QUILLEN'S WORK ON FORMAL GROUP LAWS AND COMPLEX COBORDISM THEORY

DOUGLAS C. RAVENEL

Abstract. In 1969 Quillen discovered a deep connection between complex cobordism and formal group laws which he announced in [Qui69]. Algebraic topology has never been the same since. We will describe the content of [Qui69] and then discuss its impact on the field. This paper is a writeup of a talk on the same topic given at the Quillen Conference at MIT in October 2012. Slides for that talk are available on the author's home page.

1. Quillen's cryptic and insightful masterpiece: six pages that changed algebraic topology forever

ON THE FORMAL GROUP LAWS OF UNORIENTED AND COMPLEX COBORDISM THEORY

BY DANIEL QUILLEN[1]

Communicated by Frank Peterson, May 16, 1969

Table of contents of [Qui69]:

1. Formal group laws.
2. The formal group law of complex cobordism.
3. The universal nature of cobordism group laws.
4. Typical group laws (after Cartier).
5. Decomposition of $\Omega^*_{(p)}$. *(The p-local splitting of $\Omega = MU$.)*
6. Operations in ΩT^*. *(The structure of $BP^*(BP)$.)*

1.1. **Enter the formal group law.** Quillen began by defining formal group laws just as we define them today.

Definition 1. *A formal group law over a ring R is a power series $F(X,Y) \in R[[X,Y]]$ with*

$$\begin{aligned} F(X,0) = F(0,X) &= X \\ F(Y,X) &= F(X,Y) \\ F(X,F(Y,Z)) &= F(F(X,Y),Z). \end{aligned}$$

For a thorough treatment of this topic, see Hazewinkel [Haz78]. For a shorter treatment written with this application in mind, see [Rav86, Appendix 2].

Examples:

2010 *Mathematics Subject Classification.* 55N22, 55N34, 55S25, 55T15, 14L05, 57R77.

Key words and phrases. Formal group law, complex cobordism. chromatic filtration, elliptic cohomology, topological modular forms.

- $X+Y$, the additive formal group law.
- $X+Y+XY$, the multiplicative formal group law.
- Euler's addition formula for a certain elliptic integral, the power series expansion of

$$\frac{X\sqrt{1-Y^4}+Y\sqrt{1-X^4}}{1-X^2Y^2} \in \mathbf{Z}[1/2][[X,Y]].$$

He then defined the formal group law of complex cobordism in terms of the first Conner-Floyd Chern class (defined in [CF66]) of the tensor product of two complex line bundles, just as we define it today.

Definition 2. *Let L_1 and L_2 be complex line bundles over a space X with Conner-Floyd Chern classes*

$$c_1(L_1),\, c_1(L_2) \in \Omega^2(X) = MU^2(X)$$

Then the formal group law over the complex cobordism ring is

$$F^{\Omega}(c_1(L_1), c_1(L_2)) = c_1(L_1 \otimes L_2).$$

Quillen's notation for the complex cobordism of a space or spectrum X was $\Omega^* X$; today it is commonly denoted by $MU^* X$.

His first theorem was

THEOREM 1. *Let E be a complex vector bundle of dimension n, let f: $PE' \to X$ be the associated projective bundle of lines in the dual E' of E, and let $\mathcal{O}(1)$ be the canonical quotient line bundle on PE'. Then the Gysin homomorphism f_*: $\Omega^q(PE') \to \Omega^{q-2n+2}(X)$ is given by the formula*

$$f_*(u(\xi)) = \operatorname{res} \frac{u(Z)\omega(Z)}{\prod_{j=1}^{n} F^{\Omega}(Z, I\lambda_j)} \,. \tag{4}$$

Here $u(Z) \in \Omega(X)[Z]$, $\xi = c_1^{\Omega}(\mathcal{O}(1))$, ω and I are the invariant differential form and inverse respectively for the group law F^{Ω}, and the λ_j are the dummy variables of which $c_q^{\Omega}(E)$ is the qth-elementary symmetric function.

The hardest part of this theorem is to define the residue; we specialize to dimension one an unpublished definition of Cartier, which has also been used in a related form by Tate [7].

(This paper contains several snapshots from [Qui69], showing internal theorem, equation and reference numbers, not to be confused with the ones used in this paper. His [6] and [7] are [Nov67] and [Tat68].) He never defined the residue. Fortunately this mysterious statement was only used to recalculate the logarithm of the formal group law.

Applying the theorem to the map $f\colon CP^n \to pt$, we find that the coefficient of $X^n dX$ in $\omega(X)$ is P_n, the cobordism class of CP^n in $\Omega^{-2n}(pt)$. From (2) we obtain the

COROLLARY (MYSHENKO [6]). *The logarithm of the formal group law of complex cobordism theory is*

$$l(X) = \sum_{n\geq 0} P_n \frac{X^{n+1}}{n+1}. \tag{5}$$

The *logarithm* $\ell(X)$ of a formal group law is a power series defining an isomorphism (after tensoring with the rationals) with the additive formal group law, so we have

$$\ell(F(X,Y)) = \ell(X) + \ell(Y).$$

It is related to the formal group law by the formula

$$\ell'(X) = \frac{1}{F_2(X,0)}$$

where $F_2(X,Y) = \partial F(X,Y)/\partial Y$. The above Corollary identifies the logarithm for the formal group law associated with complex cobordism theory.

1.2. **Show it is universal.** Then he showed that the formal group law for complex cobordism is universal.

THEOREM 2. *The group law F^Ω over $\Omega^{ev}(pt)$ is a universal formal (commutative) group law in the sense that given any such law F over a commutative ring R there is a unique homomorphism $\Omega^{ev}(pt) \to R$ carrying F^Ω to F.*

His proof used two previously known facts:

- Michel Lazard [Laz55] had determined the ring L over which the universal formal group law F^L is defined. The previous corollary implies that the map $L \to \Omega^{ev}(pt)$ carrying F^L to F^Ω is a rational isomorphism. The target was known to be torsion free, so it suffices to show the map is onto.
- Milnor [Mil60] and Novikov[Nov67] had independently determined the structure of the ring MU_*. It is torsion free and generated by as a ring by the cobordism classes of the Milnor hypersurfaces,

  $$H^{m,n} \subset \mathbf{C}P^m \times \mathbf{C}P^n.$$

  $H^{m,n}$ is the zero locus of a bilinear function on $\mathbf{C}P^m \times \mathbf{C}P^n$.

These imply that it suffices to show that the cobordism classes of the $H^{m,n}$ can be defined in terms of the formal group law. Denote the latter as usual by

$$\begin{aligned}
F(X,Y) &= \sum_{i,j\geq 0} a_{i,j} X^i Y^j \qquad \text{where } a_{i,j} \in MU_{2(i+j-1)} \\
\text{with} \qquad P(X) &= \sum_{n\geq 0} P_n X^n \\
&= \ell'(X) \qquad \text{where } \ell(X) \text{ is the logarithm} \\
\text{and} \qquad H(X,Y) &= \sum_{m,n\geq 0} [H^{m,n}] X^m Y^n.
\end{aligned}$$

These are related by the formula

$$H(X,Y) = P(X)P(Y)F(X,Y),$$

so the cobordism class of each Milnor hypersurface is defined in terms of the formal group law. This was Quillen's proof of his Theorem 2.

He proved a similar result about unoriented cobordism. Here there is a formal group law defined in terms of Stiefel-Whitney classes instead of Chern classes. As in the complex case, the cobordism ring is generated by real analogs of the Milnor hypersurfaces. Unlike the complex case, the tensor product square of any real line bundle is trivial. This forces the formal group law to have characteristic 2 and satisfy the relation

$$F(X,X) = 0.$$

1.3. **The Brown-Peterson theorem.** In 1966 Brown and Peterson [BP66] showed that after localization at a prime p (although they did not use this language), Ω (or MU) splits into a wedge of suspensions of a smaller spectrum now known as BP and denoted by Quillen as ΩT. This splitting is suggested by a corresponding decomposition of $H^*(MU;\mathbf{Z}/(p))$ as a module over the mod p Steenrod algebra. Their methods did not show that BP is a ring spectrum and gave little information about its internal structure.

By using some algebra developed by Pierre Cartier [Car67], Quillen gave a much cleaner form of the splitting, thereby showing that BP is a ring spectrum. A formal group law F over a ring R defines a group structure on the set of *curves over* R, meaning power series with trivial constant term. Given a curve $f(X)$ and a positive integer n, let

$$(F_n f)(X) = \sum_{i=1}^{n}{}^F f(\zeta_i X^{1/n}),$$

where the ζ_i are the nth roots of unity, and the addition on the right is defined by the formal group law F. Note that if we replace the formal sum by an ordinary one and

$$f(X) = \sum_{j>0} f_j X^j, \qquad \text{then} \qquad (F_n f)(X) = n\sum_{j>0} f_{nj} X^j.$$

The curve f is said to be *p-typical* (Quillen used the term "typical") if $F_q f = 0$ for each prime $q \neq p$. In the case of ordinary summation this means that f has the form

$$f(X) = \sum_{k\geq 0} f_{(k)} X^{p^k}.$$

The formal group law itself is said to be *p-typical* if the curve X is p-typical with respect to it. Over a torsion free ring, this is equivalent to the logarithm having the form above. Cartier showed that when R is a $\mathbf{Z}_{(p)}$-algebra, there is a canonical coordinate change that converts any formal group law into a p-typical one.

Quillen used this to define an idempotent map $\hat{\xi}$ on $\Omega_{(p)} = MU_{(p)}$ whose telescope is $\Omega T = BP$. This construction is much more convenient than that of Brown and Peterson. This process changes the logarithm from

$$\sum_{n\geq 0} \frac{[\mathbf{C}P^n]X^{n+1}}{n+1} \quad \text{to} \quad \sum_{k\geq 0} \frac{[\mathbf{C}P^{p^k-1}]X^{p^k}}{p^k}.$$

This new logarithm is much simpler than the old one.

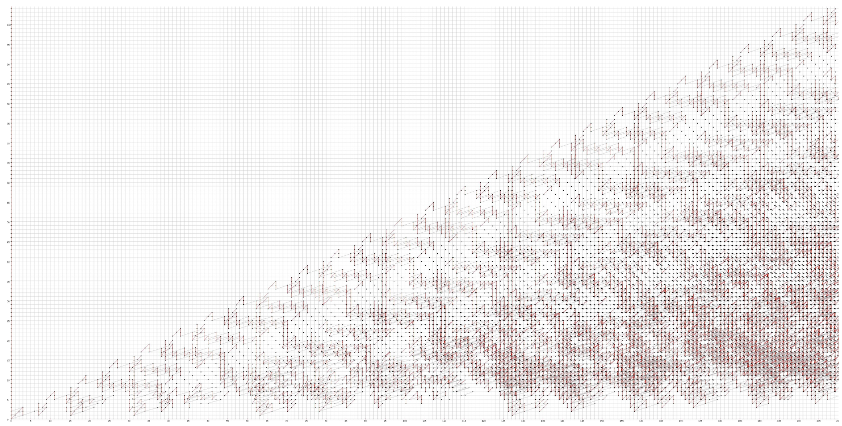

FIGURE 1. 210 dimensions worth of Adams Ext groups, as computed by Christian Nassau [Nas] in 1999.

An analogous construction converts the unoriented cobordism spectrum MO to the mod 2 Eilenberg-Mac Lane spectrum $H\mathbf{Z}/2$.

1.4. **Operations in BP-theory.** In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself. In favorable cases one can set up an E-based Adams spectral sequence and wonder about its E_2-term. This is usually some Ext group defined in terms of the algebra of operations A^E. Finding it explicitly is often a daunting task.

Here are some examples.

- For $E = H\mathbf{Z}/2$ (ordinary mod 2 cohomology), the algebra A^E is the mod 2 Steenrod algebra, which has proven to be a fertile source of theorems in algebraic topology. The corresponding Ext group has been extensively studied, first by Adams in [Ada58].
- For $E = MU$, A^E was determined in 1967 by Novikov in [Nov67]. He found a small but extremely rich portion of its Ext group, rich enough to include the denominator of the value of the Riemann zeta function at each negative odd integer!
- For $E = BP$, A^E was determined by Quillen. The details are too technical for this paper. He gave a precise description in less than two pages, with little indication of proof. The resulting Ext group is the same as Novikov's localized at the prime p. The underlying formulas are easier to calculate with than Novikov's, once one knows how to use them. *It took the rest of us about 5 years to figure out how to do it.*

2. Complex cobordism theory after Quillen

In those days the AMS Bulletin was a vehicle for announcements of new results. Detailed accounts would typically be published elsewhere at a later date. Quillen's article was unusually long. He never wrote a detailed account of it because Frank

Adams did it for him. In [Ada74, Part II] Adams explained all the proofs with great care. His book became the definitive reference for Quillen's results.

He also introduced a very helpful but counterintuitive point of view. Instead of studying the algebra A^{BP}, one should compute in terms of its suitably defined linear dual. As he put it,

> Quillen's formal variables t_i are crying out to be located in $BP_*(BP)$.

This proved to be a huge technical simplification.

Quillen's work was a bridge connecting algebraic topology with algebraic geometry and number theory. *Homotopy theorists have been expanding that bridge ever since.*

2.1. **Morava's work.** In the early 1970s Jack Morava applied deeper results (due mostly to Lazard) from the theory of formal group laws to algebraic topology. While his work [Mor85] was not published until over a decade later, he had several preprints in circulation. This author learned a great deal about them in private conversations with him. They were so influential that his name appeared in titles of several papers published before [Mor85] by other authors, such as [JW75], [MR77], [Rav82], [Rav77], [Rav76], [RW80], [Wil84] and [Yag80].

The deeper results included

- A classification of formal group laws over the algebraic closure of the field $\mathbf{F}_p$. There is a complete isomorphism invariant called the *height*, which can be any positive integer or infinity.
- The automorphism group of a height n formal group law is a certain p-adic Lie group. *It is now known as the Morava stabilizer group S_n.*
- He defined a cohomology theory associated with height n formal group laws now known as *Morava K-theory $K(n)$.*

He also studied the affine variety $Spec(MU_*)$ and defined an action on it by a group of power series substitutions. It is now known as the moduli stack of formal group laws; see [Goe04] and [Goe10]. After passage to characteristic p, the orbits under this action are isomorphism classes of formal group laws as classified by Lazard. The Zariski closures of these orbits form a nested sequence of affine subspaces of the affine variety. The isotropy group of the height n orbit is the height n automorphism group S_n, hence the name *stabilizer group.*

2.2. **Chromatic homotopy theory.** Morava's insights led to the formulation of the *chromatic point of view* in stable homotopy theory. In the late 1970s Haynes Miller, Steve Wilson and I showed that Morava's stratification of $Spec(MU_*)$ leads to a nice filtration of the Adams-Novikov E_2-term ; see [MRW77] and [Rav86, Chapter 5]. In the early 80s we learned [Rav84] that the stable homotopy category itself possesses a filtration similar to the one found by Morava in $Spec(MU_*)$. A key technical tool in defining it is Bousfield localization, defined in [Bou79]. As in the algebraic case, for each positive integer n, there is a layer of the stable homotopy category (localized at a prime p) related to height n formal group laws. Roughly speaking, its structure is controlled by the cohomology of the nth Morava stabilizer group S_n.

Homotopy groups of objects in the nth layer tend to repeat themselves every $2p^k(p^n-1)$ dimensions for various k. This is known as v_n-periodicity. The term *chromatic* refers to this separation into varying frequencies.

The first known example of this phenomenon was the *Bott Periodicity Theorem* of [Bot59], describing the homotopy of the stable unitary and orthogonal groups. It is an example of v_1-periodicity.

The motivating problem behind this work was understanding the stable homotopy groups of spheres. Research on them in the 1950s and 60s (such as [Tod62]) indicated a very disorganized picture, a zoo of erratic phenomena. It was seen then to contain one systematic pattern related to Bott periodicity. The known homotopy groups of the stable orthogonal group mapped to the unknown stable homotopy groups of spheres by the Hopf-Whitehead J-homomorphism [Whi42]. Its image was determined by Adams [Ada66]. It contained the rich arithmetic structure detected by the Novikov calculation referred to earlier.

In the early 1970s some more systematic patterns were found independently by Larry Smith [Smi77] and Hirosi Toda [Tod71]. The aim of chromatic theory was find a unified framework for such patterns. It was very successful. A milestone result in it is the Nilpotence Theorem of Ethan Devinatz, Mike Hopkins and Jeff Smith [DHS88]. Of this result Adams [Ada92, page 525] said

> At one time it seemed that homotopy theory was utterly without system; now it is almost proved that systematic effects predominate.

A unified account of these developments can be found in [Rav92].

2.3. **Elliptic cohomology and topological modular forms.** For over a century elliptic curves have stood at the center of mathematics. Every elliptic curve over a ring R has a formal group law attached to it. This means there is a homomorphism to R from MU_*. It is known that the mod p reduction (for any prime p for which the curve has good reduction) of this formal group law has height 1 or 2.

This led to the definition of the *elliptic genus* by Serge Ochanine [Och87] in 1984 and the definition of *elliptic cohomology* by Peter Landweber, Bob Stong and myself [LRS95] a few years later. Attempts to interpret the former analytically have been made by Ed Witten [Wit88] and [Wit87], and by Stephan Stolz and Peter Teichner [ST04]. The proceedings of two conferences on this topic are [Lan88] and [MR07]. A useful survey with many more references is [Lur09].

A deeper study of the connection between elliptic curves and algebraic topology led to the theory of *topological modular forms* in the past decade. The main players here are Mike Hopkins, Haynes Miller, Paul Goerss and Jacob Lurie. References include [HM], [Goe10] and [Beh].

Algebraic geometers study objects like elliptic curves by looking at *moduli spaces* for them. Roughly speaking, the moduli space (or stack) $\mathcal{M}_{ell}$ for elliptic curves is a topological space with an elliptic curve attached to each point. The theory of elliptic curves is in a certain sense controlled by the geometry of this space.

To each open subset in the moduli stack $\mathcal{M}_{ell}$ one can associate a certain commutative ring of functions related to the corresponding collection of elliptic curves. This collection is called a *sheaf of rings* $\mathcal{O}_{ell}$ over $\mathcal{M}_{ell}$.

Such a sheaf has a *ring of global sections* $\Gamma(\mathcal{O}_{ell})$, which encodes a lot of useful information. Elements of this ring are closely related to *modular forms*, which are complex analytic functions with certain arithmetic properties that have fascinated number theorists for over a century. They were a key ingredient in Wiles' proof of Fermat's Last Theorem.

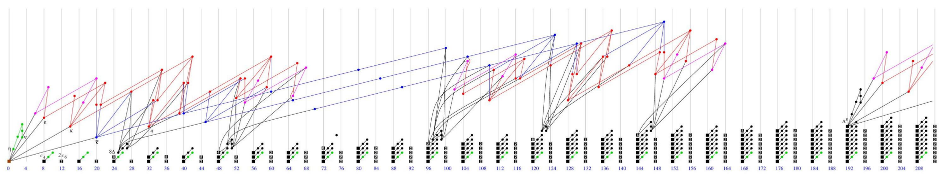

FIGURE 2. The 2-primary homotopy of *tmf* illustrated by Andre Henriques in [Hen]. See also Bauer [Bau08].

Hopkins, Lurie *et al.* have found a way to enrich this theory by replacing every ring R in sight with a *commutative ring spectrum* E with suitable formal properties. We can think of E as an iceberg whose tip is R. The one associated with $\Gamma(\mathcal{O}_{ell})$ is known as ***tmf***, the ring spectrum of topological modular forms. *This ring spectrum is an iceberg whose tip is the classical theory of modular forms.*

More recently this work has been generalized to a theory of *topological automorphic forms* by Mark Behrens and Tyler Lawson in [BL10]. Here the elliptic curves are replaced by more general abelian varieties classified by suitable moduli stacks. While an elliptic curve has a 1-dimensional formal group law attached to it, an abelian variety of dimension d has a d-dimensional one. In favorable cases it has a one dimensional formal summand whose height could be as large as $d-1$. In this way one gets spectra similar to tmf that give information about all layers of the chromatic tower.

Quillen's work on formal group laws and complex cobordism opened a new era in algebraic topology. It led to a chain of discoveries that is unabated to this day.

References

[Ada58] J. F. Adams. On the structure and applications of the Steenrod algebra. *Comment. Math. Helv.*, 32:180–214, 1958.

[Ada66] J. F. Adams. On the groups $J(X)$. IV. *Topology*, 5:21–71, 1966.

[Ada74] J. F. Adams. *Stable homotopy and generalised homology.* University of Chicago Press, Chicago, Ill., 1974. Chicago Lectures in Mathematics.

[Ada92] J. Frank Adams. *The selected works of J. Frank Adams. Vol. II.* Cambridge University Press, Cambridge, 1992. Edited and with an introduction and biographical data by J. P. May and C. B. Thomas.

[Bau08] Tilman Bauer. Computation of the homotopy of the spectrum `tmf`. In *Groups, homotopy and configuration spaces*, volume 13 of *Geom. Topol. Monogr.*, pages 11–40. Geom. Topol. Publ., Coventry, 2008.

[Beh] Mark Behrens. Notes on the construction of tmf. To appear in proceedings of 2007 Talbot Workshop, and available on the author's home page.

[BL10] Mark Behrens and Tyler Lawson. Topological automorphic forms. 2010.

[Bot59] Raoul Bott. The stable homotopy of the classical groups. *Ann. of Math. (2)*, 70:313–337, 1959.

[Bou79] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[BP66] E. H. Brown and F. P. Peterson. A spectrum whose $\mathbf{Z}_p$ cohomology is the algebra of reduced p-th powers. *Topology*, 5:149–154, 1966.

[Car67] Pierre Cartier. Modules associés à un groupe formel commutatif. Courbes typiques. *C. R. Acad. Sci. Paris Sér. A-B*, 265:A129–A132, 1967.

[CF66] P. E. Conner and E. E. Floyd. *The relation of cobordism to K-theories*. Lecture Notes in Mathematics, No. 28. Springer-Verlag, Berlin, 1966.

[DHS88] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith. Nilpotence and stable homotopy theory. I. *Ann. of Math. (2)*, 128(2):207–241, 1988.

[Goe04] Paul G. Goerss. (Pre-)sheaves of ring spectra over the moduli stack of formal group laws. In *Axiomatic, enriched and motivic homotopy theory*, volume 131 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 101–131. Kluwer Acad. Publ., Dordrecht, 2004.

[Goe10] Paul G. Goerss. Topological modular forms [after Hopkins, Miller and Lurie]. *Astérisque*, (332):Exp. No. 1005, viii, 221–255, 2010. Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011.

[Haz78] Michiel Hazewinkel. *Formal groups and applications*, volume 78 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.

[Hen] André Henriques. The homotopy groups of tmf and its localizations. To appear in proceedings of 2007 Talbot Workshop, and available on the author's home page.

[HM] Michael J. Hopkins and M. A. Mahowald. From elliptic curves to homotopy theory. Preprint in Hopf archive at `hopf.math.purdue.edu/Hopkins-Mahowald/eo2homotopy`.

[JW75] David Copeland Johnson and W. Stephen Wilson. BP operations and Morava's extraordinary K-theories. *Math. Z.*, 144(1):55–75, 1975.

[Lan88] Peter S. Landweber. Elliptic cohomology and modular forms. In *Elliptic curves and modular forms in algebraic topology (Princeton, NJ, 1986)*, volume 1326 of *Lecture Notes in Math.*, pages 55–68. Springer, Berlin, 1988.

[Laz55] Michel Lazard. Sur les groupes de Lie formels à un paramètre. *Bull. Soc. Math. France*, 83:251–274, 1955.

[LRS95] Peter S. Landweber, Douglas C. Ravenel, and Robert E. Stong. Periodic cohomology theories defined by elliptic curves. In *The Čech centennial (Boston, MA, 1993)*, volume 181 of *Contemp. Math.*, pages 317–337, Providence, RI, 1995. Amer. Math. Soc.

[Lur09] J. Lurie. A survey of elliptic cohomology. In *Algebraic topology*, volume 4 of *Abel Symp.*, pages 219–277. Springer, Berlin, 2009.

[Mil60] J. Milnor. On the cobordism ring Ω^* and a complex analogue. I. *Amer. J. Math.*, 82:505–521, 1960.

[Mor85] Jack Morava. Noetherian localisations of categories of cobordism comodules. *Ann. of Math. (2)*, 121(1):1–39, 1985.

[MR77] Haynes R. Miller and Douglas C. Ravenel. Morava stabilizer algebras and the localization of Novikov's E_2-term. *Duke Math. J.*, 44(2):433–447, 1977.

[MR07] Haynes R. Miller and Douglas C. Ravenel, editors. *Elliptic Cohomology Geometry, Applications, and Higher Chromatic Analogues*, volume 342 of *London Mathematical Society Lecture Note Series*, Cambridge, 2007. Cambridge University Press.

[MRW77] Haynes R. Miller, Douglas C. Ravenel, and W. Stephen Wilson. Periodic phenomena in the Adams-Novikov spectral sequence. *Ann. Math. (2)*, 106(3):469–516, 1977.

[Nas] Christian Nassau. Cohomology charts.

[Nov67] S. P. Novikov. Methods of algebraic topology from the point of view of cobordism theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 31:855–951, 1967.

[Och87] Serge Ochanine. Sur les genres multiplicatifs définis par des intégrales elliptiques. *Topology*, 26(2):143–151, 1987.

[Qui69] Daniel Quillen. On the formal group laws of unoriented and complex cobordism theory. *Bull. Amer. Math. Soc.*, 75:1293–1298, 1969.

[Rav76] Douglas C. Ravenel. The structure of Morava stabilizer algebras. *Inventiones Math.*, 37:109–120, 1976.

[Rav77] Douglas C. Ravenel. The cohomology of Morava stabilizer algebras. *Mathematische Zeitschrift*, 152:287–297, 1977.

[Rav82] Douglas C. Ravenel. Morava K-theories and finite groups. In S. Gitler, editor, *Symposium on Algebraic Topology in Honor of José Adem*, Contemporary Mathematics, pages 289–292, Providence, Rhode Island, 1982. American Mathematical Society.

[Rav84] Douglas C. Ravenel. Localization with respect to certain periodic homology theories. *Amer. J. Math.*, 106(2):351–414, 1984.

[Rav86] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986. Errata and second edition available online at author's home page.

[Rav92] Douglas C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith.

[RW80] Douglas C. Ravenel and W. Stephen Wilson. The Morava K-theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture. *Amer. J. Math.*, 102(4):691–748, 1980.

[Smi77] Larry Smith. On realizing complex bordism modules. IV. Applications to the stable homotopy groups of spheres. *Amer. J. Math.*, 99(2):418–436, 1977.

[ST04] Stephan Stolz and Peter Teichner. What is an elliptic object? In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 247–343. Cambridge Univ. Press, Cambridge, 2004.

[Tat68] John Tate. Residues of differentials on curves. *Ann. Sci. École Norm. Sup. (4)*, 1:149–159, 1968.

[Tod62] Hirosi Toda. *Composition methods in homotopy groups of spheres*. Annals of Mathematics Studies, No. 49. Princeton University Press, Princeton, N.J., 1962.

[Tod71] Hirosi Toda. On spectra realizing exterior parts of the Steenrod algebra. *Topology*, 10:53–65, 1971.

[Whi42] George W. Whitehead. On the homotopy groups of spheres and rotation groups. *Ann. of Math. (2)*, 43:634–640, 1942.

[Wil84] W. Stephen Wilson. The Hopf ring for Morava K-theory. *Publ. Res. Inst. Math. Sci.*, 20(5):1025–1036, 1984.

[Wit87] Edward Witten. Elliptic genera and quantum field theory. *Comm. Math. Phys.*, 109(4):525–536, 1987.

[Wit88] Edward Witten. The index of the Dirac operator in loop space. In *Elliptic curves and modular forms in algebraic topology (Princeton, NJ, 1986)*, volume 1326 of *Lecture Notes in Math.*, pages 161–181, Berlin, 1988. Springer.

[Yag80] Nobuaki Yagita. On the Steenrod algebra of Morava K-theory. *J. London Math. Soc. (2)*, 22(3):423–438, 1980.

Department of Mathematics, Rochester University, Rochester, NY
E-mail address: doug@math.rochester.edu
URL: http://www.math.rochester.edu/people/faculty/doug/