Lecture 4

A solution to the Arf-Kervaire invariant problem

Instituto Superior Técnico
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Introduction

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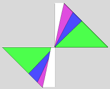

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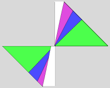

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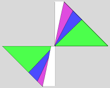

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Before we can do this, we need to introduce another concept from equivariant stable homotopy theory, that of *geometric fixed points.*

Geometric fixed points

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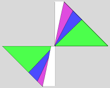

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The *homotopy fixed point set* X^{hG} is the space of based equivariant maps $EG_+ \to X_+$, where EG is a contractible free G-space.

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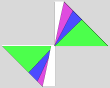

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Geometric fixed points (continued)

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The *geometric fixed set* $\Phi^G X$ is a convenient substitute that avoids these difficulties. In order to define it we need the *isotropy separation sequence*, which in the case of a finite cyclic 2-group G is

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$$E\mathbf{Z}/2_+ \to S^0 \to \tilde{E}\mathbf{Z}/2.$$

Here $E\mathbf{Z}/2$ is a G-space via the projection $G \to \mathbf{Z}/2$ and S^0 has the trivial action, so $\tilde{E}\mathbf{Z}/2$ is also a G-space.

Geometric fixed points (continued)

Under this action $E\mathbf{Z}/2^G$ is empty while for any proper subgroup H of G, $E\mathbf{Z}/2^H = E\mathbf{Z}/2$, which is contractible. For an arbitrary finite group G it is possible to construct a G-space with the similar properties.

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Definition

For a finite cyclic 2-group G and G-spectrum X, the geometric fixed point spectrum is

$$\Phi^G X = (X \wedge \tilde{E}\mathbf{Z}/2)^G.$$

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From the last property we can deduce that for $H \subset G$,

- $\Phi^H S^V = S^{V^H}$.
- $\Phi^H MU^{(g/2)} = MO^{(g/h)}$, where MO is the unoriented cobordism spectrum.

Geometric fixed points (continued)

Geometric Fixed Point Theorem

Let G be a finite cyclic 2-group and let $\overline{\rho}$ denote its reduced regular representation. Then for any G-spectrum X, $\pi_\star(\tilde{E}\mathbf{Z}/2 \wedge X) = \chi_{\overline{\rho}}^{-1}\pi_\star(X)$, where $\chi_{\overline{\rho}} \in \pi_{-\overline{\rho}}$ is the element defined in Lecture 3.

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To prove this will show that $E = \lim_{i\to\infty} S(i\overline{\rho})$ is G-equivalent to $E\mathbf{Z}/2$ by showing it has the appropriate fixed point sets.

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It follows that $\tilde{E}\mathbf{Z}/2$ is equivalent to $\lim_{i\to\infty} S^{i\,\overline{\rho}}$, which implies the result.

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Lemma

The generators r_i and y_i can be chosen so that

$$\Phi^G Nr_i = \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise.} \end{cases}$$

Toward the proof of the slice theorem

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There is a way to kill the $r_i(j)$ for any collection of is and get a new equivariant spectrum which is a module over the E_∞-ring spectrum $MU^{(g/2)}$. We let $R_G(m)$ denote the result of killing the $r_i(j)$ for $i \leq m$.

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The Reduction Theorem

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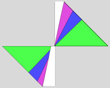

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Reduction Theorem

The map $f_G : R_G(\infty) \to H\mathbf{Z}$ is a weak G-equivalence.

The nonequivariant analog of this statement is obvious. We will prove the corresponding statement over subgroups $H \subset G$ by induction on the order of H.

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$$\Sigma^m \Phi^G R_G(m-1) \xrightarrow{\Phi^G N r_m} \Phi^G R_G(m-1) \longrightarrow \Phi^G R_G(m).$$

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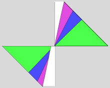

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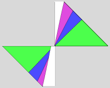
The lemma above determines the map $\Phi^G N r_m$.

The spectrum $\Phi^G R_G(\infty)$ (continued)

We know that $\Phi^G R_G(0) = MO$ and $\Phi^G N r_1$ is trivial, so $\Phi^G R_G(1) = MO \wedge (S^0 \vee S^2)$.



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$$\Sigma^2 MO \xrightarrow{y_2} MO \longrightarrow Q(2).$$

Similarly we find that $\Phi^G R_G(\infty) = \vee_{k \geq 0} \Sigma^{2k} H\mathbf{Z}/2$.

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Recall that $\Phi^H H\mathbf{Z} = (\tilde{E}\mathbf{Z}/2 \wedge H\mathbf{Z})^H$. The action of the subgroup of index 2 is trivial, so this is the same as $(\tilde{E}\mathbf{Z}/2 \wedge H\mathbf{Z})^{\mathbf{Z}/2} = \Phi^{\mathbf{Z}/2} H\mathbf{Z}$.

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The spectrum $\Phi^H H\mathbf{Z}$ (continued)

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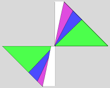

Earlier we described the computation of

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This means we have all of $\pi_\star(H\mathbf{Z})$, the $RO(\mathbf{Z}/2)$-graded homotopy of $H\mathbf{Z}$. It turns out that $\chi_\sigma^{-1}\pi_\star(H\mathbf{Z}) = \mathbf{Z}/2[u_{2\sigma}, \chi_\sigma^{\pm 1}]$,

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Hence $\pi_*(\Phi^G H\mathbf{Z})$ and $\pi_*(\Phi^G R_G(\infty))$ are abstractly isomorphic.

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Some differentials in the slice spectral sequence

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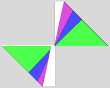

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For each $i > 0$ there is an element

$$f_i \in \pi_i(S^{i\rho_g}) \subset E_2^{(g-1)i,gi},$$

the bottom element in $\pi_*(S^{i\rho_g})$.

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Some slice differentials (continued)

It is the composite $S^{i} \xrightarrow{\chi_{i\rho_g}} S^{i\rho_g} \xrightarrow{Nr_i} MU^{(g/2)}$.

Some slice differentials (continued)

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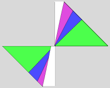

It is the composite $S^i \xrightarrow{\chi_{i\rho_g}} S^{i\rho_g} \xrightarrow{Nr_i} MU^{(g/2)}$.
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$$\pi_*(MU^{(g/2)}) \to \pi_*(\Phi^G MU^{(g/2)}) = \pi_*(MO)$$

we have

$$f_i \mapsto \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise} \end{cases}$$

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It follows that any differentials hitting the vanishing line must land in the ideal $(f_1, f_3, f_7, \dots)$. A similar statement can be made after smashing with $S^{2^k\sigma}$.

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Some slice differentials (continued)

Slice Differentials Theorem

In the slice spectral sequence for $\Sigma^{2^k\sigma} MU^{(g/2)}$ *(for* $k > 0$*) we have* $d_r(u_{2^k\sigma}) = 0$ *for* $r < 1 + (2^k - 1)g$*, and*

$$d_{1+(2^k-1)g}(u_{2^k\sigma}) = \chi_\sigma^{2^k} f_{2^k-1}.$$

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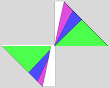

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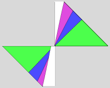
Inverting χ_σ in the slice spectral sequence will make it converge to $\pi_*(MO)$.



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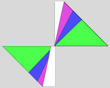
Inverting χ_σ in the slice spectral sequence will make it converge to $\pi_*(MO)$. This means each f_{2^k-1} must be killed by some power of χ_σ. The only way this can happen is as indicated in the theorem.



Some slice differentials (continued)

Let

$$\overline{\Delta}_k^{(g)} = Nr_{2^k-1} \in \pi_{(2^k-1)\rho_g}(MU^{(g/2)}).$$

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Some slice differentials (continued)

Let

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We want to invert this element and study the resulting slice spectral sequence.

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Let

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We want to invert this element and study the resulting slice spectral sequence. As explained previously, it is confined to the first and third quadrants with vanishing lines of slopes 0 and $g-1$.

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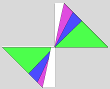

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The differential d_r on $u_{2^{k+1}\sigma}$ described in the theorem is the last one possible since its target, $\chi_\sigma^{2^{k+1}} f_{2^{k+1}-1}$, lies on the vanishing line.

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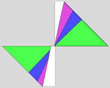

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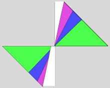
The differential d_r on $u_{2^{k+1}\sigma}$ described in the theorem is the last one possible since its target, $\chi_\sigma^{2^{k+1}} f_{2^{k+1}-1}$, lies on the vanishing line. If we can show that this target is killed by an earlier differential after inverting $\overline{\Delta}_k^{(g)}$, then $u_{2^{k+1}\sigma}$ will be a permanent cycle.



Some slice differentials (continued)

We have

$$f_{2^{k+1}-1}\overline{\Delta}_k^{(g)} = \chi_{(2^{k+1}-1)\rho_g} Nr_{2^{k+1}-1} Nr_{2^k-1}$$

Some slice differentials (continued)

We have

$$\begin{aligned} f_{2^{k+1}-1}\overline{\Delta}_k^{(g)} &= \chi_{(2^{k+1}-1)\rho_g} N r_{2^{k+1}-1} N r_{2^k-1} \\ &= \chi_{2^k\rho_g}\overline{\Delta}_{k+1}^{(g)} f_{2^k-1} \end{aligned}$$

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Some slice differentials (continued)

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Corollary

In the $RO(G)$-graded slice spectral sequence for $\left(\overline{\Delta}_k^{(g)}\right)^{-1} MU^{(g/2)}$, the class $u_{2\sigma}^{2^k}$ is a permanent cycle.

The Periodicity Theorem

The corollary shows that inverting a certain element makes a power of $u_{2\sigma}$ a permanent cycle.

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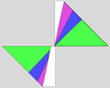

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We will get this by using the norm property of u, namely that if W is an oriented representation of a subgroup $H \subset G$ with $W^H = 0$ and induced representation W', then the norm functor N_h^g from H-spectra to G-spectra satisfies $N_H^G(u_W)u_{2\rho_{G/H}}^{|W|/2} = u_{W'}$.

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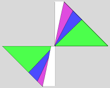
From this we can deduce that $u_{2\rho_g} = \prod_{m=1}^{n} N_{2^m}^{2^n}(u_{2^m\sigma_m})$,



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From this we can deduce that $u_{2\rho_g} = \prod_{m=1}^{n} N_{2^m}^{2^n}(u_{2^m\sigma_m})$, where σ_m denotes the sign representation on C_{2^m}.

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The Periodicity Theorem (continued)

In particular we have $u_{2\rho_8} = u_{8\sigma_3} N_4^8(u_{4\sigma_2}) N_2^8(u_{2\sigma_1})$.

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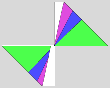
The Periodicity Theorem (continued)

In particular we have $u_{2\rho_8} = u_{8\sigma_3} N_4^8(u_{4\sigma_2}) N_2^8(u_{2\sigma_1})$.

By the Corollary we can make a power of each factor a permanent cycle by inverting some $\overline{\Delta}_{k_m}^{(2^m)}$ for $1 \leq m \leq 3$.



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- Inverting $\overline{\Delta}_2^{(4)}$ makes $u_{8\sigma_2}$ a permanent cycle.

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The Periodicity Theorem (continued)

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- Inverting the product D of the norms of all three makes $u_{32\rho_8}$ a permanent cycle.

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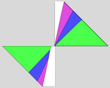

The Periodicity Theorem (continued)

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The Periodicity Theorem (continued)

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The Periodicity Theorem (continued)

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Since the inverted element is represented by a map from $S^{m\rho_8}$, the slice spectral sequence for $\pi_*(M)$ has the usual properties:

- It is concentrated in the first and third quadrants and confined by vanishing lines of slopes 0 and 7.
- It has the gap property, i.e., no homotopy between dimensions -4 and 0.

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The Periodicity Theorem (continued)

Preperiodicity Theorem

Let $\Delta_1^{(8)} = u_{2\rho_8}(\overline{\Delta}_1^{(8)})^2 \in E_2^{16,0}(D^{-1}MU^{(4)})$. *Then* $(\Delta_1^{(8)})^{16}$ *is a permanent cycle.*

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The Periodicity Theorem (continued)

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To prove this, note that $(\Delta_1^{(8)})^{16} = u_{32\rho_8\left(\overline{\Delta}_1^{(8)}\right)^{32}}$.

The Periodicity Theorem (continued)

Preperiodicity Theorem

Let $\Delta_1^{(8)} = u_{2\rho_8}(\overline{\Delta}_1^{(8)})^2 \in E_2^{16,0}(D^{-1}MU^{(4)})$*. Then* $(\Delta_1^{(8)})^{16}$ *is a permanent cycle.*

To prove this, note that $(\Delta_1^{(8)})^{16} = u_{32\rho_8\left(\overline{\Delta}_1^{(8)}\right)^{32}}$. Both $u_{32\rho_8}$ and $\overline{\Delta}_1^{(8)}$ are permanent cycles, so $(\Delta_1^{(8)})^{16}$ is also one.

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Thus we have an equivariant map $\Sigma^{256}D^{-1}MU^{(4)} \to D^{-1}MU^{(4)}$ and a similar map on the fixed point set. The latter one is invertible because $u_{2\rho_8}^{32}$ restricts to the identity.

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Periodicity Theorem

Let $M = (D^{-1}MU^{(4)})^{C_8}$. *Then* $\Sigma^{256}M$ *is equivalent to* M.

Homotopy and actual fixed points

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The slice spectral sequence computes the homotopy of the former while the Hopkins-Miller spectral sequence (which is known to detect θ_j) computes that of the latter.

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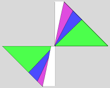

Homotopy and actual fixed points (continued)

Here is a general approach to showing that actual and homotopy fixed points are equivalent for a G-spectrum X.

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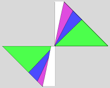

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The case of interest is $X = \tilde{M}$ and $G = C_8$. We will argue by induction on the order of the subgroups H of G, the statement being obvious for the trivial group. We will smash φ with the isotropy separation sequence

$$EG_+ \to S^0 \to \tilde{E}G.$$

Homotopy and actual fixed points (continued)

This gives us the following diagram in which both rows are cofiber sequences.

$$\begin{array}{ccccc} EG_+ \wedge \tilde{M} & \longrightarrow & \tilde{M} & \longrightarrow & \tilde{E}G \wedge \tilde{M} \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi''} \\ EG_+ \wedge F(EG_+, \tilde{M}) & \longrightarrow & F(EG_+, \tilde{M}) & \longrightarrow & \tilde{E}G \wedge F(EG_+, \tilde{M}) \end{array}$$

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Thus it suffices to show that φ''_H is an equivalence, which we will do by showing that both its source and target are contractible. Both have the form $\tilde{E}G \wedge X$ where X is a module spectrum over $\tilde{M}$, so it suffices to show that $\tilde{E}G \wedge \tilde{M}$ is contractible.

Homotopy and actual fixed points (continued)

We need to show that $\tilde{E}G \wedge \tilde{M}$ is G-equivariantly contractible.

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Homotopy and actual fixed points (continued)

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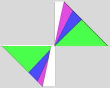

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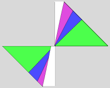
We will smash our spectrum with the cofiber sequence

$$EH_{2+} \to S^0 \to \tilde{E}H_2.$$



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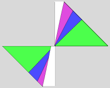

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Thus it remains to show that $EH_{2+} \wedge \tilde{E}G \wedge \tilde{M}$ is H-contractible. But this is equivalent to the H'-contractibility of $\tilde{E}G \wedge \tilde{M}$, which we have by induction.