The detection theorem

A solution to the Arf-Kervaire invariant problem

Supplement to lectures given at
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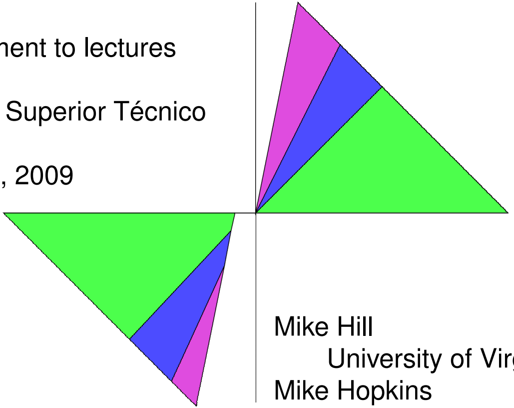

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θ_j in the Adams-Novikov spectral sequence

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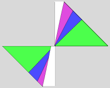
$$h_j^2 \in \mathrm{Ext}_A^{2,2^{j+1}}(\mathbf{Z}/2, \mathbf{Z}/2).$$



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It is more convenient for us to work with the Adams-Novikov spectral sequence, which maps to the Adams spectral sequence. It has a family of elements in filtration 2, namely

$$\beta_{i/j} \in \mathrm{Ext}_{MU_*(MU)}^{2,6i-2j}(MU_*, MU_*)$$

for certain values of of i and j. When $j = 1$, it is customary to omit it from the notation.

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θ_j in the Adams-Novikov spectral sequence (continued)

Here are the first few of these in the relevant bidegrees.

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\begin{array}{ll}
\theta_5: & \beta_{8/8} \text{ and } \beta_{6/2} \\
\theta_6: & \beta_{16/16},\ \beta_{12/4} \text{ and } \beta_{11} \\
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We need to show that any element mapping to h_j^2 in the classical Adams spectral sequence has nontrivial image the Adams-Novikov spectral sequence for M.

θ_j in the Adams-Novikov spectral sequence (continued)

Detection Theorem

Let $x \in \mathrm{Ext}^{2,2^{j+1}}_{MU_*(MU)}(MU_*, MU_*)$ be any element whose image in $\mathrm{Ext}^{2,2^{j+1}}_{A}(\mathbf{Z}/2, \mathbf{Z}/2)$ is h_j^2 with $j \geq 6$.

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We will prove this by showing the same is true after we map the latter to a simpler object involving another algebraic tool, *the theory of formal A-modules*, where A is the ring of integers in a suitable field.

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Formal A-modules

Recall the a formal group law over a ring R is a power series

$$F(x,y) = x + y + \sum_{i,j>0} a_{i,j} x^i y^j \in R[[x,y]]$$

with certain properties.

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$$[m+n](x) = F([m](x), [n](x)) \text{ and } [m]([n](x)) = [mn](x).$$

With these properties we can define $[m](x)$ uniquely for all integers m, and we get a homomorphism τ from $\mathbf{Z}$ to $End(F)$, the endomorphism ring of F.

Formal A-modules (continued)

If the ground ring R is an algebra over the p-local integers $\mathbf{Z}_{(p)}$ or the p-adic integers $\mathbf{Z}_p$, then we can make sense of $[m](x)$ for m in $\mathbf{Z}_{(p)}$ or $\mathbf{Z}_p$.

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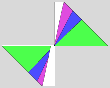
Now suppose R is an algebra over a larger ring A, such as the ring of integers in a number field or a finite extension of the p-adic numbers. We say that the formal group law F is a *formal A-module* if the homomorphism τ extends to A in such a way that

$$[a](x) \equiv ax \bmod (x^2) \text{ for } a \in A.$$



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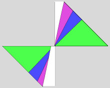
The theory of formal A-modules is well developed. Lubin-Tate used them to do local class field theory.

Formal A-modules (continued)

The example of interest to us is $A = \mathbf{Z}_2[\zeta_8]$, where ζ_8 is a primitive 8th root of unity.



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$$\log_G(G(x,y)) = \log_G(x) + \log_G(y)$$

where

$$\log_G(x) = \sum_{n \geq 0} \frac{w^{2^n - 1} x^{2^n}}{\pi^n}.$$

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The classifying map $\lambda : MU_* \to R_*$ for G factors through BP_*, where the logarithm is

$$\log_F(x) = \sum_{n \geq 0} \ell_n x^{2^n}.$$

Formal A-modules (continued)

Recall that $BP_* = \mathbf{Z}_{(2)}[v_1, v_2, \ldots]$ with $|v_n| = 2(2^n - 1)$.

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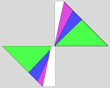
$$\begin{aligned}
\ell_1 &= \frac{v_1}{2} \\
\ell_2 &= \frac{v_2}{2} + \frac{v_1^3}{4} \\
\ell_3 &= \frac{v_3}{2} + \frac{v_1 v_2^2 + v_2 v_1^4}{4} + \frac{v_1^7}{8} \\
\ell_4 &= \frac{v_4}{2} + \frac{v_1 v_3^2 + v_2^5 + v_3 v_1^8}{4} + \frac{v_1^3 v_2^4 + v_1^9 v_2^2 + v_2 v_1^{12}}{8} + \frac{v_1^{15}}{16} \\
&\vdots
\end{aligned}$$

The relation between $MU^{(4)}$ and formal A-modules

What does all this have to do with our spectrum $M = D^{-1}MU^{(4)}$?



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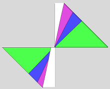

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We will prove this later.

The proof of the Detection Theorem

It follows that we have a map

$$H^*(C_8; \pi_*(D^{-1}MU^{(4)})) = H^*(C_8; \pi_*(M)) \longrightarrow H^*(C_8; R_*).$$

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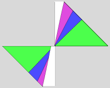
We will prove this by showing that the image of x in $H^{2,2^{j+1}}(C_8; R_*)$ is nonzero.

The proof of the Detection Theorem (continued)

We will calculate with *BP*-theory.



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There is a map from this Hopf algebroid to one associated with $H^*(C_8; R_*)$ in which t_n maps to an R_*-valued function on C_8 (regarded as the group of 8th roots of unity) determined by

$$[\zeta](x) = \sum_{n \geq 0}^{F} \langle t_n, \zeta \rangle x^{2^n}.$$

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An easy calculation shows that the function t_1 sends a primitive root in C_8 to a unit in R_*.

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Let

$$b_{1,j-1} = \frac{1}{2} \sum_{0<i<2^j} \binom{2^j}{i} \left[t_1^i | t_1^{2^j-i} \right] \in \mathrm{Ext}^{2,2^{j+1}}$$

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It is is known to be cohomologous to $\beta_{2^{j-1}/2^{j-1}}$ and to have order 2.

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It is is known to be cohomologous to $\beta_{2^{j-1}/2^{j-1}}$ and to have order 2. We will show that its image in $H^{2,2^{j+1}}(C_8; R_*)$ is nontrivial for $j \geq 2$.

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The proof of the Detection Theorem (continued)

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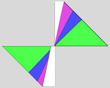
$H^*(C_8; R_*)$ is the cohomology of the cochain complex

$$R_*[C_8] \xrightarrow{\gamma-1} R_*[C_8] \xrightarrow{\text{Trace}} R_*[C_8] \xrightarrow{\gamma-1} \cdots$$

where Trace is multiplication by $1 + \gamma + \cdots + \gamma^7$.

The proof of the Detection Theorem (continued)

The cohomology groups $H^s(C_8; R_*)$ for $s > 0$ are periodic in s with period 2.



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$$H^1(C_8; R_{2m}) \quad = \quad \ker(1 + \zeta_8^m + \cdots + \zeta_8^{7m})/\operatorname{im}(\zeta_8^m - 1)$$

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The proof of the Detection Theorem (continued)

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&= \begin{cases} w^m A/(\pi) & \text{for } m \text{ odd} \\ w^m A/(\pi^2) & \text{for } m \equiv 2 \bmod 4 \\ w^m A/(2) & \text{for } m \equiv 4 \bmod 8 \\ 0 & \text{for } m \equiv 0 \bmod 8 \end{cases}
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\end{aligned}$$

An easy calculation shows that $b_{1,j-1}$ maps to $4w^{2^j}$, which is the element of order 2 in $H^2(C_8; R_{2^{j+1}})$.

The proof of the Detection Theorem (continued)

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The proof of the Detection Theorem (continued)

Hence for $k \geq 1$ and $j \geq 6$ we have

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Hence for $k \geq 1$ and $j \geq 6$ we have

$$||\beta_{c(j,k)/2^{j-1-2k}}|| = \left|\left|\frac{v_2^{c(j,k)}}{2v_1^{2^{j-1-2k}}}\right|\right|$$

The proof of the Detection Theorem (continued)

Hence for $k \geq 1$ and $j \geq 6$ we have

$$\begin{aligned} ||\beta_{c(j,k)/2^{j-1-2k}}|| &= \left|\left| \frac{v_2^{c(j,k)}}{2v_1^{2^{j-1-2k}}} \right|\right| \\ &= \frac{c(j,k)}{2} - \frac{3 \cdot 2^{j-1-2k}}{4} - 1 \end{aligned}$$

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Hence for $k \geq 1$ and $j \geq 6$ we have

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&= \frac{2^j + 2^{j-1-2k}}{6} - \frac{3 \cdot 2^{j-1-2k}}{4} - 1 \\
&= (2^{j-1} - 7 \cdot 2^{j-3-2k})/3 - 1 \\
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The proof of the Detection Theorem (continued)

Hence for $k \geq 1$ and $j \geq 6$ we have

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This means $\beta_{c(j,k))/2^{j-1-2k}}$ maps to an element that is divisible by 8 and therefore zero.

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The proof of the Detection Theorem (continued)

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This completes the proof of the Detection Theorem modulo the Lemma.

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The proof of the Lemma

Here it is again.

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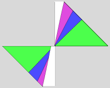
- *the homomorphism $\lambda^{(4)} : \pi_*(MU^{(4)}) \to R_*$ is equivariant, where C_8 acts on $\pi_*(MU^{(4)})$ as before, it acts trivially on A and $\gamma w = \zeta_8 w$ for a generator γ of C_8.*



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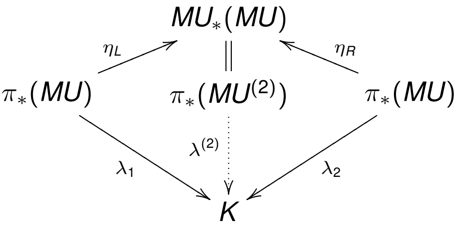

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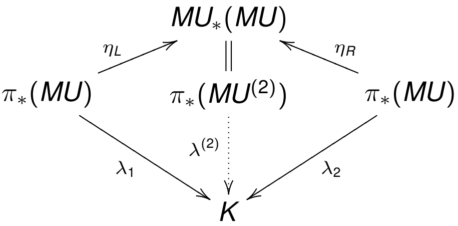

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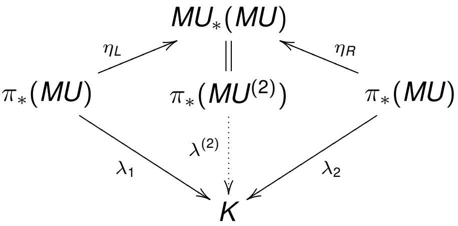

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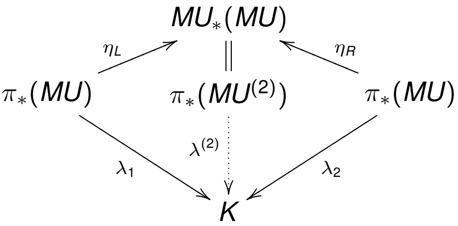

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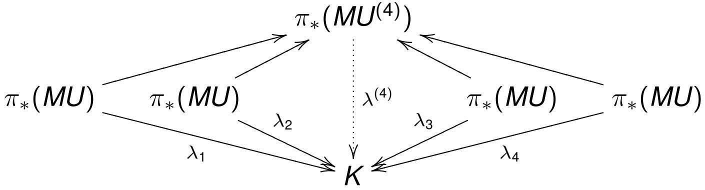
Similarly consider the diagram



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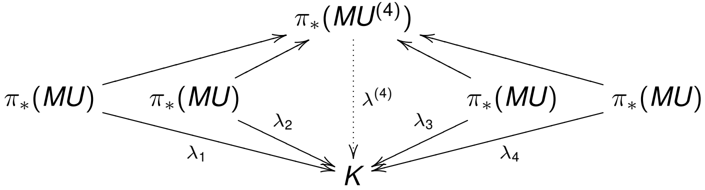
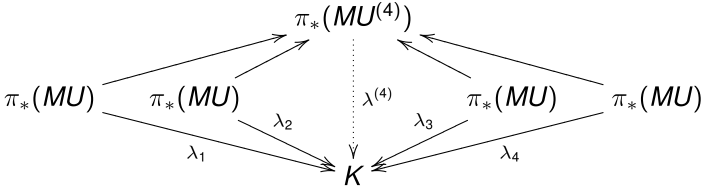


The existence of $\lambda^{(4)}$ is equivalent to that of compatible strict isomorphisms between the formal group laws F_j classified by the λ_j.

The proof of the Lemma (continued)



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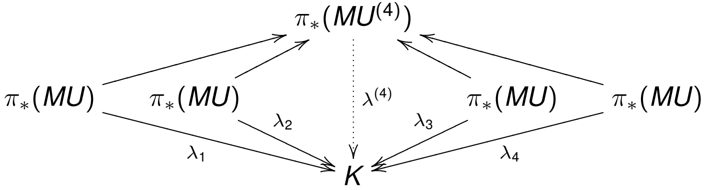

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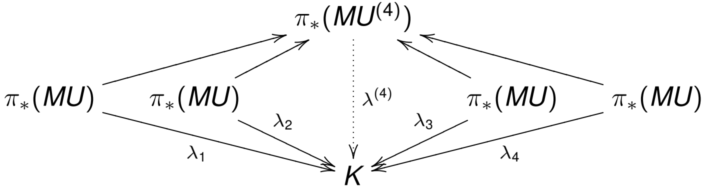

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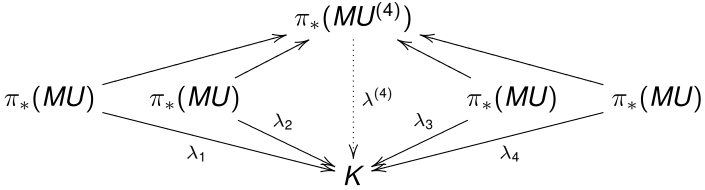

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This means that the existence of an equivariant $\lambda^{(4)}$ is equivalent to that of a formal $\mathbf{Z}[\zeta_8]$-module structure on each of the F_j, which are all isomorphic. This proves the first part of the Lemma.

The proof of the Lemma (continued)

For the second part, recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$, where

$$\overline{\Delta}_k^{(g)} = \begin{cases} x_{2^k-1} & \text{for } g = 2 \\ N_4^g(r_{2^k-1}) & \text{otherwise.} \end{cases}$$

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$$\begin{aligned} v_1 &\mapsto (-\pi^3 - 4\pi^2 - 6\pi - 4)w \\ v_2 &\mapsto (4\pi^3 + 11\pi^2 + 6\pi - 6)w^3 \\ v_3 &\mapsto (40\pi^3 + 166\pi^2 + 237\pi + 100)w^7 \\ v_4 &\mapsto (-15754\pi^3 - 56631\pi^2 - 63495\pi - 9707)w^{15}. \end{aligned}$$

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so v_4 (but not v_n for $n < 4$) and therefore $N_2^8(\overline{\Delta}_4^{(2)})$ maps to a unit.

The proof of the Lemma (continued)

We have $\overline{\Delta}_k^{(2)} = t_k$. We consider the equivariant composite

$$BP_*^{(2)} \longrightarrow BP_*^{(4)} \longrightarrow R_*$$

under which

$$\eta_R(\ell_n) \mapsto \frac{\zeta_8^2 w^{2^n-1}}{\pi^n}.$$

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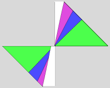

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Using the right unit formula we find that

$$\begin{aligned} t_1 &\mapsto (\pi+2)w \\ t_2 &\mapsto (\pi^3+5\pi^2+9\pi+5)w^3. \end{aligned}$$

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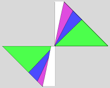

The proof of the Lemma (continued)

We have $\overline{\Delta}_k^{(2)} = t_k$. We consider the equivariant composite

$$BP_*^{(2)} \to BP_*^{(4)} \to R_*$$

under which

$$\eta_R(\ell_n) \mapsto \frac{\zeta_8^2 w^{2^n-1}}{\pi^n}.$$

Using the right unit formula we find that

$$\begin{aligned} t_1 &\mapsto (\pi + 2)w \\ t_2 &\mapsto (\pi^3 + 5\pi^2 + 9\pi + 5)w^3. \end{aligned}$$

This means t_2 (but not t_1) and therefore $N_4^8(\overline{\Delta}_2^{(4)})$ maps to a unit.

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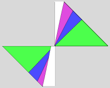
The proof of the Lemma (continued)

Finally, we have $\overline{\Delta}_n^{(8)} = t_n(1) \in BP_*^{(4)}$,



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This implies

$$\overline{\Delta}_1^{(8)} = \ell_1(2) - \ell_1(1) \mapsto w.$$

The proof of the Lemma (continued)

Finally, we have $\overline{\Delta}_n^{(8)} = t_n(1) \in BP_*^{(4)}$, where $t_n(1)$ is the analog of $r_{2^n-1}(1)$. Then we find

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Thus we have shown that each factor of

$$D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$$

and hence D itself maps to a unit in R_*, thus proving the lemma.