Topological Algebraic Geometry Workshop
Oslo, September 4th-8th, 2006

Spectra associated with Artin-Schreier curves

Doug Ravenel

University of Rochester

September 6, 2006

§1. My favorite part of the *tmf* story

The spectrum tmf gives a lot of information about the stable homotopy groups of spheres at the prime 2. The relevant computation was described by Hopkins-Mahowald in the 1995 preprint *From elliptic curves to homotopy theory* [HM].

They start with the Weierstrass equation for an elliptic curve.

$$x^2 + a_1yx + a_3x = y^3 + a_2y^2 + a_4y + a_6.$$

Under the affine coordinate change

$$y \mapsto y + r \qquad \text{and} \qquad x \mapsto x + sy + t$$

we get

$$\begin{array}{rcl} a_6 & \mapsto & a_6 + a_4\, r + a_3\, t + a_2\, r^2 \\ & & \quad +a_1\, r\, t + t^2 - r^3 \\ a_4 & \mapsto & a_4 + a_3\, s + 2\, a_2\, r \\ & & \quad +a_1(r\, s + t) + 2\, s\, t - 3\, r^2 \\ a_3 & \mapsto & a_3 + a_1\, r + 2\, t \\ a_2 & \mapsto & a_2 + a_1\, s - 3\, r + s^2 \\ a_1 & \mapsto & a_1 + 2\, s. \end{array}$$

This can be used to define a Hopf algrebroid (A, Γ) with

$$\begin{array}{rcl} A & = & \mathbf{Z}[a_1, a_2, a_3, a_4, a_6] \\ \Gamma & = & A[r, s, t] \end{array}$$

and right unit $\eta_R : A \to \Gamma$ given by the formulas above. Its Ext group is the E_2-term of a spectral sequence converging to $\pi_*(tmf)$. Tilman Bauer [Bau] has written a nice account of this calculation.

For a picture of it, see page 15 of [HM] or
`http://www.math.rochester.edu/people/faculty/doug/mypapers/HMtmf.ps`.
The elements in black are permanent cycles, while the ons in gray are killed by differentials.

The bottom row shows the subring of the ring of classical modular forms generarted by Δ, the discriminant. The notation indicates that 8Δ, $4\Delta^2$, $2\Delta^4$ and δ^8 are all permanent cycles, while 4Δ, $2\Delta^2$ and Δ^4 are not. *What is the arithmetic interpretation of this fact?* The chart extends to dimension 192, where Δ^8 is a periodicity operator.

§2. The role of 1-dimensional formal group laws

Recall the following definition.

Definition 1. *A 1-dimensional* **formal group law** *over R is a power series $G(x, y) \in R[[x, y]]$ satisfying*

(i) $G(x, 0) = G(0, x) = x$,
(ii) $G(y, x) = G(x, y)$ *and*
(iii) $G(x, G(y, z)) = G(G(x, y), z)$.

Remarks:

- A commutative 1-dimensional analytic Lie group also leads to such a power series with a positive radius of convergence, but here there is no convergence requirement.
- A commutative n-dimensional formal group law (consisting of n power series in $2n$ variables) can be defined in a similar way.

Examples for formal group laws:

(i) $G(x, y) = x + y$, the additive formal group law.
(ii) $G(x, y) = x + y + xy$, the multiplicative formal group law. Here

$$1 + G(x, y) = (1 + x)(1 + y),$$

which makes the associativity condition transparent.

(iii) There is a formal group law associated to an every elliptic curve since it is a commutative 1-dimensional analytic Lie group. For example the power series expansion of

$$G(x, y) = \frac{x\sqrt{1 - 2\delta y^2 + \epsilon y^4} + y\sqrt{1 - 2\delta x^2 + \epsilon x^4}}{1 - \epsilon x^2 y^2}$$

is a formal group law over the ring

$$R = \mathbf{Z}[1/2, \delta, \epsilon]$$

and is associated with the elliptic curve (known as the Jacobi quartic) defined by the equation

$$v^2 = 1 - 2\delta u^2 + \epsilon u^4.$$

This computation is due to Euler.

The H-space structure on $\mathbf{C}P^\infty$ leads to a 1-dimensional formal group law over the complex ocbordism ring MU_*. Quillen [Qui69] showed that it has a universal property which means that any formal group law over a ring R is induced by a homomorphism

$$\theta : MU_* \to R.$$

The functor

$$X \mapsto MU_*(X) \otimes_\theta R$$

is known to be a homology theory and hence have a representing spectrum if θ satisfies certain conditions spelled out in the Landweber Exact Functor Theorem [Lan76].

One-dimensional formal group laws in characteristic p are known to be classified by the following invariant.

Definition 2. *Let F be 1-dimensional formal group law over a field k of characteristic p. For a positive integer m, the m-series is defined inductively by*

$$[m]_F(x) = G(x, [m-1]_F(x)),$$

where $[1]_F(x) = x$. The p-series is either 0 or has the form

$$[p]_F(x) = ax^{p^n} + \cdots$$

for some nonzero $a \in k$. The **height** *of F is the integer n. It is defined to be ∞ when $[p]_F(x) = 0$, which happens when $G(x, y) = x + y$.*

There is an intimate connection between height and the chromatic filtration of the stable homotopy category. For example, the formal group law associated with the nth Morava K-theory $K(n)_*$ has height n.

It is known that the height of a formal group law associated with an ellitpic curve has height 1 or 2. This means that *elliptic cohomology and tmf can only give information about v_1- and v_2-periodic phenomena.*

§3. Toward higher chromatic analogs of tmf

We are looking for analogs of elliptic cohomolgy and tmf that can take us deeper into the chromatic filtration. This means finding

(i) A geometric object X (replacing the Weierstrass elliptic curve) defined over a suitable ring A (replacing the Weierstrass ring).

(ii) It should have attached to it a 1-dimensional formal group law G over A of larger height which is Landweber exact.

(iii) It should have an automorphism group (similar to the affine coordinate transformations in the tmf case) which leads to an interesting Hopf algebroid.

(iv) These data should satisfy the conditions of Lurie's machine so we get a spectrum analogous to tmf.

Let X be a curve of genus g over a ring A.

- It has a Jacobian $J(X)$, which is an abelian variety of dimension g, i.e., a commutative g-dimensional analytic Lie group.
- $J(X)$ has a formal completion $\widehat{J}(X)$ which is a g-dimensional formal group.
- If $\widehat{J}(X)$ has a 1-dimensional summand G (over A) which is Landweber exact, then we get a spectrum analogous to elliptic cohomology, so we have met the first two conditions above. It is known that if G has height n for some $n \geq 2$, then $g \geq n$.
- If X has a group of automorphisms which acts on G, then we can look for an analog of tmf.

§4. Artin-Schreier curves

Fix a prime p and a positive integer f, excluding the case $(p, f) = (2, 1)$.
Let

$$\begin{array}{rclrcl} e & = & p^f - 1 & q & = & p-1 \\ m & = & qe & F(x,y) & = & x - x^p - y^e \end{array}$$

The plane curve defined by $F(x, y) = 0$ is the Artin-Schreier curve $C(p, f)$. Its genus is

$$g = \frac{q(e-1)}{2}.$$

Theorem 3. *[Manin* [Man63]*] $\widehat{J}(C(p, f))$ (defined over $\mathbf{F}_p$ or the p-adic integers $\mathbf{Z}_p$) has a 1-dimensional summand of height*

$$h = (p-1)f.$$

Manin's proof requires explicit knowledge of the zeta function of the curve, which makes it vary hard to generalize to other variants of $C(p, f)$. We need to to do this because this 1-dimensional formal group law is not Landweber exact. To get Landweber exactness we need to replace $\mathbf{Z}_p$ by a ring whose Krull dimension is at least $h - 1$.

Conjecture 4. *Let $\tilde{C}(p, f)$ be the curve over over the ring*

$$A = \mathbf{Z}_p[[u_1, \ldots, u_{h-1}]][u, u^{-1}]$$

defined by

$$y^e = u^m x - x^p - \sum_{i=0}^{h-2} u_{i+1} x^{p-1-[i/f]} y^{p^{f-1}-p^{i-[i/f]f}}.$$

Then its Jacobian has a formal 1-dimensional subgroup isomorphic to the Lubin-Tate lifting of the formal group law above. In particular it is Landweber exact.

This curve does not satisfy the third condition above, namely its does not have an interesting automorphism group.

A more promising candidate is the curve defined by

$$y^e = \prod_{i=1}^{p} \left(x - \sum_{j=1}^{f} r_{i,j} y^{p^{f-1}-p^{j-1}} \right)$$

over the invariant ring

$$A = \mathbf{Z}_p[r_{i,j} \colon 1 \le i \le p,\, 1 \le j \le f]^{\Sigma_p}$$

where the symmetric group Σ_p acts on the first subscript.

Here the coordinate transformations are

$$x \quad \mapsto \quad x + \sum_{j=1}^{f} t_j y^{p^{f-1}-p^{j-1}} \qquad \text{and} \qquad y \quad \mapsto \quad y$$

where the t_j are closely related to elements of the same name in $BP_*(BP)$. This ring then has the right Krull dimension for Lurie's machine. However its explicit structure seems to be unknown for $f > 1$.

Question 5. *What is the structure of the Σ_p-invariant subring*

$$A \subset \mathbf{Z}_p[r_{i,j}\colon 1 \le i \le p,\ 1 \le j \le f]?$$

§5. Outline of a new proof of Theorem 3.

Recall our notation. Fix a prime p and a positive integer f, and let

$$\begin{array}{rclcrcl} e & = & p^f - 1 & & q & = & p-1 \\ m & = & qe & & F(x,y) & = & x - x^p - y^e \\ g & = & \dfrac{(e-1)q}{2} & & h & = & qf. \end{array}$$

- The Lagrange inversion formula of 1770 gives us a power series expansion

$$\frac{x^r}{F_x} = \sum_{n\geq 0} \binom{pn+r}{n} y^{mn+er}.$$

- Classical algebraic curve theory tells us that

$$\omega_{r,s} = \frac{x^r y^{s-1}\mathrm{d}y}{F_x} \qquad \text{for } r, s-1 \geq 0$$

  is a holomorphic 1-form or differential of the first kind if

$$er + ps < m.$$

  There are precisely g such values of (r, s).

- Honda theory (to be explained below) tells us that we can analyze the g-dimensional formal group $\widehat{J}(C(p,f))$ by studying the series expansions for the integrals of the first kind, namely

$$\psi_{er+s} \quad = \quad \int \omega_{r,s} = \sum_{n\geq 0} \binom{pn+r}{n} \frac{y^{mn+er+s}}{mn+er+s}.$$

  It leads us to the conclusion that

$$\psi_\ell \qquad \text{where } \ell = p^f - p^{f-1} - 1$$

  corresponds to a 1-dimensional summand of $\widehat{J}(C(p,f))$ of height h.

§6. Honda theory.

Given a power series ϕ in several variables over $\mathbf{Z}_p$, let $T\phi$ be the power series obtained by replacing each variable by its pth power. The operator T can be iterated. Thus a ring of power series over $\mathbf{Q}_p$ becomes a module over $\mathbf{Q}_p[[T]]$.

If $\Phi(x_1, x_2, \dots)$ is a d-dimensional column vector of power series in several variables, and

$$M = \sum_{i \geq 0} C_i T^i$$

is a $d \times d$ matrix over $\mathbf{Q}_p[[T]]$ (meaning that each C_i is a $d \times d$ matrix over $\mathbf{Q}_p$), then we write

$$M * \Phi = \sum_{i \geq 0} C_i T^i \Phi = \sum_{i \geq 0} C_i \Phi(x_1^{p^i}, x_2^{p^i}, \dots).$$

Definition 6. *For Φ and M as above, we say that M is a Honda matrix for Φ if*

(i) *$M \equiv I$ modulo (T), where I is the identity matrix;*
(ii) *pM is a matrix over $\mathbf{Z}_p[[T]]$; and*
(iii) *$M * \Phi$ is a vector of power series over $\mathbf{Z}_p$.*

Two Honda matrices are equivalent if they differ by left multiplication by an invertible matrix over $\mathbf{Z}_p[[T]]$.

Honda proved two major theorems about this. The first is a classification theorem for formal group laws over $\mathbf{Z}_p$.

Theorem 7. *[Honda, 1970,* [Hon70]*] Let G be a g-dimensional formal group law over $\mathbf{Z}_p$ with logarithm L. Hence L is a g-dimensional vector of power series in g variables*

$$x_1, x_2, \dots x_g.$$

Then G has a Honda matrix H, and two such formal group laws are isomorphic iff their Honda matrices are equivalent.

Let

$$X = (x_1, x_2, \dots x_g)^t.$$

Then G is isomorphic to the formal group law with logarithm

$$L' = H^{-1} * X,$$

and there is a bijection between isomorphism classes of formal group laws and equivalence classes of Honda matrices.

Honda's second theorem gives a shortcut for studying the formal completion of the Jacobian of an algebraic curve.

Theorem 8. *[Honda, 1973,* [Hon73]*] Let X be a curve of genus g and let Ψ be the vector of power series expansions of its integrals of the first kind. Then Ψ has a Honda matrix which is the same as the one for the logarithm of $\widehat{J}(X)$.*

As an example, we will apply this to the case $(p, f) = (2, 3)$. Here the curve is defined by the equation

$$y^7 = x - x^2$$

and its genus is 3. The integrals of the first kind are

$$\psi_s = \sum_{n\geq 0} \binom{2n}{n} \frac{y^{7n+s}}{7n+s} \qquad \text{for } s = 1, 2, 3.$$

A number theoretic exercise will reveal that there are 2-adic units c_s satisfying the following congruences modulo series with integer coefficients:

$$\begin{aligned} \psi_1 &\equiv c_1 \frac{T^2\psi_2}{2} \\ \psi_2 &\equiv c_2 \frac{T\psi_1}{2} \\ \psi_3 &\equiv c_3 \frac{T^3\psi_3}{2} \end{aligned}$$

The numbers c_s are defined in terms of Morita's 2-adic Γ-function [Mor75] and are not 2-local integers.

It follows that the desired Honda matrix is

$$H = \begin{bmatrix} 1 & -c_1T^2/2 & 0 \\ -c_2T/2 & 1 & 0 \\ 0 & 0 & 1 - c_3T^3/2 \end{bmatrix}$$

The block decomposition of this matrix implies that the formal group law splits into 1- and 2-dimensional summands. The logarithm for the former is

$$\left(1 - c_3T^3/2\right)^{-1} * x = \sum_{n\geq 0} \frac{c_3^n x^{8^n}}{2^n} = x + \frac{c_3x^8}{2} + \frac{c_3^2x^{64}}{4} + \cdots$$

and its height is 3.

References

[Bau] T. Bauer. Computation of the homotopy of the spectrum tmf. Online at `http://wwwmath.uni-muenster.de/u/tbauer/` .

[HM] M. J. Hopkins and M. A. Mahowald. From elliptic curves to homotopy theory. Preprint in Hopf archive at `http://hopf.math.purdue.edu/Hopkins-Mahowald/eo2homotopy`.

[Hon70] Taira Honda. On the theory of commutative formal groups. *J. Math. Soc. Japan*, 22:213–246, 1970.

[Hon73] Taira Honda. On the formal structure of the Jacobian variety of the Fermat curve over a p-adic integer ring. In *Symposia Mathematica, Vol. XI (Convegno di Geometria, INDAM, Rome, 1972)*, pages 271–284. Academic Press, London, 1973.

[Lan76] P. S. Landweber. Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$. *American Journal of Mathematics*, 98:591–610, 1976.

[Man63] Ju. I. Manin. Theory of commutative formal groups over fields of finite characteristic. *Uspehi Mat. Nauk*, 18(6 (114)):3–90, 1963.

[Mor75] Yasuo Morita. A p-adic analogue of the Γ-function. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 22(2):255–266, 1975.

[Qui69] D. G. Quillen. On the formal group laws of oriented and unoriented cobordism theory. *Bulletin of the American Mathematical Society*, 75:1293–1298, 1969.

University of Rochester, Rochester, NY 14627