Homotopy fixed point sets of finite subgroups of the Morava stabilizer group

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The Morava stabilizer group S_n

Fix a prime p and a positive integer n.

S_n is the automorphism group of the Honda formal group law H_n in characteristic p, which has height n.

It is the group of units in a certain division algebra D_n over the p-adic numbers $\mathbf{Q}_p$.

D_n is known to contain every degree n extension of $\mathbf{Q}_p$ as a subfield.

S_n is a pro-p-group that plays a critical role in chromatic homotopy theory.

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The Morava stabilizer group S_n, continued

There is a spectrum E_n related to the classification of lifts of H_n to characteristic zero.

There is an action of S_n on $\pi_*(E_n)$ defined by Lubin-Tate theory, which is hard to describe explicitly.

It gives an action on E_n defined up to homotopy.

The cohomology of this action controls the homotopy of the $K(n)$-local sphere spectrum $L_{K(n)}S^0$.

This was all known in the '70s.

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The Goerss-Hopkins-Miller theorem

In the '90s Goerss-Hopkins-Miller showed

- E_n is an E_∞-ring spectrum.
- The action of S_n is rigid enough to allow the existence of homotopy fixed point sets for arbitrary closed subgroups $G \subset S_n$.
- There is a spectral sequence

$$H^*(G; \pi_*(E_n)) \implies \pi_*(E_n^{hG}).$$

There are homomorphisms

$$\pi_*(S^0) \to \pi_*(L_{K(n)}S^0) \to \pi_*(E_n^{hG}).$$

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Finite subgroups of S_n

Such groups G have been classified by Hewett.

If $(p-1)p^k|n$ but $(p-1)p^{k+1} \nmid n$, then S_n has $k+1$ maximal finite subgroups. If $(p-1) \nmid n$, there is only one, and its order is prime to p.

When $p=2$ and $n \equiv 2 \bmod 4$, one 2-Sylow subgroup is the quaternion group Q_8. *We exclude this case in what follows.*

Otherwise the p-Sylow subgroup is always cyclic.

S_n has an element of order p^{k+1} iff $(p-1)p^k$ divides n.

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The maximal finite subgroup G containing such an element is metacyclic

$$0 \to \mathbf{Z}/p^{k+1} \to G \to \mathbf{Z}/m \to 0$$

where m prime to p, depends on n, and is divisible by $p-1$.

When $k = 0$ and $n = (p-1)f$, then $m = (p-1)(p^f - 1)$.

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Some previously known examples: real K-theory

Let $p = 2$, $n = 1$, and $G = \mathbf{Z}/2$.

In this case $S_1 = \mathbf{Z}_2^\times \cong \{\pm 1\} \times \mathbf{Z}_2$, the 2-adic units.

Then $E_1 = K_2$, the 2-adic completion of complex K-theory.

The group action is complex conjugation.

$E_1^{hG} = KO_2$, the 2-adic completion of real K-theory.

The behavior of the Hopkins-Miller spectral sequence is well known. It collapses from E_4.

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Let *TMF* be the spectrum associated with topological modular forms. It has been studied by Hopkins, Mahowald and Miller.

For $p = 3$, $L_{K(2)} TMF = E_2^{hG}$ where G is the Hewett group of order 12.
The Hopkins-Miller spectral sequence shows the Toda differential.

For $p = 2$, $L_{K(2)} TMF = E_2^{hG}$ where G is the semidirect product of the quaternion group Q_8 with $\mathbf{Z}/3$.
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Let p be odd and $n = p - 1$. There is a Hewett group G of order $(p-1)^2 p$.

E_n^{hG} is denoted by EO_{p-1}. The symbol O is meant to suggest the analogy with real K-theory.

EO_{p-1} has been studied by Hopkins-Miller, Gorbunov-Mahowold and Nave.

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Let $p = 2$ and $n > 0$. There is a Hewett group G of order $2(2^n - 1)$.

The Johnson-Wilson spectrum $E(n)$ has

$$\pi_*(E(n)) = \mathbf{Z}_{(2)}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}].$$

It has an action of $\mathbf{Z}/2$ by complex conjugation. The fixed point set, $ER(n)$ has been studied by Hu-Kriz and Kitchloo-Wilson. Averett has recently shown that after completion, $ER(n) = E_n^{hG}$.

There is a fibration

$$\Sigma^{\lambda(n)} ER(n) \to ER(n) \to E(n)$$

where $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$.

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Let $p = 2$ and $n > 0$. There is a Hewett group G of order $2(2^n - 1)$.
The Johnson-Wilson spectrum $E(n)$ has

$$\pi_*(E(n)) = \mathbf{Z}_{(2)}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}].$$

It has an action of $\mathbf{Z}/2$ by complex conjugation. The fixed point set, $ER(n)$ has been studied by Hu-Kriz and Kitchloo-Wilson. Averett has recently shown that after completion, $ER(n) = E_n^{hG}$.
There is a fibration

$$\Sigma^{\lambda(n)} ER(n) \to ER(n) \to E(n)$$

where $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$.

New results: the action of $\mathbf{Z}/p$

Let $n = (p-1)f$ and $|G| = p(p-1)(p^f - 1)$

$\pi_*(E_n)$ is roughly a polynomial algebra of rank $(p-1)f$.

Theorem 1

Polynomial generators can be chosen so that $\mathbf{Z}/p$ acts on them linearly via f copies of the reduced regular representation.

The quotient group $G/(\mathbf{Z}/p) = \mathbf{Z}/(p-1)(p^f-1)$ acts on $H^*(\mathbf{Z}/p; \pi_*(E_n))$ and gives it an eigenspace decomposition.

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New results: cohomology of G

Theorem 2

Modulo some elements on the 0-line,

$$H^*(G;\pi_*(E_n)) = E(h_{i,0},\dots,h_{f,0}) \otimes P(\Delta^{1/(p-1)}\beta,\Delta^{\pm1})[[x_1,\dots,x_{f-1}]]$$

where

$$h_{i,0} \in H^{1,2p^i-2} \quad \beta \in H^{2,0}$$
$$\Delta \in H^{0,2|G|} \quad x_i \in H^{0,2p(p^f-p^i)}$$

REMARK: The element $\Delta^{1/(p-1)}\beta \in H^{2,2p(p^f-1)}$ is not a product, but is written this way to simplify statements in the next theorem.

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New results: differentials

Theorem 3

The Hopkins-Miller spectral sequence has the following differentials for $1 \le i \le f$, and no others.

- $d_{2p^i-1}(\Delta^{p^{i-1}}) = h_{i,0}\beta^{p^i-1}\Delta^{p^{i-1}}$.
- $d_{1+2(p-1)(p^i-1)}(h_{i,0}\Delta^{(p-1)p^{i-1}})$

$$= \Delta^{1/(p-1)}\beta^{1+(p-1)(p^i-1)}x_i\Delta^{(p-1)p^{i-1}}$$

where $x_f = 1$. Δ^{p^f} is a permanent cycle.

REMARK: The last differential kills a unit multiple of $(\Delta^{1/(p-1)}\beta)^{1+(p-1)(p^f-1)}$ and gives a horizontal vanishing line.

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New results: the Hopkins-Miller spectral sequence, continued

Corollary

There are permanent cycles

$$
\begin{aligned}
a_i &= \Delta^{e_i} h_{i,0} \\
y_i &= \Delta^{e_i'} x_i
\end{aligned}
$$

with p-fold Massey products

$$\langle a_i, \ldots, a_i \rangle = y_i \Delta^{1/(p-1)} \beta.$$