{1})^{2k+1}(1\otimes\sigma\bar{\xi}_{1})+ (\bar{\xi}_{1}^{2k}\otimes 1+1\otimes\bar{\xi}_{1}^{2k})(\bar{\xi}_{2}\otimes 1+\bar{\xi}_{1}\otimes\bar{\xi}_{1}^{2}+1\otimes\bar{\xi}_{2}) \\ = (\bar{\xi}_{1}\otimes 1+1\otimes\bar{\xi}_{1})^{2k+1}(1\otimes\sigma\bar{\xi}_{1})+\bar{\xi}_{1}^{2k}\bar{\xi}_{2}\otimes 1+ \bar{\xi}_{2}\otimes\bar{\xi}_{1}^{2k}+\bar{\xi}_{1}^{2k+1}\otimes\bar{\xi}_{1}^{2}+\bar{\xi}_{1}^{2k+3}\otimes 1+\bar{\xi}_{1}^{2k}\otimes\bar{\xi}_{2}+1\otimes\bar{\xi}_{1}^{2k}\bar{\xi}_{2},$$

we know that

$$\nu_1(\bar{\xi}_1^{2k+1}\sigma\bar{\xi}_1) = (\bar{\xi}_1 \otimes 1 + 1 \otimes \bar{\xi}_1)^{2k+1}(1 \otimes \sigma\bar{\xi}_1) + \bar{\xi}_1^{2k+1} \otimes \bar{\xi}_1^2 + \bar{\xi}_2 \otimes \bar{\xi}_1^{2k} + \bar{\xi}_1^{2k+3} \otimes 1 + \bar{\xi}_1^{2k}\bar{\xi}_2 \otimes 1.$$

In the case of  $\bar{\xi}_1^{2k}\sigma\bar{\xi}_1$ , adding  $\sigma$ -cycles of the form  $\bar{\xi}_1^j$  does not affect the comodule structure on the source up to a change of basis, since  $\bar{\xi}_1^j$  is also in the target. However, if we add a  $\sigma$ -cycle in  $H_*(THH(H\mathbb{F}_2))$  that is not in the source, then this affects the comodule structure on the source. We therefore determine  $\phi_*$  up to this indeterminacy. In summary,  $\bar{\xi}_1^{2k}\sigma\bar{\xi}_1$  maps to  $\bar{\xi}_1^{2k}\sigma\bar{\xi}_1$  up to  $\sigma$ -cycles in  $H_*(THH(H\mathbb{F}_2))$  that are not in the image of  $H_*(THH(y(1)))$ . In Proposition 3.8, we will describe  $(\phi_n)_*$  up to the same type of indeterminacy.

**Remark 3.6.** In fact, we can actually avoid indeterminacy in the previous example because  $\bar{\xi}_1^{2k}\sigma\bar{\xi}_1$  is a  $\sigma$ -boundary. Since we know that  $\bar{\xi}_1^{2k+1}$  maps to the element of the same name, we see that  $\bar{\xi}_1^{2k}\sigma\bar{\xi}_k$  must map to the element of the same name without any indeterminacy. This argument no longer applies when studying  $(\phi_n)_*$  for  $n \geq 2$  since there will typically be additional elements in lower Bökstedt filtration.

We may choose a basis of  $H_*(THH(y(n)))$  so that  $\sigma$  behaves as a derivation at the level of symbols, i.e. there is an equality up to higher Bökstedt filtration  $\sigma(xy) = \sigma(x)y + x\sigma(y)$  for  $x, y \in H_*(y(n))$ . Indeed, we may apply [34, Prop. 3.2] to see that class  $\sigma(xy)$  is detected by  $\sigma_*(xy)$ in the  $E_2$ -page of the Bökstedt spectral sequence (where  $\sigma_* : H_*(y(n)) \to HH_*(H_*(y(n)))$ ) is defined by  $z \mapsto 1 \otimes z$ ). We have  $\sigma_*(xy) = \sigma_*(x)y + x\sigma_*(y)$  since  $\sigma_*$  is a derivation in Hochschild homology of the graded commutative ring  $H_*(y(n))$  [34, Pg. 7], and  $\sigma_*(x)y + x\sigma_*(y)$  detects the element we call  $\sigma(x)y + x\sigma(y)$  in  $H_*(THH(y(n)))$ .

The coaction on  $\sigma \xi_k$  is given by

$$\nu_n(\sigma\bar{\xi}_k) = (1\otimes\sigma)\left(\sum_{i+j=k}\bar{\xi}_i\otimes\bar{\xi}_j^{2^i}\right).$$

Since  $\sigma$  behaves like a derivation symbolically, we see that the only term that is nontrivial in the formula for  $\nu_n(\sigma \bar{\xi}_k)$  is  $1 \otimes \sigma \bar{\xi}_k$ . We therefore conclude that  $\sigma \bar{\xi}_k$  is a comodule primitive for all k.

We now proceed to the main result of this section. We thank Vigleik Angeltveit for discussions which led to a simplification of the proof of Proposition 3.8; we use some notation from [6, Prop. 4.12]. We also note once and for all that elements in  $H_*(THH(y(n)))$  are only well-defined up to lower Bökstedt filtration as in [6, Sec. 5].

**Definition 3.7.** Let bfilt(x) be the Bökstedt filtration of an element x. Define

$$J_n = (x \in H_*(THH(H\mathbb{F}_2)) \setminus \operatorname{im}(\phi_n)_* : \operatorname{bfilt}(x) \leq n \text{ and } \sigma(x) = 0)$$

to be the ideal generated by all  $\sigma$ -cycles x in  $H_*(THH(H\mathbb{F}_2))$  in Bökstedt filtration less than or equal to n that are not in the image of  $(\phi_n)_*$ . We refer to these elements as the *complementary*  $\sigma$ -cycles in the proof of the following proposition.

**Proposition 3.8.** Let  $x_i = \sigma \bar{\xi}_i \sigma \bar{\xi}_{i+1} \dots \sigma \bar{\xi}_n$  The map  $(\phi_n)_* \colon H_*(THH(y(n))) \to H_*(THH(H\mathbb{F}_2))$ 

is determined by

(5)  $(\phi_n)_*(\bar{\xi}_i x_i) = \bar{\xi}_i x_i + \bar{\xi}_{n+1}$ 

and for  $y \in H_*(y(n))$  or  $y \in E(\sigma \overline{\xi}_1, \sigma \overline{\xi}_2, \dots, \sigma \overline{\xi}_n)$ 

$$(6) \qquad \qquad (\phi_n)_*(y) = y$$

For all remaining products of elements, the map  $(\phi_n)_*$  is symbolically multiplicative modulo complementary  $\sigma$ -cycles.

*Proof.* We begin with the proof of (5). The elements  $\bar{\xi}_i x_i$  are  $\sigma$ -cycles in  $H_*(THH(y(n)))$ , but in  $H_*(THH(H\mathbb{F}_2))$  we know that  $\sigma(\bar{\xi}_i x_i) = \sigma \bar{\xi}_{n+1}$ . Since  $\bar{\xi}_n \sigma \bar{\xi}_n$  maps to  $\bar{\xi}_n \sigma \bar{\xi}_n + z$  for z in lower Bökstedt filtration, we know that  $\sigma(\bar{\xi}_i x_i + z) = 0$  and therefore that  $\sigma(z) = \sigma \bar{\xi}_{n+1}$ . In this case, the only element z in lower Bökstedt filtration such that  $\sigma z = \sigma \bar{\xi}_{n+1}$  is  $\bar{\xi}_{n+1}$  itself. We now proceed by downward induction on i to show  $\bar{\xi}_i x_i$  maps to  $\bar{\xi}_i x_i + \bar{\xi}_{n+1}$  for all  $i \leq n$ . By our inductive hypothesis,  $\bar{\xi}_i x_j$  maps to  $\bar{\xi}_i x_j + \bar{\xi}_{n+1}$  up to elements in lower Bökstedt filtration for all j > i. Now,  $\bar{\xi}_i x_i$  maps to  $\bar{\xi}_i x_i + z$  where either  $z = \bar{\xi}_{n+1}$  or

$$z \in \{\bar{\xi}_n x_n, \bar{\xi}_{n-1} x_{n-1}, \dots, \bar{\xi}_{i+1} x_{i+1}\}$$

up to  $\sigma$ -cycles in  $H_*(THH(y(n)))$ . If the former holds, then we are done. If the latter holds, then we can add z to the source and we know that  $\bar{\xi}_i x_i + z$  maps to  $\bar{\xi}_i x_i + \bar{\xi}_{n+1}$  by the inductive hypothesis. Thus we have proven that (5) holds.

We now turn to (6). For  $y \in H_*(y(n))$  it is clear that  $(\phi_n)_*(y) = y$  because all such y are in Bökstedt filtration zero. For  $y \in E(\sigma \xi_1, \sigma \xi_2, \ldots, \sigma \xi_n)$ , we know that after a possible change of basis, each of these elements y is a comodule primitive in  $H_*(THH(y(n)))$  and therefore each such y maps to the element of the same name in  $H_*(THH(H\mathbb{F}_2))$ .

We now prove the last sentence of the proposition. We first begin with products of the form  $y\bar{\xi}_i x_i$ .

Case 1: If y is a  $\sigma$ -cycle, then

(7) 
$$(\phi_n)_*(y\bar{\xi}_i x_i) = y\bar{\xi}_i x_i + y\bar{\xi}_{n+1}$$

by the same proof as the one given for  $\bar{\xi}_i x_i$ . Note that all  $\sigma$ -cycles y are in even degree so  $y\bar{\xi}_1 x_i$  must be in an odd degree. We therefore know that formula (7) holds.

Case 2: Suppose  $y \in H_*(THH(y(n)))$  is not a  $\sigma$ -cycle and therefore  $\sigma(y\bar{\xi}_i x_i) = \sigma(y)\bar{\xi}_i x_i$  in  $H_*(THH(y(n)))$ . Since  $\sigma(y\bar{\xi}_i x_i) = \sigma(y)\bar{\xi}_i x_i + y\sigma\bar{\xi}_{n+1}$  in  $H_*(THH(H\mathbb{F}_2))$ , we must add a correcting term z in the target such that  $\sigma(z) = y\sigma\bar{\xi}_{n+1}$ . The only possibility is an element of the form  $y\bar{\xi}_{n+1}+c$  where c is a  $\sigma$ -cycle.

We note that this seems to add additional terms that cause a further discrepancy since  $\sigma(y\bar{\xi}_{n+1}) = \sigma(y)\bar{\xi}_{n+1} + y\sigma\bar{\xi}_{n+1}$ , but this extra term is accounted for since  $\sigma(y)\bar{\xi}_i x_i$  is also of the form we are currently handling in Case 2 and thus  $\sigma(y)\bar{\xi}_i x_i$  maps to  $\sigma(y)\bar{\xi}_i x_i + \sigma(y)\bar{\xi}_n$ . Thus, we also have that  $y\bar{\xi}_i x_i$  maps to  $y\bar{\xi}_i x_i + y\bar{\xi}_n + c$  where c is a  $\sigma$ -cycle. We can restrict to  $c \in J_n$  by changing the element in the source of this map by a change of basis where we add on terms in lower Bökstedt filtration.

This covers all products that are divisible by  $\bar{\xi}_i x_i$ . Let  $y \in H_*(y(n))$  and let

$$w \in E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \dots \sigma\bar{\xi}_n).$$

Case 1: Suppose y, and consequently yw, is a  $\sigma$ -cycle. Then yw maps to yw + c where c is a  $\sigma$ -cycle. By the same argument as earlier, we may assume  $c \in J_n$ . This completes this case.

Case 2: Suppose y is not a  $\sigma$ -cycle. Then  $\sigma(yw) = \sigma(y)w \neq 0$  in  $H_*(THH(y(n)))$ , but  $\sigma(yw) = \sigma(y)w$  in  $H_*(THH(H\mathbb{F}_2))$ . Therefore, if yw maps to the class of the same name, then the map is well defined as a map of  $E(\sigma)$ -modules. Therefore we may only possibly add  $\sigma$ -cycles to yw in the target, and as above, we only need to consider complementary  $\sigma$ -cycles.

We now note that  $(\phi_n)_*$  is also a map of  $E(\sigma)$ -modules in  $\mathcal{A}_*$ -comodules. Since  $(\phi_n)_*$  is exotic in some cases, there is an exotic  $\mathcal{A}_*$ -coaction on some elements in  $H_*(THH(y(n)))$ .

**Corollary 3.9.** The  $\mathcal{E}_*$ -coaction on  $H_*(THH(y(n)))$  is determined by the formula

(8) 
$$\nu_n(\bar{\xi}_i x_i) = 1 \otimes \bar{\xi}_i x_i + \sum_{0 < j+k=i} \bar{\xi}_j \otimes \bar{\xi}_k^{2^j} x_i + \bar{\xi}_i \otimes x_i + \sum_{j+k=n+1} \bar{\xi}_j \otimes \bar{\xi}_k^{2^{n+j}}$$

for  $1 \leq i \leq n$  the usual coaction on  $H_*(y(n))$  and primitivity of the coaction on  $E(\sigma \bar{\xi}_1, \dots, \sigma \bar{\xi}_n)$ .

# 4. Topological periodic cyclic homology of y(n)

In many classical trace methods computations, topological periodic cyclic homology is understood using the homotopical Tate spectral sequence described by Greenlees-May [20]. In Section 4.1, we explain why this method of understanding TP(R) is not tractable when R = y(n) for  $n < \infty$ . In Section 4.2, we apply an alternative approach to understanding TP(R) inspired by foundational work of Bruner and Rognes [14], the *homological* Tate spectral sequence. We analyze this spectral sequence to compute the *continuous* homology  $H^*_*(TP(y(n)))$  in Proposition 4.5.

In Section 4.3, we use the localized Adams spectral sequence to compute the Morava Ktheory of certain truncations of TP(y(n)). In Section 4.4, we show that the continuous Morava K-theory  $K(m)^c_*(TP(y(n)))$  vanishes for  $1 \le m \le n$ .

4.1. Limitations of the homotopical Tate spectral sequence. Let R be an  $E_1$  ring spectrum. The topological periodic cyclic homology spectrum TP(R) arises in many classical trace methods computations. For example, when p is an odd prime, the spectrum  $TP(H\mathbb{F}_p)$  appears in Hesselholt and Madsen's computation of the algebraic K-theory of finite algebras over the Witt vectors of perfect fields [22]. Similarly, it plays an important role in the computation of  $TC(\mathbb{Z}_2; \mathbb{Z}/2)$  by Rognes [40]. In both cases, they analyze the mod p homotopical Tate spectral sequence

$$\widehat{E}^2 = \widehat{H}^{-*}(\mathbb{T}; \pi_*(THH(R)); \mathbb{Z}/p)) \Rightarrow \pi_*(TP(R); \mathbb{Z}/p)$$

defined in [20]. We will review the filtration used to define this spectral sequence when we define the homological Tate spectral sequence in Subsection 4.2.

When  $R = y(\infty) = H\mathbb{F}_2$ , this spectral sequence is fairly simple. By Böksedt periodicity [12],  $\pi_*(THH(H\mathbb{F}_2)) \cong P(u)$  with |u| = 2, so one has a familiar checkerboard pattern on the  $E^2$ -page and the spectral sequence collapses. On the other hand, when R = y(n) for  $n < \infty$ , this spectral sequence appears to be intractable.

**Example 4.1.** We have y(0) = S and  $THH(S) \simeq S$  as T-spectra. There is an equivalence of spectra

$$TP(S) \simeq \Sigma^2 \mathbb{C} P^{\infty}_{-\infty}$$

by [20, Thm. 16.1]. The homotopy groups of  $\mathbb{C}P_{-\infty}^{\infty}$  are less well understood than the homotopy groups of spheres.

Moreover, [11, Thm. 1] implies that

$$THH(y(n)) \simeq Th(L^{\eta}(Bf))$$

where  $f: \Omega J_{2^n-1}(S^2) \to BGL_1S$  is the map defining y(n) as Th(f) = y(n) and  $Th(L^{\eta}(Bf))$  is the Thom spectrum of the composite map  $L^{\eta}(Bf)$  defined as

$$LB\Omega J_{2^n-1}(S^2) \xrightarrow{L(Bf)} LB^2 F \simeq BGL_1 S \times B^2 GL_1 S \xrightarrow{BGL_1 S \times \eta} BG_1 S \times BGL_1 S \to BGL_1 S.$$

This spectrum has homotopy groups at least as complicated as  $\pi_*(y(n))$ , which are only known in a finite range. Since we want to understand large-scale phenomena in these homotopy groups, we will adopt a different approach.

4.2. Homological Tate spectral sequence for THH(y(n)). In notes from a talk by Rognes [42], it is shown using the homological homotopy fixed point spectral sequence and the inverse limit Adams spectral sequence [27] that there is an isomorphism of graded abelian groups

$$\pi_*(TC^-(H\mathbb{F}_2)) \cong \prod_{i \in \mathbb{Z}} \Sigma^{2i} \mathbb{Z}_2$$

We recall this calculation further in Proposition 5.1. A similar argument shows that there is an isomorphism of graded abelian groups

$$\pi_*(TP(H\mathbb{F}_2)) \cong \prod_{i \in \mathbb{Z}} \Sigma^{2i} \mathbb{Z}_2$$

Our goal in this section is to obtain similar results for TP(y(n)) when  $n < \infty$ .

**Definition 4.2** (Homological Tate spectral sequence [14]). Let R be an  $E_1$  ring spectrum. The homological Tate spectral sequence has the form

$$\widehat{E}^2 = \widehat{H}^{-*}(\mathbb{T}; H_*(THH(R))) \Rightarrow H^c_*(TP(R)).$$

It arises from the Greenlees filtration of  $THH(R)^{t\mathbb{T}} = [F(E\mathbb{T}_+, THH(R)) \wedge \widetilde{E\mathbb{T}}]^{\mathbb{T}}$  defined for  $i \ge 0$  by setting (cf. [14, Sec. 2])

$$TP(R)[i] := [F(E \mathbb{T}_+, THH(R)) \land \widetilde{E} \mathbb{T}/\widetilde{E} \mathbb{T}_i]^{\mathbb{T}}$$

where  $\widetilde{ET}_i$  is the cofiber of the map  $ET^{(i)}_+ \to S^0$ , where  $ET^{(i)}_+$  is the *i*-th skeleton of ET, if i > nand  $\widetilde{ET}_i$  is the Spanier-Whitehead dual of  $\widetilde{ET}_{-i-1}$  if i < 0 [19, p.46]. The limit

$$H^c_*(TP(R)) := \lim_{i \to \infty} H_*(TP(R)[i])$$

is called the *continuous homology* of TP(R). For  $0 \le n \le \infty$ , we will denote the  $E^r$ -page of the homological Tate spectral sequence converging to  $H^c_*(TP(y(n)))$  by  $\widehat{E}^r(n)$ .

Lemma 4.3. There is an additive isomorphism

$$\widehat{E}^{2}(n) \cong P(t, t^{-1}) \otimes H_{*}(THH(y(n))) \cong P(t, t^{-1}) \otimes P(\bar{\xi}_{1}, \bar{\xi}_{2}, \dots, \bar{\xi}_{n}) \otimes E(\sigma\bar{\xi}_{1}, \dots, \sigma\bar{\xi}_{n})$$
  
where  $|t| = (-2, 0), |\bar{\xi}_{i}| = (0, 2^{i} - 1), and |\sigma\bar{\xi}_{i}| = (0, 2^{i}).$ 

In [14, Prop. 3.2], Bruner and Rognes show that  $d^2(x) = t \cdot \sigma(x)$  in the homological Tate spectral sequence. Therefore in order to compute  $\hat{E}^3(n)$ , we need to understand the T-action on  $H_*(THH(y(n)))$ . This can be understood using Proposition 3.8 and the relation  $(\phi_n)_*(\sigma(x)) = \sigma((\phi_n)_*(x))$  which follows from naturality of  $\sigma$ .

**Definition 4.4.** [14, Prop. 6.1.(a)] Let  $k \ge 1$ . Define  $\bar{\xi}'_{k+1} \in H_*(THH(H\mathbb{F}_2))$  by

$$\bar{\xi}_{k+1}' := \bar{\xi}_{k+1} + \bar{\xi}_k \sigma \bar{\xi}_k.$$

**Proposition 4.5.** There is an isomorphism of graded  $\mathbb{F}_2$ -vector spaces

$$H^{c}_{*}(TP(y(n))) \cong P(t,t^{-1}) \otimes P(\bar{\xi}_{1}^{2},\bar{\xi}_{2}',\ldots,\bar{\xi}_{n}') \otimes E(\bar{\xi}_{n}\sigma\bar{\xi}_{n})$$
  
with  $|t| = (-2,0), \ |\bar{\xi}_{i}| = (0,2^{i}-1), \ and \ |\sigma\bar{\xi}_{i}| = (0,2^{i}).$ 

Proof. First,  $\hat{E}_{**}^2(n)$  was computed in Lemma 4.3. The homological Tate spectral sequence is not a multiplicative spectral sequence since THH(y(n)) is not a ring spectrum, but it is a module over the spectral sequence for the sphere  $\{\hat{E}_{*,*}^r(0)\}_r$ . Consequently  $d^r(t) = 0$  and the differentials are *t*-linear. We have differentials  $d^2(\bar{\xi}_k) = t\sigma\bar{\xi}_k$  and thus  $d^2(t^m\bar{\xi}_k) = t^{m+1}\sigma\bar{\xi}_k$  for  $m \in \mathbb{Z}$  by *t*-linearity.

Recall that  $x_i = \sigma \bar{\xi}_i \cdots \sigma \bar{\xi}_n$ . Any class of the form  $y \bar{\xi}_i x_i$ , where y is a  $\sigma$ -cycle, is a  $d^2$ -cycle in the homological Tate spectral sequence converging to  $H^c_*(THH(y(n))^{t^{\mathrm{T}}})$ . Many of these classes are also  $d^2$ -homologous; in particular,

$$d^2(y\bar{\xi}_i\bar{\xi}_n\sigma\bar{\xi}_i\cdots\sigma\bar{\xi}_{n-1})) = ty\bar{\xi}_ix_i + ty\bar{\xi}_nx_n.$$

Using these relations and the fact that this spectral sequence is a module over the spectral sequence for the sphere, we obtain an additive isomorphism (cf. [14, Proposition 6.1])

$$\widehat{E}^3_{**}(n) \cong P(t,t^{-1}) \otimes P(\bar{\xi}^2_1,\ldots,\bar{\xi}^2_n) \otimes P(\bar{\xi}'_2,\bar{\xi}'_3,\ldots,\bar{\xi}'_n) \otimes E(\bar{\xi}_n\sigma\bar{\xi}_n).$$

To see that there are no further differentials, we use the map of spectral sequences induced by the  $\mathbb{T}$ -equivariant map  $THH(y(n)) \rightarrow THH(H\mathbb{F}_2)$ . The homological Tate spectral sequence converging to  $H^c_*(THH(H\mathbb{F}_2)^{t\mathbb{T}})$  has  $\widehat{E}^3$ -page

$$\widehat{E}^3_{**}(\infty) \cong P(t, t^{-1}) \otimes P(\bar{\xi}^2_1, \bar{\xi}^2_2, \ldots) \otimes E(\bar{\xi}'_2, \bar{\xi}'_3, \ldots).$$

All of the generators are permanent cycles by [14, Thm. 5.1], so there are no further differentials. The map  $\widehat{E}^3_{*,*}(n) \to \widehat{E}^3_{*,*}(\infty)$  is injective by Proposition 3.8 so we can conclude that there is also an isomorphism  $\widehat{E}^3(n) \cong \widehat{E}^\infty(n)$ .

A similar proof can be used to compute the homology of the spectra TP(y(n))[i] which were used to define the filtration of TP(y(n)) giving rise to the homological Tate spectral sequence. Indeed, one may truncate the homological Tate spectral sequence to obtain a spectral sequence which converges strongly to  $H_*(TP(y(n))[i])$ .

Notation 4.6. When computing the truncated homological Tate spectral sequence or the truncated homological homotopy fixed point spectral sequence, we denote the left-most column (using Serre grading) by

$$V(i) := \left( H_*(THH(y(n)) / \operatorname{im}(d_2^{2i-2,*}) \right) \{ t^i \}$$

where the integer n is understood from the context.

We study V(i) further in Corollary 4.15. See Example 5.10 where V(i) is calculated for n = 1.

**Corollary 4.7.** There is an isomorphism of graded  $\mathbb{F}_2$ -vector spaces

$$H_*(TP(y(n))[i]) \cong \left[P(t^{-1})\{t^{i-1}\} \otimes P(\bar{\xi}_1^2, \bar{\xi}_2', \dots, \bar{\xi}_n') \otimes E(\bar{\xi}_n \sigma \bar{\xi}_n)\right] \oplus V(i)$$

with |t| = (-2,0),  $|\bar{\xi}_i| = (0,2^i-1)$ ,  $|\sigma\bar{\xi}_i| = (0,2^i)$ , and  $P(t^{-1})\{t^{i-1}\}$  is viewed as a  $P(t^{-1})$ -submodule of  $P(t,t^{-1})$ .

If  $X = \lim_{i} X_i$  is the homotopy limit of bounded below spectra  $X_i$  of finite type, then the inverse limit Adams spectral sequence

$$E_2^{*,*} = \operatorname{Ext}_{\mathcal{A}_*}^{**}(\mathbb{F}_2, H_*^c(X)) \Rightarrow \pi_*(X)$$

arises from the filtration of X obtained by taking the inverse limit of compatible Adams filtrations of the spectra  $X_i$ , where the left-hand side is computed using the continuous  $\mathcal{A}_*$ -coaction on  $H^c_*(X)$ . For details, see [28, Sec. 2]. Taking X = TP(y(n)) gives a method for calculating  $\pi_*(TP(y(n)))$ . In view of Rognes' computation of  $\pi_*(TC^-(H\mathbb{F}_2))$  [42], one might suspect that the inverse limit Adams spectral sequence could be used to compute the homotopy groups  $\pi_*(TP(y(n)))$  directly. However, this approach is significantly less tractable for  $n < \infty$  since  $\mathcal{A}_*$  coacts nontrivially on  $P(t, t^{-1}) \subset H^c_*(TP(y(n)))$ . This problem is avoided when  $n = \infty$  as follows. There is an  $\mathcal{A}_*$ comodule isomorphism

$$H^c_*(TP(H\mathbb{F}_2)) \cong P(t,t^{-1}) \otimes H_*(H\mathbb{Z}_2) \cong P(t,t^{-1}) \otimes (\mathcal{A}//E(0))_*$$

A change-of-rings isomorphism then gives

$$E_2^{*,*} \cong Ext_{E(\bar{\xi}_1)}^{*,*}(\mathbb{F}_2, P(t, t^{-1})).$$

Since  $\bar{\xi}_1$  is in an odd degree and  $P(t, t^{-1})$  is concentrated in even degrees, the  $E(\bar{\xi}_1)$ -coaction on  $P(t, t^{-1})$  is trivial. Therefore

$$E_2^{*,*} \cong P(t,t^{-1}) \otimes Ext_{E(\bar{\xi}_1)}^{*,*}(\mathbb{F}_2,\mathbb{F}_2)$$

and the spectral sequence collapses for degree reasons. The key simplification in the sequel is that we can replace the functor  $\operatorname{Ext}_{\mathcal{A}_*}^{*,*}(\mathbb{F}_2,-)$  by the functor  $\operatorname{Ext}_{E(\bar{\xi}_n)}^{*,*}(\mathbb{F}_2,-)$  if we compute connective Morava K-theory instead of stable homotopy because of the change of rings isomorphism.

4.3. Morava K-theory of truncations of TP. To determine the chromatic complexity of TP(y(n)), we need to compute  $K(m)_*(TP(y(n)))$ .

**Lemma 4.8.** The  $A_*$ -coaction (resp. continuous  $A_*$ -coaction)

 $\nu_n: H_*(TP(y(n))[i]) \to \mathcal{A}_* \otimes H_*(TP(y(n))[i]) \quad (resp. \ \nu_n: H^c_*(TP(y(n))) \to \mathcal{A}_* \widehat{\otimes} H^c_*(TP(y(n))))$ satisfies

 $(\phi_n)_*(\nu_n(x)) = \nu_{\infty}((\phi_n)_*(x))$ 

where  $(\phi_n)_*$  is the map in homology (resp. continuous homology) induced by  $\phi_n : TP(y(n))[i] \to TP(H\mathbb{F}_2)[i]$  (resp.  $\phi_n : TP(y(n)) \to TP(H\mathbb{F}_2)$ ) and  $\nu_\infty$  is the  $\mathcal{A}_*$ -coaction (resp. continuous  $\mathcal{A}_*$ -coaction) on  $H_*(TP(H\mathbb{F}_2)[i])$  (resp.  $H^*_*(TP(H\mathbb{F}_2)))$ .

We can identify a piece of the continuous homology  $H^*_*(TP(y(n)))$  computed in Proposition 4.5 with the homology  $H_*(z(n)/v_n)$ . Our identification may not hold as  $\mathcal{A}_*$ -comodules due to the possibility of complementary  $\sigma$ -cycles in the image of  $(\phi_n)_*$ , but if we restrict to a smaller sub-Hopfalgebra of  $\mathcal{A}_*$ , it does. Recall the definition of  $\mathcal{E}_*$  from Section 1.2.

**Proposition 4.9.** There is an isomorphism of continuous  $\mathcal{E}_*$ -comodules

$$H^c_*(TP(y(n))) \cong P(t^{\pm 1}) \otimes H_*(z(n)/v_n).$$

*Proof.* First, note that the  $\mathcal{E}_*$ -coaction on  $P(t^{\pm 1})$  is trivial because |t| = -2 and each  $\overline{\xi}_i$  is in an odd degree for all  $i \ge 0$ . The desired isomorphism is therefore given by a map

$$P(\bar{\xi}_1^2,\ldots,\bar{\xi}_n^2)\otimes E(\bar{\xi}_2',\ldots,\bar{\xi}_n')\otimes E(\bar{\xi}_n\sigma\bar{\xi}_n)\to H_*(z(n)/v_n)$$

which is determined by

$$\bar{\xi}_i^2 \mapsto \bar{\xi}_i^2, 1 \le i \le n, \quad \bar{\xi}_i' \mapsto \bar{\xi}_i, 2 \le i \le n, \quad \bar{\xi}_n \sigma \bar{\xi}_n \mapsto \bar{\xi}_{n+1}$$

This is clearly an additive isomorphism, so it only remains to calculate the action of  $Q_i$ .

By Lemma 4.8, we can compute the continuous  $\mathcal{A}_*$ -coaction

 $\nu_n \colon H^c_*(TP(y(n)) \to \mathcal{A}_* \widehat{\otimes} H^c_*(TP(y(n)))$ 

by comparison with the known coaction

$$\nu_{\infty} \colon H^c_*(TP(H\mathbb{F}_2)) \to \mathcal{A}_* \widehat{\otimes} H^c_*(TP(H\mathbb{F}_2)).$$

We have  $(\phi_n)_*(x) = x$  for  $x \in P(\xi_1^2, \ldots, \xi_n^2)$  since  $\operatorname{bflt}(x) = 0$ , so  $\nu_n(x) = \nu_\infty(x)$ . We then calculate  $\nu_n(\bar{\xi}'_i)$  as follows. We have  $(\phi_n)_*(\bar{\xi}'_i) = \bar{\xi}'_i + y$  where y is a  $\sigma$ -cycle with  $\operatorname{bflt}(y) = 0$ . If  $y \notin J_n$ , then we have  $\nu_n(\bar{\xi}'_i) = \nu_\infty(\bar{\xi}'_i)$  up to the addition of elements in lower Bökstedt filtration. If  $y \in J_n$ , then y is divisible by  $\bar{\xi}_{n+r}^2$  or  $\sigma \bar{\xi}_{n+r}$  for  $r \ge 1$ . In both cases,  $\nu_\infty(y)$  does not contain any terms of the form  $\bar{\xi}_i \otimes z$  for any i, so  $Q_i$  acts trivially on y for all i and the action of  $Q_i$  on  $\bar{\xi}'_i$  is unaffected by y. Finally, we have  $(\phi_n)_*(\bar{\xi}_n\sigma\bar{\xi}_n) = \bar{\xi}_n\sigma\bar{\xi}_n + \bar{\xi}_{n+1}$ . Therefore

$$\nu_n(\bar{\xi}_n\sigma\xi_n) = \nu_\infty(\bar{\xi}_n\sigma\bar{\xi}_n) + \sum_{i=1}^{n+1}\bar{\xi}_i\otimes\bar{\xi}_{n+1-i}^{2^i}$$

so  $Q_i(\bar{\xi}_n \sigma \bar{\xi}_n) = Q_i(\bar{\xi}'_{n+1})$  for all i.

**Corollary 4.10.** For any  $m \ge 0$ , there is an isomorphism

$$Ext^{*,*}_{E(\bar{\xi}_{m+1})}(\mathbb{F}_{2},H^{c}_{*}TP(y(n))) \cong Ext^{*,*}_{E(\bar{\xi}_{m+1})}(\mathbb{F}_{2},H^{c}_{*}((z(n)/v_{n})^{t\,\mathbb{T}})$$

between the  $E_2$ -page of the inverse limit Adams spectral sequence converging to  $k(m)^c_*(TP(y(n)))$ and the  $E_2$ -page of the inverse limit Adams spectral sequence converging to  $k(m)^c_*((z(n)/v_n)^{t T}))$ .

One may ask if this isomorphism of  $E_2$ -pages is, in fact, an isomorphism of spectral sequences. Note that  $z(n)/2 \simeq y(n)$ . When n = 0 we see that  $z(0)/v_0 = y(0) = S$  and there is an equivalence

$$TP(S) \simeq S^{t^{\mathbb{T}}}.$$

On the other hand, when  $n = \infty$  we use the convention that  $z(\infty)/v_{\infty} = H\mathbb{Z}$  and then there is an equivalence

$$(H\mathbb{Z})_2^{t\,\mathbb{T}} \simeq TP(H\mathbb{F}_2).$$

**Conjecture 4.11.** There is an equivalence  $(z(n)/v_n)^{t^{\mathbb{T}}} \simeq TP(y(n))$  after 2-completion.

**Remark 4.12.** To prove this conjecture, we would need to construct a  $\mathbb{T}$ -equivariant map  $z(n)/v_n \to THH(y(n))$  inducing a map

$$(z(n)/v_n)_2^{t^*} \to TP(y(n))$$

and consequently a map of inverse limit Adams spectral sequences. Then the isomorphism on  $E_2$ -pages gives an isomorphism of  $E_{\infty}$ -pages, and after resolving extensions, the result would follow.

In the case  $n = \infty$ , it is known that  $K(H\mathbb{F}_p)_p \simeq H\mathbb{Z}_p$  and therefore there is a map of  $\mathbb{T}$ equivariant  $E_{\infty}$  ring spectra  $H\mathbb{Z}_p \to THH(\mathbb{F}_p)$  [37, Cor. IV.4.13]. The case n = 0 is trivial because there is a  $\mathbb{T}$ -equivariant equivalence  $THH(S) \simeq S$  induced by tensoring the sphere spectrum with the collapse map from  $\mathbb{T}$  to a point.

By essentially the same proof as that of Proposition 4.9, we have the following corollary.

**Corollary 4.13.** There is an isomorphism of  $\mathcal{E}_*$ -comodules

$$H_*(TP(y(n))[i]) \cong H_*(z(n)/v_n) \otimes P(t^{-1})\{t^{i-1}\} \oplus V(i).$$

4.4. Chromatic complexity of TP(y(n)). We now compute the Margolis homology of TP(y(n))[i] using the computations from Section 2.

**Lemma 4.14.** The Margolis homology of TP(y(n))[i] is given by

$$H(TP(y(n))[i];Q_m) \cong \begin{cases} P(t^{-1})\{t^{i+1}\} \oplus H(V(i);Q_0) & \text{if } m = 0, \\ H(V(i);Q_m) & \text{if } 1 \le m \le n, \\ H_*(TP(y(n))[i]) & \text{if } m \ge n+1. \end{cases}$$

*Proof.* We have an isomorphism of  $\mathcal{E}_*$ -comodules

$$H_*(TP(y(n))[i]) \cong P(t^{-1})\{t^i\} \otimes H_*(z(n)/v_n) \oplus V(i)$$

by Proposition 4.9, so we can compute  $H(TP(y(n))[i];Q_i)$  in terms of  $H(z(n)/v_n;Q_i)$  and V(i) for all *i*. We have  $Q_n(t^i) = 0$  for all  $i \in \mathbb{Z}$  for degree reasons. Therefore  $H(P(t,t^{-1});Q_m) \cong P(t,t^{-1})$ for all  $m \ge 0$ . The Künneth isomorphism and Corollary 2.21 then prove the lemma for  $m \le n$ . When  $m \ge n+1$ , the proof is similar to the proof in Corollary 2.21.

**Theorem 4.15.** For  $1 \le m \le n < \infty$ , there are isomorphisms

$$\lim H(H_*(TP(y(n))[i]); Q_m) \cong 0$$

and the continuous Morava K-theory of TP(y(n)) vanishes, i.e.

$$K(m)^c_*(TP(y(n))) = 0.$$

*Proof.* Let  $1 \le m \le n$ . Since  $H_*(TP(y(n))[i])$  and therefore  $H^*(TP(y(n))[i])$  is bounded below, we may apply Lemma 2.14 to see that the localized Adams spectral sequence

$$E_2 = v_m^{-1} \operatorname{Ext}_{E(Q_m)_*}^{**}(\mathbb{F}_2, H_*(TP(y(n))[i])) \Rightarrow K(m)_*(TP(y(n))[i])$$

converges strongly. Since  $H^*TP(y(n))[i]$  is finite type, we can apply Lemma 4.14 and observe that the  $E_2$ -page has the form

$$v_m^{-1} \operatorname{Ext}_{E(Q_m)}^{**}(\mathbb{F}_2, H_*(TP(y(n))[i])) = H(V(i); Q_n) \otimes \mathbb{F}_2[v_m^{\pm 1}].$$

We claim that the map

$$K(m)_*(TP(y(n))[i]) \to K(m)_*(TP(y(n))[i-1])$$

is zero for all  $i \in \mathbb{Z}$ . To prove the claim, we show that the map is zero on the  $E_2$ -page of the localized Adams spectral sequence. It suffices to prove that no elements in the summand  $H_*(V(i-1); Q_m)$ are in the image of the map  $H(TP(y(n))[i]; Q_m) \to H(TP(y(n))[i-1]; Q_m)$  for  $1 \le m \le n$ . To see this, we note that the map

$$H_*(TP(y(n))[i]) \rightarrow H_*(TP(y(n))[i-1])$$

is induced by the map of truncated Tate spectral sequences which is defined on the (i-1)-st column (using Serre grading) by  $H_*(z(n)/v_n)\{t^{i-1}\} \to V(i-1)$ . However, on Margolis homology this induces the zero map

$$0 = H(H_*(z(n)/v_n)\{t^{i-1}\}; Q_m) \to H(V(i-1); Q_m)$$

for  $1 \leq m \leq n$ . Therefore, the induced map on homotopy groups

$$K(m)_*TP(y(n))[i] \to K(m)_*TP(y(n))[i-1]$$

is trivial.

4.5. Chromatic complexity of relative topological periodic cyclic homology. In this subsection we will compute the continuous Morava K-theory of relative topological periodic cyclic homology. In the proof of the previous theorem, we computed  $K(m)_*(TP(y(n))[i])$  for  $0 \le m \le n+1$ . We will need the following specialization to the case  $n = \infty$ .

**Lemma 4.16.** For  $1 \le m < \infty$ , there is an isomorphism

$$\lim_{i} K(m)_*(TP(H\mathbb{F}_2)[i]) \cong 0.$$

*Proof.* We saw above that

$$H_*(TP(H\mathbb{F}_2)[i]) \cong H_*(H\mathbb{Z})\{\dots, t^{-2}, t^{-1}, 1, t, \dots, t^{i-1}\} \oplus V(i)\{t^i\}.$$

Applying Lemma 2.18 with  $n = \infty$  shows that the Margolis homology of  $H\mathbb{Z}$  is given by

$$H(H\mathbb{Z};Q_m) \cong \begin{cases} \mathbb{F}_2 & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases}$$

As above,  $Q_m$  acts trivially on  $t^i$  for all i, so we have

$$H(TP(H\mathbb{F}_2)[i]; Q_m) \cong \begin{cases} P(t^{-1})\{t^{i+1}\} \oplus H(V(i); Q_0) & \text{if } m = 0, \\ H(V(i); Q_m) & \text{if } m > 0. \end{cases}$$

Since  $TP(H\mathbb{F}_2)[i]$  is bounded below, Lemma 2.14 implies convergence of the localized Adams spectral sequence computing  $K(m)_*(TP(H\mathbb{F}_2)[i])$  for m > 0. The remainder of the proof of the lemma proceeds exactly as in Theorem 4.15.

**Theorem 4.17.** For  $1 \le m \le n$ , the m-th continuous Morava K-theory of  $TP(y(n), H\mathbb{F}_2)$  vanishes, *i.e.* there is an isomorphism

$$K(m)^c_*(TP(y(n), H\mathbb{F}_2)) \cong 0.$$

*Proof.* Clearly we have  $TP(y(n), H\mathbb{F}_2) = \lim_i TP(y(n), H\mathbb{F}_2)[i]$ . For each *i* we obtain a long exact sequence in Morava K-theory

which we use to compute relative continuous topological periodic cyclic homology.

When  $1 \le m \le n$ , we have shown

$$\lim K(m)_*(TP(y(n))[i]) \cong 0 \cong \lim K(m)_*(TP(H\mathbb{F}_2)[i])$$

so we see that

$$K(m)^c_*(TP(y(n), H\mathbb{F}_2)[i]) \cong 0.$$

## 5. Topological negative cyclic homology of y(n)

In this section, we mimic the analysis from Section 4 in order to calculate the (continuous) Morava K-theory of the topological negative cyclic homology of y(n). We analyze the homological homotopy fixed point spectral sequence in Section 5.1 and use the results to calculate Margolis homology and continuous Morava K-theory in Section 5.2. The main result is Theorem 5.8.

5.1. Homological T-homotopy fixed point spectral sequence for THH(y(n)). We now analyze the homological homotopy fixed point spectral sequence converging to the continuous homology of topological negative cyclic homology of y(n). This spectral sequence has the form

$$E^{2}(n) := H^{-*}(\mathbb{T}; H_{*}(THH(y(n)))) \Rightarrow H^{c}_{*}(TC^{-}(y(n)))$$

where

$$H^{c}_{*}(TC^{-}(y(n))) = \lim_{i} H_{*}TC^{-}(y(n))[i]$$

and  $TC^{-}(y(n))[i] := F(E \mathbb{T}^{(i)}_{+}, THH(y(n)))^{\mathbb{T}}$  so that  $\lim_{n \to \infty} TC^{-}(y(n))[i] = TC^{-}(y(n)).$ 

We will first discuss the case  $n = \infty$ . This computation is entirely contained in [42], but we review it here and fill in some details for later use.

**Proposition 5.1.** [42] There is an equivalence

$$TC^{-}(H\mathbb{F}_p)_p \simeq \prod_{i\in\mathbb{Z}} \Sigma^{2i} H\mathbb{Z}_p$$

In [42], Rognes proves that  $TC^-_*(H\mathbb{F}_p) \cong \prod_{i \in \mathbb{Z}} \Sigma^{2i} \mathbb{Z}_p$ . The theorem follows from the fact that  $TC^{-}(H\mathbb{F}_p)_p$  is a commutative  $K(\mathbb{F}_p)_p$ -algebra where  $K(\mathbb{F}_p)_p \simeq \mathbb{Z}_p$ . We will just prove the case p = 2, but the case p odd is the same up to a change in notation.

Proof for p=2. The input of the homological homotopy fixed point spectral sequence  $E^2(\infty)$  is isomorphic to

$$P(t) \otimes \mathcal{A}_* \otimes P(\sigma \overline{\xi}_1).$$

As in the homological Tate spectral sequence, the differentials are t-linear and are determined by those of the form  $d^2(x) = t\sigma x$  given by [14, Prop. 3.2]. Since  $\eta$  is trivial in  $THH_*(H\mathbb{F}_2)$ ,  $\sigma$  is a derivation and therefore the only nontrivial differentials are  $d^2(\bar{\xi}_i) = t\sigma\bar{\xi}_i$ . Recall that  $(\sigma\xi_1)^{2^i} =$  $\sigma \bar{\xi}_{i+1}$ . Then  $E^3(\infty)$  is isomorphic to

 $P(t) \otimes P(\bar{\xi}_1^2, \bar{\xi}_2', \dots) \oplus P(\bar{\xi}_1^2, \bar{\xi}_2', \dots) \{ (\sigma \xi_1)^k | k \ge 1 \}$ 

and the spectral sequence then collapses by [14, Thm. 5.1].

There is an isomorphism of  $\mathcal{A}_*$ -comodules

$$P(t) \otimes P(\bar{\xi}_1^2, \bar{\xi}_2', \dots) \oplus P(\bar{\xi}_1^2, \bar{\xi}_2', \dots) \{ (\sigma\xi_1)^k | k \ge 1 \} \cong H_*(\mathbb{Z}) \otimes P(t) \oplus H_*(\mathbb{Z}) \{ (\sigma\xi_1)^k | k \ge 1 \}.$$

The isomorphism sends  $\bar{\xi}'_i$  to  $\bar{\xi}_i$  for  $i \geq 2$  and so in order for the coaction to be consistent, it must send  $\bar{\xi}_1^{2j}$  to  $\bar{\xi}_1^{2j} + (\sigma \bar{\xi}_1)^j$ . Similarly,  $\bar{\xi}_n^{2j}$  must map to  $\bar{\xi}_n^{2j} + (\sigma \bar{\xi}_1)^{j(2^{n-1}-1)}$  for all  $1 \leq n \leq \infty$ . The inverse limit Adams spectral sequence then has  $E^2$ -page

$$\operatorname{Ext}_{\mathcal{A}_{*}}^{*,*}(\mathbb{F}_{2}, H_{*}(\mathbb{Z}) \otimes P(t) \oplus H_{*}(\mathbb{Z})\{(\sigma\xi_{1})^{k} | k \geq 1\}) \cong \operatorname{Ext}_{E(Q_{0})_{*}}^{*,*}(\mathbb{F}_{2}, P(t) \oplus \mathbb{F}_{2}\{\{(\sigma\xi_{1})^{k} | k \geq 1\}).$$

Since t and  $\sigma \bar{\xi}_1$  are in even degrees, the  $Q_0$ -action is trivial and we see that this  $E^2$ -page is isomorphic to

$$P(v_0) \otimes P(t) \oplus \mathbb{F}_2\{(\sigma \xi_1)^k | k \ge 1\}$$

The usual calculation for resolving extensions in this spectral sequence produces an isomorphism

$$\pi_*(TC^-(H\mathbb{F}_2)) \simeq \prod_{i\mathbb{Z}} \pi_*(\Sigma^{2i}\mathbb{Z}_2)$$

and the the result follows by the remarks before this proof.

**Remark 5.2.** The case n = 0 is not hard to compute. The map  $THH(S) \to S$  induced by collapsing  $\mathbb{T}$  to a point is an  $\mathbb{T}$ -equivariant equivalence and consequently the  $\mathbb{T}$ -action on  $THH(S) \simeq S$  is trivial. This implies that

$$TC^{-}(S) \simeq F(\mathbb{C}P^{\infty}_{+}, S)$$

We thus expect  $TC^{-}(y(n))$  to interpolate between the Spanier-Whitehead dual of  $\mathbb{C}P^{\infty}_{+}$  and  $\prod_{i\in\mathbb{Z}}\Sigma^{2i}\mathbb{Z}$ , and indeed, our computations are consistent with this expectation.

We now discuss the case  $0 < n < \infty$ . The key difference between the T-Tate and T-homotopy fixed point spectral sequences is the presence of  $t^{-1}$ , which greatly simplified  $\widehat{E}^3(n)$  in Section 4.4.

**Proposition 5.3.** There is an isomorphism of graded  $\mathbb{F}_2$ -vector spaces

$$H^c_*(TC^-(y(n))) \cong P(\bar{\xi}_1^2, \bar{\xi}_2', \dots, \bar{\xi}_n') \otimes E(\bar{\xi}_n \sigma \bar{\xi}_n) \otimes P(t) \oplus T$$

where T is the simple t-torsion module  $T := P(\bar{\xi}_1^2, \bar{\xi}_2', \dots, \bar{\xi}_n') \otimes \mathbb{F}_2 \left\{ \prod_{i=1}^n (\sigma \bar{\xi}_i)^{\epsilon_i} : \epsilon_i \in \{0, 1\}, \sum \epsilon_i \ge 1 \right\}$ where  $|t| = (-2, 0), \ |\bar{\xi}_i| = (0, 2^i - 1), \ and \ |\sigma \bar{\xi}_i| = (0, 2^i).$ 

*Proof.* The homological homotopy fixed point spectral sequence has  $E^2$ -term

$$E^{2}(n) = H^{-*}(\mathbb{T}; H_{*}(THH(y(n)))) \cong P(t) \otimes P(\bar{\xi}_{1}, \dots, \bar{\xi}_{n}) \otimes E(\sigma\bar{\xi}_{1}, \dots, \sigma\bar{\xi}_{n})$$

where |t| = (-2,0),  $|\bar{\xi}_i| = (0,2^i-1)$  and  $|\sigma\bar{\xi}_i| = (0,2^i)$ . As in the Tate case,  $E^2(n)$  is a module over  $E^2(0) \cong P(t)$ . Therefore  $d^r(t) = 0$  for all  $r \ge 1$  and all differentials are t-linear.

The  $d^2$ -differentials in the homological homotopy fixed point spectral sequence are of the form  $d^2(x) = t\sigma x$  by [14, Prop. 3.2], and  $\sigma$  acts as a derivation symbolically as in the Tate case. We therefore obtain an additive isomorphism

$$\ker d^2 \cong P(t) \otimes P(\bar{\xi}_1^2, \bar{\xi}_2', \dots, \bar{\xi}_n') \oplus T$$

where  $T := P(\overline{\xi}_1^2, \overline{\xi}_2', \dots, \overline{\xi}_n') \otimes T_0$  and

(10) 
$$T_0 = \mathbb{F}_2\{\prod_{i=1}^n \sigma \bar{\xi}_i^{\epsilon_i} : \epsilon_i \in \{0,1\}, \sum_i \epsilon_i \ge 1\}$$

We therefore just need to compute  $\operatorname{im} d^2 \subset \ker d^2$  to identify  $E^3(n)$ . First, note that  $t\sigma \overline{\xi_i}$  is in  $\operatorname{im} d^2$  for all  $1 \leq i \leq n$  since  $d^2(\overline{\xi_i}) = t\sigma \overline{\xi_i}$ . Also, no element of the form  $x \in P(t) \otimes P(\overline{\xi_1^2}, \ldots, \overline{\xi_n^2}) \otimes P(\overline{\xi_2'}, \ldots, \overline{\xi_n'})$  is in  $\operatorname{im} d^2$ . Finally, observe that  $t\overline{\xi_i}x_i + t\overline{\xi_j}x_j \in \operatorname{im} d^2$  for  $1 \leq i < j \leq n$  as in the proof of Proposition 4.5. We conclude that

$$\operatorname{im} d^2 = \left[ P(t) \otimes E(\sigma \bar{\xi}_1, \dots, \sigma \bar{\xi}_n) \otimes \mathbb{F}_2 \{ y \cdot \bar{\xi}_i x_i + y \bar{\xi}_j x_j : 1 \le i < j \le n \text{ and } y \in \ker d^2 \} \right] \{ t \}.$$

Thus, up to a change of basis,  $t\bar{\xi}_n x_n$  survives to  $E^3(n)$ . We can therefore identify the  $E^3$ -page as

$$E^3_{**}(n) \cong P(\bar{\xi}^2_1, \bar{\xi}'_2, \dots, \bar{\xi}'_n) \otimes E(\bar{\xi}_n x_n) \otimes P(t) \oplus T$$

with  $T := P(\overline{\xi}_1^2, \overline{\xi}_2', \dots, \overline{\xi}_n') \otimes T_0$  and  $T_0$  defined in (10).

To see that there are no further differentials, we use the map of homological T-homotopy fixed point spectral sequences induced by the T-equivariant map  $THH(y(n)) \rightarrow THH(H\mathbb{F}_2)$ . The homological homotopy fixed point spectral sequence converging to  $H^c_*(TC^-(H\mathbb{F}_2))$  has  $E^3$ -page

$$E^{3}(\infty) \cong P(t) \otimes P(\bar{\xi}_{1}^{2}, \bar{\xi}_{i+1}': i \ge 1) \oplus P(\bar{\xi}_{1}^{2}, \bar{\xi}_{i+1}': i \ge 1) \otimes \mathbb{F}_{2}\{(\sigma\bar{\xi}_{1})^{k}: k \ge 1\}$$

By Proposition 3.8 and the fact that all  $d^2$ -differentials in the source also occur in the target, the map is injective on  $E^3$ -pages. Since there is an isomorphism  $E^3(\infty) \cong E^{\infty}_{*,*}(\infty)$  by [14, Thm. 5.1], there are isomorphisms  $E^3(n) \cong E^{\infty}(n)$  for all n > 0.

**Corollary 5.4.** There is an isomorphism of graded  $\mathbb{F}_2$ -vector spaces

$$H_*(TC^-(y(n))[i]) \cong V(i) \oplus [P(\bar{\xi}_1^2, \bar{\xi}_2', \dots, \bar{\xi}_n') \otimes E(\bar{\xi}_n \sigma \bar{\xi}_n) \otimes P(t)/(t^{i-1})] \oplus T$$

with the same bidegrees, where T is defined in Proposition 5.3 and V(i) is defined in Notation 4.6.

5.2. Chromatic complexity of  $TC^{-}(y(n))$ . We now modify our analysis from Section 5.1 to study the chromatic complexity of the topological negative cyclic homology  $TC^{-}(y(n))$ .

**Lemma 5.5.** There are isomorphisms of continuous  $\mathcal{E}_*$ -comodules

$$H^c_*(TC^-(y(n))) \cong (H_*(z(n)/v_n) \otimes P(t)) \oplus H_*(z(n)) \otimes T_0$$

where  $T_0$  is defined as in the statement of Proposition 5.3. Also, there are isomorphisms of  $\mathcal{E}_*$ comodules

$$H_*(TC^{-}(y(n))[i]) \cong V(i) \oplus [H_*(z(n)/v_n) \otimes P(t)/(t^{i-1})] \oplus [H_*(z(n)) \otimes T_0]$$

where the simple t-torsion classes  $\sigma \bar{\xi}_i$  are  $\mathcal{E}_*$ -comodule primitives and the coaction on  $\bar{\xi}_n \sigma \bar{\xi}_n$  has the form

$$\nu_n(\bar{\xi}_n\sigma\bar{\xi}_n) = \nu_\infty(\bar{\xi}_n\sigma\xi_n) + \sum_{i=1}^{n+1}\bar{\xi}_i\otimes\bar{\xi}_{n+1-i}^{2^i}.$$

The  $\mathcal{A}_*$ -comodule V(i) is defined in Notation 4.6.

*Proof.* A straightforward modification of the proof of Lemma 4.9 can be used to calculate the  $\mathcal{E}_*$ coaction on  $H^c_*(TC^-(y(n)))$ . The only classes which were not discussed in Lemma 4.9 are monomials
of the form  $\sigma \bar{\xi}_{i_1} \cdots \sigma \bar{\xi}_{i_k}$ , the class  $\bar{\xi}_n \sigma \bar{\xi}_n$ , the elements  $\bar{\xi}'_n$  and nontrivial products of these elements.
The classes  $\sigma \bar{\xi}_{i_1} \cdots \sigma \bar{\xi}_{i_k}$  are comodule primitives in  $H_*THH(y(n))$  by the formula  $\nu_n(\sigma \bar{\xi}_i) =$ 

 $(1 \otimes \sigma)(\nu_n(\bar{\xi}_i))$  and there are no possible hidden  $\mathcal{E}_*$ -comodule extensions in the homotopy fixed point spectral sequence because the degree of t is even. The class  $\bar{\xi}_n \sigma \bar{\xi}_n$  maps to  $\bar{\xi}_n \sigma \bar{\xi}_n + \bar{\xi}_{n+1}$  by Proposition 3.8, which implies the stated coaction. The coaction on  $\bar{\xi}'_i$  is

$$1 \otimes \bar{\xi}'_i + \bar{\xi}_{i-1} \otimes \sigma \bar{\xi}_{i-1} + \sum_{0 < j+k \le i} \bar{\xi}_j \otimes \bar{\xi}_k^{2^j} + \bar{\xi}_i \otimes 1.$$

by direct computation.

We now specify the isomorphism. The map  $P(\bar{\xi}_1^2, \bar{\xi}_2', ..., \bar{\xi}_n') \otimes E(\bar{\xi}_n \sigma \bar{\xi}_n) \to H_*(z(n)/v_n)$ sends  $\bar{\xi}_i'$  to  $\bar{\xi}_i$  for  $2 \leq i \leq n$  and  $\bar{\xi}_n \sigma \bar{\xi}_n$  to  $\bar{\xi}_{n+1}$ . It sends  $\bar{\xi}_1^2$  to  $\bar{\xi}_1^2 + \sigma \bar{\xi}_1$ ,  $\bar{\xi}_1^4$  to  $\bar{\xi}_1^4 + \sigma \bar{\xi}_2$ ,  $\bar{\xi}_1^6$  to  $\bar{\xi}_1^6 + \sigma \bar{\xi}_1 \sigma \bar{\xi}_2$ , and so on, until  $\bar{\xi}_1^{2(m+1)}$  where  $m = |\prod_{i=1}^n \sigma \xi_i|$  which maps to the element with the same name in  $H_*(z(n)/v_n)$ .

**Lemma 5.6.** The Margolis homology of  $TC^{-}(y(n))$  is isomorphic to

$$H(TC^{-}(y(n))[i];Q_{m}) \begin{cases} P(t) \oplus P(\xi_{n}^{2}) \otimes T_{0} \oplus H(V(i);Q_{0}) & \text{if } m = 0 \\ H(V(i);Q_{m}) & \text{if } 1 \leq m \leq n-1, \\ H(V(i);Q_{n}) \oplus H_{*}(z(n)) \otimes T_{0} & \text{if } m = n, \\ H_{*}(TC^{-}(y(n))[i]) & \text{if } m \geq n+1. \end{cases}$$

where the  $\mathcal{A}_*$ -comodule V(i) is defined in Notation 4.6.

*Proof.* The decomposition in Corollary 5.4 is a splitting as  $E(Q_m)$ -modules since t does not appear in the coaction of any class in the second summand and  $Q_m(t) = 0$ . Thus

$$H(TC^{-}(y(n))[i];Q_m) \cong H(V(i);Q_m) \oplus H(H_*(z(n)/v_n) \otimes P(t)/(t^{i-1})\{t\};Q_m) \oplus H(T;Q_m).$$

The Margolis homology of V(i) is not computed in our calculation, but it will not affect the continuous Margolis homology. The Margolis homology of  $M := H_*(z(n)/v_n) \otimes P(t)/(t^{i-1})\{t\}$  can be computed the same way that  $H(TP(y(n))[i]; Q_m)$  was computed in Section 5.1, giving

$$H(M;Q_m) \cong \begin{cases} 0 & \text{if } 1 \le m \le n, \\ P(\bar{\xi}_1^2, \bar{\xi}_2, \dots, \bar{\xi}_n) \otimes E(\bar{\xi}_{n+1}) \otimes P(t)/(t^{i+1}) & \text{if } m = n+1. \end{cases}$$

Finally, the Margolis homology of T can be computed as follows. For  $1 \leq m \leq n$ , we have  $H(H_*(z(n)); Q_m) = 0$  by Corollary 2.21, so  $H(T; Q_m) = H(H_*(z(n) \otimes T_0), Q_m) \cong 0$  if 0 < m < n and  $H(H_*(z(n)) \otimes T_0), Q_n) \cong H_*(z(n)) \otimes T_0$  by the Margolis homology and we computed  $H(H_*(z(n)), Q_n)$  earlier. Similarly,  $H(T; Q_0) = H(H_*(z(n)) \otimes T_0, Q_0) \cong P(\bar{\xi}_n^2) \otimes T_0$  for the same reasons. For  $m \geq n+1$ , we observe that there are no terms of the form  $\bar{\xi}_m \otimes y$  in the coproduct of any classes. Indeed, these terms could only come from the coproducts of complementary  $\sigma$ -cycles, but the coproduct of any complementary  $\sigma$ -cycle z contains only terms of the form  $\bar{\xi}_j^2 \otimes y$  where  $j \geq 0$ ; compare with the proof of Lemma 4.9.

## Corollary 5.7. There are isomorphisms

$$\lim_{i} H(TC^{-}(y(n))[i];Q_{m}) \begin{cases} P(t) \oplus P(\bar{\xi}_{n}^{2}) \otimes T_{0} & \text{if } m = 0\\ 0 & \text{if } 1 \leq m \leq n-1, \\ H_{*}(z(n)) \otimes T_{0} & \text{if } m = n, \\ H_{*}(TC^{-}(y(n))[i]) & \text{if } m \geq n+1. \end{cases}$$

**Theorem 5.8.** There are isomorphisms

$$\lim_{i} K(m)_* (TC^-(y(n), H\mathbb{F}_2)[i]) \cong 0$$

for  $1 \le m \le n - 1$ .

*Proof.* Let  $TC^{-}(y(n), H\mathbb{F}_2)[i]$  be the fiber in the fiber sequence

$$TC^{-}(y(n), H\mathbb{F}_2)[i] \to TC^{-}(y(n))[i] \to TC^{-}(H\mathbb{F}_2)[i].$$

Smashing with Morava K-theory produces a fiber sequence

$$K(m) \wedge TC^{-}(y(n), H\mathbb{F}_{2})[i] \to K(m) \wedge TC^{-}(y(n))[i] \to K(m) \wedge TC^{-}(H\mathbb{F}_{2})[i]$$

and then taking the limit again produces a fiber sequence

$$\lim_{i} K(m) \wedge TC^{-}(y(n), H\mathbb{F}_2)[i] \to \lim_{i} K(m) \wedge TC^{-}(y(n))[i] \to \lim_{i} K(m) \wedge TC^{-}(H\mathbb{F}_2)[i].$$

By Corollary 5.7 and the localized Adams spectral sequence, we know the filtered abelian groups  $\{K(m)_*(TC^-(y(n))[i])\}, \{K(m)_*(TC^-(H\mathbb{F}_2)[i])\}$ , and  $\{K(m)_*(TC^-(y(n), H\mathbb{F}_2)[i])\}$  are pro-equivalent to the constant trivial group for  $1 \le m \le n-1$ . This immediately implies the result.  $\Box$ 

**Remark 5.9.** In an earlier draft, a mistake in our homotopy fixed point spectral sequence calculation led us to claim  $K(n)^c_*(TC^-(y(n))) \cong 0$ . In the corrected calculation of  $H(TC^-(y(n))[i];Q_m)$ , the factor  $T_0$  contributes nontrivial classes in the localized Adams spectral sequence, as we will see in a special case in Example 5.10. These can be viewed as obstructions to the vanishing of  $K(n)^c_*(TC^-(y(n)))$ .

**Example 5.10.** We completely calculate  $K(1)^c_*(TC^-(y(1)))$ . The calculations above give an isomorphism of  $\mathcal{A}_*$ -comodules

$$H^{c}_{*}(TC^{-}(y(1))) \cong H_{*}(z(1)/v_{1}) \otimes P(t) \oplus P(\bar{\xi}_{1}^{2})\{\sigma\bar{\xi}_{1}\}$$

and we know  $H(H_*(z(1)/v_1), Q_1) \cong 0$ . The input of the Adams spectral sequence computing  $K(1)^c_*(TC^-(y(1))[i])$  is

 $v_1^{-1}\operatorname{Ext}_{E(Q_1)_*}^{*,*}(\mathbb{F}_2, H_*TC^-(y(1))[i]) \cong P(v_1^{\pm 1}) \otimes \left(P(\bar{\xi}_1^2)\{\sigma\bar{\xi}_1\} \oplus H(V(i), Q_1)\right)$ 

where  $V(i) = (P(\bar{\xi}_1^2) \oplus P(\bar{\xi}_1^2) \{ \bar{\xi}_1 \sigma \bar{\xi}_1 \}) \{ t^i \}$  and  $Q_1(\bar{\xi}_1^{2k+1} \sigma \bar{\xi}_1 t^i) = \bar{\xi}_1^{2k} t^i$  for all  $k \ge 0$ , so  $H(V(i), Q_1) \cong 0$ . Since all the elements in the  $E_2$ -page of the spectral sequence are concentrated in even columns, the spectral sequence collapses. Therefore

$$K(1)_*(TC^{-}(y(1))[i]) \cong K(1)_* \otimes P(\bar{\xi}_1^2) \{ \sigma \bar{\xi}_1 \}.$$

Since the sequence is constant and all maps are the identity map, we see

$$K(1)^{c}_{*}(TC^{-}(y(1))) = \lim_{i} K(1)_{*}(TC^{-}(y(1))[i]) \cong K(1)_{*} \otimes P(\bar{\xi}^{2}_{1}) \{\sigma\bar{\xi}_{1}\} \not\cong 0.$$

We emphasize that this does not imply (non-continuous) Morava K-theory is nontrivial. Indeed, if we consider the cofiber sequence

$$K(1) \wedge TC^{-}(y(1)) \rightarrow \lim K(1) \wedge TC^{-}(y(1)))[i] \rightarrow C,$$

then it is still possible that  $K(1)_*(C) \cong K(1)_* \otimes P(\bar{\xi}_1^2) \{\sigma \bar{\xi}_1\}^3$ .

With that said, there would be some surprising consequences if  $K(1)_*(TC^-(y(1))) \neq 0$ . Consider the fiber sequence

$$\Sigma THH(y(1))_{h\mathbb{T}} \to TC^{-1}(y(1)) \xrightarrow{\operatorname{can}} TP(y(1))$$

In the sequel, we will see that  $K(1)_*(TP(y(1))) \cong 0$ . This implies that

$$K(1)_*(\Sigma T H H(y(1))_{h\mathbb{T}}) \cong K(1)_*(T C^-(y(1)))$$

and since homology commutes with homotopy colimits,

$$K(1)_*(\Sigma T H H(y(1))_{h\mathbb{T}}) \cong \pi_*(\Sigma (K(1) \wedge T H H(y(1)))_{h\mathbb{T}}).$$

If this were nonvanishing, we would need either the K(1)-based Adams spectral sequence

$$HH_*^{K(1)*}(K(1)*y(1)) \Rightarrow K(1)*THH(y(1))$$

or the homotopy orbit spectral sequence

$$H_*(\mathbb{T}, K(1)_*THH(y(1))) \Rightarrow \pi_*((K(1) \wedge THH(y(1))_{h\mathbb{T}}))$$

to have a pattern of differentials that kills everything in sight. We have not been able to rule out this interesting possibility.

## 6. Topological cyclic homology and algebraic K-theory of y(n)

In this section, we prove that  $K(y(n), H\mathbb{F}_2)$  has chromatic complexity at least n using a recent theorem of the first author and Andrew Salch [5]. We proceed in three steps. First, we will show that the filtered spectra  $\{TP(y(n))[i]\}$  satisfy the hypotheses of [5, Thm. 3.5] and consequently continuous and ordinary Morava K-theory K(m) agree for TP(y(n)) when  $1 \leq m \leq n$ . This implies an analogous result for  $TC^-(y(n))$  when  $1 \leq m \leq n-1$ . Second, the Nikolaus-Scholze description [37, Prop. II.1.9] of  $TC(y(n), H\mathbb{F}_2)$  as the fiber of a map from  $TC^-(y(n), H\mathbb{F}_2)$  to  $TP(y(n), H\mathbb{F}_2)$  yields a long exact sequence in Morava K-theory which implies vanishing for TC. Finally, the Dundas-Goodwillie-McCarthy Theorem [16, Thm. 7.0.0.2] is used to deduce vanishing for algebraic K-theory from vanishing for TC.

<sup>&</sup>lt;sup>3</sup>Since  $K(1)^{c}(TC^{-}(y(1)))$  is non-vanishing, the main theorem of [5] does not apply.

6.1. Passage to ordinary Morava K-theory. We first recall the main theorem of [5]. We will write  $\tau_{\leq M}X$  to indicate a Postnikov truncation of X with  $\pi_j(X) \cong \pi_j(\tau_{\leq M}X)$  for j < M and  $\pi_j(\tau_{\leq M}X) = 0$  for  $j \geq M$ .

**Theorem 6.1.** [5, Thm. 3.5] Fix m > 0. Suppose M is an integer and

 $\cdots \to Y[2] \to Y[1] \to Y[0] \to \dots$ 

is a sequence of bounded below finite-type spectra which are  $H\mathbb{F}_p$ -nilpotently compete, such that the largest grading degree of a comodule primitive in  $H_*(Y[i])$  is strictly less than M, the homology groups  $H_*(Y[i])$  and  $H_*(\tau_{\leq M}Y[i])$  are finitely generated for each i, and the limit

$$\lim_{i \to \infty} H(H_*(Y[i]), Q_m)$$

of the Margolis homologies vanishes. Then the m-th Morava K-theory of  $\operatorname{holim}_{i} Y[i]$  is trivial. That is, we have an isomorphism

$$K(m)_*(\operatorname{holim}_i Y[i]) \cong 0.$$

We will show that the filtered spectra  $\{TP(y(n))[i]\}\$  satisfy the hypotheses of Theorem 6.1 in the following lemmas.

**Lemma 6.2.** The homotopy groups  $\pi_*(TP(y(n))[i])$  are bounded below finite type graded  $\mathbb{Z}_2$ -modules.

*Proof.* First, we note that the vanishing line in the truncated Tate spectral sequence converging to  $\pi_*(TP(y(n))[i])$  implies that only finitely many bidigrees contribute to each homotopy degree in the abutment, and moreover, the abutment is bounded below. It therefore suffices to show that  $\pi_*(THH(y(n)))$  is a finite type graded  $\mathbb{F}_2$ -vector space. To see that the graded  $\mathbb{F}_2$ -vector space  $\pi_*(THH(y(n)))$  is finite type, we consider the Künneth spectral sequence

$$Tor_{*,*}^{\pi_*(y(n) \land y(n)^{\circ p})}(y(n)_*, y(n)_*) \Rightarrow \pi_*(THH(y(n))).$$

By the Thom isomorphism, we have that  $\pi_*(y(n) \wedge y(n)^{\text{op}}) \cong y(n)_*(\Omega J_{2^n-1}(S^2))$  is a connective graded  $\mathbb{F}_2$ -algebra, so there is a resolution of  $y(n)_*$  by free graded  $\pi_*(y(n) \wedge y(n)^{\text{op}})$ -modules that gives a vanishing line in  $Tor_{*,*}^{\pi_*(y(n) \wedge y(n)^{\text{op}})}(y(n)_*, y(n)_*)$ . Thus, the spectral sequence strongly converges and has a vanishing line so that only finitely many bidegrees contribute to  $\pi_k(THH(y(n))$  for each k. Since  $y(n)_*$  is finite type by [30],<sup>4</sup> and consequently  $y(n)_*(\Omega J_{2^n-1}(S^2))$  is finite type, we know that

$$Tor_{*,*}^{\pi_*(y(n)\wedge y(n)^{\rm op}}(y(n)_*, y(n)_*)$$

is finite type as a bigraded  $\mathbb{F}_2$ -module. Therefore,  $\pi_*(THH(y(n)))$  is a finite type  $\mathbb{F}_2$ -module and consequently  $\pi_*(TP(y(n)[i]))$  is a finite type bounded below  $\mathbb{Z}_2$ -module.

**Corollary 6.3.** The homology groups  $H_*(TP(y(n))[i])$  and  $H_*(\tau_{\leq \ell}TP(y(n))[i])$  are finitely generated for each  $i, \ell \in \mathbb{Z}$ .

*Proof.* The previous lemma implies that both  $\pi_*(TP(y(n))[i])$  and  $\pi_*(\tau_{<\ell}TP(y(n))[i]))$  are finite type bounded below  $\mathbb{Z}_2$ -modules. In general, when  $\pi_*(X)$  is bounded below and finite type as a  $\mathbb{Z}_2$ -module, a simple argument using the *Tor* spectral sequence

$$Tor_{*,*}^{\pi_*(S)}(\mathbb{F}_2,\pi_*(X)) \Rightarrow H_*(X)$$

implies that  $H_*(X)$  is a finite type graded  $\mathbb{F}_2$ -module as well.

We now prove that the filtered spectra  $\{TP(y(n))[i]\}\$  satisfy the main technical hypothesis needed for Theorem 6.1.

**Lemma 6.4.** For all  $n \ge 0$  and  $i \in \mathbb{Z}$ , the largest grading degree of a comodule primitive in  $H_*(TP(y(n))[i])$  is strictly less than  $2^{n+1}$ . In particular, this upper bound depends only on n.

<sup>&</sup>lt;sup>4</sup>In [30], they show that the Adams spectral sequence for y(n) has a vanishing line of slope  $1/|v_n|$  and their description of the  $E_2$ -page of the Adams spectral sequence is clearly finitely generated in each bidegree.

*Proof.* We note that even though  $H_*(TP(y(n))[i]; \mathbb{F}_2)$  is not necessarily a ring, it is a module over  $H^c_*(TP(S); \mathbb{F}_2)$  in the category of (continuous)  $\mathcal{A}_*$ -comodules, and therefore the coaction

$$\nu_n: H_*(TP(y(n))[i]; \mathbb{F}_2) \to \mathcal{A}_* \otimes H_*(TP(y(n))[i]; \mathbb{F}_2)$$

satisfies  $\nu_n(tx) = \nu_n(t)\nu_n(x)$  where |t| = -2. Also, as long as i < -1 the element t is not a comodule primitive. In particular, if we write

$$\psi \colon H^*(B\mathbb{T}) \to H^*(B\mathbb{T}) \otimes \mathcal{A}_*$$

for the  $\mathcal{A}_*$ -coaction on  $H^*(B\mathbb{T}) = H^*_{ap}(\mathbb{T}; \mathbb{F}_2)$  then

$$\psi(t) = 1 \otimes t + \bar{\xi}_1^2 \otimes t^2 + \bar{\xi}_2^2 \otimes t^4 + \dots$$

by [44, Lem. 2.5]. So in the truncated Tate spectral sequence

$$P(t^{-1})\{t^i\} \Rightarrow \pi_*(TP(S)[i])$$

for the sphere spectrum, the  $\mathcal{A}_*$ -cocaction on the abutment is given by the formula

$$\psi(t)\psi(t^{-1}) = 1 \otimes 1$$

and by

$$\psi(t^a) = \psi(t)^a \mod (t^{i+1})$$
$$\psi(t^{-a}) = \psi(t^{-1})^a \mod (t^{i+1})$$

for  $a \ge 0$  as well.

In particular, note that  $\bar{\xi}_k^2$  and  $t^a$  are not zero divisors for any  $k \ge 1$  or  $a \in \mathbb{Z}$ . So by the formula

$$\nu_n(tx) = \psi_n(t)\nu_n(x)$$

we note that whenever  $x \neq 0$  there will always be a term

 $\bar{\xi}_1^2 \otimes xt^2$ 

in the coaction of tx. Consequently tx cannot be a comodule primitive. The same argument applies to  $t^a x$  for any  $a \neq 0$  such that 2a < i. If  $2a \ge i$ , then we could have comodule primitives of the form  $t^a x$ , but since in that case  $|xt^a| = |x| - 2a$ , these comodule primitives will be in degrees less than over equal to |x|. Note that we need to choose a final sequence such that elements that are not divisible by t always appear, but this is possible.

We now show that the comodule primitives in  $H_*(THH(y(n)))$  are bounded above. By Proposition 3.2, there is an isomorphism

$$H_*(THH(y(n))) \cong H_*(y(n)) \otimes E(\sigma \overline{\xi}_j | 0 \le j \le n)$$

of  $\mathcal{A}_*$ -comodules. Since  $H_*(y(n))$  is a sub-Hopf algebra of  $\mathcal{A}_*$  and the coaction is the restriction of the coproduct on the dual Steenrod algebra, the only nonzero comodule primitive is 1. The elements in  $H_*(y(n)) \otimes E(\sigma \bar{\xi}_j | 0 \leq j \leq n)$  that are comodule primitives must be of the form of an element in  $E(\sigma \bar{\xi}_j | 0 \leq j \leq n)$  plus an element in  $H_*(y(n)) \otimes E(\sigma \bar{\xi}_j | 0 \leq j \leq n)$  of the same degree. Therefore, the degree of the comodule primitives is bounded above by  $2^{n+1} - 1 = |\prod_{j=1}^n \sigma \bar{\xi}_j|$ .

It remains to show that no additional comodule primitives can be obtained by adding terms in different filtrations in the truncated Tate spectral sequence converging to  $H_*(TP(y(n))[i])$ . To prove this we split into two cases. When

$$x \in H_*(y(n)) \otimes E(\sigma \overline{\xi}_j | 0 \le j \le n)$$

is not a comodule primitive, then we know

$$\nu_n(x) = 1 \otimes x + x_1^{(1)} \otimes x_1^{(2)} + \sum_{j>1} x_j^{(1)} \otimes x_j^{(2)}$$

where  $x_1^{(1)} \otimes x_1^{(2)}$  is nontrivial and t does not divide  $x_1^{(2)}$ . Then we claim that we cannot add terms in higher filtration to  $t^a x$  and get a comodule primitive. In particular, there will always be a term of the form  $x_1^{(1)} \otimes t^a x_2^{(2)}$  and this cannot be canceled by the coaction on an element of the form  $t^{a+b}y$  for some element y and b > 0. On the other hand, if

$$x \in H_*(y(n)) \otimes E(\sigma \bar{\xi}_j | 0 \le j \le n)$$

is a comodule primitive, then we note that if a > 0 such that 2a < i, then  $t^a x$  has a term of the form  $\bar{\xi}_1^{2a} \otimes t^{2a} x$  and this cannot be canceled by the coaction on an element  $t^{a+b}y$  for b > 0 because the comodule primitives are of the form x = z + w where  $z \in E(\sigma \bar{\xi}_j | 0 \le j \le n)$  and there are no other coactions that have this same z in their coaction which are in a higher degree than x. Therefore, there are no comodule primitives above degree  $2^{\ell+1} - 1 = |\prod_{j=1}^{\ell} \sigma \bar{\xi}_j|$ . Since this is the case for all  $\ell \le n$ , we see that  $M := 2^{n+1}$  is a uniform upper bound on the degree of the comodule primitives in  $H_*(TP(y(n))[i])$ .

Theorem 6.5. There are isomorphisms

(11) 
$$K(m)^c_*TP(y(n)) \cong K(m)_*TP(y(n)) \cong 0 \text{ and}$$

(12) 
$$K(\ell)^c_* TC^-(y(n)) \cong K(\ell)_* TC^-(y(n)) \cong 0$$

for  $1 \le m \le n$  and  $1 \le \ell \le n - 1$ .

*Proof.* To prove isomorphism (11) it suffices to prove that the filtered spectra  $\{TP(y(n))[i]\}_{i\in\mathbb{Z}}$  satisfy the hypotheses of Theorem 6.1. This was the content of Lemma 6.2, Corollary 6.3, and Lemma 6.4 and Theorem 4.15.

By Theorem 5.8, the isomorphism (12) follows if we can show that  $K(\ell)_*TC^-(y(n)) \cong 0$ for  $1 \leq \ell \leq n-1$ . By the fiber sequence

$$\Sigma THH(y(n))_{h\mathbb{T}} \to TC^{-}(y(n)) \to TP(y(n)),$$

it suffices to show that  $K(\ell)_*(\Sigma THH(y(n))_{h\mathbb{T}}) \cong 0$  for  $1 \leq \ell \leq n-1$ . Since smashing with  $K(\ell)$  commutes with homotopy colimits, it suffices to show that  $K(m)_*(THH(y(n))) \cong 0$  for  $1 \leq \ell \leq n-1$ . Since  $K(\ell)_*y(n) \cong 0$  for  $1 \leq \ell \leq n-1$  by Proposition 2.22, the  $K(\ell)$ -based Bókstedt spectral sequence

$$HH^{K(\ell)}_{*}(K(\ell)_{*}(y(n))) \Rightarrow K(\ell)_{*}THH(y(n))$$

implies that  $K(\ell)_*THH(y(n))$  vanishes for  $1 \le \ell \le n-1$ .

TZ(0)

Corollary 6.6. There are isomorphisms

$$K(m)_*(TP(y(n), H\mathbb{F}_2)) \cong 0 \cong K(\ell)_*(TC^-(y(n), H\mathbb{F}_2))$$

for all  $1 \le n < \infty$ ,  $1 \le m \le n$ , and  $1 \le \ell \le n - 1$ .

*Proof.* The result follows easily from Theorem 6.5.

6.2. Chromatic complexity of TC(y(n)). We now apply a result of Nikolaus-Scholze [37] to analyze topological cyclic homology. If  $A \to B$  is a map of *p*-complete connective  $E_1$  ring spectra, then [37, Prop. II.1.9] implies that there is a fiber sequence

$$TC(A, B) \to TC^{-}(A, B) \to TP(A, B)$$

in the  $\infty$ -category of spectra, where TC here is implicitly p-complete topological cyclic homology.

**Theorem 6.7.** If  $0 \le m \le n-1$ , then the relative topological cyclic homology  $TC(y(n), H\mathbb{F}_2)$  is K(m)-acyclic, i.e.

$$K(m)_*(TC(y(n), H\mathbb{F}_2)) \cong 0.$$

*Proof.* The cases  $1 \le m \le n-1$  follow from the long exact sequence in K(m)-homology associated to the homotopy fiber sequence

$$TC(y(n), H\mathbb{F}_2) \to TC^-(y(n), H\mathbb{F}_2) \to TP(y(n), H\mathbb{F}_2)$$

along with Corollary 6.6.

The case m = 0 follows from two classical results. By the Dundas-Goodwillie-McCarthy theorem [16, Thm. 7.0.0.2], there is a weak equivalence  $TC(y(n), H\mathbb{F}_2) \simeq K(y(n), H\mathbb{F}_2)$ . By a theorem of Waldhausen [45, Prop. 2.2], the map  $K(y(n)) \to K(H\mathbb{F}_2)$  is a rational equivalence since  $y(n) \to H\mathbb{F}_2$  is a rational equivalence. Combing these two theorems produces isomorphisms

$$H\mathbb{Q}_*(TC(y(n), H\mathbb{F}_2)) \cong H\mathbb{Q}_*(K(y(n), H\mathbb{F}_2)) \cong 0.$$

6.3. Chromatic complexity of K(y(n)). We can now show that relative algebraic K-theory preserves chromatic complexity for y(n).

**Theorem 6.8.** For  $0 \le m \le n-1$ , the m-th Morava K-theory of the relative algebraic K-theory  $K(y(n), H\mathbb{F}_2)$  vanishes, i.e.

$$K(m)_*(K(y(n), H\mathbb{F}_2)) \cong 0$$

*Proof.* This follows from Theorem 6.7 and the equivalence  $TC(y(n), H\mathbb{F}_2) \simeq K(y(n), H\mathbb{F}_2)$  implied by the Dundas-Goodwillie-McCarthy theorem [16, Thm. 7.0.0.2].

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