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Periodic phenomena in the Adams-Novikov spectral sequence

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Introduction

The problem of understanding the stable homotopy ring has long been one of the touchstones of algebraic topology. Low dimensional computation has proceeded slowly and has given little insight into the general structure of $\pi_*^S(S^0)$. In recent years, however, infinite families of elements of $\pi_*^S(S^0)$ have been discovered, generalizing the image of the Whitehead J-homomorphism. In this work we indicate a general program for the detection and description of elements lying in such infinite families. This approach shows that every homotopy class is, in some attenuated sense, a member of such a family.

For our algebraic grip on homotopy theory we shall employ S. P. Novikov's analogue of the Adams spectral sequence converging to the stable homotopy ring. Its E_2-term can be described algebraically as the cohomology of the Landweber-Novikov algebra of stable operations in complex cobordism. In his seminal work on the subject, Novikov computed the first cohomology group and showed that it was canonically isomorphic to the image of J away from the prime 2. When localized at an odd prime p these elements occur only every $2(p-1)$ dimensions; so this first cohomology group has a periodic character. Our intention here is to show that the entire cohomology is built up in a very specific way from periodic constituents. Our central application of these ideas is the computation of the second cohomology group at odd primes.

By virtue of the Adams-Novikov spectral sequence this information has a number of homotopy-theoretic consequences. The homotopy classes β_t, $t \geqq 1$, in the p-component of the $\big(2(p^2-1)t-2(p-1)-2\big)$-stem for $p>3$, constructed by L. Smith, are detected here. Indeed, it turns out that all elements with Adams-Novikov filtration exactly 2 are closely related to the β family. The lowest dimensional elements of filtration 2 aside from the β family itself are the elements denoted ε_j by Toda. The computation of the

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second cohomology provides an upper bound on the number of elements generalizing the ε_j's. We also show that p never divides β_t.

The formalism of our approach is remarkably convenient. It shows for example that the nontriviality of β_t for all $t \geqq 1$ and all $p > 3$ follows immediately from a slight reformulation of Novikov's calculation of the first cohomology group. Similarly, and more importantly, using the second cohomology we are able to prove that H. Toda's elements γ_t in the p-component of the $(2(p^3 - 1)t - 2(p^2 - 1) - 2(p - 1) - 3)$-stem are nontrivial for all $t \geqq 1$ and all $p > 5$. Products are also quite easily studied; for example, we give a condition on t guaranteeing that $\alpha_1\beta_t \neq 0$ in $\pi_*^S(S^0)$.

Since Novikov's work a number of advances in our understanding of complex cobordism and its operations have occurred. A remarkable and useful connection between complex cobordism and formal groups was discovered by D. Quillen. Quillen used this to split up the localization of complex cobordism at a prime p. The summands in this splitting are suspensions of the Brown-Peterson theory BP, which thus contains all the information of complex cobordism at the prime p. Quillen's work was put on a firm computational basis by M. Hazewinkel's construction of canonical polynomial generators v_n of dimension $2(p^n - 1)$ for the coefficient ring $\mathrm{BP}_* = \mathbf{Z}_{(p)}[v_1, v_2, \cdots]$. As a result of these advances it is easier to use the smaller theory BP when dealing with a problem one prime at a time, and we do so. It is also more convenient to use the dual of the algebra of BP operations; and for any comodule M over the resulting "coalgebra" $\mathrm{BP}_*\mathrm{BP}$, we shall write $H^s(M)$ for the graded cohomology group $\mathrm{Ext}^{s,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, M)$. Thus our object of study is $H^*(\mathrm{BP}_*)$, and our main computation gives $H^2(\mathrm{BP}_*)$.

The discovery which motivated the present research is due to Jack Morava, and the program described here is an outgrowth of his work. Morava proved a "localization theorem" identifying the cohomology group $H^*(v_n^{-1}\mathrm{BP}_*/(p, v_1, \cdots, v_{n-1}))$ with the continuous cohomology of a certain p-adic Lie group already familiar in local algebraic number theory. This relation led to a striking finiteness result for these cohomology groups, and to their computation in various cases. In a sense, the machine described in this paper reconstructs the Adams-Novikov E_2-term $H^*(\mathrm{BP}_*)$ from these localized groups.

More specifically, we construct a long exact sequence of comodules

$$0 \longrightarrow \mathrm{BP}_* \longrightarrow M^0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow \cdots$$

in which M^n is "v_n-local," in the sense that v_n acts bijectively and M^n is v_i-torsion for all $i < n$. Thus for example $M^0 = p^{-1}\mathrm{BP}_* = \mathbf{Q} \otimes \mathrm{BP}_*$. The cohomology $H^*(M^n)$ can be recovered from Morava's cohomology groups by a chain of n exact couples of Bockstein type, since M^n can be obtained from $v_n^{-1}\mathrm{BP}_*/(p, v_1, \cdots, v_{n-1})$ by introducing higher torsion one generator at a time. Application of $H^*(-)$ to the displayed long exact sequence now leads in the usual way to a spectral sequence converging to $H^*(\mathrm{BP}_*)$. We call it the "chromatic" spectral sequence. In it, $E_1^{n,*} = H^*(M^n)$ is "monochromatic," and differentials may be thought of as "interference." The entire spectral sequence is of course an algebraically defined object. The subquotient of $H^*(\mathrm{BP}_*)$ associated with M^n constitutes the "n^{th} order periodic part" of the Adams-Novikov E_2-term. Thus Im J exemplifies first order periodicity, the β family is second order, and the γ family is third order. Morava's finiteness theorems imply that the chromatic spectral sequence exhibits a broken vanishing parabola: if $p-1$ does not divide n then $H^i(M^n) = 0$ for $i > n^2$. In this case n^{th} order periodicity can occur in $H^i(\mathrm{BP}_*)$ only for $n \leqq i \leqq n^2 + n$. One of the most useful features of the spectral sequence is that such elements of $H^i(\mathrm{BP}_*)$ are represented by classes in $H^{i-n}(M^n)$. Since computational difficulties tend to increase with cohomological degree, this reduction results in substantial simplifications. In fact, the computation of $H^2(\mathrm{BP}_*)$ reduces to the determination of $H^0(M^2)$, which is a subgroup of M^2 and hence can be studied relatively easily.

Before the appearance of Morava's localization theorem it appeared that one should study periodic families in homotopy by means of periodic homology theories, since the Adams-Novikov E_2-term itself appeared to be formidably complex. For each n, Morava had constructed a theory $K(n)$ on which v_n acted as a periodicity operator. $K(1)$ is a factor of mod p complex K-theory, and one hoped to detect higher periodic families by means of these higher K-theories in analogy with Adams' detection of Im J by means of ordinary K-theory. This more geometric approach is still likely to bear fruit. It is closely related to the algebraic program initiated here; for example, the p-adic groups appearing in the localization theorem act as stable operations on the $K(n)$'s.

The choice of BP as a detecting theory for infinite families is not entirely a matter of personal preference. To describe the favored role it appears to play in homotopy theory we must briefly describe the method of constructing the periodic families in $\pi_*^S(S^0)$ studied here. This program as well as the realization of the special utility of complex cobordism is due to Larry Smith.

Periodic families arise from a self-map $\phi: S^d V \rightarrow V$ of a finite complex V such that ϕ is neither nilpotent nor a homotopy equivalence. Such self-maps appear to be rare in nature. All known examples share the property that they induce nontrivial and in fact non-nilpotent maps in BP homology. Homotopy elements can be constructed by means of the composition

$$S^{dt} \longrightarrow S^{dt}V \xrightarrow{\phi^t} V \longrightarrow S^k$$

where the first map is the inclusion of the bottom cell and the last map is the projection to the top cell.

The fact that these self-maps have BP-filtration zero leads one to hope that the resulting homotopy classes in $\pi_*^S(S^0)$ have BP filtration which is small and (at least for t large) independent of t. This has in fact always proved to be the case. The detection of these elements thus reduces to the algebraic problem of showing that they are nontrivial in the E_2-term $H^*(\mathrm{BP}_*)$ of the Adams-Novikov spectral sequence. This is in contrast to the situation in the classical Adams spectral sequence, in which these elements occur in filtration increasing with t. The algebra rapidly becomes prohibitively difficult, and information at the E_2 level will no longer suffice. In a sense, the present work provides machinery for maximizing the homotopy-theoretic consequences one may deduce from the existence of such self-maps. There is still much to be done in this program, and many more homotopy-theoretic questions could be answered by pressing these calculations further.

The paper is divided into ten sections.

1. Recollections
2. Statement of results
3. The chromatic spectral sequence
4. $H^*M_0^1$
5. $H^0M_n^1$
6. $H^0M_0^2$
7. Computation of the differential
8. On certain products
9. The Thom reduction
10. Concluding remarks

After recalling conventions regarding BP, we state in Section 2 our principal results on the Novikov E_2-term and deduce from them a variety of homotopy-theoretic consequences. Next we construct the chromatic spectral sequence and outline our program for computing its E_1-term. The succeeding three sections are devoted to this computation in the range we need. Then in Section 7 we derive $H^2(\mathrm{BP}_*)$ for $p \geqq 3$ and prove that γ_t survives to the Novikov E_2-term; combined with Section 2 this completes the proof of the nontriviality of γ_t, and furthermore shows that p does not divide β_t for any t.

Next we present a selection of results on products of alphas and betas; this implies the nontriviality of a large collection of hitherto inaccessible products in stable homotopy; these are described in Section 2. Finally we compare $\mathrm{Ext}^2_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*)$ with $\mathrm{Ext}^2_{A_*}(\mathbf{F}_p, \mathbf{F}_p)$, A_* the dual Steenrod algebra at the odd prime p, and deduce an upper bound on the survivors in that group. We conclude with a discussion of various questions which are raised or made accessible by this work.

Readers interested only in the detection of the γ_t's can avoid a good deal of the paper. After reading Sections 1 and 3 and the relevant part of 2 the gamma hunter need read only the very beginnings of Sections 4 and 5, all of 6 and the first part of 7.

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1. Recollections on BP and the Adams-Novikov spectral sequence

Let p be a prime number and let BP denote the Brown-Peterson [4] ring-spectrum at p constructed by Quillen [24]. Thus $H_*(\mathrm{BP}) = \mathbf{Z}_{(p)}[m_1, m_2, \cdots]$ for canonical generators m_i of dimension $2(p^i - 1)$. Under the Hurewicz map $\pi_*(\mathrm{BP})$ is embedded in $H_*(\mathrm{BP})$ as $\mathbf{Z}_{(p)}[v_1, v_2, \cdots]$, where the v_i are Hazewinkel's generators [6], [7], described inductively by

$$v_n = pm_n - \textstyle\sum_{i=1}^{n-1} v_{n-i}^{p^i} m_i \ . \tag{1.1}$$

These allow one to translate into homotopy the formulae of Quillen [24] (as interpreted by Adams [3]) for the diagonal $\Delta\colon \mathrm{BP}_*\mathrm{BP} \to \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP}$ and the right unit $\eta_R\colon \mathrm{BP}_* \to \mathrm{BP}_*\mathrm{BP}$:

$$\textstyle\sum_{i+j=n} m_i(\Delta t_j)^{p^i} = \sum_{i+j+k=n} m_i t_j^{p^i} \otimes t_k^{p^{i+j}} \ , \tag{1.2}$$

$$\eta_R m_n = \textstyle\sum_{i+j=n} m_i t_j^{p^i} \tag{1.3}$$

where $m_0 = t_0 = 1$ and $\mathrm{BP}_*\mathrm{BP} = \mathrm{BP}_*[t_1, t_2, \cdots]$, $|t_i| = 2(p^i - 1)$. We also have the left unit $\eta_L\colon \mathrm{BP}_* \to \mathrm{BP}_*\mathrm{BP}$ by $\eta_L v_n = v_n$, the augmentation $\varepsilon\colon \mathrm{BP}_*\mathrm{BP} \to \mathrm{BP}_*$ by $\varepsilon v_n = v_n$, $\varepsilon t_n = 0$, and the Hopf conjugation $c\colon \mathrm{BP}_*\mathrm{BP} \to \mathrm{BP}_*\mathrm{BP}$ defined inductively by

$$\sum_{i+j+k=n} m_i t_j^{p^i}(ct_k)^{p^{i+j}} = m_n . \tag{1.4}$$

Definition 1.5 An ideal $I \subset \mathrm{BP}_*$ is *invariant* if and only if $I \cdot \mathrm{BP}_*\mathrm{BP} = \mathrm{BP}_*\mathrm{BP} \cdot I$. $a \in \mathrm{BP}_*$ is *invariant mod I* if and only if $\eta_R a \equiv \eta_L a \bmod I \cdot \mathrm{BP}_*\mathrm{BP}$.

Landweber ([10]; see also [18], [9]) showed that the only invariant *prime* ideals are (with $v_0 = p$)

$$I_n = (p, v_1, \cdots, v_{n-1}) , \quad 0 \leqq n \leqq \infty . \tag{1.6}$$

Many results concerning η_R and Δ mod I_n may be found in [25], among them:

$$\eta_R v_{n+1} \equiv v_{n+1} + v_n t_1^{p^n} - v_n^p t_1 \bmod I_n . \tag{1.7}$$

For a direct proof of this formula see [16]; it was also discovered by Landweber.

For a spectrum X, $M = \mathrm{BP}_*X$ carries a natural associative, unitary coaction $\psi\colon M \to \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} M$. In general a BP_*-module M together with such a map ψ will be called a left $\mathrm{BP}_*\mathrm{BP}$-*comodule*; right comodules are defined analogously.

It will be a notational convenience in the sequel to have $\mathrm{BP}_*\mathrm{BP}$ coacting from the right. Note that the categories of left comodules and of right comodules are naturally equivalent, by means of the following standard device. Let (M, ψ_L) be a left comodule. Regard M as a right BP_*-module (BP_* is commutative), and define $\psi_R = \tilde{T} \circ \psi_L$ where $\tilde{T}$ makes the diagram

$$\begin{array}{ccc} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathbf{Z}_{(p)}} M & \xrightarrow{T(c \otimes M)} & M \otimes_{\mathbf{Z}_{(p)}} \mathrm{BP}_*\mathrm{BP} \\ \downarrow & & \downarrow \\ \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} M & \xrightarrow{\tilde{T}} & M \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} \end{array}$$

commute; here c is the Hopf conjugation. Then ψ_R is a right coaction on M.

Henceforth all left comodules will be tacitly converted to right comodules via this device.

Given a comodule M we may define

$$H^*M = \mathrm{Ext}^*_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, M) = H^*(\Omega^*M, d) \tag{1.8}$$

where the *cobar complex* Ω^*M is the $DG\mathbf{Z}_{(p)}$-module with

$$\Omega^t M = M \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} \tag{1.9}$$

(t factors of $\mathrm{BP}_*\mathrm{BP}$) and differential d of degree $+1$ given by

$$\begin{aligned} d(m \otimes x_1 \otimes \cdots \otimes x_t) = {} & \sum m' \otimes m'' \otimes x_1 \otimes \cdots \otimes x_t \\ & + \sum_{i=1}^t (-1)^i m \otimes x_1 \otimes \cdots \otimes x_i' \otimes x_i'' \otimes \cdots \otimes x_t \\ & - (-1)^t m \otimes x_1 \otimes \cdots \otimes x_t \otimes 1 \end{aligned} \tag{1.10}$$

where $\Delta x_i = \sum x'_i \otimes x''_i$ and $\psi m = \sum m' \otimes m''$.

Then for a spectrum X there is a spectral sequence, due to Adams [1], [3], and Novikov [21], with $E_2 = H^*(\mathrm{BP}_* X)$, which converges if X is connective to the localization at p of the stable homotopy $\pi_*(X)$.

Remark 1.11. Note that $H^0 M \subset M$, so cycles are unique and induced maps are easy to evaluate. One motivation for the program of this paper is to reduce the computation of higher Ext groups to an H^0 computation. In the case $M = \mathrm{BP}_*/I$ where I is an invariant ideal (see 1.5), $H^0 M$ is just the mod I reduction of the group of all $a \in \mathrm{BP}_*$ which are invariant mod I.

Note 1.12. Note that if $M_i = 0$ for $i \not\equiv 0 \bmod q$, $q = 2(p-1)$, then $H^{t,i} M = 0$ also for $i \not\equiv 0 \bmod q$. This holds for example if $M = \mathrm{BP}_*/I_n$.

Note 1.13. Since Ω^t is exact, an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of comodules induces a natural long exact sequence

$$0 \longrightarrow H^0 M' \longrightarrow H^0 M \longrightarrow H^0 M'' \xrightarrow{\delta} H^1 M' \longrightarrow \cdots$$

in Ext.

Note. 1.14. $\Omega^*(\mathrm{BP}_*/I_n)$ is a tensor algebra, and the differential clearly acts as a derivation. More generally, if the comodule M is annihilated by I_n, then $\Omega^* M$ is a $DG\Omega^*(\mathrm{BP}_*/I_n)$-module. Thus $H^* M$ is a module over the algebra $H^*(\mathrm{BP}_*/I_n)$.

Note 1.15. The cobar complex contains a smaller *normalized cobar complex* $\tilde{\Omega}^* M$ given by

$$\tilde{\Omega}^t M = M \otimes_{\mathrm{BP}_*} \ker(\varepsilon) \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} \ker(\varepsilon)$$

with t factors of $\ker(\varepsilon) \subset \mathrm{BP}_*\mathrm{BP}$. The inclusion is as usual a chain-equivalence (see [5]). Now $\ker(\varepsilon)$ is $(q-1)$-connected, so if M is $(m-1)$-connected then $\Omega^t M$ is $(qt + m - 1)$-connected, and we have:

LEMMA 1.16 [21]. *If M is an $(m-1)$-connected comodule then $H^t M$ is $(qt + m - 1)$-connected.*

Remark 1.17. In fact [36], $H^{2t} M$ is $(pqt + m - 1)$-connected and $H^{2t+1} M$ is $((pt+1)q + m - 1)$-connected; but we do not need this stronger vanishing line here.

2. Statement of results

This section contains a statement of our main results, some of which were announced in [15]. We begin with our results on E_2 of the Novikov spectral sequence for the sphere.

For the purpose of establishing notation, we recall first the structure of $H^1\mathrm{BP}_*$ at an odd prime p. (For remarks on $p = 2$, see § 4.) If $n \geqq 0$ and $s \geqq 1$ then $v_1^{sp^n}$ is invariant mod p^{n+1}: that is, $v_1^{sp^n} \in H^0(\mathrm{BP}_*/p^{n+1})$. Define

$$\alpha_{sp^n/n+1} = \delta(v_1^{sp^n}) \in H^1\mathrm{BP}_* \tag{2.1}$$

where δ is the boundary-homomorphism associated as in Note 1.13 to the short exact sequence

$$0 \longrightarrow \mathrm{BP}_* \xrightarrow{p^{n+1}} \mathrm{BP}_* \longrightarrow \mathrm{BP}_*/p^{n+1} \longrightarrow 0 .$$

THEOREM 2.2. *Let p be odd.*

a) (*Novikov* [21]) *$H^1\mathrm{BP}_*$ is generated by $\alpha_{sp^n/n+1}$ for $n \geqq 0$, $p \nmid s \geqq 1$. $\alpha_{sp^n/n+1}$ has order p^{n+1}.*

b) *For $m, n \geqq 0$ and $s, t \geqq 1$,*

$$\alpha_{sp^m/m+1}\alpha_{tp^n/n+1} = 0 .$$

Theorem 2.2 a) is proved in Section 4 (Remark 4.9), and Theorem 2.2 b) in Section 8 (Theorem 8.18). We shall abbreviate $\alpha_{sp^m/1}$ to α_{sp^m}.

Remark 2.3. Novikov proved a) and showed moreover that the p-primary component of the image of the J-homomorphism maps isomorphically to $H^1\mathrm{BP}_*$ for an odd prime p. Since it is known that products of elements of odd order in $\mathrm{Im}\, J$ are 0, b) follows from the multiplicativity of the Novikov spectral sequence. Our proof of Theorem 2.2 however is purely algebraic.

Turn now to $H^2\mathrm{BP}_*$. Let $a_0 = 1$ and $a_n = p^n + p^{n-1} - 1$ for $n \geqq 1$. The results of [16] (proved also in § 5 below) imply that certain classes $\beta_{sp^n/j}$ for $n \geqq 0$, $p \nmid s \geqq 1$, $1 \leqq j \leqq a_n$ with $j \leqq p^n$ if $s = 1$, generate the submodule of $H^2\mathrm{BP}_*$ of elements of order p. Our basic algebraic result is simply that $\beta_{sp^n/j}$ is divisible by p^i if and only if $p^i \mid j \leqq a_{n-i}$.

To be precise, define elements $x_n \in v_2^{-1}\mathrm{BP}_*$ by

$$\begin{aligned} x_0 &= v_2 , \\ x_1 &= x_0^p - v_1^p v_2^{-1} v_3 , \\ x_2 &= x_1^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3 , \\ x_n &= x_{n-1}^p - 2v_1^{b_n} v_2^{p^n - p^{n-1}+1} \quad \text{for } n \geqq 3 , \end{aligned} \tag{2.4}$$

with $b_n = (p + 1)(p^{n-1} - 1)$ for $n > 1$. Now if $s \geqq 1$ and $p^i \mid j \leqq a_{n-i}$ with $j \leqq p^n$ if $s = 1$, then (p^{i+1}, v_1^j) is invariant and $x_n^s \in H^0\mathrm{BP}_*/(p^{i+1}, v_1^j)$. The element x_n^s lies in

$$\mathrm{BP}_*/(p^{i+1}, v_1^j) \subset v_2^{-1}\mathrm{BP}_*/(p^{i+1}, v_1^j)$$

despite the v_2^{-1} which appears in its definition. Let

$$\beta_{sp^n/j,i+1} = \delta'\delta'' x_n^s \in H^2\mathrm{BP}_* \tag{2.5}$$

where δ' (resp. δ'') is the boundary associated to E' (resp. E''):

$$E' : 0 \longrightarrow \mathrm{BP}_* \xrightarrow{p^{i+1}} \mathrm{BP}_* \longrightarrow \mathrm{BP}_*/p^{i+1} \longrightarrow 0 ,$$

$$E'' : 0 \longrightarrow \mathrm{BP}_*/p^{i+1} \xrightarrow{v_1^j} \mathrm{BP}_*/p^{i+1} \longrightarrow \mathrm{BP}_*/(p^{i+1}, v_1^j) \longrightarrow 0 .$$

We shall abbreviate $\beta_{sp^n/j,1}$ to $\beta_{sp^n/j}$ and $\beta_{sp^n/1}$ to β_{sp^n}.

THEOREM 2.6. *Let p be odd. $H^2\mathrm{BP}_*$ is a direct sum of cyclic subgroups generated by $\beta_{sp^n/j,i+1}$ for $n \geqq 0$, $p \nmid s \geqq 1$, $j \geqq 1$, $i \geqq 0$, subject to*

i) *$j \leqq p^n$ if $s = 1$,*

ii) *$p^i \mid j \leqq a_{n-i}$, and*

iii) *$a_{n-i-1} < j$ if $p^{i+1} \mid j$.*

$\beta_{sp^n/j,i+1}$ has order p^{i+1}.

Theorem 2.2 shows that $\beta_{sp^n/j,i+1}$ is indecomposable. The internal dimension of $\beta_{sp^n/j,i+1}$ is $2(p^2 - 1)sp^n - 2(p - 1)j$.

For $H^3\mathrm{BP}_*$ we have only partial information, of which we state only the highlights here. For $t \geqq 1$, let $\gamma_t \in H^3\mathrm{BP}_*$ denote the obvious triple boundary of $v_3^t \in H^0\mathrm{BP}_*/(p, v_1, v_2)$.

THEOREM 2.7. *Let p be odd. Then $\gamma_t \neq 0$ for all $t \geqq 1$.*

The proofs of Theorems 2.6 and 2.7 are completed in Section 7.

We remark that the second author has shown that $p \nmid \gamma_t$ for $t \not\equiv 0, 1 \bmod p$; see Section 10.

THEOREM 2.8. *Let p be odd, $n \geqq 0$, $p \nmid s \geqq 1$, $1 \leqq j \leqq a_n$ with $j \leqq p^n$ if $s = 1$.*

a) *$\alpha_1\beta_{p-1} = -\gamma_1$.*

b) *In $H^3\mathrm{BP}_*$, $\alpha_1\beta_{sp^n/j} \neq 0$ if and only if one of the following conditions holds.*

i) *$j = 1$ and either $s \not\equiv -1(p)$ or $s \equiv -1(p^{n+2})$,*

ii) *$j = 1$ and $s = p - 1$,*

iii) *$j > 1 + a_{n-\nu(j-1)-1}$.*

The only linear relations among these elements are

$$\alpha_1\beta_{sp^2/2+p} = s\alpha_1\beta_{sp^2-1} ,$$
$$\alpha_1\beta_{sp^{2n+2}/2+a_{n+1}} = 2s\alpha_1\beta_{sp^{2n+2}-p^n} , \qquad n \geqq 1.$$

c) *If $0 \neq \alpha_1\beta_{sp^n/j} \in H^3\mathrm{BP}_*$ then it is divisible by at least p^i whenever $0 < i \leqq n$ and $j \leqq a_{n-i}$.*

d) *For all $m \geqq 0$ and $t \geqq 1$, $\alpha_{tp^m/m+1}\beta_{sp^n/j} = t\alpha_1\beta_{sp^n/j-tp^m+1}$.*

Here $\nu(a)$ is the largest ν such that $p^\nu | a$, and by convention $\beta_{sp^n/j} = 0$ for $j \leqq 0$.

These results are proved in Section 8.

We may now appeal to the Novikov spectral sequence to deduce facts about the p-component of the stable homotopy ring $\pi_*(S^0)$. We shall use two accessory facts about this spectral sequence: its multiplicative structure and its behavior with respect to certain cofiber sequences. We recall the latter from [8].

Let

$$X' \longrightarrow X \longrightarrow X'' \tag{2.9}$$

be a cofiber sequence, and let

$$\partial\colon \pi_i X'' \longrightarrow \pi_{i-1} X'$$

be the associated "geometric" boundary homomorphism, induced from $h\colon X'' \to \sum X'$. Suppose that $\mathrm{BP}_*(h) = 0$. Then (2.9) induces a short exact sequence in BP-homology and there results from Note 1.13 an "algebraic" boundary homomorphism

$$\delta\colon H^s\mathrm{BP}_*X'' \longrightarrow H^{s+1}\mathrm{BP}_*X' .$$

LEMMA 2.10 [8]. *If $\bar{x} \in H^s\mathrm{BP}_*X''$ survives to $x \in \pi_*(X'')$, then $\delta\bar{x} \in H^{s+1}\mathrm{BP}_*X'$ survives to $\partial x \in \pi_*(X')$.*

We also rely upon the following geometric input. Let $V(-1) = S^0$ be the sphere spectrum. For $n = 0, 1$ (Adams [2]), 2 (Smith [29]), and 3 (Toda [34]), there is for $p > 2n$ a cofibration sequence

$$\textstyle\sum^{2(p^n-1)} V(n-1) \xrightarrow{\phi_n} V(n-1) \longrightarrow V(n) \tag{2.11}$$

such that ϕ_n induces multiplication by v_n in BP-homology of $V(n-1)$. The self-maps ϕ_n induce periodic families of elements in $\pi_*(S^0)$ as follows. Let $\partial_n\colon \pi_i(V(n)) \to \pi_{i-2(p^n-1)-1}(V(n-1))$ be the boundary homomorphism induced by (2.11), and let $\iota\colon S^0 \to V(n-1)$ be the inclusion of the bottom cell. Then define, for $t \geqq 1$,

$$\begin{aligned} \alpha_t &= \partial_0(\phi_1^t\iota) , & p &\geqq 3 , \\ \beta_t &= \partial_0\partial_1(\phi_2^t\iota) , & p &\geqq 5 , \\ \gamma_t &= \partial_0\partial_1\partial_2(\phi_3^t\iota) , & p &\geqq 7 . \end{aligned}$$

Now ϕ_n induces an injection in BP-homology, so Lemma 2.10 applies. Thus, by their construction, the classes α_t, β_t, γ_t survive in the Novikov spectral sequence to the stable homotopy element of the same name. Furthermore, none of these elements can be hit by a Novikov differential because their homological degree is too low (see Note 1.12). Since Theorems 2.2a), 2.6, and 2.7 insure that they are nonzero in E_2, we have:

THEOREM 2.12. a) (*Toda* [33]) *For* $p \geqq 3$ *and* $t \geqq 1$, $\alpha_t \neq 0$ *in* $\pi^*(S^0)$.
b) (*Smith* [29]]) *For* $p \geqq 5$ *and* $t \geqq 1$, $\beta_t \neq 0$ *in* $\pi_*(S^0)$.
b)′ *Furthermore p does not divide* β_t *in* $\pi_*(S^0)$.
c) *For* $p \geqq 7$ *and* $t \geqq 1$, $\gamma_t \neq 0$ *in* $\pi_*(S^0)$.

Partial results on the nontriviality of γ_t have been obtained by Thomas and Zahler [31] and Oka and Toda [23] ($t = 1$), Thomas and Zahler [32] ($t = ap + b$, $0 \leqq a < b < p$), Johnson, Miller, Wilson, and Zahler [8] ($t = sp^n$, $1 < s < p$, $n > 0$), and Ravenel (unpublished; see §10 below) ($t \not\equiv 0, 1 \pmod p$).

Now the work of Moss [20] shows that the Novikov spectral sequence for the sphere is multiplicative. At E_∞ the product is associated to the composition pairing; at E_2 it agrees with the multiplication defined in Note 1.14.

By sparseness (Note 1.12), no differential in the Novikov spectral sequence can hit $H^3\mathrm{BP}_*$. Thus Theorem 2.8 b) immediately results in:

THEOREM 2.13. *Let* $p \geqq 5$. *For* $n \geqq 0$ *and* $p \nmid s \geqq 1$, $\alpha_1\beta_{sp^n} \neq 0$ *in* π_*S^0 *if one of the following holds.*

i) $s \not\equiv -1 \pmod p$,
ii) $s \equiv -1 \pmod{p^{n+2}}$,
iii) $s = p - 1$.

Smith [30] and Zahler [36] have shown that for $p \geqq 5$ and $t \geqq 1$, $\beta_{tp/j}$ survives in the Novikov spectral sequence for $1 \leqq j \leqq p - 1$. Oka has also obtained this result and extended it to include $\beta_{tp^2/j}$ for $1 \leqq j \leqq 2p - 2$ [22], and $\beta_{tp/p}$ for $t \geqq 2$ [39], $\beta_{tp^2/j}$ for $t \geqq 2$ and $j \leqq 2p$ [40], and $\beta_{tp^2/p,2}$ (elements of order p^2) for $t \geqq 2$ [40]. We remark that once they construct the appropriate complex and stable self-map, the survival of these elements is immediate from the above considerations. In any case, we have from Theorems 2.6 and 2.8:

THEOREM 2.14. *None of the elements of Oka, Smith and Zahler described in the preceding paragraph are divisible by p except possibly for* $\beta_{tp^2/p}$, $t \geqq 1$, *and* $\beta_{tp^3/2p}$, $t > 1$, *and these cannot be divisible by* p^2.

Proof. Multiplication by p can never lower the filtration of a homotopy element in the Novikov spectral sequence. There are no elements in $H^0\mathrm{BP}_*$ or $H^1\mathrm{BP}_*$ in these degrees so the nondivisibility follows from the nondivisibility in $H^2\mathrm{BP}_*$ from Theorem 2.6. □

THEOREM 2.15. *Let* $p \geqq 5$ *and* $p \nmid s \geqq 1$. *In* $\pi_*(S^0)$,

a) $\alpha_1\beta_{sp/j} \neq 0$ *for* $3 \leqq j \leqq p-1$, *and*

b) $\alpha_1\beta_{sp^2/j} \neq 0$ *for* $p+1 \leqq j \leqq 2p-2$.

Finally we study the Thom map $H^*\mathrm{BP}_* \to \mathrm{Ext}^*_{A_*}(\mathbf{F}_p, \mathbf{F}_p)$ to the E_2-term of the classical Adams spectral sequence. This has the following corollary; for notation, see Section 9.

THEOREM 2.16. *Let* $p > 2$. *In the classical Adams spectral sequence for the sphere,*

a) (*Liulevicius* [12], *Shimada-Yamanoshita* [28]) *Of the generators* (9.2) *of* $\mathrm{Ext}^1_{A_*}(\mathbf{F}_p, \mathbf{F}_p)$ *only* a_0, h_0 *can survive, and*

b) *Of the generators* (9.3) *of* $\mathrm{Ext}^2_{A_*}(\mathbf{F}_p, \mathbf{F}_p)$, *only the following can survive in the Adams spectral sequence:* a_1, $b_i (i \geqq 0)$, k_1, a_0^2, $h_0h_i (i \geqq 2)$; *and if* $p = 3$, a_0h_1.

3. The chromatic spectral sequence and the cohomology of the Morava stabilizer algebras

In this section we describe the key tool of this paper, a spectral sequence converging to the Novikov E_2-term. We then link the E_1-term of this spectral sequence to the cohomology of Morava's stabilizer algebras by a sequence of Bockstein exact couples.

A. *The chromatic spectral sequence.* Let M be a $\mathrm{BP}_*\mathrm{BP}$-comodule. If M is I_n-torsion, i.e., for all $x \in M$ there exists $k \geqq 0$ such that $I_n^k x = 0$, then ([14]) $v_n^{-1}M$ has a unique comodule structure such that the localization map $M \to v_n^{-1}M$ is a map of comodules.

In particular let $N_n^0 = \mathrm{BP}_*/I_n$. Assuming N_n^s has been defined, set $M_n^s = v_{n+s}^{-1}N_n^s$, and let

$$0 \longrightarrow N_n^s \xrightarrow{\ j\ } M_n^s \xrightarrow{\ k\ } N_n^{s+1} \longrightarrow 0 \tag{3.1}$$

be exact. Thus one might write

$$N_n^s = \mathrm{BP}_*/(p, \cdots, v_{n-1}, v_n^\infty, \cdots, v_{n+s-1}^\infty)\ ,$$
$$M_n^s = v_{n+s}^{-1}\mathrm{BP}_*/(p, \cdots, v_{n-1}, v_n^\infty, \cdots, v_{n+s-1}^\infty)\ .$$

Let $E = (e_0, e_1, \cdots)$, $e_i \in \mathbf{Z}$, with all but a finite number of the e_i equal to 0. The element $v^E \in M_n^s$, where

$$e_i = 0 \quad \text{for } 0 \leqq i < n\ ,$$
$$e_i < 0 \quad \text{for } n \leqq i < n+s\ ,$$
$$e_i \geqq 0 \quad \text{for } n+s < i\ ,$$

will consistently be written as

$$\frac{v_{n+s}^{e_{n+s}} v_{n+s+1}^{e_{n+s+1}} \cdots}{v_n^{-e_n} \cdots v_{n+s-1}^{-e_{n+s-1}}} .$$

We shall use the convention $v_0 = p$.

We shall also consistently regard these objects as *right* $\mathrm{BP}_*\mathrm{BP}$-comodules. The coactions on M_n^s and N_n^s are then induced in an evident way from the right coaction $\mathrm{BP}_* \xrightarrow{\eta_R} \mathrm{BP}_*\mathrm{BP} \cong \mathrm{BP}_* \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP}$ on BP_*.

Let

$$A_1^{s,t}(n) = H^t N_n^s ,$$
$$E_1^{s,t}(n) = H^t M_n^s .$$

Then the long exact sequences induced by (3.1) give rise to an exact couple of $H^*(\mathrm{BP}_*/I_n)$-modules

(3.2)
$$\begin{array}{ccc} A_1(n) & \xrightarrow{\delta'} & A_1(n) \\ {\scriptstyle k_*}\nwarrow & & \swarrow{\scriptstyle j_*} \\ & E_1(n) & \end{array}$$

with maps of bidegree $|\delta'| = (-1, 1)$, $|j_*| = (0, 0)$, $|k_*| = (1, 0)$.

Associated to this exact couple is a first quadrant cohomological spectral sequence, with an internal degree preserved by the differentials.

The following alternative construction of this spectral sequence will be useful. The short exact sequences (3.1) splice together into a complex

$$M_n^*\colon 0 \longrightarrow M_n^0 \xrightarrow{d_e} M_n^1 \xrightarrow{d_e} \cdots$$

such that

$$\begin{aligned} H^s(M_n^*, d_e) &= \mathrm{BP}_*/I_n && \text{if } s = 0 \\ &= 0 && \text{otherwise.} \end{aligned}$$

Application of Ω^* gives rise to a double complex $\Omega^* M_n^*$. Form the total complex C_n^* with

(3.3)
$$C_n^u = \bigoplus_{s+t=u} \Omega^t M_n^s$$

and differential $dx = d_e^* x + (-1)^s d_i x$ for $x \in \Omega^t M_n^s$, where d_e^* is induced from $d_e\colon M_n^s \to M_n^{s+1}$ and d_i is the differential in $\Omega^* M_n^s$.

Filter first by

$$'F^t C_n^u = \bigoplus_{\substack{s+t'=u \\ t' \geq t}} \Omega^{t'} M_n^s .$$

In the associated spectral sequence

$$\begin{aligned} 'E_1^{*,s} &= \Omega^*(\mathrm{BP}_*/I_n) && \text{for } s = 0 \\ &= 0 && \text{otherwise} \end{aligned}$$

since Ω^t is exact; and so

$$H^*(C_n^*, d) = {}'E_2^{*,0} = H^*(\mathrm{BP}_*/I_n) .$$

Now filter by

(3.4)
$$F^sC_n^u = \bigoplus_{\substack{s'+t=u \\ s'\geqq s}} \Omega^t M_n^{s'} .$$

The associated exact couple clearly agrees with (3.2), and the associated spectral sequence converges to $H^*(\mathrm{BP}_*/I_n)$.

Remarks 3.5. a) By Note 1.14, $E_*(n)$ is a spectral sequence of $H^*(\mathrm{BP}_*/I_n)$-modules, and in particular of $H^0(\mathrm{BP}_*/I_n) = k(n)_*$-modules, where $k(0)_* = \mathbf{Z}_{(p)}$ and, for $n > 0$, $k(n)_* = \mathbf{F}_p[v_n]$ ([10], [18], [9]). For $s > 0$, M_n^s is v_n-torsion, so $E_r^{s,*}(n)$ is v_n-torsion for all $r \geqq 1$.

b) The edge-homomorphism

$$H^t\mathrm{BP}_*/I_n \twoheadrightarrow E_\infty^{0,t}(n) \hookrightarrow E_1^{0,t}(n) = H^tM_n^0$$

is clearly induced by the localization map

$$\mathrm{BP}_*/I_n \longrightarrow v_n^{-1}\mathrm{BP}_*/I_n = M_n^0 .$$

c) The spectral sequence constructed in this section will be called the *chromatic spectral sequence.* In the sequel, we limit our interest mostly to the case $n = 0$, and write E_r for $E_r(0)$.

B. *Greek letters.* Now consider the following method of producing elements in $H^*\mathrm{BP}_*$. Let

$$A: a_0, a_1, \cdots, a_{s-1}$$

be an *invariant* sequence of elements of BP_*; that is, a_i is invariant mod $J_i(A)$ for $0 \leqq i \leqq s-1$, where $J_0(A) = 0$ and $J_i(A) = (a_0, \cdots, a_{i-1})$. Suppose further that A is *regular*; that is, multiplication by a_i is injective on $\mathrm{BP}_*/J_i(A)$ for $0 \leqq i \leqq s-1$. Then

$$0 \longrightarrow \mathrm{BP}_*/J_i(A) \xrightarrow{a_i} \mathrm{BP}_*/J_i(A) \longrightarrow \mathrm{BP}_*/J_{i+1}(A) \longrightarrow 0$$

is an exact sequence of comodules. Let $\delta_i: H^t\,\mathrm{BP}_*/J_{i+1}(A) \to H^{t+1}\,\mathrm{BP}_*/J_i(A)$ be the induced boundary map, and let $\eta_A: H^t\,\mathrm{BP}_*/J_s(A) \to H^{t+s}\,\mathrm{BP}_*$ be the composite $\delta_0 \cdots \delta_{s-1}$.

The elements figuring in Section 2 were defined in this manner. For example, $\gamma_t = \eta_A(v_3^t)$ for $A: p, v_1, v_2$.

This construction is related to our spectral sequence in the following way. Let

(3.6)
$$\eta: H^t N_0^s \longrightarrow H^{t+s}\mathrm{BP}_*$$

denote the composite

$$H^t N_0^s \xrightarrow{\delta'} H^{t+1}N_0^{s-1} \xrightarrow{\delta'} \cdots \xrightarrow{\delta'} H^{t+s}N_0^0$$

with δ' as in (3.2). Then:

LEMMA 3.7. *Let A be an invariant regular sequence of length s. Then there is a canonical comodule map i_A: $\mathrm{BP}_*/J_s(A) \to N_0^s$ such that $\eta_A = \eta \cdot i_A^*$.*

Proof. The construction of $i_A = 1/a_0 \cdots a_{s-1}$ is based on the observation, due to Peter Landweber [11], that the radical of $J_s = J_s(A)$ is the invariant prime ideal I_s.

We shall inductively construct comodule maps $1/a_0 \cdots a_{n-1}$: $\mathrm{BP}_*/J_n \to N_0^n$, beginning with $\mathrm{BP}_*/J_0 \xrightarrow{\simeq} N_0^0$. So suppose $1/a_0 \cdots a_{n-1}$ has been defined. Now $v_n^t \in J_{n+1}$ for some $t \geqq 1$, so $v_n^{-1}\mathrm{BP}_*/J_{n+1} = 0$. Since localization is exact this shows that

$$a_n\colon v_n^{-1}\mathrm{BP}_*/J_n \longrightarrow v_n^{-1}\mathrm{BP}_*/J_n$$

is bijective. Let a_n^{-1} be the element which maps to 1. BP_*/J_n is I_n-torsion, so [14] $v_n^{-1}\mathrm{BP}_*/J_n$ is a comodule. Since 1 is invariant, a_n^{-1} is invariant. We now complete the diagram of comodules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{BP}_*/J_n & \xrightarrow{a_n} & \mathrm{BP}_*/J_n & \longrightarrow & \mathrm{BP}_*/J_{n+1} & \longrightarrow & 0 \\
 & & \Big\downarrow \dfrac{1}{a_0 \cdots a_{n-1}} & & \Big\downarrow a_n^{-1} & & \Big\downarrow \dfrac{1}{a_0 \cdots a_n} & & \\
 & & & & v_n^{-1}\mathrm{BP}_*/J_n & & & & \\
 & & & & \Big\downarrow \dfrac{1}{a_0 \cdots a_{n-1}} & & & & \\
0 & \longrightarrow & N_0^n & \longrightarrow & M_0^n & \longrightarrow & N_0^{n+1} & \longrightarrow & 0\,.
\end{array}$$

□

Thus η is the "universal Greek letter" map. For example, we may redefine the elements $\beta_{sp^n/j,i+1}$ of Section 2 as follows. For the stated values of n, s, j, i, $x_n^s/p^{i+1}v_1^j \in H^0 N_0^2$; and

$$\beta_{sp^n/j,i+1} = \eta(x_n^s/p^{i+1}v_1^j)\,.$$

$\mathrm{BP}_*/I_s = N_s^0 \subset N_0^s$ (by i_A, $A = p, v_1, \cdots, v_{s-1}$), and we will denote the composite $H^t\mathrm{BP}_*/I_s \to H^t N_0^s \to H^{t+s}\mathrm{BP}_*$ equally by η. Thus

$$\gamma_t = \eta(v_3^t)\,.$$

Remark 3.8. The map η also has an interpretation as the bottom edge-homomorphism in the chromatic spectral sequence. In view of the diagram

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \Big\downarrow \\
0 & \longrightarrow & H^0 N_0^s & \longrightarrow & H^0 M_0^s & \xrightarrow{k_*} & H^0 N_0^{s+1} \\
 & & & & & \searrow^{d_1} & \Big\downarrow j_* \\
 & & & & & & H^0 M_0^{s+1}
\end{array}$$

we have a natural surjection $H^0 N_0^s \to E_2^{s,0}(0)$. Then

$$\begin{array}{ccccc} H^0 N_0^s & & & & \\ \downarrow & \searrow^{(-1)^{\left[\frac{s+1}{2}\right]}\eta} & & & \\ E_2^{s,0}(0) & \twoheadrightarrow & E_\infty^{s,0}(0) & \hookrightarrow & H^s \mathrm{BP}_* \end{array} \tag{3.9}$$

commutes.

To account for this sign, let $x \in \Omega^0 N_0^s = N_0^s$ be a cycle. In terms of the double complex $\Omega^* M_0^*$, $\eta(x)$ is computed by picking elements $x_t \in \Omega^{t-1} M_0^{s-t}$, $0 < t \leqq s$, such that $d_e x_1 = j(x)$ and $d_i x_t = d_e x_{t+1}$ for $0 < t < s$; then $d_i x_s = j(y)$ for a cycle $y \in \Omega^s N_0^0$, and $\eta(x) = \{y\}$. But in virtue of our sign conventions (3.3), $d_i x_s$ is homologous to $(-1)^{[s+1/2]} d_e x_1$ in the total complex C_0^*, and the result follows.

C. *Bockstein exact couples and the Morava stabilizer algebras.* We turn now to techniques for computing $E_1^{s,*}(n) = H^* M_n^s$. Notice that we have short exact sequences of comodules for $s > 0$:

$$0 \longrightarrow M_{n+1}^{s-1} \xrightarrow{\ i\ } M_n^s \xrightarrow{\ v_n\ } M_n^s \longrightarrow 0 \tag{3.10}$$

where $i(x) = x/v_n$. These give rise to Bockstein spectral sequences in the usual way, leading from $H^* M_{n+1}^{s-1}$ to $H^* M_n^s$.

Remark 3.11. In practice we shall proceed more directly. We shall construct a partial map of exact couples

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & E^0 & \longrightarrow & B^0 & \xrightarrow{v_n} & B^0 & \longrightarrow & E^1 & \longrightarrow & \cdots & \longrightarrow & E^t \\ & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle g^0} & & \downarrow{\scriptstyle g^0} & & \downarrow{\scriptstyle f^1} & & & & \downarrow{\scriptstyle f^t} \\ 0 & \longrightarrow & H^0 M_{n+1}^{s-1} & \longrightarrow & H^0 M_n^s & \longrightarrow & H^0 M_n^s & \longrightarrow & H^1 M_{n+1}^{s-1} & \longrightarrow & \cdots & \longrightarrow & H^t M_{n+1}^{s-1} \end{array} \tag{3.12}$$

such that f^* is an isomorphism and B^* is v_n-torsion. An inductive diagram chase then shows that g^* is an isomorphism.

Using these sequences we are in principle reduced to computing

$$H^* M_n^0 = H^*(v_n^{-1} \mathrm{BP}_*/I_n) .$$

The motivation behind our entire program is the computability of these groups, which was first perceived by Jack Morava and was the subject of our previous papers [14], [26], [27]. We now recall certain parts of this work.

First there is a change of rings theorem. Let $K(0)_* = \mathbf{Q}$ and $K(n)_* = \mathbf{F}_p[v_n, v_n^{-1}]$ for $n > 0$. Map $\mathrm{BP}_* \to K(n)_*$ by sending v_n to v_n and v_i to 0 for $i \neq n$. Then

(3.13) $$K(n)_*K(n) = K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_*$$

is a commutative Hopf algebra over the graded field $K(n)_*$. It follows from [26] that

(3.14) $$K(n)_*K(n) = K(n)_*[t_1, \cdots]/(v_n t_i^{p^n} - v_n^{p^i} t_i : i \geqq 1)$$

as algebras.

THEOREM 3.15 [14]. *The natural map*

$$H^* M_n^0 \longrightarrow \mathrm{Ext}_{K(n)_*K(n)}(K(n)_*, K(n)_*)$$

is an isomorphism.

Using essentially this identification, Morava [19] proved the following finiteness theorem. (See also [27] for the case $n < p - 1$.)

THEOREM 3.16 (Morava). *If $p - 1$ does not divide n, then $H^*M_n^0$ is a Poincaré duality algebra over $K(n)_*$ of formal dimension n^2.*

Note the resulting vanishing theorem for our spectral sequence. Call a $k(n)_*$-module L *co-torsion-free* if and only if $\mathbf{F}_p \otimes_{k(n)_*} L = 0$; i.e., if and only if $v_n | L$ is surjective.

COROLLARY 3.17. *If $(p - 1)$ does not divide n then, for $0 \leqq s \leqq n$, the $k(n)_*$-module $E_1^{s,n^2}(n - s) = H^{n^2}(M_{n-s}^s)$ is co-torsion-free, and, for $t > n^2$, $E_1^{s,t}(n - s) = H^t(M_{n-s}^s) = 0$.*

Proof. By induction on s, using the long exact sequence associated to (3.10) and the fact that a torsion $k(n)_*$-module on which v_n is injective is trivial. □

We shall also need the following results, which were proved in [26] and [27].

PROPOSITION 3.18. a) *For $p > 2$, $H^* M_1^0$ is the exterior algebra over $K(1)_*$ on one generator $h_0 = \{t_1\}$. For $p = 2$, $H^* M_1^0$ is the commutative $K(1)_*$-algebra generated by h_0 and $\rho_1 = \{v_1^{-3}(t_2 - t_1^3) + v_1^{-4}v_2t_1\}$, subject to $\rho_1^2 = 0$.*

b) *$H^0 M_n^0 = K(n)_*$ for all $n \geqq 0$.*

c) *Let $n > 1$. $H^1 M_n^0$ is the $K(n)_*$-vector space generated by elements $h_i = \{t_1^{p^i}\}$, $0 \leqq i < n$, ζ_n, and, if $p = 2$, ρ_n, where $|\zeta_n| = 0 = |\rho_n|$. $\zeta_2 \in H^1 M_2^0$ is represented by*

$$\zeta_2 = v_2^{-1}t_2 + v_2^{-p}(t_2^p - t_1^{p^2+p}) - v_2^{-p-1}v_3t_1^p \in \Omega^1 M_2^0 .$$

Proof. a) is contained in [27] Theorem 3.1; the explicit cycle given reduces to $t_1 + t_2$ in $K(1)*K(1)$ (ignoring powers of v_1), and this represents ρ_1 according to the example following Theorem 2.2 of [27]. b) follows from

Theorem 3.15 since the zero dimensional homology of a Hopf algebra is the ground field. c) is contained in [27] Theorem 2.2; the explicit cycle given reduces to $t_2 + t_2^p - t_1^{1+p}$ in $K(2)_*K(2)$ (where $t_1^{p^2} = t_1$), and this represents ζ_2 according to the same example in [27]. $\square$

Finally, we shall use the

LEMMA 3.19. *Let $p \geqq 2$, $n \geqq 0$. The elements ζ_2 and $\zeta_2^{p^n}$ of $\Omega^0 M_2^0$ are homologous.*

Proof. Since we are working mod p it suffices to consider the case $n = 1$. We use the change of rings Theorem 3.15. In $K(2)_*K(2)$, ζ_2 reduces to

$$\bar{\zeta}_2 = v_2^{-1}t_2 + v_2^{-p}(t_2^p - t_1^{p^2+p}) .$$

It follows easily from the relation (3.14) $v_2 t_i^{p^2} = v_2^{p^i} t_i$, $i \geqq 1$, that $\bar{\zeta}_2^p = \bar{\zeta}_2$, so the result is true on the chain-level in $K(2)_*K(2)$.

Alternatively, one may compute in $\Omega^* M_2^0$:

$$d\big(v_2^{-p^2-1}(v_4 - v_2^{-p}v_3^{p+1})\big) = \zeta_2^p - \zeta_2 . \tag{3.20}$$ $\square$

4. $H^* M_0^1$

In this section we shall compute $H^* M_0^1$ for all primes p by means of the "Bockstein" long exact sequence associated to the following case of (3.10):

$$0 \longrightarrow M_1^0 \xrightarrow{\ i\ } M_0^1 \xrightarrow{\ p\ } M_0^1 \longrightarrow 0 . \tag{4.1}$$

We then compute the differentials on this part of the main spectral sequence, and so determine the subquotient of the Novikov E_2-term exhibiting first-order periodicity.

This also determines $H^1 \mathrm{BP}_*$ (Theorem 2.2a)) and proves that $\beta_t \neq 0$ for all $t \geqq 1$, and hence completes the proof of Theorem 2.12b).

We begin with the case of an odd prime because it is simpler and because only this case is needed in the remainder of the paper. The case $p = 2$ is very different because then $H^n M_1^0 \neq 0$ for all $n \geqq 0$. Indeed, we use this fact to produce an element of $H^* \mathrm{BP}_*$ of order 2 in nearly every possible bidegree (4.23). We also show that for $p = 2$, $0 = \eta(v_n) \in H^n(\mathrm{BP}_*)$ for all $n > 1$ (4.22).

THEOREM 4.2. *Let $p > 2$.*

a) *$H^0 M_0^1$ is the direct sum of*

i) *cyclic $\mathbf{Z}_{(p)}$-modules generated by $v_1^{sp^i}/p^{i+1}$, of order p^{i+1}, for $i \geqq 0$, $p \nmid s \in \mathbf{Z}$; and*

ii) *$\mathbf{Q}/\mathbf{Z}_{(p)}$, whose subgroup of order p^j is generated by $1/p^j$, $j \geqq 1$.*

b) $H^1M_0^1 = \mathbf{Q}/\mathbf{Z}_{(p)}$ *concentrated in dimension* 0. *The subgroup of order* p^j *is generated by*

$$y_j = -\sum_{k>0} \frac{(-1)^k v_1^{-k} t_1^k}{kp^{j+1-k}} .$$

c) $H^t M_0^1 = 0$ *for* $t > 1$.

Proof. By Proposition 3.18 a), $H^*M_1^0$ is the exterior algebra over $K(1)_*$ on one generator $h_0 \in H^1M_1^0$ represented by $t_1 \in \Omega^1 M_1^0$. Thus (4.1) induces a long exact sequence as in (3.12):

(4.3)

$$0 \longrightarrow H^0M_1^0 \longrightarrow H^0M_0^1 \xrightarrow{p} H^0M_0^1 \xrightarrow{\delta} H^1M_1^0 \longrightarrow H^1M_0^1 \xrightarrow{p} H^1M_0^1 \longrightarrow 0 .$$

From $\eta_R v_1 = v_1 + pt_1$ and its consequence (for $p > 2$),

(4.4) $$dv_1^{sp^i} \equiv sp^{i+1} v_1^{sp^i - 1} t_1 \mod p^{i+2} ,$$

we see that $v_1^{sp^i}/p^{i+1} \in \Omega^0 M_0^1$ is a cycle and

(4.5) $$\delta(v_1^{sp^i}/p^{i+1}) = s v_1^{sp^i - 1} h_0 \neq 0 .$$

$1/p^j$ is clearly a cycle for all $j > 0$, with $\delta(1/p^j) = 0$. Thus (i) the submodule of $H^0M_0^1$ generated by the classes $v_1^{sp^i}/p^{i+1}$ and $1/p^j$ includes the image of $H^0M_1^0$; and (ii) the reduction-boundaries (4.5) are linearly independent. Part a) thus follows from Remark 3.11.

By (4.5), only $\mathbf{F}_p$-multiples of $v_1^{-1}h_0 \in H^1M_1^0$ map nontrivially to $H^1M_0^1$. The first statement of b) then follows from Remark 3.11. Since $p^{j-1}y_j = v_1^{-1}t_1/p$, the second assertion holds if y_j is a cycle. Formally let

$$y = -\sum_{k=1}^{\infty} \frac{(-pv_1^{-1}t_1)^k}{k}$$

so $y_j = y/p^{j+1}$. We have

(4.6) $$\eta_R v_1^{-1} = v_1^{-1}(1 + pv_1^{-1}t_1)^{-1} ,$$

(4.7) $$(1 - X)^{-k} = \sum_{i=0}^{\infty} \binom{i+k-1}{k-1} X^i ,$$

so

$$d(v_1^{-k}t_1^k) = v_1^{-k}\left(\sum_{i=1}^{\infty} \binom{i+k-1}{k-1} (-pv_1^{-1}t_1)^i \otimes t_1^k - \sum_{i=1}^{k-1} \binom{k}{i} t_1^i \otimes t_1^{k-i}\right) .$$

Collecting the coefficient of $t_1^i \otimes t_1^j$ in dy, we find $dy = 0$. Hence $dy_j = 0$ for all $j \geqq 1$.

Part c) is contained in Corollary 3.17. □

COROLLARY 4.8. *Let* $p > 2$. *In the chromatic spectral sequence*:

a) $E_2^{0,*} = \mathbf{Z}_{(p)}$ *concentrated in degree* 0, *and* $E_2^{0,*} = E_\infty^{0,*}$.

b) $E_2^{1,0}$ *is a direct sum of cyclic* $\mathbf{Z}_{(p)}$*-modules generated by* $v_1^{sp^i}/p^{i+1}$, *of order* p^{i+1}, *for* $i \geqq 0$ *and* $p \nmid s \geqq 1$.

c) $E_2^{1,0} = E_\infty^{1,0}$ *and* $E_\infty^{1,t} = 0$ *for* $t > 0$.

Proof. $d_1: E_1^{s,*} \to E_1^{s+1,*}$ is induced from $d_e: M_0^s \to M_0^{s+1}$. Part a) is thus clear from Theorem 4.2 a) (ii) and the fact that $H^*M_0^0 = \mathbf{Q}$. b) follows from Theorem 4.2 and c) from the vanishing line Lemma 1.16. □

Remarks 4.9. a) Theorem 2.2 a) is immediate from this corollary: for $n \geqq 0$ and $p \nmid s > 0$, $v_1^{sp^n}/p^{n+1} \in H^0N_0^1$, and the classes

$$\alpha_{sp^n, n+1} = \eta(v_1^{sp^n}/p^{n+1})$$

generate $H^1\mathrm{BP}_*$ for p odd.

b) Theorem 2.12 b) also follows, since $v_2^t/pv_1 \in E_1^{2,0}$ survives to $-\beta_t$.

c) Corollary 4.8 represents all the data from this section needed in the rest of this paper. The reader will find the principal results of the remainder of this section stated in Corollaries 4.22 and 4.23 b).

d) We will show in Lemma 8.10 that $d_1y_j \neq 0$, so $E_2^{1,*} = E_\infty^{1,*}$ for p odd.

We turn now to $p = 2$; (4.4) no longer holds for $i \geqq 2$. We shall begin by stating a replacement for (4.5). Define $x_{1,i} \in v_1^{-1}\mathrm{BP}_*$ by

$$(4.10)\qquad \begin{aligned} x_{1,0} &= v_1 , \\ x_{1,1} &= v_1^2 - 4v_1^{-1}v_2 , \\ x_{1,i} &= x_{1,i-1}^2 \quad \text{for } i \geqq 2 , \end{aligned}$$

and let

$$(4.11)\qquad \begin{aligned} a_{1,0} &= 1 , \\ a_{1,i} &= i + 2 \qquad \text{for } i \geqq 1 . \end{aligned}$$

Recall from Proposition 3.18 a) the element

$$\rho_1 = v_1^{-3}(t_2 - t_1^3) + v_1^{-4}v_2t_1 \in v_1^{-1}\mathrm{BP}_*\mathrm{BP} .$$

Then we have:

LEMMA 4.12. *Let* $p = 2$.

a) *For* $i \geqq 0$, $x_{1,i}$ *is invariant* mod $(2^{a_{1,i}})$.

b) Mod $2^{1+a_{1,i}}$:

$$\begin{aligned} dx_{1,i} &\equiv 2t_1 \qquad \text{for} \quad i = 0 , \\ &\equiv 2^{a_{1,i}}v_1^{2^i}\rho_1 \quad \text{for} \quad i \geqq 1 . \end{aligned}$$

Proof. Clearly b) includes a), and b) for $i = 0$ is obvious from $\eta_R v_1 = v_1 + 2t_1$. For $i = 1$ we use

$$(4.13)\qquad \eta_R v_1^{-1} \equiv v_1^{-1} - 2v_1^{-2}t_1 \mod 4,$$

$$(4.14)\qquad \eta_R v_2 \equiv v_2 - v_1t_1^2 + v_1^2t_1 + 2t_2 \mod 4$$

to compute

$$dx_{1,1} = 8v_1^{-1}(v_1^{-1}v_2t_1 + t_2 + t_1^3) \mod 16 \tag{4.15}$$

and the result follows. For $i > 1$, use (4.15) and the binomial theorem. □

THEOREM 4.16. *Let $p = 2$.*

a) *$H^0M_0^1$ is the direct sum of* (i) *cyclic $\mathbf{Z}_{(2)}$-submodules generated by $x_{1,i}^s/2^{a_{1,i}}$, of order $2^{a_{1,i}}$, for $i \geqq 0$ and odd $s \in \mathbf{Z}$; and* (ii) *$\mathbf{Q}/\mathbf{Z}_{(2)}$, generated by $1/2^j$ for $j \geqq 1$.*

b) *If $t \geqq 1$ and $u \in \mathbf{Z}$ is even,*

$$\begin{aligned} H^{t,u}M_0^1 &= \mathbf{Z}/2 \oplus \mathbf{Q}/\mathbf{Z}_{(2)} \quad \text{for} \quad (t, u) = (1, 0) \\ &= \mathbf{Z}/2 \qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

The elements of order 2 are generated by

$$\frac{\rho_1}{2}, \quad \frac{v_1^s t_1^{\otimes t}}{2}, \quad \frac{v_1^s \rho_1 \otimes t_1^{\otimes(t-1)}}{2}$$

for odd $s \in \mathbf{Z}$.

Proof. Again we shall use the long exact sequence associated to (4.1). By Proposition 3.18 a), $H^*M_1^0$ is a polynomial algebra on h_0 tensored with an exterior algebra on ρ_1 (all over $K(1)_*$).

Apply the binomial theorem to Lemma 4.12 to see that mod $2^{1+a_{1,i}}$:

$$\begin{aligned} dx_{1,i}^s &\equiv 2sv_1^{s-1}t_1 , & i = 0 , \\ &\equiv 2^{a_{1,i}}sv_1^{2^i s}\rho_1 , & i \geqq 1 . \end{aligned}$$

Hence

$$\begin{aligned} \delta(v_1^s/2) &= v_1^{s-1}t_1 , & \\ \delta(x_{1,i}^s/2^{a_{1,i}}) &= v_1^{2^i s}\rho_1 , & i \geqq 1 \end{aligned} \tag{4.17}$$

and a) follows as before from Proposition 3.18 a).

By (4.17) the generators of $H^1M_1^0$ not in the image of $\delta: H^0M_0^1 \to H^1M_1^0$ are ρ_1, $v_1^s t_1$, and $v_1^s\rho_1$ for odd $s \in \mathbf{Z}$. Using (4.13), (4.14), and

$$\Delta t_2 = t_2 \otimes 1 + t_1 \otimes t_1^2 + 1 \otimes t_2 - v_1 t_1 \otimes t_1$$

we find

$$d\left(\frac{\rho_1}{4}\right) = \frac{v_1^{-2}t_1 \otimes t_1 + v_1^{-4}(t_1 \otimes t_1^3 + t_1 \otimes t_2 + t_2 \otimes t_1)}{2} .$$

The numerator of the right side represents $\delta(\rho_1/2) \in H^2M_1^0$. To see it is homologous to zero use Theorem 3.15 to reduce to the cobar construction of

$$K(1)_*K(1) = K(1)_*[t_1, \cdots]/(v_1 t_i^2 - v_1^{2^i} t_i : i \geqq 1)$$

where our cycle is

$$v_1^{-4}(t_1 \otimes t_2 + t_2 \otimes t_1) = d(v_1^{-4} t_1 t_2) .$$

Once we show that $z_0 = \rho_1/2$ is infinitely 2-divisible, b) follows by an easy induction using the long exact sequence associated to (4.1).

Consider the portion

$$H^{1,0}M_0^1 \xrightarrow{2} H^{1,0}M_0^1 \xrightarrow{\delta} H^{2,0}M_1^0 \xrightarrow{i_*} H^{2,0}M_0^1$$

of the long exact sequence of (4.1). Recall that $H^{2,0}M_1^0$ is spanned by the classes of $v_1^{-2}t_1 \otimes t_1$ and $v_1^{-1}\rho_1 \otimes t_1$. We have seen that $i_*\{v_1^{-1}\rho_1 \otimes t_1\} \neq 0$, so $\delta\{v_1^{-1}t_1/2\} = \{v_1^{-2}t_1 \otimes t_1\}$ spans $\operatorname{Im}\delta$. Suppose inductively that $z_i \in H^{1,0}M_0^1$ such that $2^i z_i = \{\rho_1/2\}$ has been found. Then

$$\delta z_i = a\delta\{v_1^{-1}t_1/2\}$$

for some $a \in \mathbf{F}_2$. Thus there exists z_{i+1} such that $2z_{i+1} = z_i - a\{v_1^{-1}t_1/2\}$; hence $2^{i+1}z_{i+1} = \{\rho_1/2\}$. □

Before stating the analogue of Corollary 4.8, we make the following observation.

LEMMA 4.18. *Let $p = 2$. Then*

a) $H^{t,u}\mathrm{BP}_* = 0$ *for* $u < 2t$,
b) $H^{t,2t}\mathrm{BP}_* = \mathbf{Z}/2$ *for* $t \geqq 1$,
c) $H^{t,2t+2}\mathrm{BP}_* = 0$ *for* $t \geqq 2$.

Proof. We use the normalized cobar complex as in Note 1.15. Thus for $t \geqq 0$ ($\langle\ \rangle$ means "span of")

$$\tilde{\Omega}^{t,u}\mathrm{BP}_* = 0 \qquad \text{for} \qquad u < 2t\,,$$
$$\tilde{\Omega}^{t,2t}\mathrm{BP}_* = \langle t_1^{\otimes t}\rangle\,,$$
$$\tilde{\Omega}^{t,2t+2}\mathrm{BP}_* = \langle v_1 t_1^{\otimes t},\, e_j\colon 0 \leqq j < t\rangle\,,$$

where $e_j = t_1^{\otimes j} \otimes t_1^2 \otimes t_1^{\otimes(t-j-1)}$. Since $d\big((1/3)t_1^3\big) = -(t_1^2 \otimes t_1 + t_1 \otimes t_1^2)$, the e_j's are all homologous (up to sign). Thus b) follows from

$$de_1 = d(v_1 t_1^{\otimes t}) = 2t_1^{\otimes(t+1)}\,,$$

and c) follows from

$$d(t_2 \otimes t_1^{\otimes(t-2)}) = v_1 t_1^{\otimes t} - e_1\,. \qquad \square$$

PROPOSITION 4.19. *Let $p = 2$. In the chromatic spectral sequence:*

a) $E_2^{0,*} = \mathbf{Z}_{(2)}$ *concentrated in degree* 0, *and* $E_2^{0,*} = E_\infty^{0,*}$.

b) $E_\infty^{1,t,u} = 0$ *for* $u < 2t + 2$.

c) *For $u \geqq 2t + 2$ we have exactly the following nontrivial differentials on $E_*^{1,t,u}$:*

$$d_1\left(\frac{x_{1,1}}{8}\right) = \frac{v_2}{2v_1}\,, \tag{4.20}$$

$$d_{r+1}\left(\frac{v_1^3\rho_1 \otimes t_1^{\otimes(r-1)}}{2}\right) = \frac{v_{r+2}}{2 \cdots v_{r+1}}\,, \qquad r \geqq 1\,. \tag{4.21}$$

Proof. a) is as in Corollary 4.8, and b) follows from the vanishing line Lemma 4.18 a). For $u \geqq 2t + 2$, only the listed generators can support nontrivial differentials since all the others are cycles in the total complex C_0^* (3.3). (4.20) follows from the definition of $x_{1,1}$.

We have in C_0^*, for $r \geqq 1$:

$$d\left(\frac{v_1^3\rho_1 \otimes t_1^{\otimes(r-1)}}{2} + \sum_{i=1}^{r} \frac{(v_{i+1}^2 - v_i^2 v_{i+1}^{-1} v_{i+2})t_1^{\otimes(r-i)}}{2 \cdots v_{i-1} v_i^3}\right)$$

$$= \frac{v_{r+2}}{2 \cdots v_{r+1}} .$$

It follows that $v_1^3\rho_1 \otimes t_1^{\otimes(r-1)}/2$ survives to E_{r+1} and that (4.21) holds. If (4.21) were trivial, i.e., if $v_{r+1}/2 \cdots v_r = 0$ in E_{r+1}, then $v_1^3\rho_1 \otimes t_1^{\otimes(r-1)}/2$ would survive to $E_\infty^{1,*}$ because $E_1^{s,t} = 0$ for $s \leqq 0$, $t > 0$. This contradicts Lemma 4.18 c) and completes the proof. □

Notice that we have also shown:

COROLLARY 4.22. *Let $p = 2$. Then $0 = \eta(v_n) \in H^n \mathrm{BP}_*$ for all $n \geqq 2$.*

COROLLARY 4.23. *Let $p = 2$.*

a) [21] *$H^1\mathrm{BP}_*$ is a sum of cyclic $\mathbf{Z}_{(2)}$-submodules generated by*

i) $\eta(v_1^s/2)$, $2 \nmid s \geqq 1$, *of order* 2

ii) $\eta(x_{1,1}/4)$ *of order* 4

iii) $\eta(x_{1,i}^s/2^{i+2})$ *for other $i \geqq 1$, $2 \nmid s \geqq 1$, of order 2^{i+2}.*

b) *For $t \geqq 2$ and u such that $2u \geqq 2t$ and $2u \neq 2t + 2$, $H^{t,2u}\mathrm{BP}_*$ contains the nontrivial class*

$$\eta(v_1^{u-t+1} t_1^{\otimes(t-1)}/2) \qquad \textit{for } u - t \textit{ even},$$

$$\eta(v_1^{u-t+2}\rho_1 \otimes t_1^{\otimes(t-2)}/2) \qquad \textit{for } u - t \textit{ odd}.$$

5. $H^0 M_n^1$

In this section we determine the groups $H^0M_n^1$ for $n \geqq 1$ and for all primes p. This computation is closely related to [16]. We use the short exact sequence (3.10)

$$0 \longrightarrow M_{n+1}^0 \longrightarrow M_n^1 \xrightarrow{v_n} M_n^1 \longrightarrow 0$$

which gives rise to the long exact sequence (3.12)

$$0 \longrightarrow H^0 M_{n+1}^0 \longrightarrow H^0 M_n^1 \xrightarrow{v_n} H^0 M_n^1 \xrightarrow{\delta} H^1 M_{n+1}^0 \longrightarrow \cdots .$$

We know $H^0M_{n+1}^0$ (3.18) so we need only push these elements into $H^0M_n^1$ and divide by v_n until we can reduce them to linearly independent elements in $H^1M_{n+1}^0$. Doing this requires an explicit construction of the generating cycles in $H^0M_n^1$. The case $n = 1$, $p > 2$, will concern us for the rest of the paper, so we treat it first.

We begin by defining certain elements of $v_2^{-1}\mathrm{BP}_*$, $p > 2$. Let

(5.1)
$$\begin{aligned} x_0 &= v_2 , \\ x_1 &= x_0^p - v_1^p v_2^{-1} v_3 , \\ x_2 &= x_1^p - v_1^{p^2-1} v_2^{(p-1)p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3 , \\ x_i &= x_{i-1}^p - 2v_1^{b_i} v_2^{(p-1)p^{i-1}+1} , \quad i \geqq 3, \end{aligned}$$

where $b_i = (p+1)(p^{i-1}-1)$ for $i > 1$. Next define integers

(5.2)
$$a_0 = 1,\ a_i = p^i + p^{i-1} - 1 , \quad i \geqq 1 .$$

THEOREM 5.3. *Let $p > 2$. $H^0 M_1^1$ is the direct sum of*

i) *cyclic $\mathbf{F}_p[v_1]$-modules isomorphic to $\mathbf{F}_p[v_1]/(v_1^{a_i})$ generated by $x_i^s/v_1^{a_i}$ for $i \geqq 0$ and $p \nmid s \in \mathbf{Z}$; and*

ii) *$\mathbf{F}_p[v_1, v_1^{-1}]/\mathbf{F}_p[v_1]$, generated by $1/v_1^j$ for $j \geqq 1$.*

It is important to note that s may be negative in this theorem. To interpret $x_i^s/v_1^{a_i}$ for $s < 0$, notice that $x_i = v_2^{p^i}(1 - v_1 z)$ for some $z \in v_2^{-1}\mathrm{BP}_*$. Then formally

$$x_i^{-1} = v_2^{-p^i} \textstyle\sum_{k \geqq 0} v_1^k z^k ;$$

but in $x_i^{-1}/v_1^{a_i}$ only terms with $k < a_i$ are nonzero. This procedure also gives meaning to $x_i^s/v_1^{a_i}$ for $s < -1$, and to similar expressions which occur later in the paper.

The proof will use part b) of the following computation. The reader will recall from Section 3 the element $\zeta_2 = v_2^{-1} t_2 + v_2^{-p}(t_2^p - t_1^{p^2+p}) - v_2^{-p-1} v_3 t_1^p \in v_2^{-1}\mathrm{BP}_*\mathrm{BP}$.

PROPOSITION 5.4. *Let $p > 2$.*

a) *For $i \geqq 0$, x_i is invariant* mod $(p, v_1^{a_i})$.

b) Mod $(p, v_1^{1+a_i})$,

$$\begin{aligned} dx_i &\equiv v_1 t_1^p , && i = 0 \\ &\equiv v_1^p v_2^{p-1} t_1 , && i = 1 \\ &\equiv 2v_1^{a_i} v_2^{(p-1)p^{i-1}} t_1 , && i \geqq 2 . \end{aligned}$$

c) Mod $(p, v_1^{2+a_i})$,

$$\begin{aligned} dx_i &\equiv v_1 t_1^p , && i = 0 \\ &\equiv v_2^{p-1}\bigl(v_1^p t_1 + v_1^{p+1}\bigl(v_2^{-1}(t_2 - t_1^{p+1}) - \zeta_2\bigr)\bigr) , && i = 1 \\ &\equiv v_2^{(p-1)p^{i-1}}(2v_1^{a_i} t_1 - v_1^{1+a_i} \zeta_2^{p^{i-1}}) , && i \geqq 2 . \end{aligned}$$

Proof. Clearly c) includes a) and b). For $i = 0$, c) follows from

(5.5)
$$\eta_R v_2 \equiv v_2 + v_1 t_1^p - v_1^p t_1 \qquad \bmod p ,$$

and for $i = 1$ we use also

(5.6) $$\eta_R v_2^{-1} \equiv v_2^{-1} - v_1 v_2^{-2} t_1^p \qquad \mathrm{mod}\,(p, v_1^2)\,,$$

(5.7) $$\eta_R v_3 \equiv v_3 + v_2 t_1^{p^2} - v_2^p t_1 + v_1 t_2^p \qquad \mathrm{mod}\,(p, v_1^2)\,.$$

For $i \geqq 2$ we note the following consequence of the binomial theorem.

Observation 5.8. Let $x \in \mathrm{BP}_*$, $y \in \mathrm{BP}_*\mathrm{BP}$, and $I \subset \mathrm{BP}_*$ be an ideal. If $dx \equiv y \bmod (p, I)\mathrm{BP}_*\mathrm{BP}$, then $dx^p \equiv y^p \bmod (p, I^p)\mathrm{BP}_*\mathrm{BP}$.

Thus the case $i = 1$ implies

$$dx_1^p \equiv v_2^{p^2-p}\big(v_1^{p^2} t_1^p + v_1^{p^2+p}\big(v_2^{-p}(t_2^p - t_1^{p^2+p}) - \zeta_2^p\big)\big) \qquad \mathrm{mod}\,(p, v_1^{p(p+2)})$$

and hence $\mathrm{mod}\,(p, v_1^{2+a_2})$. Also, by (5.5)

$$d(v_1^{p^2-1} v_2^{p^2-p+1}) \equiv v_1^{p^2-1}\big(v_1 v_2^{p^2-p} t_1^p - v_1^p v_2^{p^2-2p}(v_2^p t_1 + v_2 t_1^{p^2}) - v_1^{p+1} v_2^{p^2-2p} t_1^{p^2+p}\big)\,,$$

and by (5.5) and (5.7),

$$d(v_1^{p^2+p-1} v_2^{p^2-2p} v_3) \equiv v_1^{p^2+p-1} v_2^{p^2-2p}(v_2 t_1^{p^2} - v_2^p t_1 + v_1 t_2^p)\,,$$

both $\mathrm{mod}\,(p, v_1^{2+a_2})$. Collect terms now to obtain the case $i = 2$.

Now proceed by induction. Let $i \geqq 3$. Since $p(2 + a_{i-1}) \geqq 2 + a_i$, Obervation 5.8 and the case $i - 1$ yield

$$dx_{i-1}^p \equiv v_2^{(p-1)p^{i-1}}(2v_2^{pa_{i-1}} t_1^p - v_1^{p(1+a_{i-1})} \zeta_2^{p^{i-1}})\,,$$

and by (5.5) (since $a_i = b_i + p$),

$$d(2v_1^{b_i} v_2^{p^i - p^{i-1}+1}) \equiv 2v_1^{b_i} v_2^{(p-1)p^{i-1}}(v_1 t_1^p - v_1^p t_1)\,,$$

both $\mathrm{mod}\,(p, v_1^{2+a_i})$. Collect terms and use the fact that $pa_{i-1} = 1 + b_i$ for $i > 1$ to complete the induction. □

Proof of Theorem 5.3. We use the exact sequence

$$0 \longrightarrow H^0 M_2^0 \longrightarrow H^0 M_1^1 \xrightarrow{v_1} H^0 M_1^1 \xrightarrow{\delta} H^1 M_2^0\,.$$

Recall from Proposition 3.18 b) and c) that $H^0 M_2^0 = K(2)_*$ and $H^1 M_2^0$ is (freely) generated over $K(2)_*$ by h_0, h_1, ζ_2. By Remark 3.11 we must show

i) the image of $H^0 M_2^0$ is contained in the sub $\mathbf{F}_p[v_1]$-module generated by $\{x_i^s/v_1^{a_i}\colon i \geqq 0,\ s \not\equiv 0(p)\}$, and

ii) these generators map to linearly independent elements in $H^1 M_2^0$.

i) follows from the congruence

$$x_i^s \equiv v_2^{sp^i} \qquad \mathrm{mod}\,(p, v_1)\,.$$

For ii), we deduce from Proposition 5.4 b) and the binomial theorem that

(5.9) $$\begin{aligned} \delta(x_0^s/v_1) &= s v_2^{s-1} h_1\,, \\ \delta(x_1^s/v_1^p) &= s v_2^{sp-1} h_0\,, \\ \delta(x_i^s/v_1^{a_i}) &= 2v_2^{(sp-1)p^{i-1}} h_0\,, \qquad i > 1. \end{aligned}$$

These classes are clearly independent. □

Turn now to $H^0 M_n^1$ for $n > 1$ or $p = 2$. This work will not be used in the remainder of this paper.

We shall define $x_{n,i} \in v_n^{-1}\mathrm{BP}_*$ and $a_{n,i} \geqq 1$ for all primes p and all $n \geqq 1$, $i \geqq 0$, in such a way that the following uniform theorem holds.

THEOREM 5.10. *As a $k(n-1)_*$-module, $H^0 M_{n-1}^1$ is the direct sum of*
i) *the cyclic submodules generated by $x_{n,i}^s / v_{n-1}^{a_{n,i}}$ for $i \geqq 0$, $s \not\equiv 0$ (p); and*
ii) *$K(n-1)_*/k(n-1)_*$, generated by $1/v_{n-1}^j$, $j \geqq 1$.*

So let

(5.11)
$$\begin{aligned}
x_{1,0} &= v_1 , && \\
x_{1,1} &= v_1^2 - 4v_1^{-1}v_2 && \text{for } p = 2 ,\\
x_{1,i} &= x_{1,i-1}^p && \text{otherwise,}\\
x_{2,i} &= x_i \quad \text{(see (5.1))} && \text{for } p > 2 \text{ or } i < 3 ,\\
x_{2,i} &= x_{2,i-1}^2 && \text{for } p = 2 \text{ and } i \geqq 3 ,\\
x_{n,0} &= v_n && n > 2 ,\\
x_{n,1} &= v_n^p - v_{n-1}^p v_n^{-1} v_{n+1} , && \\
x_{n,i} &= x_{n,i-1}^p && \text{for } 1 < i \not\equiv 1\ (n-1) ,\\
x_{n,i} &= x_{n,i-1}^p - v_{n-1}^{b_{n,i}} v_n^{p^i - p^{i-1}+1} && \text{for } 1 < i \equiv 1\ (n-1) ,
\end{aligned}$$

where for $i \equiv 1 \pmod{(n-1)}$,

(5.12)
$$b_{n,i} = \frac{(p^{i-1}-1)(p^n-1)}{(p^{n-1}-1)} .$$

Also let

(5.13)
$$\begin{aligned}
a_{1,0} &= 1 , && \\
a_{1,i} &= i + 2 && \text{for } p = 2 \text{ and } i \geqq 1 ,\\
a_{1,i} &= i + 1 && \text{for } p > 2 \text{ and } i \geqq 1 ,\\
a_{2,i} &= a_i \quad \text{(see (5.2))} && \text{for } p > 2 \text{ or } i < 2 ,\\
a_{2,i} &= 3 \cdot 2^{i-1} && \text{for } p = 2 \text{ and } i \geqq 2 ,\\
a_{n,0} &= 1 , && n > 2 ,\\
a_{n,1} &= p , && \\
a_{n,i} &= p a_{n,i-1} && \text{for } 1 < i \not\equiv 1\ (n-1) ,\\
a_{n,i} &= p a_{n,i-1} + p - 1 && \text{for } 1 < i \equiv 1\ (n-1) .
\end{aligned}$$

For $n = 1$, Theorem 5.10 is equivalent to Theorem 4.2 for $p > 2$ and to Theorem 4.16 for $p = 2$. For $n = 2$, Theorem 5.10 is equivalent to Theorem 5.3 for $p > 2$, and for $p = 2$ it follows from Lemma 3.19 and the next proposition.

PROPOSITION 5.14. *Let $p = 2$.*
a) *For $i \geqq 0$, $x_{2,i}$ is invariant* mod $(2, v_1^{a_{2,i}})$.

b) Mod $(2, v_1^{1+a_{2,i}})$,

$$dx_{2,i} \equiv v_1 t_1^2 , \qquad i = 0$$
$$\equiv v_1^2 v_2 t_1 , \qquad i = 1$$
$$\equiv v_1^{a_{2,i}} v_2^{2^{i-1}} \zeta_2^{2^{i-1}} , \qquad i \geqq 2 .$$

Proof. b) contains a), and $i = 0$ is as in Proposition 5.4. Just as in 5.4 c) one shows that

$$dx_{2,1} \equiv v_2(v_1^2 t_1 + v_1^3(v_2^{-1}(t_2 - t_1^3) - \zeta_2)) \qquad \bmod (2, v_1^4) , \tag{5.15}$$

and this includes the case $i = 1$. (5.15) and Observation 5.8 imply

$$dx_{2,1}^2 \equiv v_2^2(v_1^4 t_1^2 + v_1^6(v_2^{-2}(t_2^2 - t_1^6) - \zeta_2^2)) \qquad \bmod (2, v_1^8)$$

and this together with (5.7) gives the case $i = 2$. Then $i \geqq 3$ follows by induction using Observation 5.8. □

Thus in case $p = 2$, for s odd

$$\begin{aligned} \delta(x_{2,0}^s/v_1) &= v_2^{s-1} h_1 , \\ \delta(x_{2,1}^s/v_1^2) &= v_2^{2s-1} h_0 , \\ \delta(x_{2,i}^s/v_1^{3\cdot 2^{i-1}}) &= v_2^{(2s-1)2^{i-1}} \zeta_2 , \qquad i \geqq 2 . \end{aligned} \tag{5.16}$$

For $n > 2$ we have the following proposition. The proof is analogous to the proof of Proposition 5.14.

PROPOSITION 5.17. *For all p and for $n \geqq 3$,*

a) *For $i \geqq 0$, $x_{n,i}$ is invariant* mod $(I_{n-1}, v_{n-1}^{a_{n,i}})$.

b) Mod $(I_{n-1}, v_{n-1}^{1+a_{n,i}})$,

$$dx_{n,i} \equiv v_{n-1} t_1^{p^{n-1}} , \qquad i = 0 ,$$
$$\equiv v_{n-1}^{a_{n,i}} v_n^{(p-1)p^{i-1}} t_1^{p^j} , \qquad i \geqq 1$$

where $j \equiv i - 1 \bmod (n - 1)$ and $0 \leqq j < n - 1$. □

Hence for $n \geqq 3$

$$\begin{aligned} \delta(v_n^s/v_{n-1}) &= s v_n^{s-1} h_{n-1} , \\ \delta(x_{n,i}^s/v_{n-1}^{a_{n,i}}) &= s v_n^{(sp-1)p^{i-1}} h_j , \qquad i \geqq 1 . \end{aligned} \tag{5.18}$$

Theorem 5.10 follows now in these cases in the usual way. □

Remark 5.19. We invite the reader at this point to compute $d_1: E_1^{1,0}(n) \to E_1^{2,0}(n)$ in the chromatic spectral sequence. The principal results of [16] are then immediate corollaries.

6. $H^0 M_0^2$

In this section we carry out the computation of

$$H^0 M_0^2 = \mathrm{Ext}^0_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, v_2^{-1}\mathrm{BP}_*/(p^\infty, v_1^\infty))$$

for an odd prime p. This group is central to all our main results. In outline we proceed as follows.

We will use the short exact sequence (3.10)

$$0 \longrightarrow M_2^0 \longrightarrow M_1^1 \xrightarrow{v_1} M_1^1 \longrightarrow 0$$

which gives rise to the long exact sequence (3.12)

$$0 \longrightarrow H^0 M_2^0 \longrightarrow H^0 M_1^1 \xrightarrow{v_1} H^0 M_1^1 \xrightarrow{\delta} H^1 M_2^0 \longrightarrow H^1 M_1^1 \xrightarrow{v_1} H^1 M_1^1 \longrightarrow \cdots .$$

Our knowledge of $H^1 M_2^0$ (3.18) and of $H^0 M_1^1$ (5.3) allows us to determine an $\mathbf{F}_p$ basis for $\ker v_1 | H^1 M_1^1$. This will in fact enable us to detect many other elements in this group. Next we use the exact sequence (3.10)

$$0 \longrightarrow M_1^1 \longrightarrow M_0^2 \xrightarrow{p} M_0^2 \longrightarrow 0$$

and the associated long exact sequence (3.12):

$$0 \longrightarrow H^0 M_1^1 \longrightarrow H^0 M_0^2 \xrightarrow{p} H^0 M_0^2 \xrightarrow{\delta} H^1 M_1^1 \longrightarrow \cdots .$$

Recall that we know $H^0 M_1^1$ (5.3) and from the above we know something about $H^1 M_1^1$. We take our generators for $H^0 M_1^1$ and push them into $H^0 M_0^2$. Here we divide by powers of p until we can show they reduce to a linearly independent set in $H^1 M_1^1$. This gives us $H^0 M_0^2$.

Recall (Theorem 5.3) that $H^0 M_1^1$ is generated over $\mathbf{F}_p$ by certain classes x_n^s/v_1^j. To determine their p-divisibility in $H^0 M_0^2$, we first obtain in Proposition 6.4 an invariant ideal smaller than (p, v_1^j) modulo which x_n is invariant. This implies in particular that $x_n^s/p^{i+1}v_1^j \in \Omega^0 M_0^2$ is a cycle for $p^i | j \leqq a_{n-i}$. Our main theorem is then:

THEOREM 6.1. *Let* $p \geqq 3$. $H^0 M_0^2$ *is the direct sum of cyclic* $\mathbf{Z}_{(p)}$*-submodules generated by*

i) $x_n^s/p^{i+1}v_1^j$ *for* $n \geqq 0$, $p \nmid s \in \mathbf{Z}$, $i \geqq 0$, $j \geqq 1$, *such that* $p^i | j \leqq a_{n-i}$ *and either* $p^{i+1} \nmid j$ *or* $a_{n-i-1} < j$; *and*

ii) $1/p^{i+1}v_1^j$ *for* $i \geqq 0$ *and* $p^i | j \geqq 1$.

By Theorem 5.3 these elements certainly span a $\mathbf{Z}_{(p)}$-submodule of $H^0 M_0^2$ containing the image of $H^0 M_1^1$, i.e., the submodule of elements of order p. So by Remark 3.11 what remains is to show that the set R of mod p reduction-boundaries of these elements is linearly independent in $H^1 M_1^1$. It turns out that it suffices to know $v_1^{l-2}z$ where v_1^l is the lowest power of v_1 killing $z \in R$. Thus the computations modulo $(p, v_1^{2+a_k})$ contained in Propostion 5.4 c) suffice. The linear independence argument is given at the end of this section. It relies upon a particular choice of cycles representing a basis for $\ker (v_1 | H^1 M_1^1)$, given in Lemma 6.12.

Throughout this section p will be an odd prime.

We must study the integral invariance of the element $x_n^s \in v_2^{-1}\mathrm{BP}_*$. We shall begin with some generalities on certain I_2-primary invariant ideals. Let $c_0, c_1, \cdots$ be a strictly increasing sequence of positive integers, and for $n \geqq 0$ let

$$J_{2,n} = (p^{n+1}, p^n v_1^{c_0}, \cdots, pv_1^{c_{n-1}}, v_1^{c_n}) \subset \mathrm{BP}_* .$$

LEMMA 6.2. *Let $n \geqq 0$.*

a) *$J_{2,n}$ is invariant.*

b) *If $x \equiv y$ mod $J_{2,n}$ then $x^p \equiv y^p$ mod $(pJ_{2,n} + J_{2,n}^p)$.*

c) *$pJ_{2,n} + J_{2,n}^p = (p^{n+2}, p^{n+1}v_1^{c_0}, \cdots, pv_1^{c_n}, v_1^{pc_n})$.*

d) *If $c_n \geqq pc_{n-1}$ then*

$$J_{2,n} = (p, v_1^{c_n}) \cap (pJ_{2,n-1} + J_{2,n-1}^p) .$$

Proof. a) follows from $\eta_R v_1 = v_1 + pt_1$ and b) from the binomial theorem. c) and d) are easy observations. □

In particular take $c_i = a_i$ as in (5.2), and let

$$I_{2,n} = (p^{n+1}, p^n v_1^{a_0}, \cdots, v_1^{a_n}) . \tag{6.3}$$

$I_{2,n}$ is thus invariant.

PROPOSITION 6.4. *Let $n \geqq 0$, $s \in \mathbf{Z}$. Then x_n^s is invariant* mod $I_{2,n}$.

Proof. Clearly we may take $s = 1$. For $n = 0$ we have v_2 invariant mod (p, v_1), cf. (1.7). Assume by induction that x_{n-1} is invariant mod $I_{2,n-1}$. Then x_{n-1}^p is invariant mod $(pI_{2,n-1} + I_{2,n-1}^p)$ by Lemma 6.2 b), and $x_n - x_{n-1}^p$ is invariant mod this ideal by c) and the definition (5.1) of x_n. Thus x_n is also invariant mod this ideal. By Proposition 5.4 a) x_n is invariant mod $(p, v_1^{a_n})$, so the result follows from Lemma 6.2 d). □

COROLLARY 6.5. *Let $n \geqq 0$, $s \in \mathbf{Z}$, $i \geqq 0$, $j \geqq 1$. Then $x_n^s/p^{i+1}v_1^j \in \Omega^0 M_0^2$ is a cycle if $i \leqq n$ and $p^i \mid j \leqq a_{n-i}$.*

Proof. Since $p^i \mid j$, (p^{i+1}, v_1^j) is invariant, we are claiming that x_n^s is invariant mod (p^{i+1}, v_1^j). Since $j \leqq a_{n-i}$, $(p^{i+1}, v_1^j) \supseteqq I_{2,n}$, so the result follows from Proposition 6.4. □

Remark 6.6. Observe that $p^i \mid j \leqq a_{n-i}$ if and only if $2i \leqq n$ and $j = mp^i$ for $m \leqq a_{n-2i}$.

Let $b_k = (p+1)(p^{k-1} - 1)$ for $k > 1$ as in Section 5, and let $b_1 = p$.

LEMMA 6.7. *Let $i \geqq 0$, $k > 1$. Then*

$$x_{i+k} \equiv x_{k-1}^{p^{i+1}} \bmod (p^i v_1^{b_k}, p^{i-1}v_1^{pb_k}, \cdots, v_1^{p^i b_k}) ;$$

$$x_{i+1} \equiv x_0^{p^{i+1}} \bmod (p^i v_1^p, p^{i-1}v_1^{b_2}, \cdots, v_1^{p^{i-1}b_2}) .$$

Proof. Since $x_k \equiv x_{k-1}^p \bmod v_1^{b_k}$,

$$x_k^{p^i} \equiv x_{k-1}^{p^{i+1}} \bmod (p^i v_1^{b_k}, p^{i-1} v_1^{pb_k}, \cdots, v_1^{p^i b_k}) .$$

The result follows from the observation that

$$\begin{aligned} \min\{b_{k+j}, pb_{k+j-1}, \cdots, p^j b_k\} &= p^j b_k && \text{if } k > 1 , \\ &= p^{j-1} b_2 && \text{if } k = 1 . \end{aligned}$$ □

LEMMA 6.8. *Let* $s \in \mathbf{Z}$, $i \geqq 0$. *Then* $\bmod (p^{i+1}, v_1^{2+a_k})$ *we have*

$$\begin{aligned} dx_k^{sp^i} &\equiv sp^i v_2^{sp^i-1} v_1 t_1^p + \binom{sp^i}{2} v_2^{sp^i-2} v_1^2 t_1^{2p} && \text{if } k = 0 \\ &\equiv sp^i v_2^{sp^{i+1}-1}(v_1^p t_1 + v_1^{p+1}(v_2^{-1}(t_2 - t_1^{p+1}) - \zeta_2)) && \text{if } k = 1 \\ &\equiv sp^i v_2^{(sp^{i+1}-1)p^{k-1}}(2v_1^{a_k} t_1 - v_1^{1+a_k} \zeta_2^{p^{k-1}}) && \text{if } k \geqq 2 . \end{aligned}$$

Proof. This follows from Proposition 5.4 c). We leave the case $k = 0$ to the reader. Note that if L is any ideal such that $y^2 \in L$, and if $dx \equiv y \bmod (p, L)$, then $dx^{sp^i} \equiv sp^i x^{sp^i-1} y \bmod (p^{i+1}, L)$. Apply this, with $L = (v_1^{2+a_k})$, to Proposition 5.4 c). Then use the definition of x_k to replace $x_k^{sp^i-1}$ by $v_2^{(sp^i-1)p^k}$. □

PROPOSITION 6.9. *Let* $p \nmid s \in \mathbf{Z}$, $0 \leqq i \leqq k$, $1 \leqq m \leqq a_{k-i}$, *and write* $j = mp^i$. *For* $\delta: H^0 M_0^2 \to H^1 M_1^1$, *we have*

i) $$\begin{aligned} \delta\left(\frac{x_{i+k}^s}{p^{i+1} v_1^j}\right) &= -\frac{v_2^s t_1}{v_1^2} + \frac{s v_2^{s-1}(t_2 - t_1^{p+1})}{v_1} && \text{if } k = 0 \\ &= -\frac{m x_{i+1}^s t_1}{v_1^{j+1}} + \frac{s v_2^{sp^{i+1}-1} t_1^p}{v_1^{j-1}} + \cdots && \text{if } k = 1 \\ &= -\frac{m x_{i+2}^s t_1}{v_1^{j+1}} + \frac{s v_2^{sp^{i+2}-1}(t_1 + v_1 v_2^{-1}(t_2 - t_1^{p+1}) - v_1 \zeta_2)}{v_1^{j-p}} + \cdots && \text{if } k = 2 \\ &= -\frac{m x_{i+k}^s t_1}{v_1^{j+1}} + \frac{s v_2^{(sp^{i+2}-1)p^{k-2}}(2t_1 - v_1 \zeta_2^{p^{k-2}})}{v_1^{j-a_{k-1}}} + \cdots && \text{if } k \geqq 3 , \end{aligned}$$

where $\cdots$ *denotes an element of* $\Omega^1 M_1^1$ *killed by a lower power of* v_1 *than those shown; and*

ii) $$\delta\left(\frac{1}{p^{i+1} v_1^{sp^i}}\right) = -\frac{s t_1}{v_1^{1+sp^i}} \qquad \text{for } s \geqq 1 .$$

Proof. Since

$$\eta_R(v_1^{-j}) \equiv v_1^{-j} - mp^{i+1} v_1^{-(j+1)} t_1 (\bmod p^{i+2})$$

(remember, $j = mp^i$), we have

$$\eta_R\left(\frac{x_{i+k}^s}{p^{i+2} v_1^j}\right) = \frac{\eta_R(x_{i+k}^s)}{p^{i+2} v_1^j} - \frac{m \eta_R(x_{i+k}^s) t_1}{p v_1^{j+1}} . \tag{6.10}$$

Equation (ii) follows immediately: take $s = 0$.

We leave the case $k = 0$ to the reader.

Now $\eta_R x_{i+k}^s \equiv x_{i+k}^s \bmod (p, v_1^{a_{i+k}})$. Since $j \leqq a_{i+k}$ (with equality if and only if $i = 0$ and $j = a_k$) the second term in (6.10) is

$$-\frac{m x_{i+k}^s t_1}{p v_1^{j+1}} \quad \text{mod ker } v_1$$

(mod 0 unless $i = 0$ and $j = a_k$).

We turn now to the first term of (6.10) in case $k > 0$. By Lemma 6.7,

$$\eta_R x_{k+i}^s \equiv \eta_R x_{k-1}^{sp^{i+1}} \bmod (p^{i+2}, p^{i+1} v_1^{b_{k-1}}, \cdots, v_1^{b_{k+i}}) \tag{6.11}$$

since the indicated ideal is invariant by Lemma 6.2. (6.11) holds in particular mod$(p^{i+2}, v_1^{b_k - 1})$. Except when $p = 3$ and $k = 1$, $b_k - 1 \geqq 2 + a_{k-1}$, so (6.11) holds also mod$(p^{i+2}, v_1^{2+a_{k-1}})$.

In the exceptional case we need to compute $\delta(x_{1+i}^s / 3^{i+1} v_1^{3^i m})$ mod ker $v_1^{3^i m - 2}$, so it suffices to compute $\eta_R(x_{1+i}^s)$ mod $(3^{1+i}, v_1^2)$. Thus Lemma 6.8 gives

$$\begin{aligned}
\frac{\eta_R x_{i+k}^s}{p^{i+2} v_1^j} &= \frac{x_0^{sp^{i+1}} + sp^{i+1} v_2^{sp^{i+1}-1} v_1 t_1^p}{p^{i+2} v_1^j} + \cdots && \text{if } k = 1 \\
&= \frac{x_1^{sp^{i+1}} + sp^{i+1} v_2^{sp^{i+2}-1}\left(v_1^p t_1 + v_1^{p+1}\left(v_2^{-1}(t_2 - t_1^{p+1}) - \zeta_2\right)\right)}{p^{i+2} v_1^j} + \cdots && \text{if } k = 2 \\
&= \frac{x_{k-1}^{sp^{i+1}} + sp^{i+1} v_2^{(sp^{i+2}-1)p^{k-2}}(2 v_1^{a_{k-1}} t_1 - v_1^{1+a_{k-1}} \zeta_2^{p^{k-2}})}{p^{i+2} v_1^j} + \cdots && \text{if } k \geqq 3\,.
\end{aligned}$$

The result now follows upon using Lemma 6.7 again to convert $x_{k-1}^{sp^{i+1}}$ to x_{i+k}^s.

□

LEMMA 6.12. *The following cycles represent the elements of a basis for* ker v_1 *in* $H^1 M_1^1$: *for* $t \in \mathbf{Z}$ *and* $p \nmid s \in \mathbf{Z}$ *such that either* $s \not\equiv -1(p)$ *or* $s \equiv -1(p^2)$,

a) $\dfrac{v_2^{sp^k} t_1}{v_1}$ $\qquad k \geqq 0$

b) $\dfrac{v_2^{pt-1} t_1}{v_1^2} + \dfrac{v_2^{pt-2}(t_2 - t_1^{p+1})}{v_1}$ $\qquad p \nmid t$

c) $\dfrac{v_2^{(pt-1)p^k} t_1}{v_1^2}$ $\qquad k \geqq 1,\ p \nmid t$

d) $\dfrac{v_2^{pt-1} t_1^p}{v_1}$

e) $\dfrac{v_2^{sp^k} \zeta_2}{v_1}$ $\qquad k \geqq 0$

f) $\dfrac{t_1}{v_1}, \dfrac{\zeta_2}{v_1}$.

Proof. By virtue of the short exact sequence

$$0 \longrightarrow M_2^0 \xrightarrow{\ i\ } M_1^1 \xrightarrow{\ v_1\ } M_1^1 \longrightarrow 0\,,$$

we must find Coker $(\delta: H^0M_1^1 \to H^1M_2^0)$. By Proposition 3.18 c), Theorem 5.3, and (5.9), this $\mathbf{F}_p$-vector space is spanned by:

$$\frac{v_2^{sp^k}t_1}{v_1}, \quad \frac{v_2^{pt-1}t_1^p}{v_1}, \quad \frac{v_2^t\zeta_2}{v_1}, \quad \frac{t_1}{v_1}$$

for $k \geqq 0$, $t \in \mathbf{Z}$, and s as in the lemma. Now Lemmas 6.8 and 3.19 together give homologies

$$\frac{v_2^{pt-1}\zeta_2}{v_1} \sim \frac{v_2^{pt-1}t_1}{v_1^2} + \frac{v_2^{pt-2}(t_2 - t_1^{p+1})}{v_1},$$

and for $k \geqq 1$

$$\frac{v_2^{(pt-1)p^k}\zeta_2}{v_1} \sim \frac{v_2^{(pt-1)p^k}\zeta_2^{p^{k-1}}}{v_1} \sim \frac{2v_2^{(pt-1)p^k}t_1}{v_1^2}.$$

The result follows. □

Proof of Theorem 6.1. As we have said, it suffices to prove that the set R of mod p reduction-boundaries of the elements listed in the theorem is linearly independent in $H^1M_1^1$. We do this by induction on the (v_1)-adic filtration on $H^1M_1^1$. Write $l(z)$ for the lowest power of v_1 killing $z \in R$, and let

$$F_lR = \{z \in R: l(z) \leqq l\}.$$

Say that $z \in R$ is of *type* (x) if $v_1^{l(z)-1}z$ occurs in part (x) of the list of generators of $\ker(v_1 | H^1M_1^1)$ given in Lemma 6.12. Further say $z = \delta(x_{i+k}^s/p^{i+1}v_1^{mp^i})$ of type (a) has type $(\mathrm{a})_1$ if and only if $p \nmid m$ and type $(\mathrm{a})_2$ if and only if $p | m$. Let $r(z)$ denote the power to which v_2 occurs in the leading term of $v_1^{l(z)-1}z$. Among elements $z \in R$ of fixed type with $l(z) = l$, $r(z)$ and dim z determine each other.

Proposition 6.9 now results in the following table, all but the last line of which refers to $z = \delta(x_{i+k}^s/p^{i+1}v_1^j) \in R$, $j = mp^i$.

$l = 1,\ r = s,$	type (b),	if $k = 0$ and $s = pt - 1,\ p \nmid t$.
$l = j,\ r = sp^{i+k},$	type (c),	if $k > 0,\ p \nmid m$, and $s = pt-1,\ p \nmid t$.
$l = j + 1,\ r = sp^{i+k},$	type $(\mathrm{a})_1$,	if $k \geqq 0,\ p \nmid m$, and s otherwise.
$l = j - 1,\ r = sp^{i+1} - 1,$	type (d),	if $k = 1$ and $p \mid m$.
$l = j - a_{k-1},\ r = sp^{i+k} - p^{k-2},$	type $(\mathrm{a})_2$,	if $k > 1$ and $p \mid m$.
$l = j,\ r = 0,$	type (f),	if $z = 1/p^{i+1}v_1^j$.

Thus F_0R is empty, and this starts the induction. So suppose $F_{l-1}R$ is linearly independent. Now it is very easy to see from the table that there is for fixed l and r at most one element $z \in R$ of each type. Thus a homogeneous linear relation among elements of F_lR must be of the form $\sum_{\nu=1}^N \alpha_\nu z_\nu = 0$ with z_1 of type $(\mathrm{a})_1$, z_2 of type $(\mathrm{a})_2$, and $z_\nu \in F_{l-1}R$ for $\nu > 2$.

Consider $v_1^{l-2}\sum \alpha_\nu z_\nu$. According to Proposition 6.9, the leading term of $\alpha_1 z_1 + \alpha_2 z_2$ is

$$-\alpha_2 \frac{s v_2^{(sp^{i+2}-1)p^{k-2}} \zeta_2^{p^{k-2}}}{v_1} \sim -\alpha_2 \frac{s v_2^{(sp^{i+2}-1)p^{k-2}} \zeta_2}{v_1} \tag{6.13}$$

(using Lemma 3.19), and each element of $F_{l-1}R$ contributes its leading term, which is described in the table. Since (6.13) occurs in Lemma 6.12 (e) while (e) is absent from the table, the coefficient $\alpha_2 = 0$. Thus $\alpha_1 = 0$ also, and we have a relation among elements of $F_{l-1}R$. Thus by induction it is trivial, so $F_l R$ is independent. This completes the induction and the proof of Theorem 6.1. □

7. Computation of the differential

In this section we complete our computation of $H^2\mathrm{BP}_*$ (for p odd) and construct a plethora of nonzero classes in $H^3\mathrm{BP}_*$. Among these classes are γ_t for $t > 0$.

We have already done all the hard work. We have computed $H^0 M_0^2 = E_1^{2,0}$ in the chromatic spectral sequence, and we have seen:

i) $0 = d_1 \colon E_1^{1,0} \to E_1^{2,0}$ in positive dimensions (Corollary 4.8).

ii) $E_\infty^{0,t} = E_1^{0,t} = 0$ for $t > 0$ (Corollary 3.17).

iii) $E_\infty^{1,1} = E_1^{1,1} = 0$ in positive dimensions (Theorem 4.2).

These facts together with Remark 3.8 show that the sequence

$$0 \longrightarrow H^2\mathrm{BP}_* \longrightarrow H^0 M_0^2 \xrightarrow{k_*} H^0 N_0^3 \xrightarrow{\eta} H^3\mathrm{BP}_* \tag{7.1}$$

is exact in positive dimensions.

LEMMA 7.2. *Let $x_n^s/p^{i+1}v_1^j$ be one of the generators of $H^0M_0^2$ listed in Theorem* 6.1. *Then $0 = k_*(x_n^s/p^{i+1}v_1^j) \in H^0 N_0^3$ unless*

i) $s < 0$ *or*

ii) $n \geqq 2$, $s = 1$, $i = 0$, *and* $p^n < j \leqq a_n$.

In case ii),

$$k_*\left(\frac{x_n}{p v_1^j}\right) = -\frac{v_3^{p^{n-1}}}{p v_1^{j-p^n} v_2^{p^{n-1}}}. \tag{7.3}$$

Proof. i) is clear, so suppose $s > 0$. Let $n = k + i$. Lemma 6.7 implies that $x_{k+i}^s \equiv x_k^{sp^i} \bmod (v_1^{b_{k+1}})$. Since $b_{k+1} \geqq a_k \geqq j$, it suffices to consider $x_k^{sp^i}/p^{i+1}v_1^j$. Mod p, $x_k \equiv v_2^{p^k} - v_1^{p^k} v_2^{-p^{k-1}} v_3^{p^{k-1}} +$ terms $v_1^a v_2^b v_3^c$ with $a \geqq p^k - p^{k-2}$ and $b > 0$. Since $x \equiv y(p)$ implies $x^{p^i} \equiv y^{p^i}(p^{i+1})$, it suffices to consider the sp^ith power of this sum. Because $2(p^k - p^{k-2}) \geqq a_k \geqq j$, the only term in $x_k^{sp^i}/p^{i+1}v_1^j$ which can map nontrivially is thus

$$\frac{sp^i v_2^{(sp^i-1)p^k}(-v_1^{p^k} v_2^{-p^{k-1}} v_3^{p^{k-1}})}{p^{i+1} v_1^j}.$$

The power of v_2 here is negative if and only if $s = 1$ and $i = 0$, and the power of v_1 is negative if and only if $j > p^k$. This completes the proof. $\square$

Remark 7.4. Since

$$\begin{array}{ccc} H^0 M_0^2 & \xrightarrow{d_1} & H^0 M_0^3 \\ \cap & & \cap \\ M_0^2 & \xrightarrow{k} & M_0^3 \end{array}$$

commutes this lemma determines $d_1\colon E_1^{2,0} \to E_1^{3,0}$ in positive dimensions.

Remark 7.5. From Theorem 6.1 and Lemma 7.2 it is now easy to read off the structure of $H^2\mathrm{BP}_*$ given in Theorem 2.6. Furthermore, since $v_3^t/pv_1v_2 \notin \mathrm{Im}(k_*)$, it survives in (7.1) to $H^3\mathrm{BP}_*$, and this is Theorem 2.7. This completes the proof of Theorem 2.12.

The reader may easily construct many elements of $H^0N_0^3$ which survive to $H^3\mathrm{BP}_*$. We give some examples of elements of order p, based on the next lemma.

LEMMA 7.6 (Baird). *Let $s_1, \cdots, s_n$ be a sequence of positive integers, and let p^{e_i} be the largest power of p dividing s_i. Then the sequence*

$$p,\ v_1^{s_1},\ \cdots,\ v_n^{s_n}$$

is invariant if and only if $s_i \leqq p^{e_{i+1}}$ for $1 \leqq i < n$.

Proof. We argue by induction on n. The case $n = 0$ is clear, so suppose that $p, v_1^{s_1}, \cdots, v_{n-1}^{s_{n-1}}$ is invariant and that $s_i \leqq p^{e_{i+1}}$ for $1 \leqq i < n-1$.

If also $s_{n-1} \leqq p^{e_n}$ then

$$I = (p, v_1^{s_1}, \cdots, v_{n-1}^{s_{n-1}}) \supseteq J = (p, v^{p^{e_n}}, \cdots, v_{n-1}^{p^{e_n}})\ .$$

Now v_n is invariant mod I_n, so $v_n^{p^{e_n}}$ (and hence $v_n^{s_n}$) is invariant mod J and so also mod I.

If on the other hand $v_n^{s_n}$ is invariant mod I then it is also invariant mod the larger ideal $(I_{n-1}, v_{n-1}^{s_{n-1}})$. But

$$\eta_R v_n^{mp^e} \equiv v_n^{mp^e} + mv_n^{(m-1)p^e} v_{n-1}^{p^e} t_1^{p^{n-1+e}} \bmod (I_{n-1}, v_{n-1}^{2p^e})\ ,$$

so $v_n^{mp^e}$ is invariant mod I only if $s_{n-1} \leqq p^{e_n}$, as desired. $\square$

Consequently $v_3^{s_3}/pv_1^{s_1}v_2^{s_2} \in \Omega^0 N_0^3$ is a cycle for $1 \leqq s_1 \leqq p^{e_2}$, $1 \leqq s_2 \leqq p^{e_3}$, $1 \leqq s_3$. Write

$$\gamma_{s_3/s_2,s_1} = \eta(v_3^{s_3}/pv_1^{s_1}v_2^{s_2}) \in H^3\mathrm{BP}_*\ . \tag{7.7}$$

From Lemma 7.2 and (7.1) we have:

COROLLARY 7.8. $0 \neq \gamma_{s_3/s_2,s_1} \in H^3\mathrm{BP}_*$ *unless $s_1 < s_2 = p^{e_3} = s_3$. In fact, these elements are linearly independent.* $\square$

Remarks 7.9. a) We shall see in Theorem 8.1 that $\gamma_1 = \gamma_{1/1,1} = -\alpha_1\beta_{p-1}$ and in Theorem 8.6 that $\gamma_{p^n/p^n,p^n} = -2\alpha_1\beta_{(p-1)p^n}$ for $n \geqq 1$.

b) There are of course many other nonzero elements of $H^0N_0^3$. For example, by Theorem 5.10, $x_{3,n}^s/pv_1v_2^k \in H^0N_0^3$ for $s \geqq 1$, $1 \leqq k \leqq a_{3,n}$ with $k \leqq p^n$ if $s = 1$. This provides many elements not included in Lemma 7.6. By Lemma 7.2, most of these give rise to nonzero elements in $H^3\mathrm{BP}_*$.

c) There is an exact sequence

$$0 \longrightarrow E_2^{3,0} \longrightarrow H^3\mathrm{BP}_* \longrightarrow E_3^{2,1} \longrightarrow 0 .$$

In Section 8 we shall investigate the third term in this sequence.

8. On certain products

In this section we exploit the $H^*\mathrm{BP}_*$-module structure of the chromatic spectral sequence (Remark 3.5a) to study products in $H^*\mathrm{BP}_*$. We regard this module structure as one of the most powerful computational features of the spectral sequence and expect to see further applications of it in the future. It enables one to obtain a representative for a product in the chromatic E_1 term by replacing one factor by its "chromatic representative"; for example, $\beta_t\alpha_1$ is represented by $-v_2^t t_1/pv_1 \in \Omega^1 M_0^2$ since β_t is represented by $-v_2^t/pv_1 \in \Omega^0 M_0^2$ and α_1 is represented by $t_1 \in \Omega^1\mathrm{BP}_*$.

Many of the results of this section are collected in Theorem 2.8.

We have made no attempt to be systematic; indeed we have restricted ourselves to results closely related to the work of earlier sections. In particular we leave open the question of the decomposability of γ_t for $t > 1$ and the question of the nontriviality of $\beta_t\beta_s$ products. However, we do attempt to demonstrate all of the basic techniques for dealing with products in the chromatic spectral sequence and many of our results have immediate homotopy theoretic consequences.

The section is divided into six subsections. We begin by showing $\eta(v_n) \in H^n\mathrm{BP}_*$ is decomposable for $n \geqq 3$. As a consequence of our decomposition, we have $\alpha_1\beta_{p-1} = -\gamma_1$ in stable homotopy; and using a result of Thomas and Zahler, we also find that $\alpha_1\gamma_{p-1} \neq 0$ in $H^4\mathrm{BP}_*$ and in stable homotopy. The next subsection is devoted to an analysis of the products $\alpha_1\beta_{t/j} \in H^3\mathrm{BP}_*$. This is our principal result on products and it has numerous homotopy-theoretic corollaries; see Theorem 2.15.

Next we give an algebraic generalization of the first element of order p^2 in the cokernel of the J-homomorphism; we show *inter alia* that t divides $\alpha_1\beta_t$ in $H^3\mathrm{BP}_*$. In part D we show that all products of the form $\alpha_{s/i}\beta_{t/j}$ reduce to those studied in part B. We then give an algebraic proof of the known result that the product of any two elements in $H^1\mathrm{BP}_*$ is trivial. Finally we show that for $p = 2$, $\alpha_1\beta^2_{2^s-1/2^s-1} = 0$ in $H^5\mathrm{BP}_*$. The result dampens any hope that the mod 2 Arf invariant question might be resolved using the Adams-Novikov spectral sequence.

A. *Decomposability of* $\eta(v_n)$. To begin with we prove a generalization of the relation

$$\alpha_1\beta_{p-1} = -\gamma_1$$

(see [23]). Recall the Greek letter map (from before 3.8)

$$\eta\colon H^0(\mathrm{BP}_*/I_n) \longrightarrow H^n(\mathrm{BP}_*)\ .$$

THEOREM 8.1. *Let p be arbitrary and $n \geqq 2$. In $H^{n+1}(\mathrm{BP}_*)$,*

$$\alpha_1\eta(v_n^{sp-1}) = 0 \qquad \textit{for } p \nmid s > 1$$

and

$$\alpha_1\eta(v_n^{p-1}) = (-1)^{n+1}\eta(v_{n+1})\ .$$

Proof. Let

$$z = \frac{v_n^{sp} - sv_{n-1}^p v_n^{(s-1)p-1}v_{n+1}}{p\cdots v_{n-2}v_{n-1}^{p+1}} \in \Omega^0 M_0^n\ .$$

In the double complex $\Omega^* M_0^*$ we have, using the congruence $dv_{n+1} \equiv v_n t_1^{p^n} - v_n^p t_1 \bmod I_n$ of (1.7),

$$d_i z = \frac{sv_n^{sp-1}t_1}{p\cdots v_{n-1}} = (-1)^{\left[\frac{n+1}{2}\right]}\eta\Big(\frac{sv_n^{sp-1}}{p\cdots v_n}\Big)\alpha_1 \qquad \text{from 3.9}\ ,$$

$$d_e z = -\frac{v_{n+1}}{p\cdots v_n} = -(-1)^{\left[\frac{n+2}{2}\right]}\eta\Big(\frac{v_{n+1}}{p\cdots v_n}\Big)\ , \qquad s = 1\ ,$$

$$d_e z = 0 \text{ (all the powers of } v_n \text{ are positive)}\ , \qquad s > 1\ .$$

From 3.3, $dz = d_e z + (-1)^n d_i z$ so in the double complex $(-1)^{[n+1/2]}\eta(sv_n^{sp-1}/p\cdots v_n)\alpha_1$ is homologous to 0 if $s > 1$ and to $(-1)^n(-1)^{[n+2/2]}\eta(v_{n+1}/p\cdots v_n)$ if $s = 1$. By our convention concerning the use of η (see before 3.8), this says that in $H^{n+1}\mathrm{BP}_*$

$$(-1)^{\left[\frac{n+1}{2}\right]}\eta(sv_n^{sp-1})\alpha_1 = 0 \qquad \text{if } s > 1\ ,$$

$$= (-1)^n(-1)^{\left[\frac{n+2}{2}\right]}\eta(v_{n+1}) \qquad \text{if } s = 1\ .$$

This proves the first statement. For $s = 1$, an analysis of the signs involved gives

$$-\eta(v_n^{p-1})\alpha_1 = \eta(v_{n+1}) \text{ always.}$$

However, $\eta(v_n^{p-1})$ is in cohomological degree n and α_1 in degree 1 so

$$\eta(v_{n+1}) = -\eta(v_n^{p-1})\alpha_1 = (-1)^{n+1}\alpha_1\eta(v_n^{p-1})\ .$$

This concludes the proof of 8.1. The signs can be somewhat confusing. In an attempt to give the reader faith in these signs we offer a more direct proof of this last result.

Let $\delta_k: H^* \mathrm{BP}_*/I_{k+1} \to H^{*+1}\mathrm{BP}/I_k$ denote the boundary homomorphism associated with the exact sequence

$$0 \longrightarrow \mathrm{BP}_*/I_k \xrightarrow{v_k} \mathrm{BP}_*/I_k \longrightarrow \mathrm{BP}_*/I_{k+1} \longrightarrow 0 .$$

Then $\eta(v_n^t) = \delta_0 \cdots \delta_{n-1}(v_n^t) \in H^n \mathrm{BP}_*$, where $v_n^t \in H^0 \mathrm{BP}_*/I_n = \mathbf{F}_p[v_n]$. It follows from $dv_{n+1} = v_n t_1^{p^n} - v_n^p t_1 \bmod I_n$ that

$$\delta_n(v_{n+1}) = t_1^{p^n} - v_n^{p-1} t_1 \text{ in } H^1 \mathrm{BP}_*/I_n .$$

Now $t_1^{p^n}$ is an element in $H^1 \mathrm{BP}_*/I_{n-1}$ so by exactness

$$\delta_{n-1}\delta_n(v_{n+1}) = \delta_{n-1}(-v_n^{p-1} t_1) .$$

An elementary computation in the cobar complex shows that this is $-\delta_{n-1}(v_n^{p-1})t_1$. The same computation shows that

$$\delta_0 \cdots \delta_n(v_{n+1}) = -\delta_0 \cdots \delta_{n-1}(v_n^{p-1})t_1,$$
$$\text{i.e., } \eta(v_{n+1}) = -\eta(v_n^{p-1})t_1 = -\eta(v_n^{p-1})\alpha_1 .$$

The argument proceeds from here as in the first proof. □

Remarks 8.2. a) Thus $\alpha_1\beta_{p-1} \equiv -\gamma_1 \bmod F^{2p+1}\pi_*(S^0)$ (using Note 1.12); but in this dimension $F^{2p+1} = 0$ by Remark 1.17. So we have recovered the relation in $\pi_*(S^0)$.

b) Thomas and Zahler [32] have shown that for $p > 5$, $0 \neq \eta(v_4) \in H^4(\mathrm{BP}_*)$. We have just seen that $\eta(v_4) = \alpha_1\gamma_{p-1}$. Combining these two results with the fact that $\alpha_1\gamma_{p-1}$ cannot be a boundary in the Novikov spectral sequence (Note 1.12), we find that $\alpha_1\gamma_{p-1} \neq 0$ in $\pi_*(S^0)$.

B. *The products $\alpha_1\beta_{t/j}$.* Throughout part B, p will be an odd prime.

We turn now to the study of products of the form $\alpha_1\beta_{t/j} = \beta_{t/j}\alpha_1$. The element $\alpha_1 \in H^1\mathrm{BP}_*$ is represented by $t_1 \in \Omega^1 \mathrm{BP}_*$. The element

$$\eta\left(\frac{x_n^s}{pv_1^j}\right) = \beta_{sp^n/j} \in H^2\mathrm{BP}_*$$

is represented by $-x_n^s/pv_1^j \in \Omega^0 M_0^2$ in the double complex C_0^* (3.3) (see 3.9 for the sign). By the module structure, $-x_n^s t_1/pv_1^j \in \Omega^1 M_0^2$ in the double complex represents $\alpha_1\beta_{sp^n/j} \in H^3\mathrm{BP}_*$. We first decide which of these is zero in $H^1M_0^2$. From the definition of x_n (5.1) we see that $x_n^s \equiv v_2^{sp^n} \bmod (p, v_1^2)$. Thus from 6.12 a) and c) for $p \nmid s \in Z$, $x_n^s t_1/v_1$ is a generator of $\ker(v_1 | H^1M_1^1)$ if $n \geqq 0$ and either $s \not\equiv -1(p)$ or $s \equiv -1(p^2)$, and $x_n^s t_1/v_1^2$ is a generator if $n \geqq 1$ and $s \equiv -1(p)$ but $s \not\equiv -1(p^2)$.

Consequently, the set $T \subset H^1 M_1^1$ of classes $x_n^s t_1/v_1^j$ for $n \geqq 0$, $1 \leqq j \leqq a_n$, and $p \nmid s \in \mathbf{Z}$ such that if $j = 1$ then either $s \not\equiv -1(p)$ or $s \equiv -1(p^2)$, is linearly independent.

Consider the exact sequence

$$H^0 M_0^2 \xrightarrow{\delta} H^1 M_1^1 \xrightarrow{i_*} H^1 M_0^2 \,. \tag{8.3}$$

Proposition 6.9 shows that (with $j = mp^i$)

$$\begin{aligned}
\delta\left(\frac{x_{i+k}^s}{p^{i+1} v_1^j}\right) &= -\frac{m x_{i+k}^s t_1}{v_1^{1+j}}\,, && k \geqq 1,\ j \leqq a_{k-1},\ p \nmid m \\
&= -\frac{x_2^s t_1}{v_1^{2+p}} + \frac{s v_2^{sp^2-1} t_1}{v_1}\,, && k = 2,\ i = 0,\ m = p+1 \\
&= \frac{2s v_2^{sp^{i+k}-p^{k-2}} t_1}{v_1}\,, && k \geqq 3,\ j = 1 + a_{k-1},\ p \mid m \\
&= -\frac{x_{2k-2}^s t_1}{v_1^{2+a_{k-1}}} + \frac{2s v_2^{(sp^k-1)p^{k-2}} t_1}{v_1}\,, \\
& && k \geqq 3,\ i = k-2,\ m = p+1\,.
\end{aligned} \tag{8.4}$$

An excruciatingly painful inspection of the proof of Theorem 6.1 can convince the reader that the images under δ of the remaining generators (for $H^0 M_0^2$) are linearly independent modulo the span of T. Combining this with the exact sequence (8.3) and recalling that $i_*(x) = x/p$, we have:

LEMMA 8.5. *Let $n \geqq 0$, $p \nmid s \in \mathbf{Z}$, $1 \leqq j \leqq a_n$. In $H^1 M_0^2$, $x_n^s t_1/pv_1^j \neq 0$ if and only if either* i) *or* ii) *holds.*

i) *$j = 1$ and either $s \not\equiv -1(p)$ or $s \equiv -1(p^{n+2})$.*

ii) *$j > 1 + a_{n-\nu(j-1)-1}$.*

Furthermore all linear relations among these classes are given by

$$\frac{x_2^s t_1}{pv_1^{2+p}} = s\frac{x_0^{sp^2-1} t_1}{pv_1}\,,$$

$$\frac{x_{2n+2}^s t_1}{pv_1^{2+a_{n+1}}} = 2s\frac{x_n^{sp^{n+2}-1} t_1}{pv_1}\,, \qquad n \geqq 1\,.$$

Here $\nu(a)$ is the largest integer ν such that $p^\nu \mid a$.

We can now prove the central result of this section.

THEOREM 8.6. *Let $n \geqq 0$, $p \nmid s \geqq 1$, and $1 \leqq j \leqq a_n$ with $j \leqq p^n$ if $s = 1$. In $H^3 \mathrm{BP}_*$, $\alpha_1 \beta_{sp^n/j} \neq 0$ if and only if one of the following conditions holds.*

i) *$j = 1$ and either $s \not\equiv -1(p)$ or $s \equiv -1(p^{n+2})$.*

ii) *$j = 1$ and $s = p-1$.*

iii) *$j > 1 + a_{n-\nu(j-1)-1}$.*

In case ii), *we have $\alpha_1 \beta_{p-1} = -\gamma_1$ and for $n \geqq 1$, $2\alpha_1 \beta_{(p-1)p^n} = -\gamma_{p^n/p^n, p^n}$* (*see* (7.7)). *Finally, the only linear relations among these classes are*

$$\alpha_1\beta_{sp^2/2+p} = s\alpha_1\beta_{sp^2-1}\,,$$
$$\alpha_1\beta_{sp^{2n+2}/2+a_{n+1}} = 2s\alpha_1\beta_{sp^{2n+2}-p^n}\,, \qquad n \geqq 1\,.$$

Proof. As noted above, $x_n^s t_1/pv_1^j$ survives to $-\alpha_1\beta_{sp^n/j}$ in the chromatic spectral sequence. So to show this element is nontrivial it suffices to see that it is never a boundary. The chromatic spectral sequence lies in the first quadrant, so the result follows from Theorem 4.2 ($E_1^{1,1}=0$ in positive dimensions) and Corollary 3.17 ($E_2^{0,2}=0$) for the non-zero elements of Lemma 8.5.

All of the other $x_n^s t_1/pv_1^j$ represent zero in $H^1M_0^2$. Thus for these s, n, and j, $\alpha_1\beta_{sp^n/j}$ lies in filtration 3 of $H^3\mathrm{BP}_*$. To find a representative for it, one must find $z\in\Omega^0M_0^2$ such that $d_iz = x_n^s t_1/pv_1^j$ and evaluate $d_ez\in\Omega^0M_0^3$; for d_ez and $-d_iz$ are homologous in the total complex.

There are two distinct types of $x_n^s t_1/pv_1^j$ which we must check. All of them represent elements in $H^1M_1^1$ (as $x_n^s t_1/v_1^j$). Our first type, x_n^s/pv_1, $s = pt-1$, $p\nmid t$, is already zero here (6.12 c)). Our second type, listed in 8.4, is nonzero in $H^1M_1^1$ but goes to zero in $H^1M_0^2$ as in 8.3. We compute for the first type now.

From (5.9) we see that

$$(8.7)\qquad d_i\left(\frac{x_n^t}{pv_1^{1+a_n}}\right) = \frac{tv_2^{tp-1}t_1}{pv_1} \qquad \text{for } n=1\,,$$
$$= \frac{2tv_2^{(tp-1)p^{n-1}}t_1}{pv_1} \qquad \text{for } n>1$$

in $\Omega^*M_0^2$. Mod p,

$$x_n \equiv v_2^{p^n} - v_1^{p^n}v_2^{-p^{n-1}}v_3^{p^{n-1}} + \text{terms } v_1^a v_2^b v_3^c$$

with $a \geqq p^n - p^{n-2}$ and $b>0$. Since $2(p^n - p^{n-2}) \geqq 1 + a_n$, we find:

$$d_e\left(\frac{x_n^t}{pv_1^{1+a_n}}\right) = -\frac{v_3^{p^{n-1}}}{pv_1^{p^{n-1}}v_2^{p^{n-1}}}\,, \qquad t=1$$
$$= 0\,, \qquad t>1\,.$$

Thus the element (8.7) survives, and this gives condition ii) and the relation $2\alpha_1\beta_{(p-1)p^n} = -\gamma_{p^n/p^n,p^n}$ for $n\geqq 1$; for the sign, see Remark 3.8.

The elements of the second type are just i_* (as in (8.3) $i_*(x) = x/p$) of those listed in 8.4. Since on the chain level, $i_*\delta(z) = d_i(z/p)$,

$$(8.8)\qquad d_e(x_{i+k}^s/p^{i+2}v_1^{mp^i}) = 0$$

for values of k, i, m as in (8.4). The proof of Lemma 7.2 applies since in all cases of (8.4) $j\leqq 1 + a_{k-1}$, and $b_k \geqq 1 + a_{k-1}$, and the result follows. □

C. *Divisibility of $\alpha_1\beta_{t/j}$.* These products are more highly p-divisible than the bare $\beta_{t/j}$ elements. To see this, define cycles $y_{j,i+1}\in\Omega^1M_0^2$ for

$j \geqq 1$, $i \geqq 0$, by

(8.9) $$y_{j,i+1} = \sum_{k>0} \frac{c_{j,k} t_1^k}{kp^{i+2-k} v_1^{j-1+k}}$$

where $c_{j,k} = (-1)^{k-1}(j-1, k-1)$. Note that

$$y_{1,i+1} = d_e y_{i+1}$$

for y_{i+1} as in Theorem 4.2 b), and that for $j > 1$,

$$y_{j,i+1} = -d\left(\frac{1}{p^{i+2}(j-1)v_1^{j-1}}\right);$$

so $y_{j,i+1}$ is indeed a cycle. Notice also that for $i \geqq 1$,

$$py_{j,i+1} = y_{j,i}$$

and that if p^i divides j then

$$y_{j,i+1} = \frac{t_1}{p^{i+1} v_1^j}.$$

It is convenient to prove here the assertion of Remark 4.9 d).

LEMMA 8.10. *For $i \geqq 0$, $y_{1,i+1} \neq 0$ in $H^1 M_0^2$.*

Proof. Since $py_{1,i+1} = y_{1,i}$, it suffices to see that $t_1/pv_1 = y_{1,1}$ is nonzero in $H^1 M_0^2$. Now in the diagram (with $q = 2(p-1)$)

$$\begin{array}{ccccc} H^{0,q} M_1^1 & & H^{0,0} M_0^2 & & \\ \downarrow & & \downarrow & & \\ H^{1,q} M_2^0 & \xrightarrow{\delta} & H^{1,0} M_1^1 & \xrightarrow{\delta} & H^{1,0} M_0^2, \end{array}$$

the two L-shaped pieces are exact, and $0 \neq \{t_1\} = h_0 \in H^{1,q} M_2^0$ is carried to $\{y_{1,1}\}$. But by Theorems 5.3 and 6.1 the top groups are 0, so $\{y_{1,1}\} \neq 0$. □

Recall from Corollary 6.5 that $x_n^s/p^{i+1} v_1^j \in \Omega^0 M_0^2$ is a cycle for $i \leqq n$ and $p^i | j \leqq a_{n-i}$. In contrast, we have

LEMMA 8.11. *Let $p > 2$. Then $x_n^s y_{j,i+1} \in \Omega^1 M_0^2$ is a cycle for $0 \leqq i \leqq n$ and $j \leqq a_{n-i}$.*

Proof. Let l be so large that all terms of $x_n^s y_{j,i+1}$ lie in $\Gamma_l = v_2^{-1}\mathrm{BP}_*\mathrm{BP}/(p^{l+1}, v_1^{p^l}) \subset \Omega^1 M_0^2$. Γ_l is a Hopf algebroid with coefficient algebra $v_2^{-1}\mathrm{BP}_*/(p^{l+1}, v_1^{p^l})$, and it suffices to show $x_n^s y_{j,i+1}$ is a cycle in its cobar construction.

Recall (Proposition 6.4) that x_n (and hence x_n^s) is invariant modulo the ideal $I_{2,n} = (p^{n+1}, \cdots, p^{n-t} v_1^{a_t}, \cdots)$. We claim that $I_{2,n} y_{j,i+1} = 0$. Clearly $(\bigcap_{k>0} J_{j,i,k}) y_{j,i+1} = 0$, where $J_{j,i,k} = (kp^{i+2-k}, v_1^{j+k-1})$; and we claim that $I_{2,n} \subseteq J_{j,i,k}$ for all $k > 0$ and the stated values of n, i, and j.

The condition $n \geqq i$ insures that $p^{n+1} \in J_{j,i,k}$ for all $k > 0$.

Now $p^{n-t}v_1^{a_t} \in J_{j,i,k}$ if either $n - t \geqq i + 2 + \nu(k) - k$ (where $\nu(k)$ is the largest integer ν such that $p^\nu | k$) or $a_t \geqq j + k - 1$. Thus we claim that $n - t < i + 2 + \nu(k) - k$ implies

(8.12) $$a_t \geqq j + k - 1 .$$

The first statement is equivalent to

(8.13) $$t \geqq (n - i) + m$$

where $m = k - 1 - \nu(k)$. Notice that $m > 0$ and $a_m \geqq k$. If $n - i = 0$, then $j = 1$ and (8.12) follows from $a_m \geqq k$. Otherwise (8.13) implies that $a_t \geqq a_{n-i} + a_m$; and (8.12) follows since $a_{n-i} \geqq j$ and $a_m \geqq k$.

Write $x = x_n^s$, $y = y_{j,i+1}$. Then $\Delta y = y \otimes 1 + 1 \otimes y$, and we have

$$\begin{aligned} d(xy) &= xy \otimes 1 - x(y \otimes 1 + 1 \otimes y) + 1 \otimes xy \\ &= 1 \otimes xy - x \otimes y \\ &= (\eta_R(x) - x) \otimes y \in I_{2,n} \otimes y = 0 \end{aligned}$$

since $I_{2,n}$ is invariant and $I_{2,n} y = 0$. □

We may now apply the technique of proof of Lemma 7.2 to show that $d_e(x_n^s y_{j,i+1}) \in \Omega^1 M_0^3$ is nonzero if and only if $s < 0$ or $s = 1$, $i = 0$, and $j > p^n$. Thus with these exceptions $x_n^s y_{j,i+1}$ is a cycle in $\Omega^1 N_0^2$. Define

(8.14) $$\phi_{sp^n/j,i+1} = \eta(x_n^s y_{j,i+1}) \in H^3\mathrm{BP}_* ,$$

so that for $i > 0$

(8.15) $$p\phi_{sp^n/j,i+1} = \phi_{sp^n/j,i}$$

and if p^i divides j then

(8.16) $$\phi_{sp^n/j,i+1} = \alpha_1\beta_{sp^n/j,i+1} .$$

In particular $\phi_{p/1,2}$ is the well-known element such that $p\phi_{p/1,2} = \alpha_1\beta_p$; it survives in the Novikov spectral sequence to the first element of order p^2 not in the image of the J-homomorphism. Notice that more generally s divides $\alpha_1\beta_s$ in $H^3\mathrm{BP}_*$.

This completes the proof of Theorem 2.8 c).

D. *Products with other elements of* $H^1\mathrm{BP}_*$. By convention let $\beta_{tp^n/j} = 0$ for $j \leqq 0$.

PROPOSITION 8.17. *Let p be odd. Let s and t be prime to p, let $m, n \geqq 0$, and let $1 \leqq j \leqq a_n$ with $j \leqq p^n$ if $t = 1$. Then in* $H^3\mathrm{BP}_*$,

$$\alpha_{sp^m/m+1}\beta_{tp^n/j} = s\alpha_1\beta_{tp^n/j-sp^m+1} .$$

Proof. $H^*\mathrm{BP}_*$ acts on $H^*N_1^1$ through $H^*(\mathrm{BP}_*/p)$. The element $\alpha_{sp^m/m+1} \in H^1\mathrm{BP}_*$ reduces to $sv_1^{sp^m-1}t_1 \in H^1(\mathrm{BP}_*/p)$. Hence, since η is $H^*\mathrm{BP}_*$-linear,

$$\alpha_{sp^m/m+1}\beta_{tp^n/j} = \alpha_{sp^m/m+1}\eta(x_n^t/pv_1^j) = \eta(sx_n^t t_1/pv_1^{j-sp^m+1}) = s\alpha_1\beta_{tp^n/j-sp^m+1} . \quad \square$$

E. *Triviality of* $H^1(\mathrm{BP}_*)\cdot H^1(\mathrm{BP}_*)$. We next prove Theorem 2.2 b):

THEOREM 8.18. *Let p be odd. Then* $H^1(\mathrm{BP}_*)\cdot H^1(\mathrm{BP}_*) = 0$.

Proof. By Theorem 2.2 a) it suffices to show that

$$\alpha_{sp^m/m+1}\alpha_{tp^n/n+1} = 0$$

for s and t positive and prime to p, and $m, n \geqq 0$. In $\Omega^1\mathrm{BP}_*$,

$$\alpha_{tp^n/n+1} = \sum\nolimits_{i\geqq 1}\binom{tp^n}{i}p^{i-n-1}v_1^{tp^n-i}t_1^i .$$

Thus in $\Omega^1 M_0^1$,

$$\frac{v_1^{sp^m}}{p^{m+1}}\cdot\alpha_{tp^n/n+1} = \sum\nolimits_{i\geqq 1}\binom{tp^n}{i}\frac{v_1^{sp^m+tp^n-i}t_1^i}{p^{m+n+2-i}} . \tag{8.19}$$

We claim that this is a boundary in the total complex C_0^*, and the theorem follows. In fact,

$$d\left(\frac{tp^n}{sp^m+tp^n}\,\frac{v_1^{sp^m+tp^n}}{p^{m+n+2}}\right) = \frac{v_1^{sp^m}}{p^{m+1}}\cdot\alpha_{tp^n/n+1} .$$

To see this, note that this boundary is by definition

$$\left(\frac{tp^n}{sp^m+tp^n}\right)\sum\nolimits_{i\geqq 1}\binom{sp^m+tp^n}{i}\frac{v_1^{sp^m+tp^n-i}t_1^i}{p^{m+n+2-i}} . \tag{8.20}$$

We claim that (8.19) and (8.20) are equal term-by-term. That is, for $1\leqq i\leqq m+n+1$,

$$\binom{tp^n}{i} \equiv \left(\frac{tp^n}{sp^m+tp^n}\right)\binom{sp^m+tp^n}{i} \bmod p^{m+n+2-i} . \tag{8.21}$$

For $i = 1, 2$ this is clear (for p odd) so suppose $i > 2$. For $p = 3$ and $i = 3$, compute directly. Otherwise (again for p odd), p^{i-2} does not divide $i!$, so (8.21) follows from the obvious congruence

$$(tp^n)\cdots(tp^n-i+1) \equiv tp^n(sp^m+tp^n-1)\cdots(sp^m+tp^n-i+1) \mod p^{m+n} . \quad \square$$

F. *On the Arf invariant.* The element $v_2^{2^j}/2v_1^{2^j}\in\Omega^0 M_0^2$, $p = 2$, survives by Section 4 in the chromatic spectral sequence to a nontrivial element $\beta_{2^j/2^j}\in H^2\mathrm{BP}_*$. If it survives in the Novikov spectral sequence then it represents an element in $\pi_*^s(S^0)$ of Arf invariant 1 ([35]; see also § 9 below).

The work of Milgram and others leads one to expect

$$d_3\beta_{2^j/2^j} = \alpha_1\beta^2_{2^{j-1}/2^{j-1}}$$

in the Novikov spectral sequence. It is well-known that the analogous product in the Adams E_2-term is trivial, but it might be hoped that it is nonzero for large j in the Novikov E_2-term. This is unfortunately not the case:

PROPOSITION 8.22. *In* $H^4\mathrm{BP}_*$ *for* $p = 2$, $\beta^2_{2^j/2^j} = 0$ *for* $j > 0$.

Recall (Corollary 4.22) that $\beta_{1/1} = 0$.

Proof. First compute the mod 2 reduction of $\beta_{2^j/2^j}$:

$$\delta_1'\left(\frac{v_2^{2^j}}{2v_1^{2^j}}\right) = \frac{v_1^{2^j}t_1^{2^{j+1}} - v_1^{2^{j+1}}t_1^{2^j}}{2v_1^{2^j}} = \frac{t_1^{2^{j+1}} - v_1^{2^j}t_1^{2^j}}{2},$$

$$\delta_0'\left(\frac{t_1^{2^{j+1}} - v_1^{2^j}t_1^{2^j}}{2}\right) \equiv t_1^{2^j} \otimes t_1^{2^j} + v_1^{2^j}t_1^{2^{j-1}} \otimes t_1^{2^{j-1}} \quad (\mathrm{mod}\ 2)\ .$$

So $\beta^2_{2^j/2^j}$ is represented in the chromatic spectral sequence by $v_2^{2^j}t_1^{2^j}\otimes t_1^{2^j}/2v_1^{2^j} \in \Omega^2 M_0^2$. Now an easy calculation shows that this is the boundary of

$$\frac{v_2^{2^{j-1}\cdot 3}t_1^{2^j}}{2v_1^{2^{j-1}\cdot 3}} + \frac{v_3^{2^{j-1}}t_1^{2^j}}{2v_1^{2^{j-1}}} \in \Omega^1 M_0^2\ .$$

Furthermore, this element is killed by the external differential d_e, so $\beta^2_{2^j/2^j} = 0$ as desired. □

9. The Thom reduction

In this section we study the map

$$\Phi\colon \mathrm{Ext}^*_{\mathrm{BP_*BP}}(\mathrm{BP}_*, \mathrm{BP}_*) \longrightarrow \mathrm{Ext}^*_{A_*}(\mathbf{F}_p, \mathbf{F}_p)$$

induced by the Thom map $\phi\colon \mathrm{BP} \to H$. Here H is the mod p Eilenberg-MacLane spectrum and $A_* = H_*H$ is the dual Steenrod algebra. We restrict ourselves to p odd. By studying the I-adic filtration on $\mathrm{Ext}_{\mathrm{BP_*BP}}(\mathrm{BP}_*, \mathrm{BP}_*)$ (where $I = (p, v_1, \cdots) = \ker(\mathrm{BP}_* \to \mathbf{F}_p)$), we evaluate Φ on Ext^1 and Ext^2. This puts a strong upper bound on the possible survivors in $\mathrm{Ext}^2_{A_*}(\mathbf{F}_p, \mathbf{F}_p)$ in the Adams spectral sequence, analogous to the mod 2 results of [13] (as corrected by [38] Prop. 3.3.7).

Recall that

$$A_* = E[e_0, e_1, \cdots] \otimes \mathbf{F}_p[t_1, t_2, \cdots] \text{ with } t_0 = 1,$$
$$\Delta t_n = \sum_{i=0}^n t_i \otimes t_{n-i}^{p^i}\ ,$$
$$\Delta e_n = \sum_{i=1}^n e_i \otimes t_{n-i}^{p^i} + 1 \otimes e_n\ .$$

Thus e_n is the Hopf conjugate of Milnor's τ_n [17], and t_n is the conjugate of ξ_n. Zahler [35] showed that $t_n \mapsto t_n$ under

$$\phi_*\phi\colon \mathrm{BP}_*\mathrm{BP} \longrightarrow A_*\ .$$

From the work of Liulevicius [12], we recall the following facts.

(9.1) $$\mathrm{Ext}^0_{A_*}(\mathbf{F}_p, \mathbf{F}_p) = \mathbf{F}_p .$$

(9.2) $\mathrm{Ext}^1_{A_*}(\mathbf{F}_p, \mathbf{F}_p)$ has generators

$$a_0 = \{e_0\} ,$$
$$h_i = \{t_1^{p^i}\} , \qquad i \geqq 0 .$$

(9.3) $\mathrm{Ext}^2_{A_*}(\mathbf{F}_p, \mathbf{F}_p)$ has generators

$$a_1 = \{2e_1 \otimes t_1 + e_0 \otimes t_1^2\} ,$$
$$b_i = \left\{\sum_{j=1}^{p-1} \frac{\binom{p}{j}}{p} t_1^{p^i(p-j)} \otimes t_1^{p^i j}\right\} , \qquad i \geqq 0 ,$$
$$g_i = \{2t_1^{p^i} \otimes t_2^{p^i} + t_1^{2p^i} \otimes t_1^{p^{i+1}}\} , \qquad i \geqq 0 ,$$
$$k_i = \{2t_2^{p^i} \otimes t_1^{p^{i+1}} + t_1^{p^i} \otimes t_1^{2p^{i+1}}\} , \qquad i \geqq 0 ,$$
$$a_0^2 ,$$
$$a_0 h_i , \qquad i > 0 ,$$
$$h_i h_j , \qquad j - 1 > i \geqq 0 .$$

THEOREM 9.4. *Let $p > 2$.* a) (*Novikov* [21]) Φ *maps the generators of* $H^1\mathrm{BP}_*$ *given in Theorem* 2.2 *to zero with the following single exception*:

$$\Phi\alpha_1 = h_0 .$$

b) Φ *maps the generators of* $H^2\mathrm{BP}_*$ *given in Theorem* 2.6 *to zero with the following exceptions*:

$$\Phi\beta_2 = k_0 ,$$
$$\Phi\beta_{p^i/p^i-1} = h_0 h_{i+1} , \qquad i > 0 ,$$
$$\Phi\beta_{p^i/p^i} = -b_i , \qquad i \geqq 0 .$$

Proof. Let $I = (p, v_1, \cdots)$ be the augmentation ideal of BP_*. Give the comodule M_0^s the I-adic filtration: that is, for $k \in \mathbf{Z}$, $v^E \in F^k M_0^s$ if and only if $\sum e_i \geqq k$. This induces a filtration on the subcomodule N_0^s. Since I is invariant, these are filtrations by subcomodules.

Now the exact sequence

$$0 \longrightarrow N_0^s \longrightarrow M_0^s \longrightarrow N_0^{s+1} \longrightarrow 0$$

has the property that for all k,

(9.5) $$0 \longrightarrow F^k N_0^s \longrightarrow F^k M_0^s \longrightarrow F^k N_0^{s+1} \longrightarrow 0$$

is exact. If we filter $\Omega^* N_0^s$ by

$$F^k \Omega^* N_0^s = \Omega^* F^k N_0^s$$

then it follows from (9.5) that the connecting homomorphism

$$\delta_s' \colon H^t N_0^{s+1} \longrightarrow H^{t+1} N_0^s$$

is filtration-preserving. Thus

$$\eta\colon H^t N_0^s \longrightarrow H^{t+s}\mathrm{BP}_*$$

is filtration-preserving.

Now [16] $\Omega^*(\mathrm{BP}_*/I)$ is isomorphic to the unnormalized cobar construction of the Hopf algebra $\mathbf{F}_p[t_1, t_2, \cdots]$; and $\Omega^*\mathrm{BP}_* \to \Omega^*(\mathrm{BP}_*/I)$ induces the Thom reduction Φ. Thus Φ kills $F^1H^*\mathrm{BP}_*$. So for $x \in H^t N_0^s$, $\Phi(\eta x) \neq 0$ only if $x \notin F^1H^tN_0^s$.

To prove a), note that for $n \geqq 0$, $p \nmid s$, $v_1^{sp^n}/p^{n+1} \notin F^1N_0^1$ if and only if

$$sp^n - (n+1) < 1\;;$$

i.e., if and only if $n = 0$, $s = 1$. Further, $\eta(v_1/p) = t_1$, and this proves the result.

In b) we are concerned with $\eta(x_n^s/p^{i+1}v_1^j)$ for $n \geqq 0$, $p \nmid s \geqq 1$, $i \geqq 0$, $j \geqq 1$, $p^i \mid j \leqq a_{n-i}$ with $j \leqq p^n$ if $s = 1$. Since the term in x_n having lowest filtration is $v_2^{p^n}$, $x_n^s/p^{i+1}v_1^j \notin F^1N_0^2$ if and only if

$$sp^n - (i+1) - j < 1\,.$$

For $i \geqq 1$, sp^n is minimized by $s = 1$ and j is maximized by $j = p^i a_{n-2i}$, but still $p^n - (i+1) - p^i a_{n-2i} \geqq 1$. For $i = 0$, we may have either

i) $s = 1$ and $p^n - 1 \leqq j \leqq p^n$, or

ii) $s = 2$, $n = 0$, $j = 1$.

We compute $\Phi\eta(x_n/pv_1^{p^n})$ and leave the other cases to the reader. Note that $x_n/pv_1^{p^n} = v_2^{p^n}/pv_1^{p^n}$. Now

$$\delta_1'\Big(\frac{v_2^{p^n}}{pv_1^{p^n}}\Big) \equiv \frac{v_1^{p^n}t_1^{p^{n+1}}}{pv_1^{p^n}} = \frac{t_1^{p^{n+1}}}{p} \qquad \bmod F^1\,,$$

$$\delta_0'\Big(\frac{t_1^{p^{n+1}}}{p}\Big) = -\sum_{i=1}^{p^{n+1}-1}\frac{\binom{p^{n+1}}{i}}{p}t_1^{p^{n+1}-i}\otimes t_1^i$$

$$\equiv -b_n \qquad \bmod F^1\,. \quad \square$$

COROLLARY 9.6. *Let $p > 2$. In the classical Adams spectral sequence for the sphere,*

a) *(Liulevicius [12], Shimada-Yamanoshita [28]) Of the generators (9.2) of $\mathrm{Ext}^1_{A_*}(\mathbf{F}_p, \mathbf{F}_p)$, only a_0, h_0 can survive, and*

b) *Of the generators (9.3) of $\mathrm{Ext}^2_{A_*}(\mathbf{F}_p, \mathbf{F}_p)$, only the following can survive in the Adams spectral sequence: a_1, b_i $(i \geqq 0)$, k_1, a_0^2, h_0h_i $(i \geqq 2)$, and if $p = 3$, a_0h_1.*

Remark 9.7. a) Of course a_0 survives to p and h_0 to α_1. This is Novikov's proof of the mod p Hopf invariant 1 theorem. It works also for $p = 2$; we leave the details to the reader.

b) a_1 survives to α_2, b_0 to β_1, k_1 to β_2, h_0h_2 to $\beta_{p/p-1}$, and if $p = 3$, a_0h_1 survives to $\alpha_{3/2}$. The element b_1 supports the "Toda differential." The second author has used the stabilizer algebras to show that for $p \geqq 3$, $d_{2p-1}\beta_{p^n/p^n} \neq 0$ in the Novikov spectral sequence for all $n \geqq 1$. This implies that b_n dies in the Adams spectral sequence for all $n \geqq 1$, $p > 3$ (see [37]).

Proof of Corollary 9.6. The map $\mathrm{BP} \to H$ induces a map from the Novikov spectral sequence to the Adams spectral sequence. Thus for any survivor in $\mathrm{Ext}^2_{A_*}(\mathbf{F}_p, \mathbf{F}_p)$ there corresponds a survivor in $H^i\mathrm{BP}_*$ with $0 \leqq i \leqq 2$. $H^0(\mathrm{BP}_*)$ survives to $\pi_0(S^0)_{(p)} \cong \mathbf{Z}_{(p)}$; so a_0^2 can survive. The image of the J-homomorphism maps isomorphically to $H^1\mathrm{BP}_*$, and the only elements of $\mathrm{Ext}^2_{A_*}(\mathbf{F}_p, \mathbf{F}_p)$ surviving to $\mathrm{Im}\, J$ are a_1 and, if $p = 3$, a_0h_1. Any other survivor must be in the image of Φ and the result follows from Theorem 9.4. □

10. Concluding remarks

The computability of the cohomology of the Morava stabilizer algebras, $H^*M_n^0$, was the motivating force behind this entire project. However, in retrospect, the attentive reader may observe that we needed very little information about $H^*M_n^0$ in this work. In fact we have only used the structure of $H^*M_1^0$ and the nontriviality of h_0, h_1 and $\zeta_2 \in H^1M_2^0$. Stronger use of $H^*M_n^0$ will presumably lead to other homotopy-theoretic results.

For example, the second author has computed $H^*M_3^0$ for $p > 3$ and used it to detect elements in $H^*\mathrm{BP}_*$. The natural map $\mathrm{BP}_* \to M_3^0$ induces a reduction map $H^*\mathrm{BP}_* \to H^*M_3^0$, and he proved that $\gamma_t \in H^3\mathrm{BP}_*$ reduces nontrivially if and only if $t \not\equiv 0, 1 \pmod p$. This shows not only that $\gamma_t \neq 0$ but also that $p \nmid \gamma_t$ for these t.

In our view the next step in this program should be the computation of the second column $H^*M_0^2$ of the chromatic E_1-term, at least for $p > 3$. Since $H^*M_2^0$ is a 12-dimensional vector space over $K(2)_*$ for $p > 3$, this problem appears to be tractable using our Bockstein spectral sequences. We have computed $H^0M_0^2$ here and obtained $H^2\mathrm{BP}_*$ and $\gamma_t \neq 0$ as essentially immediate corollaries. Our partial computation of $H^1M_0^2$ has given us considerable information on products of α's and β's. Complete information on these products and the decomposability of the γ's could come from having all of $H^1M_0^2$. Similarly, the proper partial information about $H^2M_0^2$ could give results about products of the form $\beta_s\beta_t$. The spectral sequence behaves very well with respect to products. Although we have restricted our attention to products that used computations we had already made we have attempted to demonstrate all of the basic techniques needed to handle products in the spectral sequence.

Many of the elements of stable homotopy in the various programs for constructing infinite families show up in $H^0 M_0^n$. These groups are also the most accessible because $H^0 M_0^n \subset M_0^n$ naturally. So perhaps there is a real hope that they can be computed. In particular we have not computed $H^0 M_0^2$ for the prime 2.

Finally, $H^2 \mathrm{BP}_*$ provides an enormous supply of potential homotopy. It would be very interesting to understand the subquotient of stable homotopy represented by this line as well as we understand the image of the J-homomorphism.

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