Lecture 1

A solution to the Arf-Kervaire invariant problem

Instituto Superior Técnico
Lisbon
May 5, 2009

Mike Hill
University of Virginia
Mike Hopkins
Harvard University 1.1
Doug Ravenel
University of Rochester

1 Background and history

A poem by Frank Adams written in the 70s

The school at Northwestern
is as fertile as manure,
full of deep insights,
some rather obscure.

Mark loves those damn thetas
like a sister or brother,
and if you don't like one proof, 1.2
he'll give you another.

Our main result

Our main theorem can be stated in two different but equivalent ways:

- It says that a certain algebraically defined invariant $\Phi(M)$ (the Arf-Kervaire invariant, to be defined later) on certain manifolds M is always zero.
- It says that a related invariant (having to do with secondary cohomology operations) defined on maps between high dimensional spheres is always zero.

The question answered by our theorem is nearly 50 years old. It is known as the Arf-Kervaire invariant problem. There were several unsuccessful attempts to solve it in the 1970s. They were all aimed at proving the opposite of what we have proved. 1.3

Our main result (continued)

Some homotopy theorists, most notably Mark Mahowald, conjectured the opposite of what we have proved, and had derived numerous consequences about homotopy groups of spheres. We now know that the world of homotopy theory is different from what they had envisioned. Barratt and Mahowald called the possible nonexistence of the θ_j for large j the *Doomsday Hypothesis*.

After 1980, the problem faded into the background because it was thought to be too hard. Our proof is two giant steps away from anything that was attempted in the 70s.

Main Theorem. *The Arf-Kervaire elements* $\theta_j \in \pi_{2^{j+1}-2}(S^0)$ *do not exist for* $j \geq 7$.

The θ_j in the theorem is the name given to a hypothetical manifold or map between spheres for which the Arf-Kervaire invariant is nontrivial. It has long been known that such things can exist only in dimensions that are 2 less than a power of 2. 1.4

Our main result (continued)

θ_1, θ_2 and θ_3 are the squares of the Hopf maps $\eta \in \pi_1$, $\nu \in \pi_3$ and $\sigma \in \pi_7$. These maps were constructed by Hopf in 1930. Their advent marked the beginning of modern homotopy theory.

Here is Hopf's definition of the map $\eta : S^3 \to S^2$.

- Think of S^3 as the unit sphere in a 2-dimensional complex vector space $\mathbf{C}^2$.
- Think of S^2 as the one-point compactification of the complex numbers, $\mathbf{C} \cup \{\infty\}$.
- For $(z_1, z_2) \in S^3$, define

$$\eta(z_1, z_2) = \begin{cases} z_1/z_2 & \text{for } z_2 \neq 0 \\ \infty & \text{for } z_2 = 0 \end{cases}$$

- Maps $\nu : S^7 \to S^4$ and $\sigma : S^{15} \to S^8$ can be defined in a similar way using quaternions and Cayley numbers or octonions. 1.5

Our main result (continued)

Main Theorem. *The Arf-Kervaire elements* $\theta_j \in \pi_{2^{j+1}-2}(S^0)$ *do not exist for* $j \geq 7$.

- $\theta_4 \in \pi_{30}$ and $\theta_5 \in \pi_{62}$ were constructed in the '60s and '70s. The latter is the subject of a paper by Barratt-Jones-Mahowald published in 1984.
- The status of $\theta_6 \in \pi_{126}$ is still open. 1.6

1.1 Exotic spheres

The work of Kervaire and Milnor

Fifty years ago the topological community was startled by two results.

Milnor's Theorem (1956). Existence of exotic spheres. *There are manifolds homeomorphic to* S^7 *but not diffeomorphic to it.*

Kervaire's Theorem (1960). Existence of nonsmoothable manifolds. *There is a 10-dimensional topological manifold with no differentiable structure.*

These theorems are opposite sides of the same coin. 1.7

The classification of exotic spheres

Let Θ_k denote the group (under connected sum) of diffeomorphism classes of smooth manifolds Σ^k homotopy equivalent to S^k.

By the Poincaré Conjecture in dimensions ≥ 5 (proved by Smale in 1962), homotopy equivalence here implies homeomorphism. (The Poincaré Conjecture in dimensions 3 and 4 was not solved until much later. Kervaire-Milnor did not treat those cases.)

Such a Σ^k, when embedded in Euclidean space, has a (nonunique) framing on its normal bundle and thus represents an element in the framed cobordism group Ω_k^{framed}. By the Pontrjagin-Thom construction, Ω_k^{framed} is isomorphic to the stable k-stem $\pi_k(S^0)$. 1.8

1.1.1 The Pontrjagin-Thom construction

The Pontrjagin-Thom construction

A k-dimensional framed manifold M (such as Σ^k) can be embedded in a Euclidean space $\mathbf{R}^{n+k}$ in such a way that it has a tubular neighborhood V homeomorphic to $M \times D^n$. The framing gives a projection

$$(V, \partial V) \xrightarrow{f_M} (D^n, \partial D^n) \longrightarrow D^n/\partial D^n \cong S^n$$

We can extend this map to $\mathbf{R}^{n+k}$ and its one-point compactification S^{n+k} by sending everything outside of V to the base point (or point at ∞) in S^n.

The resulting map $\tilde{f}_M : S^{n+k} \to S^n$ represents an element in the homotopy group $\pi_{n+k}(S^n)$, which for large n is isomorphic to the stable k-stem π_k. Pontrjagin showed that a cobordism between M_1 and M_2 leads to a homotopy between $\tilde{f}_{M_1}$ and $\tilde{f}_{M_2}$. 1.9

The Pontrjagin-Thom construction (continued)

He also showed the converse:

- Any map $S^{n+k} \to S^n$ is homotopic to one associated to a framed k-manifold in this way.
- Any homotopy between two such maps is induced by a cobordism.

This means that π_k is isomorphic to the cobordism group of framed k-manifolds.

Thom used transversality to prove similar theorems in which the framing is replaced by a weaker structure on the normal bundle of a manifold M. This is the basis of modern cobordism theory. *He won the Fields Medal for this work in 1958.* 1.10

1.1.2 The J-homomorphism

The classification of exotic spheres (continued)

Two framings on a framed k-manifold $M^k \subset \mathbf{R}^{n+k}$ differ by a map $M^k \to SO(n)$, the special orthogonal group of $n \times n$ real matrices. When M^k is a sphere (or a homotopy sphere) we get an element in $\pi_k(SO(n))$. This leads to the definition of the Hopf-Whitehead J-homomorphism

$$J : \pi_k(SO(n)) \to \pi_{n+k}(S^n).$$

Both of these groups are independent of n when n is large, so we get

$$J : \pi_k(SO) \to \pi_k(S^0).$$

The group of the left and its image on the right are known for all k. 1.11

The classification of exotic spheres (continued)

Hence we have a homomorphism

$$\tau_k : \Theta_k \to \operatorname{coker}_k J = \pi_k(S^0)/\operatorname{im} J$$

It sends an exotic sphere Σ^k to its framed cobordism class, modulo the indeterminacy related to the nonuniqueness of the framing. Kervaire-Milnor denote this homomorphism by p' in their Lemma 4.5.

An element in the kernel of τ_k is represented by an exotic sphere Σ^k bounding a framed manifold M^{k+1}. Milnor's original Σ^7 was such an example, bounding a D^4-bundle over S^4, which can be framed.

An element in the cokernel of τ_k is a framed k-manifold which is not framed cobordant to a sphere. 1.12

The classification of exotic spheres (continued)

We are studying the homomorphism

$$\tau_k : \Theta_k \longrightarrow \mathrm{coker}_k J = \pi_k(S^0)/\mathrm{im}\, J$$

There are framings on $S^1 \times S^1$, $S^3 \times S^3$ and $S^7 \times S^7$ which are not framed cobordant to spheres.

Let $\eta : S^3 \to S^2$ be the Hopf map and consider the composite

$$S^4 \xrightarrow{\Sigma\eta} S^3 \xrightarrow{\eta} S^2.$$

The preimage of a typical point in S^2 is an exotically framed torus $S^1 \times S^1$ in S^4. 1.13

1.1.3 The use of surgery

The use of surgery

Let M^n be a framed manifold, either closed or bounded by a sphere Σ^{n-1}.

By using surgery (which was originally invented for this purpose!) one can convert M to another framed manifold in the same cobordism class which is roughly $n/2$-connected, without disturbing the boundary.

When n is odd, we can surger M^n into a sphere Σ^n or a disk D^n, whose boundary must be an ordinary sphere S^{n-1}.

This implies $\tau_k : \Theta_k \to \mathrm{coker}_k J$ is onto when k is odd and one-to-one when k is even. 1.14

Obstructions in the middle dimension

When $n = 2m$, we can surger our framed manifold M^{2m} into an $(m-1)$-connected manifold, but we may not be able to get rid of $H^m(M)$.

When $m = 2\ell$ is even, the cup product gives us a pairing

$$H^{2\ell}(M;\mathbf{Z}) \otimes H^{2\ell}(M;\mathbf{Z}) \to H^{4\ell}(M, \partial M;\mathbf{Z})$$

represented by a symmetric unimodular matrix B with even diagonal entries.

Such matrices have been classified over the real numbers up to the appropriate equivalence relation. The key invariant is the signature $\sigma(B)$, the difference between the number of positive and negative eigenvalues over $\mathbf{R}$, which is always divisible by 8. 1.15

An interesting matrix

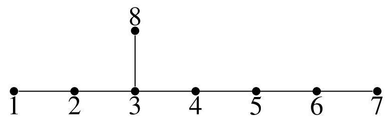
Here is a symmetric matrix with even diagonal entries and signature 8.

$$B = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

1.16

The Dynkin diagram for E_8

The matrix on the previous page is related to the following Dynkin diagram.



The nodes on the graph correspond to the rows/columns of the matrix.

Nodes i and j are connected by an edge iff $b_{i,j} \neq 0$. 1.17

1.1.4 The Hirzebruch signature theorem

Implications of the Hirzebruch signature theorem

The Hirzebruch signature theorem relates the signature of a smooth oriented closed 4ℓ-manifold M to its Pontrjagin numbers.

If M is framed (as in our case), its Pontrjagin numbers and therefore its signature vanish. This means when M is closed we can surger it into a sphere, so $\tau_{4\ell}$ is onto.

If our $M^{4\ell}$ is bounded by a sphere diffeomorphic to $S^{4\ell-1}$, then we can close M by attached a 4ℓ-ball. We get a new manifold $N^{4\ell}$ that is framed at every point except the center of that ball.

Such a manifold is said to be *almost framed*. 1.18

The signature of an almost framed manifold

Hirzebruch's formula implies that our signature $\sigma(N^{4\ell})$ is divisible by a certain integer related to the numerator of a Bernoulli number. For $\ell = 2$, this integer is $224 = 8 \cdot 28$.

On the other hand, there is a way to construct a framed 4ℓ-manifold bounded by a sphere $\Sigma^{4\ell-1}$ such that $\sigma(B)$ is any multiple of 8. This gives us 28 distinct differentiable structures on S^7.

The kernel of $\tau_{4\ell-1}$ is a large cyclic group whose order was determined by Kervaire-Milnor. 1.19

1.2 The Arf invariant

The one remaining case

To recap, we have a homomorphism

$$\tau_k : \Theta_k \to \mathrm{coker}_k J$$

It is onto when k is odd or divisible by 4.
It is one-to-one when k is even, and has a known kernel when $k = 4\ell - 1$.

We have not yet discussed the kernel for $k = 4\ell + 1$ or the cokernel for $k = 4\ell + 2$.

It turns out that the two groups are related. For each ℓ, one is trivial iff the other is $\mathbf{Z}/2$. 1.20

Framed $4\ell+2$-manifolds

We have a framed $4\ell+2$-manifold, possibly bounded by a sphere $\Sigma^{4\ell+1}$. We can surger it into a 2ℓ-connected manifold.

We have a pairing

$$H^{2\ell+1}(M;\mathbf{Z}/2)\otimes H^{2\ell+1}(M;\mathbf{Z}/2)\to H^{4\ell+2}(M,\partial M;\mathbf{Z}/2)$$

Evaluation on the fundamental class gives us a quadratic form

$$\lambda : H^{2\ell+1}\otimes H^{2\ell+1}\to \mathbf{Z}/2.$$

There is a map (not a homomorphism) $\mu : H^{2\ell+1}\to \mathbf{Z}/2$ such that

$$\lambda(x,y)=\mu(x)+\mu(y)+\mu(x+y).$$

1.21

The Arf invariant

This map $\mu : H^{2\ell+1}(M;\mathbf{Z}/2)\to \mathbf{Z}/2$ is either 1 most of the time or 0 most of the time.

This value is its *Arf invariant* $\Phi(M)$, which is the obstruction to doing surgery in the middle dimension.

The *Arf-Kervaire invariant* $\Phi(M)$ of a framed $(4\ell+2)$-manifold is defined to be the Arf invariant of its quadratic form. 1.22

The Arf-Kervaire invariant question

Is there a closed framed $(4\ell+2)$-manifold with nontrivial Arf-Kervaire invariant?

If there is, then $\tau_{4\ell+2}$ has a cokernel of order 2 and $\tau_{4\ell+1}$ is one-to-one.

If there is not, then $\tau_{4\ell+1}$ has a kernel of order 2 and $\tau_{4\ell+2}$ is onto.

Kervaire answered the question in the negative for $\ell=2$. He constructed a framed 10-manifold bounded by an exotic 9-sphere. By coning off its boundary, he got his nonsmoothable closed topological 10-manifold. 1.23

1.3 Browder's theorem

Enter stable homotopy theory

Algebraic topologists attacked this question vigorously in the 1960s. The best result was the following.

Browder's Theorem (1969). Relation to the Adams spectral sequence. *A framed $(4\ell+2)$-manifold with nontrivial Arf-Kervaire invariant can exist only when $\ell=2^{j-1}-1$ for some integer j. In that case it exists iff the Adams spectral sequence element*

$$h_j^2\in E_2^{2,2^{j+1}}=\mathrm{Ext}_A^{2,2^{j+1}}(\mathbf{Z}/2,\mathbf{Z}/2)$$

is a permanent cycle.

1.24

2 Spectral sequences in stable homotopy theory

2.1 The Adams spectral sequence

The classical Adams spectral sequence

Here A denotes the mod 2 Steenrod algebra and

$$h_j \in E_2^{2,2^j} = \mathrm{Ext}_A^{1,2^j}(\mathbf{Z}/2, \mathbf{Z}/2)$$

is the element corresponding to Sq^{2^j}. It is defined for all integers $j \geq 0$.

θ_j denotes any element in $\pi_{2^{j+1}-2}$ that is detected by h_j^2.

Adams showed that h_j is a permanent cycle only for $0 \leq j \leq 3$.
These h_j represent 2ι (twice the fundamental class) and the Hopf maps $\eta \in \pi_1$, $\nu \in \pi_3$ and $\sigma \in \pi_7$.

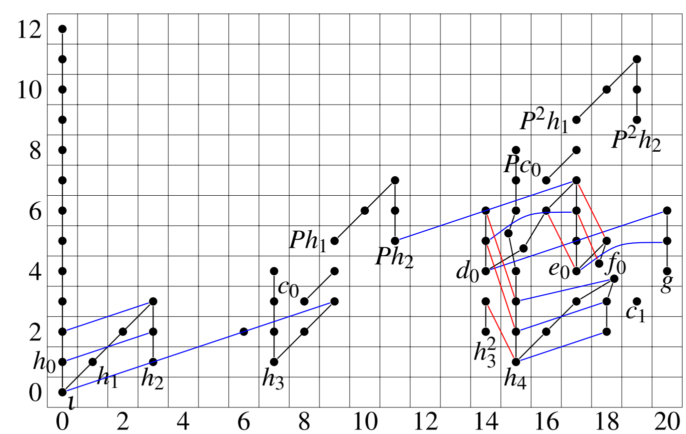
For $j \geq 4$ there is a nontrivial differential

$$d_2(h_j) = h_0 h_{j-1}^2.$$

1.25

The classical Adams spectral sequence (continued)

Here is a picture of the Adams spectral sequence for the prime 2 in low dimensions.



1.26

Spectral sequences in stable homotopy theory

Spectral sequences have been used to study the stable homotopy groups of spheres for the past 50 years.

The classical Adams spectral sequence of the previous slide is based on ordinary mod 2 cohomology and the Steenrod algebra and was introduced by Adams in 1959.

It is possible to use other cohomology or homology theories for the same purpose.

Complex cobordism theory has proven to be extremely useful. The corresponding spectral sequence was first studied by Novikov in 1967 and is known as the *Adams-Novikov spectral sequence*.

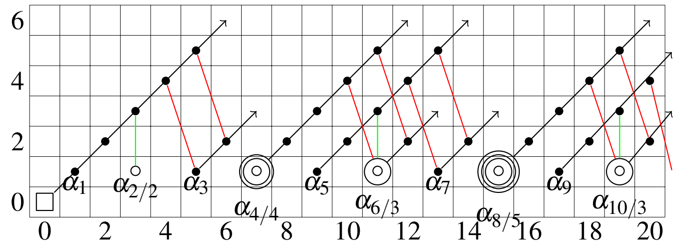
1.27

2.2 The Adams-Novikov spectral sequence

The Adams-Novikov spectral sequence for $p = 2$ in low dimensions

It is helpful to separate it into two parts having to do with v_1-periodic and v_1-torsion elements. These are related to the image and cokernel of J.

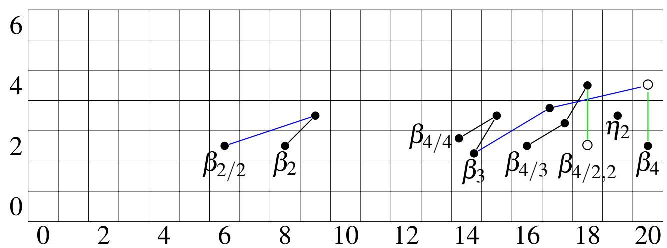
Here is the v_1-periodic part.



The box indicates a copy of $\mathbf{Z}_{(2)}$. Circles indicate group orders. Black lines indicate α_1-multiplication. Red lines indicate differentials. Green lines indicate group extensions. 1.28

The Adams-Novikov spectral sequence for $p = 2$ in low dimensions

Here is the v_1-torsion part.



$\beta_{2/2} = h_2^2 = \theta_2$ and $\beta_{4/4} = h_3^2 = \theta_3$.

Color coding is as before. Blue lines indicate multiplication by $\nu = h_2 = \alpha_{2,2/2}$. Green lines indicate group extensions.

The first differential in this spectral sequence occurs in dimension 26. 1.29

The Arf-Kervaire invariant in the Adams-Novikov spectral sequence

These spectral sequences are very complicated and have been studied very closely. Fortunately our proof does not involve these complexities.

The Arf-Kervaire invariant question translates to the following:

> In the Adams-Novikov spectral sequence, is the element $\theta_j = \beta_{2^{j-1}/2^{j-1}} \in E_2^{2,2^{j+1}}$ a permanent cycle?

It cannot be the target of a differential because its filtration is too low. We will show that it is the source of a nontrivial differential for $j \geq 7$. 1.30

3 Our strategy

Our strategy

We will produce a map $S^0 \to M$, where M is a nonconnective spectrum with the following properties.

(i) It has an Adams-Novikov spectral sequence in which the image of each θ_j is nontrivial.
(ii) It is 256-periodic, meaning $\Sigma^{256} M \cong M$.
(iii) $\pi_{-2}(M) = 0$.

(ii) and (iii) imply that $\pi_{254}(M) = 0$.

If θ_7 exists, (i) implies it has a nontrivial image in this group, so it cannot exist.

The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$. 1.31