THE METHOD OF INFINITE DESCENT IN STABLE HOMOTOPY THEORY II

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Abstract. This paper is a continuation of [Rav02] and the second in a series of papers intended to clarify and expand results of [Rav04, Chapter 7], and it will give the foundation for a program to compute the p-components of $\pi_*(S^0)$ for a prime greater than 2 through roughly dimension $p^3|v_2|$. We will refer to results from [Rav02] freely as if they were in the first four sections of this paper, which begins with section 5.

5. Introduction

In [Rav86] the second author described a method for computing the Adams-Novikov E_2-term for spheres and used it to determine the stable homotopy groups through dimension 108 for $p = 3$ and 999 for $p = 5$. The latter computation was a substantial improvement over prior knowledge, and neither has been improved upon since. It is generally agreed among homotopy theorists that it is not worthwhile to try to improve our knowledge of stable homotopy groups by a few stems, but that the prospect of increasing the known range by a factor of p would be worth pursuing. This possibility may be within reach now, due to a better understanding of the methods of [Rav04, Chapter 7] and improved computer technology. This paper should be regarded as laying the foundation for a program to compute $\pi_*(S^0)_{(p)}$ through roughly dimension $p^3|v_2|$, i.e., 432 for $p = 3$ and 6,000 for $p = 5$.

5.1. **Summary of I.** First we review [Rav02] briefly. The method referred to in the title involves the connective p-local ring spectra $T(m)$ (cf. [Rav04, §6.5]) satisfying

$$BP_*(T(m)) = BP_*[t_1, \dots, t_m] \subset BP_*(BP)$$

and the natural map $T(m) \to BP$ which is an equivalence below dimension $|t_{m+1}|$. In particular, we have $T(0) = S^0_{(p)}$ and $T(\infty) = BP$.

We also defined the quotient module

$$\Gamma(k) = BP_*(BP)/(t_1, \dots, t_{k-1}) \cong BP_*[t_k, t_{k+1}, \dots].$$

In particular, $\Gamma(1) = BP_*(BP)$. Then the pair $(BP_*, \Gamma(k))$ forms a Hopf algebroid, whose structure maps are inherited from $(BP_*, BP_*(BP))$.

For a Hopf algebroid (A, Γ) and Γ-comodule M, we will often drop the first variable of Ext for short, i.e., $\mathrm{Ext}_\Gamma(A, M)$ will be denoted by $\mathrm{Ext}_\Gamma(M)$. By the change-of-rings isomorphism [Rav04, A1.3.12], the Adams-Novikov E_2-term for $T(m)$ is reduced to $\mathrm{Ext}^*_{\Gamma(m+1)}(BP_*)$.

It is not difficult to find the structure of $\mathrm{Ext}^*_{\Gamma(m+1)}(BP_*)$ in low dimensions. For example, it is known below dimension $|v_m^p|$ [Rav86, 7.1.9] and below dimension $|v_{m+1}^p|$ [Rav86, 7.1.13].

Moreover, the structure of $\mathrm{Ext}^*_{\Gamma(m+1)}(BP_*)$ below dimension $|v_{m+1}^{p^2}|$ was algebraically determined in Theorem 4.5. To explain it, recall some machinery developed in [Rav02]. We have constructed the short exact sequence of $\Gamma(m+1)$-comodules

$$0 \longrightarrow BP_* \xrightarrow{i_1} D^0_{m+1} \xrightarrow{j_1} E^1_{m+1} \longrightarrow 0 \qquad \text{for } m \geq 0 \tag{5.1}$$

where the map i_1 induces an isomorphism of Ext^0 (Theorem 3.7), and D_{m+1} is a weak injective $\Gamma(m+1)$-comodule. So, we have isomorphisms

$$\mathrm{Ext}^t_{\Gamma(m+1)}(E^1_{m+1}) \cong \mathrm{Ext}^{t+1}_{\Gamma(m+1)}(BP_*) \quad \text{for } t \geq 0.$$

We may also assume the existence of the short exact sequence

$$0 \longrightarrow E^1_{m+1} \xrightarrow{i_2} D^1_{m+1} \xrightarrow{j_2} E^2_{m+1} \longrightarrow 0. \tag{5.2}$$

where D^1_{m+1} is weak injective. In fact, for $m = 0$ and odd p, an inclusion $E^1_1 \to D^1_1$ to a weak injective inducing an isomorphism in Ext^0 are constructed in Lemma 4.1. For $m > 0$, it was shown in Lemma 3.18 that $v_1^{-1}E^1_{m+1}$ is weak injective with

$$\mathrm{Ext}^0_{\Gamma(m+1)}(v_1^{-1}E^1_{m+1}) \cong v_1^{-1}\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*).$$

Thus, we may regard D^1_{m+1} as $v_1^{-1}E^1_{m+1}$ at worst. Of course, it is desirable to define D^1_{m+1} to make its Ext^0 as small as possible. If we assume that the map i_2 induces an isomorphism of Ext^0, then we have isomorphisms

$$\mathrm{Ext}^t_{\Gamma(m+1)}(E^2_{m+1}) \cong \mathrm{Ext}^{t+2}_{\Gamma(m+1)}(BP_*) \quad \text{for } t \geq 0.$$

In [Rav02] the second author shows that we have such isomorphisms[1] below dimension $p^2|v_{m+1}|$ by producing the subcomodule E^2_{m+1} of $E^1_{m+1}/(v_1^\infty)$ satisfying some desirable conditions and the comodule D^1_{m+1} as the induced extension:

$$\begin{tikzcd}
0 \arrow[r] & E^1_{m+1} \arrow[r, "i_2"] \arrow[d, equal] & D^1_{m+1} \arrow[r, "j_2"] \arrow[d, hook] & E^2_{m+1} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & E^1_{m+1} \arrow[r] & v_1^{-1}E^1_{m+1} \arrow[r] & E^1_{m+1}/(v_1^\infty) \arrow[r] & 0.
\end{tikzcd}$$

Then, $\mathrm{Ext}^*_{\Gamma(m+1)}(E^2_{m+1})$ is computed in Theorem 4.5. Since there is no Adams-Novikov differential and no nontrivial group extension in this range (except in the case $m = 0$ and $p = 2$), $\mathrm{Ext}^*_{\Gamma(m+1)}(BP_*)$ determines $\pi_*(T(m))$ below dimension $p^2|v_{m+1}| - 2$. This was the goal of [Rav02].

5.2. **Introduction to II.** To descend from $T(m+1)$ to $T(m)$, we can consider some interpolating spectra $T(m)_{(i)}$ introduced in Lemma 1.15. Each $T(m)_{(i)}$ is the $T(m)$-module spectrum satisfying

$$BP_*(T(m)_{(i)}) = BP_*(T(m))\{t^\ell_{m+1} \mid 0 \leq \ell < p^i\}$$

and the natural equivalence $T(m)_{(i)} \to T(m+1)$ below dimension $p^i|t_{m+1}|$. In particular, we have $T(m)_{(0)} = T(m)$ and $T(m)_{(\infty)} = T(m+1)$. Then, we can

[1] Unfortunately, i_2 induces an isomorphism of Ext^0 only below dimension $p|v_{m+2}|$ for $m > 0$. See Remark 7.4.

descend from $T(m)_{(i+1)}$ to $T(m)_{(i)}$ by making use of a topological small descent spectral sequence (Theorem 1.21), whose E_1-term is

$$E(h_{m+1,i}) \otimes P(b_{m+1,i}) \otimes \pi_*(T(m)_{(i+1)})$$

where $h_{m+1,i} \in E_1^{1,2p^i(p^{m+1}-1)}$ and $b_{m+1,i} \in E_1^{2,2p^{i+1}(p^{m+1}-1)}$ are permanent cycles.

We also have another way for descending from $T(m)_{(i+1)}$ to $T(m)_{(i)}$ algebraically. Denoting the BP_*-module generated by

$$\{t_{m+1}^{\ell} \mid 0 \le \ell < p^i\}$$

by $T_m^{(i)}$, the Adams-Novikov E_2-term for $T(m)_{(i)}$ is reduced to $\mathrm{Ext}^{*,*}_{\Gamma(m+1)}(BP_*, T_m^{(i)})$ by Lemma 1.15. To compute this, we have the 3-term resolution of $T_m^{(i)}$ by tensoring the short exact sequence (5.1) with $T_m^{(i)}$, and the associated spectral sequence $\{E_r^{n,t}, d_r\}_{r\ge 1}$ converging to $\mathrm{Ext}^*_{\Gamma(m+1)}(T_m^{(i)})$ with

$$E_1^{n,t} = \begin{cases} \mathrm{Ext}^0_{\Gamma(m+1)}(T_m^{(i)} \otimes_{BP_*} D^0_{m+1}) & \text{for } (n,t) = (0,0), \\ \mathrm{Ext}^t_{\Gamma(m+1)}(T_m^{(i)} \otimes_{BP_*} E^1_{m+1}) & \text{for } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have only one nontrivial differential $d_1 : E_1^{0,0} \to E_1^{1,0}$ induced by j_1 (5.1), and the spectral sequence collapses from E_2-term. Thus we have

Proposition 5.3. *The Adams-Novikov E_2-term for $T(m)_{(i)}$ is*

$$\mathrm{Ext}^n_{\Gamma(m+1)}(T_m^{(i)}) \cong \begin{cases} \ker d_1 & \text{for } n = 0, \\ \mathrm{coker}\ d_1 & \text{for } n = 1, \\ \mathrm{Ext}^{n-1}_{\Gamma(m+1)}(T_m^{(i)} \otimes_{BP_*} E^1_{m+1}) & \text{for } n \ge 2. \end{cases}$$

□

Note that the 0-line and the 1-line were determined in [Na08, 2.5, 4.1 and §5]; see also Proposition 6.8. The purpose of this paper is to determine the second and higher lines of the Adams-Novikov E_2-term, and the stable homotopy groups of $T(m)_{(2)}$ (Corollary 9.8) and $T(m)_{(1)}$ (Theorem 10.13) for $m > 0$ below dimension $p|v_{m+3}|$. In this range there is still no room for Adams-Novikov differentials, so the homotopy and Ext calculations coincide [2]. It is only when we pass from $T(m)_{(1)}$ to $T(m)_{(0)} = T(m)$ that we encounter Adams-Novikov differentials below dimension $p^2|v_{m+2}|$. For $m = 0$, the first of these is the Toda differential $d_{2p-1}(\beta_{p/p}) = \alpha_1\beta_1^p$ of [Tod67] and [Tod68], and the relevant calculations were the subject of [Rav04, Chapter 7]. An analogous differential for $m > 0$ was also established in [Rav], and we will discuss it somewhere else in the future.

6. A variant of Cartan-Eilenberg spectral sequence

Assume that M is a $\Gamma(k)$-comodule for some k. Once we know the structure of $\mathrm{Ext}^*_{\Gamma(k)}(M)$, then there is an inductive step reducing the value of k. Set

$$A(m) = \mathbb{Z}_{(p)}[v_1, \dots, v_m] \quad \text{and} \quad G(m) = A(m)[t_m]$$

[2]For $m = 0$, the second author determined the structure of $\mathrm{Ext}^{*+2}_{\Gamma(1)}(T_0^{(1)})$ in [Rav04, 7.5.1] for $p > 2$ below dimension $(p^3 + p)|v_1|$.

(Definition 1.1). The pair $(A(m), G(m))$ is a Hopf algebroid. Then we have an extension of Hopf algebroids (cf. Proposition 1.2)

$$(A(m), G(m)) \longrightarrow (BP_*, \Gamma(m)) \longrightarrow (BP_*, \Gamma(m+1))$$

and the associated Cartan-Eilenberg spectral sequence

$$\mathrm{Ext}^*_{G(m)}(\mathrm{Ext}^*_{\Gamma(m+1)}(M)) \quad \Longrightarrow \quad \mathrm{Ext}^*_{\Gamma(m)}(M).$$

We will approach $\mathrm{Ext}^n_{\Gamma(m+1)}(T_m^{(i)} \otimes_{BP_*} E^1_{m+1})$ using a Cartan-Eilenberg spectral sequence associated with the above Hopf algebroid extension. For a given $\Gamma(m+2)$-comodule M, we will denote $\mathrm{Ext}^0_{\Gamma(m+2)}$ by $\overline{M}$ for short. In particular, we have

$$\overline{T}_m^{(i)} = A(m+1)\{t^\ell_{m+1} \mid 0 \le \ell < p^i\}.$$

The Cartan-Eilenberg spectral sequence converging to $\mathrm{Ext}^*_{\Gamma(m+1)}(T_m^{(i)} \otimes_{BP_*} E^1_{m+1})$ has E_2-term

$$\begin{aligned} \tilde{E}_2^{s,t} &= \mathrm{Ext}^s_{G(m+1)}(\mathrm{Ext}^t_{\Gamma(m+2)}(T_m^{(i)} \otimes_{BP_*} E^1_{m+1})) \\ &\cong \mathrm{Ext}^s_{G(m+1)}(\overline{T}_m^{(i)} \otimes_{A(m+1)} \mathrm{Ext}^t_{\Gamma(m+2)}(E^1_{m+1})) \end{aligned} \tag{6.1}$$

and differentials $\tilde{d}_r : \tilde{E}_r^{s,t} \to \tilde{E}_r^{s+r,t-r+1}$. Since the case of $s = t = 0$ is not interesting, we may consider only for $s + t \ge 1$.

For simplicity, we will hereafter omit the subscript in $\overline{T}_m^{(i)} \otimes_{A(m+1)}$, and we will also denote $\mathrm{Ext}^t_{\Gamma(m+2)}(BP_*)$ by U^t_{m+1}. As we will see in Corollary 8.1, U^t_{m+1} for $t \ge 3$ is a certain suspension of U^2_{m+1} below dimension $p|\widehat{v}_3|$.

Since D^0_{m+1} is weak injective, we have isomorphisms $\mathrm{Ext}^t_{\Gamma(m+2)}(E^1_{m+1}) \cong U^{t+1}_{m+1}$ and

$$\tilde{E}_2^{s,t} \cong \mathrm{Ext}^s_{G(m+1)}(\overline{T}_m^{(i)} \otimes U^{t+1}_{m+1}) \qquad \text{for } t \ge 1. \tag{6.2}$$

The structure of U^*_{m+1} has been given in Theorem 4.5 below dimension $p^2|v_{m+1}|$. This will be discussed again in Corollary 8.1.

To describe $\tilde{E}_2^{s,0}$, we need a resolution of $\overline{E}^1_{m+1} = \mathrm{Ext}^0_{\Gamma(m+2)}(E^1_{m+1})$. The obvious one is obtained by applying $\mathrm{Ext}^0_{\Gamma(m+2)}(-)$ to (5.2). In practice, there is a "smaller resolution." Recall some notations used in [Rav02].

For a fixed positive integer m, we will set $\widehat{v}_i = v_{m+i}$ and $\widehat{t}_i = t_{m+i}$ (3.3), and define

$$\widehat{\beta}_{i/e_1,e_0} = \frac{\widehat{v}_2^i}{p^{e_0} v_1^{e_1}}, \qquad \widehat{\beta}_{i/e_1} = \widehat{\beta}_{i/e_1,1}, \qquad \widehat{\beta}_i = \widehat{\beta}_{i/1},$$
$$\widehat{\beta}'_{i/e_1} = \frac{1}{i}\widehat{\beta}_{i/e_1}, \qquad \widehat{\beta}'_i = \widehat{\beta}'_{i/1}, \qquad \text{and } \widehat{\gamma}_i = \frac{\widehat{v}_3^i}{pv_1v_2}.$$

Then we have

Theorem 6.3. *Let B_{m+1} be the $A(m+1)$-module generated by $\widehat{\beta}'_{i/i}$ for $i > 0$. Then B_{m+1} is a sub $G(m+1)$-comodule of $E^1_{m+1}/(v_1^\infty)$ and it is invariant over $\Gamma(m+2)$. Its Poincaré series is*

$$g(B_{m+1}) = g_{m+1}(t) \sum_{k \ge 0} \frac{x^{p^{k+1}}(1 - y^{p^k})}{(1 - x^{p^{k+1}})(1 - x_2^{p^j})}$$

where $y = t^{|v_1|}$, $x = t^{|\widehat{v}_1|}$, $x_2 = t^{|\widehat{v}_2|}$ *and*

$$g_{m+1}(t) = \prod_{i=1}^{m+1} \frac{1}{1-y_i} \qquad \textit{where } y_i = t^{|v_i|}.$$

Proof. This is [NR09, 2.4]. To clarify that $\widehat{\beta}'_{i/i}$ are in $E^1_{m+1}/(v_1^\infty)$, note that an element in N^2 lies in $E^1_{m+1}/(v_1^\infty)$ if and only if it has trivial image in $(M^0/D^0_{m+1})/(v_1^\infty)$. This can be shown by the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & E^1_{m+1} \arrow[r] \arrow[d, tail] & v_1^{-1}E^1_{m+1} \arrow[r] \arrow[d, tail] & E^1_{m+1}/(v_1^\infty) \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & N^1 \arrow[r] \arrow[d, two heads] & M^1 \arrow[r] \arrow[d, two heads] & N^2 \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & M^0/D^0_{m+1} \arrow[r] & v_1^{-1}(M^0/D^0_{m+1}) \arrow[r] & (M^0/D^0_{m+1})/(v_1^\infty) \arrow[r] & 0
\end{tikzcd}$$

where M^i and N^i are usual chromatic comodules. If we define $w \in D^0_{m+1}$ by

$$w = (1-p^{p-1})\widehat{\lambda}_1^p - v_1^{p^{m+1}-1}\lambda_1 \tag{6.4}$$

then we have $\widehat{v}_2 = p(\widehat{\lambda}_2 + \lambda_1 w)$ and

$$\widehat{\beta}'_{i/i} = \frac{p^i(\widehat{\lambda}_2 + \lambda_1 w)^i}{ipv_1^i} = \frac{p^{i-1}(\widehat{\lambda}_2 + \lambda_1 w)^i}{iv_1^i}$$

which is in $(M^0/D^0_{m+1})/(v_1^\infty)$ as desired. □

Remark 6.5. For $m = 0$ and $p > 2$, $E^1_1/(v_1^\infty)$ is isomorphic to N^2.

Let W_{m+1} be the $G(m+1)$-comodule defined by the induced extension in the following commutative diagram ([NR09, (1.4)]):

$$\begin{tikzcd}
0 \arrow[r] & \overline{E}^1_{m+1} \arrow[r, "\iota"] \arrow[d, equal] & W_{m+1} \arrow[r, "\rho"] \arrow[d, hook] & B_{m+1} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \overline{E}^1_{m+1} \arrow[r] & v_1^{-1}\overline{E}^1_{m+1} \arrow[r] & \overline{E}^1_{m+1}/(v_1^\infty) \arrow[r] & 0
\end{tikzcd}$$

Remark 6.6. For $m = 0$ and $p > 2$, we have a weak injective comodule D^1_1 and an inclusion $E^1_1 \to D^1_1$ ([Rav04, 7.2.1]) which induces an isomorphism in $\mathrm{Ext}^0_{\Gamma(1)}$. So, we could simply set $W_1 = \mathrm{Ext}^0_{\Gamma(2)}(D^1_1)$ (cf. [Rav04, (7.2.17)]).

We can describe W_{m+1} explcitly. Recall that

$$\mathrm{Ext}^1_{\Gamma(m+2)}(BP_*) \cong A(m+1)\left\{\frac{\widehat{v}_2^i}{ip} \mid i > 0\right\}.$$

Applying $\mathrm{Ext}_{\Gamma(m+2)}$ to (5.1) we have the short exact sequence

$$0 \longrightarrow A(m)[\widehat{\lambda}_1]/A(m+1) \longrightarrow \overline{E}^1_{m+1} \xrightarrow{\delta} U^1_{m+1} \longrightarrow 0.$$

Then, a lift of the element $\widehat{v}_2^i/ip \in U^1_{m+1}$ to $\overline{E}^1_{m+1}$ is given by

$$b_i = \frac{\widehat{v}_2^i - (v_1 w)^i}{ip} \quad \text{where } w \text{ is as in (6.4).}$$

and a lifting of the generator $\widehat{\beta}'_{i/i} \in B_{m+1}$ is

$$v_1^{-i} b_i = \sum_{0<j\le i} \binom{i-1}{j-1} \frac{(pv_1^{-1}\widehat{\lambda}_2)^j}{pj} w^{i-j}$$

W_{m+1} is the subcomodule of M^1 obtained by adjoining $v_1^{-i}b_i$ $(i>0)$ to $\overline{E}^1_{m+1}$.

The following properties of W_{m+1} can be read off in [NR09, 2.4].

Proposition 6.7. *W_{m+1} is weak injective with $\mathrm{Ext}^0_{G(m+1)}(W_{m+1}) \cong U^1_{m+1}$, i.e., the map $\iota : \overline{E}^1_{m+1} \to W_{m+1}$ induces an isomorphism in Ext^0.* □

Now we have a 3-term resolution of $\overline{E}^1_{m+1}$

$$0 \longrightarrow \overline{E}^1_{m+1} \xrightarrow{\iota} W_{m+1} \xrightarrow{\rho} B_{m+1} \longrightarrow 0.$$

Let $C^{*,s}$ denote the cochain complex obtained by applying $\mathrm{Ext}^s_{G(m+1)}(T_m^{(j)} \otimes -)$ to the sequence

$$\overline{D}^0_{m+1} \xrightarrow{\iota\circ j_1} W_{m+1} \xrightarrow{\rho} B_{m+1}$$

and let $H^{*,s}(C)$ be the associated cohomology group. Then we have

Proposition 6.8. *For $n=0$ and 1, $H^{n,0}(C)$ is isomorphic to the Adams-Novikov E_2-term $\mathrm{Ext}^n_{\Gamma(m+1)}(T_m^{(j)})$.*

Proof. Since W_{m+1} is weak injective over $G(m+1)$, $\overline{T}_m^{(i)} \otimes W_{m+1}$ is also weak injective by Lemma 1.14 and we have $C^{1,s}=0$ for $s\ge 1$. So, we have the commutative diagram

$$\begin{tikzcd}
 & C^{0,0} \arrow[r] \arrow[d, "(j_1)_*"] & C^{1,0} \arrow[r] \arrow[d, equal] & C^{2,0} \arrow[d, equal] & & \\
0 \arrow[r] & \tilde{E}_2^{0,0} \arrow[r, "\iota_*"] & C^{1,0} \arrow[r, "\rho_*"] & C^{2,0} \arrow[r] & \tilde{E}_2^{1,0} \arrow[r] & 0 \quad \text{(exact)}
\end{tikzcd}$$

and isomorphisms $C^{2,s-1} \cong \tilde{E}_2^{s,0}$ for $s\ge 2$. The map $(j_1)_*$ coincides with the differential $d_1 : E_1^{0,0} \to E_1^{1,0}$ of the resolution spectral sequence of §5.2, so we have

$$\begin{aligned}
H^{0,0}(C) &= \ker\,(j_1)_* = \ker\, d_1, \\
H^{1,0}(C) &= \ker \rho_*/\mathrm{im}\,(j_1)_* \cong \tilde{E}_2^{0,0}/\mathrm{im}\,(j_1)_* = \mathrm{coker}\, d_1.
\end{aligned}$$ □

The structure of $H^{n,0}(C)$ for $n=0,1$ was determined in [Na08, 2.5, 4.1 and §5] . We can also read the following result from the above proof.

Proposition 6.9. *For the Cartan-Eilenberg spectral sequence of (6.1) we have*

$$\tilde{E}_2^{s,0} \cong \begin{cases} \ker\rho_* & \text{for } s=0, \\ \mathrm{coker}\,\rho_* \quad (= H^{2,0}(C)) & \text{for } s=1, \\ \mathrm{Ext}^{s-1}_{G(m+1)}(\overline{T}_m^{(j)} \otimes B_{m+1}) & \text{for } s\ge 2. \end{cases}$$

□

Combining this with (6.2), we have the following chart of E_2-term, in which all Ext groups are over $G(m+1)$.

TABLE 1. The Cartan-Eilenberg E_2-term of (6.1).

	$\vdots$	$\vdots$	$\vdots$	
$t=2$	$\mathrm{Ext}^0(\overline{T}_m^{(j)} \otimes U_{m+1}^3)$	$\mathrm{Ext}^1(\overline{T}_m^{(j)} \otimes U_{m+1}^3)$	$\mathrm{Ext}^2(\overline{T}_m^{(j)} \otimes U_{m+1}^3)$	$\cdots$
$t=1$	$\mathrm{Ext}^0(\overline{T}_m^{(j)} \otimes U_{m+1}^2)$	$\mathrm{Ext}^1(\overline{T}_m^{(j)} \otimes U_{m+1}^2)$	$\mathrm{Ext}^2(\overline{T}_m^{(j)} \otimes U_{m+1}^2)$	$\cdots$
$t=0$	$\ker \rho_*$	$\operatorname{coker} \rho_*$	$\mathrm{Ext}^1(\overline{T}_m^{(j)} \otimes B_{m+1})$	$\cdots$
	$s=0$	$s=1$	$s=2$	

Note that the case of $s=t=0$ is not interesting here, as we stated before. For coker ρ_*, we need to recall some results from the other papers. For a $G(m+1)$-comodule M, denote the subgroup $\bigcap_{n \geq p^j} \ker \widehat{r}_n$ of M by $L_j(M)$. Then, it was proved in [Rav02, 1.12] that the map

$$(c \otimes 1)\psi : L_j(M) \longrightarrow \mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(j)} \otimes M)$$

is an isomorphism between $A(m+1)$-modules. Thus, to obtain the structure of $\tilde{E}_2^{1,0}$, we may alternatively examine the map

$$\rho_* : L_j(W_{m+1}) \longrightarrow L_j(B_{m+1}).$$

The following can be read from [Na08, 4.3].

Lemma 6.10. *The* coker ρ_* *is isomorphic to the quotient*

$$L_j(B_{m+1})/A(m+1)\left\{\widehat{\beta}'_{i/i} \mid 0 < i \leq p^{j-1}\right\}.$$

The structure of $L_j(B_{m+1})$ is defermined in [NR09] for all m and j. In particular, the following is the results for $j=2$.

Lemma 6.11 ([NR09, Theorem 6.1])**.** *Assume that $m > 0$. Below dimension $p^3|\widehat{v}_2|$, $L_2(B_{m+1})$ is the $A(m+1)$-module generated by*

$$\left\{\widehat{\beta}'_{i/t} \mid i \geq 1,\ 0 < t \leq \min(i,p)\right\} \cup \left\{\widehat{\beta}_{ap^2+b/t} \mid p < t \leq p^2,\ a > 0 \text{ and } 0 \leq b < p\right\}.$$

In particular, below dimension $|\widehat{v}_2^{p^2+1}/v_1^{p^2}|$, the comodule B_{m+1} is 2-free and $L_2(B_{m+1})$ is the $A(m+1)$-module generated by

$$\left\{\widehat{\beta}'_{i/\min(i,p)} \mid i > 0\right\} \cup \left\{\widehat{\beta}_{i/t} \mid p < t \leq p^2 \leq i < p^2+p\right\}. \tag{6.12}$$

7. Extending the range of $\mathrm{Ext}^{2+*}_{\Gamma(m+1)}$

In Theorem 4.5 the structure of $\mathrm{Ext}^*_{\Gamma(m+1)}(BP_*)$ was determined below dimension $p^2|\widehat{v}_1|$. Here we extend this range to $p|\widehat{v}_2|$. This is the dimension where the subcomodule E^2_{m+1} of $E^1_{m+1}/(v_1^\infty)$ starts to behave badly for $m > 0$.

By Lemma 4.2 the Poincaré series of E^2_{m+1} below dimension $p|\widehat{v}_2|$ is

$$g_{m+2}(t)\left(\frac{x^p(1-y)}{(1-x^p)(1-x_2)}+\frac{x^{p^2}(1-y^{p+1})}{(1-x^{p^2})(1-x_3)}\right)$$
$$= g(BP_*/I_2)\frac{x^p}{(1-x^p)(1-x_2)}+g(BP_*/I_2)\frac{x^{p^2}(1-y^{p+1})}{1-y}. \tag{7.1}$$

The first term corresponds to the module described in Theorem 4.5, and the second term presumably corresponds to

$$BP_*/(p,v_1)\left\{\widehat{\beta}_{p/j,p+2-j} \mid 0<j\le p\right\}.$$

We see that

$$\widehat{\beta}_{p/j,p+2-j}=\frac{\widehat{v}_2^p}{p^{p+2-j}v_1^j}=\sum_{0\le k<j}\binom{p}{k}\frac{p^{j-2-k}}{v_1^{j-k}}\widehat{\lambda}_2^{p-k}w^k \quad \in E^1_{m+1}/(v_1^\infty)$$

(where w is as in (6.4)) for $j\ge 2$, but $\widehat{\beta}_{p/1,p+1}\notin E^1_{m+1}/(v_1^\infty)$. We get around this problem by replacing $\widehat{\beta}_{p/1,p+1}$ with

$$\widetilde{\widehat{\beta}}_{p/1,p+1}=\frac{\widehat{v}_2^p}{p^{p+1}v_1}-\frac{\widehat{v}_3}{pv_1^2}+\frac{v_2\widehat{v}_2^p}{pv_1^{p+2}}-\frac{v_2^{p^{m+1}}\widehat{v}_1}{p^2v_1^2} \quad \in E^1_{m+1}/(v_1^\infty).$$

Then, our extension of Theorem 4.5 ([Rav04, 7.2.6]) is the following.

Theorem 7.2. *For $m>0$, let E^2_{m+1} be the $A(m+2)$-module generated by the set*

$$\left\{\widehat{\beta}_{i/j,k} \mid i,j,k,i+1\ge j+k\right\}\cup\left\{\widehat{\beta}_{p/j,p+2-j} \mid 2\le j\le p\right\}\cup\left\{\widetilde{\widehat{\beta}}_{p/1,p+1}\right\}.$$

Below dimension $p|\widehat{v}_2|$, it has the Poincaré series specified in (7.1), it is a sub $\Gamma(m+1)$-comodule of $E^1_{m+1}/(v_1^\infty)$, and its Ext group is isomorphic to

$$E(\widehat{h}_{1,0})\otimes P(\widehat{b}_{1,0})\otimes \mathrm{Ext}^0_{\Gamma(m+1)}(E^2_{m+1})$$

where

$$\mathrm{Ext}^0_{\Gamma(m+1)}(E^2_{m+1})\cong A(m+1)/I_2\otimes\left\{\widehat{\beta}'_i,\ \widehat{\beta}_{p/k} \mid i>0, 2\le k\le p\right\}.$$

In particular Ext^0 *maps monomorphically to* $\mathrm{Ext}^2_{\Gamma(m+1)}(BP_*)$ *in that range.*

Proof. Define a decreasing filtration on $BP_*/(p^\infty,v_1^\infty)$ by $\widehat{v}_2^a/p^bv_1^c\in F^n$ if and only if $a-b-c\ge n$. Then it was shown in Theorem 4.5 that the submodule generated by the first set (i.e., the elements of F^{-1}) is a subcomodule. We also see that the reduced expansion of $\widehat{\beta}_{p/j,p+2-j}$ is in F^{-1} (though $\widehat{\beta}_{p/j,p+2-j}$ itself is belonging to F^{-2}) and the reduced expansion of $\widetilde{\widehat{\beta}}_{p/1,p+1}$ is in F^{-2}. Thus the module generated by the assigned set is a comodule as desired.

The Ext group can be shown similarly to the proof of Theorem 4.5. $\square$

Remark 7.3. When $m = 0$, we have

$$\eta_R\left(\frac{v_2^p}{p^2v_1}\right) = \frac{v_2^p}{p^2v_1} - \frac{v_2^pt_1}{pv_1^2}$$

which means that the element β'_p does not exist and that $h_{1,0}\beta_{p/2} = 0$.

Remark 7.4. Applying Ext to the short exact sequence (5.2), we have an associated long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^0(E^1_{m+1}) \arrow[r, "(i_2)_*"] & \mathrm{Ext}^0(D^1_{m+1}) \arrow[r, "(j_2)_*"] & \mathrm{Ext}^0(E^2_{m+1}) \arrow[dll, "\delta_1"'] \\
& \mathrm{Ext}^1(E^1_{m+1}) \arrow[r, "(i_2)_*"] & \cdots,
\end{tikzcd}$$

where we have omitted the subscript $\Gamma(m+1)$ in Ext to save space. We have seen that $(i_2)_*$ is an isomorphism in Ext^0 for $m = 0$ (Lemma 4.1), however, it doesn't for $m > 0$ since we have an element $pv_1\tilde{\hat{\beta}}_{p/1,p+1}(= -v_2^{p^{m+1}}\hat{v}_1/pv_1)$, which is actually the first nontrivial element in $\ker\delta$. Below this dimension (i.e., $p|\hat{v}_2|$), the map $(i_2)_*$ is still isomorphic and $\mathrm{Ext}^0_{\Gamma(m+1)}(E^2_{m+1})$ is isomorphic to $\mathrm{Ext}^2_{\Gamma(m+1)}(BP_*)$ and it is justified that we use D^1_{m+1} and E^2_{m+1} instead of $v_1^{-1}E^1_{m+1}$ and $E^1_{m+1}/(v_1^\infty)$.

8. Some elements in $E^1_{m+1}/(v_1^\infty)$

To determine the Cartan-Eilenberg E_2-term we need to know the structure of U^*_{m+1} (6.2). A translation of Theorem 7.2 to the present context is the following.

Corollary 8.1. *Below dimension $p|\hat{v}_3|$, we have an isomorphism*

$$U^{*+2}_{m+1} \cong E(\hat{h}_{2,0}) \otimes P(\hat{b}_{2,0}) \otimes U^2_{m+1} \qquad \textit{for } m > 0$$

where U^2_{m+1} is isomorphic to the $A(m+1)/I_2$-module generated by

$$\left\{\delta^0\delta^1\left(\frac{\hat{v}_2^j\hat{v}_3^i}{i!\,pv_1}\right),\ \delta^0\delta^1\left(\frac{\hat{v}_3^p}{pv_1^k}\right) \mid 0 < i \le p,\ j \ge 0,\ 2 \le k \le p\right\}. \tag{8.2}$$

where δ^0 and δ^1 are the connecting homomorphisms for the short exact sequences

$$0 \to BP_* \to M^0 \to N^1 \to 0 \qquad \textit{and} \qquad 0 \to N^1 \to M^1 \to N^2 \to 0$$

respectively. The bidegrees of elements are $|\hat{h}_{2,0}| = (1, |\hat{t}_2|)$ and $|\hat{b}_{2,0}| = (2, |\hat{t}_2^p|)$.

In particular, we have

$$U^{2a+\varepsilon}_{m+1} \cong \hat{b}_{2,0}^{a-1} \otimes \hat{h}_{2,0}^\varepsilon \otimes U^2_{m+1} \qquad \text{for } a \ge 1 \text{ and } \varepsilon = 0, 1,$$

and hereafter we denote U^k_{m+1} by $\Sigma^{k-2}U^2_{m+1}$ when we restrict ourselves below dimension $p|\hat{v}_3|$. We also denote the elements listed in Corollary 8.1 by $\hat{u}_{i,j}$ and $\hat{u}_{p/k}$ respectively. If we replace $\delta^0\delta^1$ of Corollary 8.1 with the composition

$$\mathrm{Ext}^0_{\Gamma(m+2)}(E^1_{m+1}/(v_1^\infty)) \xrightarrow{\delta^1} \mathrm{Ext}^1_{\Gamma(m+2)}(E^1_{m+1}) \xrightarrow[\cong]{\delta^0} \mathrm{Ext}^2_{\Gamma(m+2)}(BP_*) \tag{8.3}$$

then we need to pull back elements $\hat{u}_{i,j}$ and $\hat{u}_{p/k}$ to $\mathrm{Ext}^0_{\Gamma(m+2)}(E^1_{m+1}/(v_1^\infty))$. The corresponding elements will be denoted by $\hat{\theta}_{i,j}$ and $\hat{\theta}_{p/k}$, and the rest of this section is devoted to determine these elements explicitly.

Remark 8.4. The composition (8.3) is an epimorphism and the choise of $\widehat{\theta}_{i,j}$ is not unique, i.e., the definition of $\widehat{\theta}_{i,j}$ has ambiguity up to elements of $\ker\delta^1$. In particular, the comodule B_{m+1} is involved in $\ker\delta^1$ and we may tack any element of B_{m+1} to $\widehat{v}_2^j\widehat{v}_3^i/pv_1$.

Recall the recursive formula for $\widehat{\ell}_i$ (3.10), which are independent on m.

$$\widehat{\ell}_1 = \widehat{\lambda}_1, \quad \widehat{\ell}_2 = \widehat{\lambda}_2 + \ell_1\widehat{\lambda}_1^p, \quad \widehat{\ell}_3 = \widehat{\lambda}_3 + \ell_1\widehat{\lambda}_2^p + \ell_2\widehat{\lambda}_1^{p^2}$$

On the other hand, the expression of $\widehat{v}_i$ in terms of $\widehat{\lambda}_i$ depends on m. For small values of i, we have

Lemma 8.5. *In D_{m+1}^0 we have*

$$\widehat{v}_1 = p\widehat{\lambda}_1,$$
$$\widehat{v}_2 = p\widehat{\lambda}_2 + (1-p^{p-1})v_1\widehat{\lambda}_1^p - v_1^{p^{m+1}}\widehat{\lambda}_1 \quad (m \geq 1),$$
$$\widehat{v}_3 \equiv p\widehat{\lambda}_3 - p^{p^2-1}v_2\widehat{\lambda}_1^{p^2} + \zeta \mod (v_1) \quad \textit{where} \quad \zeta = v_2\widehat{\lambda}_1^{p^2} - \begin{cases} 0 & (m=1), \\ v_2^{p^{m+1}}\widehat{\lambda}_1 & (m \geq 2). \end{cases}$$

In particular the subtraction $\widehat{v}_3 - p\widehat{\lambda}_3$ modulo (v_1) is $\Gamma(m+2)$-invariant.

Proof. By (3.9) we have

$$\begin{aligned} p\widehat{\ell}_1 &= \widehat{v}_1, \\ p\widehat{\ell}_2 &= \widehat{v}_2 + \ell_1\widehat{v}_1^p + \widehat{\ell}_1\widehat{v}_1^{p^{m+1}} \quad (m \geq 1), \\ p\widehat{\ell}_3 &= \widehat{v}_3 + \ell_1\widehat{v}_2^p + \ell_2\widehat{v}_1^{p^2} + \begin{cases} v_1^{p^{m+2}}\widehat{\ell}_2 & (m=1), \\ v_1^{p^{m+2}}\widehat{\ell}_2 + v_2^{p^{m+}}\widehat{\ell}_1 & (m \geq 2). \end{cases} \end{aligned}$$

Then the result follows from (3.10). □

For $m = 1$, we have $v_2 = p\widehat{\lambda}_1$ and $\widehat{v}_3 \bmod (v_1 D_{m+1}^0)$ is in pD_{m+1}^0. On the other hand, $\widehat{v}_3 - \zeta \bmod (v_1 D_{m+1}^0)$ is in pD_{m+1}^0 for $m \geq 2$.

Define the element ξ in D_{m+1}^0 by

$$\xi = v_2\widehat{v}_2^p - \begin{cases} 0 & (m=1) \\ v_1^p v_2^{p^{m+1}}\widehat{\lambda}_1 & (m \geq 2) \end{cases}$$

Then we have

Lemma 8.6. *For $m \geq 1$, the congruences $v_1^p\zeta \equiv \xi \bmod (p^2, v_1^{p^{m+1}})$ hold in E_{m+1}^1.*

Proof. Notice that $\widehat{v}_2 \equiv v_1\widehat{\lambda}_1^p \bmod (p, v_1^{p^{m+1}})$, and for $m \geq 2$

$$v_1^p\zeta = v_2(v_1\widehat{\lambda}_1^p)^p - v_1^p v_2^{p^{m+1}}\widehat{\lambda}_1 \equiv v_2\widehat{v}_2^p - v_1^p v_2^{p^{m+1}}\widehat{\lambda}_1 = \xi$$

mod $(p^2, v_1^{p^{m+1}})$. The case for $m = 1$ is similarly proved. □

Proposition 8.7. *Define $\widehat{\theta}_{p,j}$ $(j \geq 0)$ by*

$$\widehat{\theta}_{p,j} = \widehat{v}_2^j\left(\frac{\widehat{v}_3^p}{p!\cdot pv_1} - \frac{\xi^p}{p!\cdot pv_1^{1+p^2}}\right). \tag{8.8}$$

Then it is in $\mathrm{Ext}_{\Gamma(m+2)}^0(E_{m+1}^1/(v_1^\infty))$ and satisfies $\delta^0\delta^1(\widehat{\theta}_{p,j}) = \widehat{u}_{p,j}$.

Proof. By Lemma 8.6 we see that

$$\frac{\widehat{v}_2^j\widehat{v}_3^p}{p!\cdot pv_1} \equiv \frac{\widehat{v}_2^j(p\widehat{\lambda}_3 - p^{p^2-1}v_2\widehat{\lambda}_1^{p^2}+\zeta)^p}{p!\cdot pv_1} \equiv \frac{\widehat{v}_2^j(v_1^p\zeta)^p}{p!\cdot pv_1^{1+p^2}} \equiv \frac{\widehat{v}_2^j\xi^p}{p!\cdot pv_1^{1+p^2}}$$

mod $E^1_{m+1}/(v_1^\infty)$. Direct calculations show that $\widehat{\theta}_{p,j}$ is invariant over $\Gamma(m+2)$. Since $v_1^{-p^2-1}\xi^p/p^2$ is in $\ker\delta^1$, the second statement follows. □

In Definition 1.9 we have defined a Quillen operation $\widehat{r}_j : M \to \Sigma^{j|\widehat{t}_1|}M$ by

$$\psi(x) = \sum_j \widehat{t}_1^j \otimes \widehat{r}_j(x) + \cdots$$

where the missing terms involve $\widehat{t}_i$ for $i > 1$. The following lemma on the Quillen operation is useful.

Lemma 8.9. *The k-fold iteration of $\widehat{r}_{p^j}$ is congruent to $k!\,\widehat{r}_{kp^j}$ modulo p^j*

Proof. The relation $r_s r_t = \binom{s+t}{s} r_{s+t}$ holds and the k-fold iteration of $\widehat{r}_{p^j}$ is equal to

$$\frac{(kp^j)!}{(p^j!)^k}\widehat{r}_{kp^j},$$

where the coefficient is congruent to $k!$ modulo p^j. □

Then, we have

Proposition 8.10. *Define $\widehat{\theta}_{i,j}$ $(0 < i \le p, j \ge 0)$ by (8.8) and the downward induction on i:*

$$\widehat{\theta}_{i,j} = v_2^{-1}\widehat{r}_{p^2}(\widehat{\theta}_{i+1,j}) \qquad \text{for } 0 < i < p.$$

Then they are in $\mathrm{Ext}^0_{\Gamma(m+2)}(E^1_{m+1}/(v_1^\infty))$ and satisfy $\delta^0\delta^1(\widehat{\theta}_{i,j}) = \widehat{u}_{i,j}$.

Proof. The first statement is obvious since $\mathrm{Ext}^0_{\Gamma(m+2)}(E^1_{m+1}/(v_1^\infty))$ is a subcomodule of $E^1_{m+1}/(v_1^\infty)$. Since the second term of (8.8) is in $\ker\delta^1$ and each Quillen operation commutes with the connecting homomorphism, the second statement follows. □

Proposition 8.11. *Quillen operations on $\widehat{\theta}_{1,j}$ $(0 \le j \le p^2 - p)$ are given by*

$$\widehat{r}_{p^2}(\widehat{\theta}_{1,j}) = 0 \quad \text{and} \quad \widehat{r}_p(\widehat{\theta}_{1,j}) = jv_2\widehat{\beta}_{j+p-1/p}$$

up to unit scalar multiplication.

Proof. By Lemma 8.9 $\widehat{r}_{p^2}(\widehat{\theta}_{1,j})$ is a unit multiple of $v_2^{-p+1}\widehat{r}_{p^3}(\widehat{\theta}_{p,j})$, and we can check $\widehat{r}_{p^3}(\widehat{\theta}_{p,j}) = 0$. Similarly, $\widehat{r}_p(\widehat{\theta}_{1,j})$ is a unit multiple of $v_2^{-p+1}\widehat{r}_{p^3-p^2+p}(\widehat{\theta}_{p,j})$, which can be computed by direct calculation. □

Proposition 8.12. *We have*

$$\psi(\widehat{\theta}_{i,j}) \equiv \sum_{0\le k<i} \widehat{t}_1^{kp^2} \otimes \frac{v_2^k\widehat{\theta}_{i-k,j}}{k!} \qquad \mathrm{mod}\ (v_2^i).$$

Proof. By Lemma 8.9 we have $v_2^{p-i}\widehat{\theta}_{i,j} = (p-i)!\,\widehat{r}_{(p-i)p^2}(\widehat{\theta}_{p,j})$, and the result follows from the computation of $\psi(v_2^{p-i}\widehat{\theta}_{i,j})$ mod (v_2^p). We also notice that $k!\,\widehat{r}_{kp^2}(\widehat{\theta}_{i,j}) = v_2^k\widehat{\theta}_{i-k,j}$ since $\widehat{r}_{p^2}(\widehat{\theta}_{i+1,j}) = v_2\widehat{\theta}_{i,j}$ by Lemma 8.9. □

In particular $\widehat{\theta}_{p,0}$ is an element between dimension $p^2|\widehat{v}_2|$ and $p|\widehat{v}_3|$. It is also true for the following elements.

Proposition 8.13. *Define $\widehat{\theta}_{p/k}$ $(0 < k \le p)$ by*

$$\widehat{\theta}_{p/k} = \frac{\widehat{v}_3^p}{pv_1^k} - \frac{v_2^p \widehat{v}_2^{p^2}}{pv_1^{p^2+k}} + \frac{v_2^{p^{m+2}} \widehat{v}_2}{pv_1^{k+1}}.$$

Then it is in $\mathrm{Ext}^0_{\Gamma(m+2)}(E^1_{m+1}/(v_1^\infty))$ and satisfies $\delta^0\delta^1(\widehat{\theta}_{p/k}) = \widehat{u}_{p/k}$. Moreover, it is $G(m+1)$-invariant, i.e., we have $\widehat{r}_j(\widehat{\theta}_{p/k}) = 0$ for all $j \ge 1$.

Proof. By Lemma 8.5 we see that modulo $E^1_{m+1}/(v_1^\infty)$

$$\begin{aligned}\widehat{\theta}_{p/k} &\equiv \frac{v_1^p \widehat{\lambda}_2^{p^2} + v_2^p \widehat{\lambda}_1^{p^3}}{pv_1^k} - \frac{v_2^p \cdot v_1^{p^2} \widehat{\lambda}_1^{p^3}}{pv_1^{p^2+k}} + \frac{v_2^{p^{m+2}} \cdot v_1 \widehat{\lambda}_1^p}{pv_1^{k+1}} \\ &\equiv \frac{v_1^p \widehat{\lambda}_2^{p^2}}{pv_1^k} + \frac{(p\widehat{\lambda}_1)^{p^{m+2}} \cdot \widehat{\lambda}_1^p}{pv_1^k} \equiv 0\end{aligned}$$

for $m = 1$, and

$$\widehat{\theta}_{p/k} \equiv \frac{v_1^p \widehat{\lambda}_2^{p^2} + v_2^p \widehat{\lambda}_1^{p^3} - v_2^{p^{m+2}} \widehat{\lambda}_1^p}{pv_1^k} - \frac{v_2^p \cdot v_1^{p^2} \widehat{\lambda}_1^{p^3}}{pv_1^{p^2+k}} + \frac{v_2^{p^{m+2}} \cdot v_1 \widehat{\lambda}_1^p}{pv_1^{k+1}} \equiv 0$$

for $m \ge 2$. All terms in $\widehat{\theta}_{p/k}$ except for the leading term are in $\ker \delta^1$, and thus its $\delta^0\delta^1$-image is the desired one. Direct calculations show that $\widehat{\theta}_{p/k}$ is invariant over $\Gamma(m+1)$. □

9. The homotopy groups of $T(m)_{(2)}$

In this section we determine the homotopy groups of $T(m)_{(2)}$ in dimensions less than $p|\widehat{v}_3|$. It requires the analysis of the Cartan-Eilenberg E_2-term of Table 1 for $j = 2$. By Lemma 6.10 and 6.11 we have

Proposition 9.1. *Assume that $m > 0$. Below dimension $|\widehat{v}_2^{p^2+1}/v_1^{p^2}|$, the Cartan-Eilenberg E_2-term of Table 1 for $j = 2$ satisfies $\tilde{E}_2^{s,0} = 0$ for $s \ge 2$, and $\tilde{E}_2^{1,0}$ is isomorphic to the $A(m+1)$-module generated by*

$$\left\{\widehat{\beta}'_{i/t} \mid i \ge 2,\ 0 < t \le \min(i-1, p)\right\} \cup \left\{\widehat{\beta}_{p^2/t} \mid p < t \le p^2\right\}.$$

Note that $|\widehat{v}_2^{p^2+1}/v_1^{p^2}|$ is larger than $p|\widehat{v}_3|$ if $m > 0$.

Thus our remaining task is to determine the structure of

$$\tilde{E}_2^{s,t} \cong \mathrm{Ext}^s_{G(m+1)}(\overline{T}_m^{(2)} \otimes \Sigma^{t-1} U^2_{m+1}) \quad \text{for } t \ge 1.$$

Since this is a certain suspension of $\tilde{E}_2^{s,1}$ (i.e., tensored object with some power of $\widehat{b}_{2,0}$ and $\widehat{h}_{2,0}$), we may consider only for $\tilde{E}_2^{s,1}$. Below dimension $p|\widehat{v}_3|$, define the v_2-torsion free $A(m+1)$-submodule U^0 of $v_2^{-1}U^2_{m+1}$ by adjoining the elements

$$\left\{v_2^{-i}\widehat{u}_{i,j} \mid 0 < i \le p, j \ge 0\right\} \cup \left\{v_2^{-p}\widehat{u}_{p/k} \mid 2 \le k \le p\right\}$$

to U^2_{m+1}. Note that U^0 is a comodule since the congruence in Proposition 8.12 is modulo v_2^i and the ignored elements have non-negative v_2-exponent after applying v_2^{-i}. We also define the quotient comodule U^1 by the following short exact sequence

$$0 \longrightarrow U^2_{m+1} \longrightarrow U^0 \longrightarrow U^1 \longrightarrow 0 \tag{9.2}$$

The Quillen operations on $v_2^{-p}\widehat{u}_{p/k} \in U^0$ are trivial by Proposition 8.13. On the other hand, the behavior of Quillen operations on $v_2^{-i}\widehat{u}_{i,j} \in U^0$ follows from Proposition 8.10. The following diagram for $p = 5$ may be helpful, where each diagonal arrow represents the action of $\widehat{r}_{p^2}$ up to unit scalar multiplication and the elements in the rightmost column are out of our range except for $j = 0$.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\widehat{u}_{1,j} & \widehat{u}_{2,j} & \widehat{u}_{3,j} & \widehat{u}_{4,j} & \widehat{u}_{5,j} \\
v_2^{-1}\widehat{u}_{1,j} & v_2^{-1}\widehat{u}_{2,j} \arrow[ul] & v_2^{-1}\widehat{u}_{3,j} \arrow[ul] & v_2^{-1}\widehat{u}_{4,j} \arrow[ul] & v_2^{-1}\widehat{u}_{5,j} \arrow[ul] \\
 & v_2^{-2}\widehat{u}_{2,j} \arrow[ul] & v_2^{-2}\widehat{u}_{3,j} \arrow[ul] & v_2^{-2}\widehat{u}_{4,j} \arrow[ul] & v_2^{-2}\widehat{u}_{5,j} \arrow[ul] \\
 & & v_2^{-3}\widehat{u}_{3,j} \arrow[ul] & v_2^{-3}\widehat{u}_{4,j} \arrow[ul] & v_2^{-3}\widehat{u}_{5,j} \arrow[ul] \\
 & & & v_2^{-4}\widehat{u}_{4,j} \arrow[ul] & v_2^{-4}\widehat{u}_{5,j} \arrow[ul] \\
 & & & & v_2^{-5}\widehat{u}_{5,j} \arrow[ul]
\end{tikzcd} \tag{9.3}$$

Proposition 9.4. *U^0 is 2-free, and we have an isomorphism of $A(m+1)$-modules*

$$\mathrm{Ext}^0_{G(m+1)}(\overline{T}^{(2)}_m \otimes U^0) \cong A(m+1) \otimes \left\{ v_2^{-1}\widehat{u}_{1,j}, v_2^{-p}\widehat{u}_{p/k} \mid j \geq 0, 2 \leq k \leq p \right\}.$$

Proof. By Lemma 1.12, $\mathrm{Ext}^0_{G(m+1)}(\overline{T}^{(2)}_m \otimes U^0)$ is additively isomorphic to

$$L_2(U^0) = \bigcap_{\ell \geq p^2} \ker \widehat{r}_\ell.$$

In (9.3) the only possible elements with trivial action of $\widehat{r}_{p^2}$ are $v_2^{-1}\widehat{u}_{1,j}$. Note that

$$\widehat{r}_\ell(v_2^{-1}\widehat{u}_{1,j}) = \delta^0\delta^1(v_2^{-1}\widehat{r}_\ell(\widehat{\theta}_{1,j}))$$

and $v_2^{-1}\widehat{r}_\ell(\widehat{\theta}_{1,j}) = 0$ for $\ell \neq 1, p^2$ because

$$\psi\left(\frac{\widehat{v}_2^j \widehat{v}_3}{pv_1}\right) = \frac{\widehat{v}_2^j(\widehat{v}_3 + v_2\widehat{t}_1^{p^2} - v_2^{p^{m+1}}\widehat{t}_1)}{pv_1}.$$

We have $\widehat{r}_\ell(v_2^{-1}\widehat{u}_{1,j}) = 0$ even for $\ell = 1$ or p^2 since

$$v_2^{-1}\widehat{r}_1(\widehat{\theta}_{1,j}) = v_2^{p^{m+1}-1}\widehat{\beta}_j \quad \text{and} \quad v_2^{-1}\widehat{r}_{p^2}(\widehat{\theta}_{1,j}) = \widehat{\beta}_j$$

are in $\ker \delta^1$. Thus all Quillen operations on $v_2^{-1}\widehat{u}_{1,j}$ are trivial and there is a bijection between $\mathrm{Ext}^0_{G(m+1)}(\overline{T}^{(2)}_m \otimes U^0)$ and $\mathrm{Ext}^0_{G(m+1)}(U^0)$.

The diagram (9.3) also suggests the equality of Poincaré series

$$g(U^0) = \frac{g(\mathrm{Ext}^0(U^0))}{1 - x^{p^2}} \qquad \text{where} \quad x = t^{|\widehat{v}_1|}$$

and we have

$$\begin{aligned} g(\overline{T}_m^{(2)} \otimes U^0) &= g(U^0) \cdot \frac{1 - x^{p^2}}{1 - x} = \frac{g(\mathrm{Ext}^0(U^0))}{1 - x} \\ &= g(\mathrm{Ext}^0(U^0)) \cdot g(G(m+1)/I) \\ &= g(\mathrm{Ext}^0(\overline{T}_m^{(2)} \otimes U^0)) \cdot g(G(m+1)/I) \end{aligned}$$

which means that U^0 is 2-free. □

Proposition 9.5. *U^1 is 2-free, and we have an isomorphism of $A(m+1)$-modules*

$$\mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(2)} \otimes U^1) \cong A(m+1)/I_3 \otimes \{\widehat{u}_{i,j}/v_2 \mid i \geq 1,\, j \geq 0\}\,.$$

Proof. For the v_2-torsion comodule U^1, the analogous diagram to (9.3) for $p = 5$ is

$$\begin{array}{ccccccccc}
\widehat{u}_{1,j}/v_2 & & \widehat{u}_{2,j}/v_2 & & \widehat{u}_{3,j}/v_2 & & \widehat{u}_{4,j}/v_2 & & \widehat{u}_{5,j}/v_2 \\
 & \nwarrow & & \nwarrow & & \nwarrow & & \nwarrow & \\
 & & \widehat{u}_{2,j}/v_2^2 & & \widehat{u}_{3,j}/v_2^2 & & \widehat{u}_{4,j}/v_2^2 & & \widehat{u}_{5,j}/v_2^2 \\
 & & & \nwarrow & & \nwarrow & & \nwarrow & \\
 & & & & \widehat{u}_{3,j}/v_2^3 & & \widehat{u}_{4,j}/v_2^3 & & \widehat{u}_{5,j}/v_2^3 \\
 & & & & & \nwarrow & & \nwarrow & \\
 & & & & & & \widehat{u}_{4,j}/v_2^4 & & \widehat{u}_{5,j}/v_2^4 \\
 & & & & & & & \nwarrow & \\
 & & & & & & & & \widehat{u}_{5,j}/v_2^5.
\end{array}$$

In this case Ext^0 is generated by the elements in the top row. The 2-freeness of U^1 is similarly shown to U^0. □

Proposition 9.6. *Assume that $m > 0$. Below dimension $p|\widehat{v}_3|$, the Cartan-Eilenberg E_2-term of Table 1 for $j = 2$ satisfies*

$$\tilde{E}_2^{s,*+1} \cong E(\widehat{h}_{2,0}) \otimes P(\widehat{b}_{2,0}) \otimes \mathrm{Ext}^s_{G(m+1)}(\overline{T}_m^{(2)} \otimes U^2_{m+1})$$

and

$$\begin{aligned} \tilde{E}_2^{s,1} &= \mathrm{Ext}^s_{G(m+1)}(\overline{T}_m^{(2)} \otimes U^2_{m+1}) \\ &\cong \begin{cases} A(m+1)/I_2 \otimes \left\{\widehat{u}_{1,i},\, \widehat{u}_{p/k} \mid i \geq 0, 2 \leq k \leq p\right\} & \text{for } s = 0, \\ A(m+2)/I_3 \otimes \{\widehat{\gamma}_\ell \mid \ell \geq 2\} & \text{for } s = 1, \\ 0 & \text{for } s \geq 2 \end{cases} \end{aligned}$$

where $\widehat{\gamma}_i = \delta^2\left(\widehat{u}_{i,0}/v_2\right)$ and δ^2 is the connecting homomorphism associated to (9.2). The operators behave as if they had bidegree $\widehat{h}_{2,0} \in \tilde{E}_2^{0,1}$ and $\widehat{b}_{2,0} \in \tilde{E}_2^{0,2}$.

Proof. By Proposition 9.4 and 9.5, we have the 4-term exact sequence[3]

$$0 \longrightarrow \tilde{E}_2^{0,1} \longrightarrow \mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(2)} \otimes U^0) \longrightarrow \mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(2)} \otimes U^1) \longrightarrow \tilde{E}_2^{1,1} \longrightarrow 0$$

[3]The $m = 0$ case was described in [Rav04, 7.3.5].

and $\tilde{E}_2^{s,1} = 0$ for $s \geq 2$. Since the image of the middle map is

$$A(m+1)/I_2 \otimes \{\widehat{u}_{1,i}/v_2 \mid j \geq 0\} \cong A(m+2)/I_3 \otimes \{\widehat{u}_{1,0}/v_2\}$$

we have the desired result. □

By Proposition 9.1 and 9.6, Table 1 is reduced to the following one:

TABLE 2. The Cartan-Eilenberg E_2-term of (6.1) for $j = 2$.

	$\vdots$	$\vdots$	$\vdots$	
$t=2$	$\mathrm{Ext}^0(\overline{T}_m^{(2)} \otimes \Sigma U_{m+1}^2)$	$\mathrm{Ext}^1(\overline{T}_m^{(2)} \otimes \Sigma U_{m+1}^2)$	0	$\cdots$
$t=1$	$\mathrm{Ext}^0(\overline{T}_m^{(2)} \otimes U_{m+1}^2)$	$\mathrm{Ext}^1(\overline{T}_m^{(2)} \otimes U_{m+1}^2)$	0	$\cdots$
$t=0$	$\ker \rho_*$	described in Proposition 9.1	0	$\cdots$
	$s=0$	$s=1$	$s=2$	

Proposition 9.7. *Assume that $m > 0$. Below dimension $p|\widehat{v}_3|$, the Cartan-Eilenberg spectral sequence of Table 1 for $j = 2$ collapses, and we have the short exact sequence*

$$0 \longrightarrow \tilde{E}_\infty^{1,t} \longrightarrow \mathrm{Ext}_{\Gamma(m+1)}^{t+2}(T_m^{(2)}) \longrightarrow \tilde{E}_\infty^{0,t+1} \longrightarrow 0$$

which splits for $t \geq 1$, but for $t = 0$.

Proof. In Table 2 we have only two columns, and so the spectral sequence collapses. The middle groups is isomorphic to $\mathrm{Ext}_{\Gamma(m+1)}^{t+1}(T_m^{(2)} \otimes E_{m+1}^1)$, and the short exact sequences follow by inspection of Table 2. For $t \geq 1$, it splits since the $\tilde{E}_2^{1,t}$ is v_2-torsion while the $\tilde{E}_2^{0,t+1}$ is v_2-torsion free by Proposition 9.6. For $t = 0$, for example, an element

$$\widehat{u}_{1,0} \in \mathrm{Ext}_{G(m+1)}^0(\overline{T}_m^{(2)} \otimes U_{m+1}^2) \cong \tilde{E}_2^{0,1}$$

is killed by v_1, but its lift

$$\delta^0\delta^1(\widehat{\theta}_{1,0}) = \delta^0\delta^1\left(\frac{\widehat{v}_3}{pv_1} - \frac{v_2\widehat{v}_2^p}{pv_1^{1+p}}\right) \in \mathrm{Ext}_{\Gamma(m+1)}^2(T_m^{(2)})$$

is not killed by v_1. Thus, it does not split. □

Corollary 9.8. *Assume that $m > 0$. Then, the Adams-Novikov spectral sequence for $\pi_*(T(m)_{(2)})$ collapses below dimension $p|\widehat{v}_3|$.*

Proof. We can rule out differentials originating in filtrations 0 or 1 by the usual arguments, and the shortest possible one is

$$d_{2p-1} : E_2^{2,*} \longrightarrow E_2^{2p+1,*}.$$

The first element in the target is $\widehat{b}_{2,0}^{p-1}\widehat{h}_{2,0}\widehat{\theta}_{1,0} \in E_2^{2p+1,*}$, and the total degrees are $|\widehat{\theta}_{1,0}| = |\widehat{b}_{2,0}| = 2p^{m+3} - 2p - 2$ and $|\widehat{h}_{2,0}| = 2p^{m+2} - 3$, so we have

$$|\widehat{b}_{2,0}^{p-1}\widehat{h}_{2,0}\widehat{\theta}_{1,0}| = p|\widehat{v}_3| + 2p^2(p^m - 1) - 3 \ > \ p|\widehat{v}_3|$$

which is out of our range. □

10. The homotopy groups of $T(m)_{(1)}$

In this section we determine the homotopy groups of $T(m)_{(1)}$ in dimensions less than $p|\widehat{v}_3|$. To determine the Cartan-Eilenberg E_2-term for $j = 1$, we recall the algebraic small descent spectral sequence (Theorem 1.17). For a $G(m+1)$-comodule M and non-negative integer i, it is a spectral sequence converging to $\mathrm{Ext}_{G(m+1)}(\overline{T}_m^{(i)} \otimes_{A(m+1)} M)$ with

$$E_1^{*,t} \cong E(\widehat{h}_{1,j}) \otimes P(\widehat{b}_{1,j}) \otimes \mathrm{Ext}^t_{G(m+1)}(\overline{T}_m^{(i+1)} \otimes_{A(m+1)} M)$$

with $\widehat{h}_{1,j} \in E_1^{1,0}$, $\widehat{b}_{1,j} \in E_1^{2,0}$, and $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$. In particular, d_1 is induced by the action on M of r_{p^j} for s even and $r_{(p-1)p^j}$ for s odd. Note that $r_{(p-1)p^j}$ is congruent to the $(p-1)$-fold iteration of r_{p^j} up to unit scalar multiplication.

The case for $M = U^2_{m+1}$ is easy.

Proposition 10.1. *Below dimension $p|\widehat{v}_3|$ the small descent spectral sequence for U^2_{m+1} collapses from the E_2-term, and*

$$\mathrm{Ext}^{*+k}_{G(m+1)}(\overline{T}_m^{(1)} \otimes U^2_{m+1}) \cong E(\widehat{h}_{1,1}) \otimes P(\widehat{b}_{1,1}) \otimes \mathrm{Ext}^k_{G(m+1)}(\overline{T}_m^{(2)} \otimes U^2_{m+1}).$$

Proof. Below dimension $p|\widehat{v}_3|$ the action of $\widehat{r}_p$ on U^2_{m+1} is trivial by Corollary 8.1, and the E_1-term coincides with the E_2-term. The differentials $d_2 : E_2^{s,1} \to E_2^{s+2,0}$ are also trivial since the source is v_2-torsion while the target is v_2-torsion free. By Proposition 9.6 the small descent spectral sequence has only two rows, and so we have $d_r = 0$ for $r \geq 3$. □

Hereafter we will denote $\widehat{u}_{1,i}$ by $\widehat{u}_i$ for short. Since

$$\tilde{E}_2^{s,t} \cong \Sigma^{t-1}\mathrm{Ext}^s_{G(m+1)}(\overline{T}_m^{(1)} \otimes U^2_{m+1}) \quad \text{for } t \geq 1.$$

we have the following result.

Corollary 10.2. *Below dimension $p|\widehat{v}_3|$, the Cartan-Eilenberg E_2-term of Table 1*

$$\tilde{E}_2^{*+s,*+1} \cong \Sigma^*\mathrm{Ext}^{*+s}_{G(m+1)}(\overline{T}_m^{(1)} \otimes U^2_{m+1})$$

is isomorphic to

$$E(\widehat{h}_{2,0}, \widehat{h}_{1,1}) \otimes P(\widehat{b}_{2,0}, \widehat{b}_{1,1}) \otimes \left\{ \begin{array}{c} A(m+1)/I_2\Big\{\widehat{u}_i,\ \widehat{u}_{p/k} \mid i \geq 0, 2 \leq k \leq p\Big\} \\ \oplus \\ A(m+2)/I_3\Big\{\widehat{\gamma}_\ell \mid \ell \geq 2\Big\} \end{array} \right.$$

where the bidegree of elements are $\widehat{u} \in \tilde{E}_2^{0,1}$ and $\widehat{\gamma} \in \tilde{E}_2^{1,1}$ and the operators behave as if they had the bidegree $\widehat{h}_{2,0} \in \tilde{E}_2^{0,1}$, $\widehat{b}_{2,0} \in \tilde{E}_2^{0,2}$, $\widehat{h}_{1,1} \in \tilde{E}_2^{1,0}$ and $\widehat{b}_{1,1} \in \tilde{E}_2^{2,0}$.

Next we consider the small descent spectral sequence for $M = B_{m+1}$. Below dimension $|\widehat{v}_2^{p^2+1}/v_1^{p^2}|$ it collapses from E_2-term since B_{m+1} is 2-free by Lemma 6.11 and we need only to compute d_1. On the elements of $\mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(2)} \otimes B_{m+1})$ (6.12), we have

$$\widehat{r}_p(\widehat{\beta}'_{i/e_1}) = \widehat{\beta}_{i-1/e_1-1}, \quad \widehat{r}_p(\widehat{\beta}_{pi/e_1}) = 0 \quad \text{and} \quad \widehat{r}_{p^2-p}(\widehat{\beta}'_{i/p}) = \widehat{\beta}_{i-p+1/1}$$

up to unit scalar multiplication (cf. [NR09, B.2]). It may be helpful to demonstrate the behavior of d_1 for $p = 3$. The following diagrams describes d_1 related to the first set of (6.12):

$$
\begin{tikzcd}
\widehat{\beta}'_{3/1} & \widehat{\beta}'_{3/2} \arrow[dl, "\widehat{r}_3"'] & \widehat{\beta}'_{3/3} \arrow[dl, "\widehat{r}_3"'] \arrow[ddll, bend left, "\widehat{r}_6"] & & & & \widehat{\beta}_{5/3} \arrow[dl, "\widehat{r}_3"] \arrow[ddll, bend right, "\widehat{r}_6"'] \\
\widehat{\beta}'_{2/1} & \widehat{\beta}'_{2/2} \arrow[dl, "\widehat{r}_3"'] & & & & \widehat{\beta}_{4/2} \arrow[dl, "\widehat{r}_3"] & \widehat{\beta}_{4/3} \arrow[dl, "\widehat{r}_3"] \\
\widehat{\beta}'_{1/1} & & & & \widehat{\beta}_{3/1} & \widehat{\beta}_{3/2} & \widehat{\beta}_{3/3}
\end{tikzcd}
\tag{10.3}
$$

Corresponding to the diagonal containing $\widehat{\beta}'_{1/1}$, the subgroup of E_1 generated by

$$E(\widehat{h}_{1,1}) \otimes P(\widehat{b}_{1,1}) \otimes \{\widehat{\beta}'_{1/1}, \ldots, \widehat{\beta}'_{p/p}\}$$

reduces to simply $\{\widehat{\beta}'_{1/1}\}$ on passage to E_2. The similar argument is true for the diagonal containing $\widehat{\beta}_{p/1}$. On the other hand, corresponding to the diagonal containing $\widehat{\beta}'_{i/1}$ $(2 \le i \le p)$ is the subgroup generated by

$$E(\widehat{h}_{1,1}) \otimes P(\widehat{b}_{1,1}) \otimes \{\widehat{\beta}'_{i/1}, \ldots, \widehat{\beta}'_{p/p-i+1}\}$$

which is reduced to $P(\widehat{b}_{1,1}) \otimes \{\widehat{\beta}'_{i/1}, \widehat{h}_{1,1}\widehat{\beta}'_{p/p-i+1}\}$. The similar argument is true for the diagonal containing $\widehat{\beta}_{p/i}$ $(2 \le i \le p)$; the subgroup generated by

$$E(\widehat{h}_{1,1}) \otimes P(\widehat{b}_{1,1}) \otimes \{\widehat{\beta}_{p/i}, \ldots, \widehat{\beta}_{2p-i/p}\}$$

reduces to $P(\widehat{b}_{1,1}) \otimes \{\widehat{\beta}_{p/i}, \widehat{h}_{1,1}\widehat{\beta}_{2p-i/p}\}$. In particular, the subgroups corresponding to $\widehat{\beta}'_{p/1}$ and $\widehat{\beta}_{p/p}$ survive to E_2 entirely.

Remark 10.4. In the diagram (10.3) we can read off the existence of certain Massey products. For example, if we have a relation $\widehat{r}_p(b) = a$, then it we have the Massey product $\langle \widehat{h}_{1,1}, \widehat{h}_{1,1}, a\rangle$, as we will explain in Appendix A. In general, if we have a sequence

$$a_i \xrightarrow{\widehat{r}_p} a_{i-1} \xrightarrow{\widehat{r}_p} \cdots \xrightarrow{\widehat{r}_p} a_1 \qquad (0 < i < p)$$

then we would have the Massey product $\langle \widehat{h}_{1,1}, \ldots, \widehat{h}_{1,1}, a_1\rangle$ with i-factors of $\widehat{h}_{1,1}$ whose representative has the leading term $\widehat{t}_1^p \otimes a_i$. In this paper we will denote this Massey product by $\mu_i(a_1)$. Note that it is denoted by $\underline{pi}a_1$ in [Rav04, 7.4.12].

Note that the entire configuration is $\widehat{v}_2^p$-periodic. The diagram containing $\widehat{\beta}_{p^2/1}$ corresponding to the right one of (10.3) is combined with the diagram for the second set of (6.12):

$$\begin{tikzcd}
 & & \widehat{\beta}_{11/3} \arrow[dl, "\widehat{r}_3"] \arrow[ddll, bend right, "\widehat{r}_6"'] & \cdots & \cdots & \widehat{\beta}_{11/9} \arrow[dl, "\widehat{r}_3"] \arrow[ddll, bend right, "\widehat{r}_6"'] \\
 & \widehat{\beta}_{10/2} \arrow[dl, "\widehat{r}_3"] & \cdots & \cdots & \widehat{\beta}_{10/8} \arrow[dl, "\widehat{r}_3"] & \widehat{\beta}_{10/9} \arrow[dl, "\widehat{r}_3"] \\
\widehat{\beta}_{9/1} & \cdots & \cdots & \widehat{\beta}_{9/7} & \widehat{\beta}_{9/8} & \widehat{\beta}_{9/9}
\end{tikzcd} \tag{10.5}$$

Then, the summand corresponding to $\widehat{\beta}_{p^2/k}$ $(1 \leq k \leq p^2 - p + 1)$ reduces to $\{\widehat{\beta}_{p^2/k}\}$, and the summand corresponding to $\widehat{\beta}_{p^2/p^2-\ell}$ $(0 \leq \ell \leq p-2)$ reduces to $P(\widehat{b}_{1,1}) \otimes \{\widehat{\beta}_{p^2/p^2-\ell}, \widehat{h}_{1,1}\widehat{\beta}_{p^2+\ell/p^2}\}$.

By these observations we have

Proposition 10.6 ([NR09, 7.3])**.** *Below dimensions $|\widehat{v}_2^{p^2+1}/v_1^{p^2}|$, the Cartan-Eilenberg E_2-term of Table 1*

$$\tilde{E}_2^{*+1,0} = \mathrm{Ext}^*_{G(m+1)}(\overline{T}_m^{(1)} \otimes B_{m+1})$$

has the following $A(m+1)/I_2$-basis:

$$\begin{gathered}
P(\widehat{v}_2^p) \otimes \{\widehat{\beta}'_1, \widehat{\beta}_{p/1}\} \oplus \{\widehat{\beta}_{p^2/k} \mid 1 \leq k \leq p^2 - p + 1\} \\
\oplus \\
P(\widehat{b}_{1,1}) \otimes \left(\begin{gathered} P(\widehat{v}_2^p) \otimes \{\widehat{\beta}'_{i/1}, \widehat{h}_{1,1}\widehat{\beta}'_{p/p-i+1}, \widehat{\beta}_{p/i}, \widehat{h}_{1,1}\widehat{\beta}_{2p-i/p} \mid 2 \leq i \leq p\} \\ \oplus \\ \{\widehat{\beta}_{p^2/p^2-\ell},\ \widehat{h}_{1,1}\widehat{\beta}_{p^2+\ell/p^2} \mid 0 \leq \ell \leq p-2\} \end{gathered} \right)
\end{gathered}$$

subject to the caveat that $\widehat{v}_2\widehat{\beta}_{k/e} = \widehat{\beta}_{k+1/e}$. The bigrading of elements are (omitting unnecessary subscripts) $\widehat{\beta} \in \tilde{E}_2^{1,0}$ and the operators $\widehat{h}_{1,1}$ and $\widehat{b}_{1,1}$ behave as if they had the bidegrees given in Corollary 10.2.

Note that the range of dimensions (i.e., $|\widehat{v}_2^{p^2+1}/v_1^{p^2}|$) exceeds $p|\widehat{v}_3|$ for $m > 0$.

Now we have determined the Cartan-Eilenberg E_2-term for $j = 1$. In the followings we will see that this spectral sequence has a rich pattern of differentials for $m > 0$[4], which is essentially independent of m.

For the differential

$$\tilde{d}_2 : \tilde{E}_2^{s,1} = \mathrm{Ext}^s_{G(m+1)}(\overline{T}_m^{(1)} \otimes U^2_{m+1}) \longrightarrow \tilde{E}_2^{s+2,0} = \mathrm{Ext}^{s+1}_{G(m+1)}(\overline{T}_m^{(1)} \otimes B_{m+1}).$$

notice that we may ignore the v_2-torsion part of the source (i.e.,γ-elements) since the target is v_2-torsion free. We have

Lemma 10.7. *Assume that $m > 0$. The Cartan-Eilenberg spectral sequence of Table 1 for $j = 1$ has the following differentials*[5]

(i) $\tilde{d}_2(\widehat{u}_i) = iv_2\widehat{h}_{1,1}\widehat{\beta}_{i+p-1/p}$

(ii) $\tilde{d}_2(\widehat{h}_{1,1}\widehat{u}_i) = \binom{i}{p-1} v_2\widehat{b}_{1,1}\widehat{\beta}_{i+1/2}$

[4]It was described in [Rav04, §7.3].

[5]The result for $m = 0$ is described in [Rav04, 7.3.12].

for all $i \geq 0$. All differentials commute with multiplication by $\widehat{b}_{1,1}$.

Proof. We are considering the Cartan-Eilenberg spectral sequence for $T_m^{(1)} \otimes E_{m+1}^1$, and its Ext^s for $s > 0$ is a quotient of (isomorphic to for $s > 1$) Ext^{s-1} for $T_m^{(1)} \otimes E_{m+1}^1/(v_1^\infty)$, so we can work in the cobar complex over $G(m+1)$ for the latter comodule.

For (i), it follows from Proposition 8.11 that $\widehat{r}_p(\widehat{u}_i) = iv_2\widehat{\beta}_{i+p-1/p}$ up to a unit scalar multiplication. Then, by Lemma 8.9, we have $\widehat{r}_{p^2-p}(\widehat{u}_i) = \binom{i}{p-1} v_2 \widehat{\beta}_{i+1/2}$ and the differential (ii). ☐

Adding the information of $\tilde{d}_2$, the behavior of the Quillen operations and differentials are described in the following diagram for $p = 3$ (cf. (10.3)):

$$
\begin{tikzcd}[column sep=small]
\widehat{\beta}'_{3/1} & \widehat{\beta}'_{3/2} \arrow[dl, "\widehat{r}_3"'] & \widehat{\beta}'_{3/3} \arrow[dl, "\widehat{r}_3"'] \arrow[ddll, bend left, "\widehat{r}_6"] \\
\widehat{\beta}'_{2/1} & \widehat{\beta}'_{2/2} \arrow[dl, "\widehat{r}_3"'] & \\
\widehat{\beta}'_{1/1} & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & \widehat{\beta}_{5/3} \arrow[dl, "\widehat{r}_3"', "\tilde{d}_2"] \arrow[ddll, bend right, "\widehat{r}_6"'] & v_2^{-1}\widehat{u}_2 \arrow[dl, "\tilde{d}_2"] \arrow[ddll, bend right] \\
 & \widehat{\beta}_{4/2} \arrow[dl, "\widehat{r}_3"'] & \widehat{\beta}_{4/3} \arrow[dl, "\widehat{r}_3"] & v_2^{-1}\widehat{u}_1 \arrow[dl, "\tilde{d}_2"] \\
\widehat{\beta}_{3/1} & \widehat{\beta}_{3/2} & \widehat{\beta}_{3/3} & v_2^{-1}\widehat{u}_0
\end{tikzcd}
\tag{10.8}
$$

In each case the graph now has $2p+1$ instead of $2p$ components, three of which are maximal. We can also refine (10.5) and we have the p^2 components, $p^2 - p + 2$ of which are maximal.

In fact, each d_1 in the small descent spectral sequence behaves as it were the Cartan-Eilenberg $\tilde{d}_2$. In general, the bigrading of elements in the small descent spectral sequence are $\widehat{\beta} \in E_r^{0,2}$, $\widehat{u} \in E_r^{0,2}$ and $\widehat{\gamma} \in E_r^{0,3}$, and each operator has the same bigrading as that for Cartan-Eilenberg spectral sequence. So, the small descent d_r correspond to the Cartan-Eilenberg $\tilde{d}_{r+1}$ for $r \geq 1$. See Table 3.

Remark 10.9. Note that in the picture (10.8) the "virtual" element $v_2^{-1}\widehat{u}_i$ lives in $\mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(1)} \otimes U^0)$ but not in $\mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(1)} \otimes U^2_{m+1})$ (Proposition 9.6). This means that $\widehat{h}_{1,1}\widehat{b}_{1,1}^k\widehat{\beta}_{i+p-1/p}$ is actually nontrivial and it is chromatically renamed $\widehat{v}_2^i\widehat{b}_{1,1}^k\widehat{\gamma}_1$ (thus, it is the v_2-torsion element). This situation does not happen for $m = 0$ (cf. [Rav04, 7.3.12]). For example, in the chromatic spectral sequence we have

$$
d_e(v_2^{-1}\widehat{u}_1) = d_e\left(\frac{v_2^{-1}\widehat{v}_2\widehat{v}_3}{pv_1} - \frac{\widehat{v}_2^{p+1}}{pv_1^{p+1}}\right) = \frac{\widehat{v}_2\widehat{v}_3}{pv_1v_2} = \widehat{v}_2\widehat{\gamma}_1
$$

$$
\text{and} \qquad d_i(v_2^{-1}\widehat{u}_1) = -\frac{\widehat{v}_2^p\widehat{t}_1^p}{pv_1^p} - \frac{v_2^{p^{m+1}-1}\widehat{v}_2\widehat{t}_1}{pv_1} \equiv -\widehat{h}_{1,1}\widehat{\beta}_{p/p}.
$$

The computation of d_i is the same as Lemma 10.7 (i), and the second term in d_i is the product of $\widehat{t}_1$ with an invariant element x. It is ignored because we are working in $T(m)_{(1)}$, i.e., it is the coboundary of $\widehat{t}_1 \otimes x$.

Table 3. Bigradings of elements. Some subscripts have been omitted.

Cartan-Eilenberg spectral sequence for $j=1$

$t=3$	$\widehat{b}_{2,0}\widehat{u}$	$\widehat{h}_{1,1}\widehat{b}_{2,0}\widehat{u}$	$\widehat{b}_{1,1}\widehat{b}_{2,0}\widehat{u}$	$\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{b}_{2,0}\widehat{u}$
$t=2$	$\widehat{h}_{2,0}\widehat{u}$	$\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{u}$	$\widehat{b}_{1,1}\widehat{h}_{2,0}\widehat{u}$	$\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{h}_{2,0}\widehat{u}$
$t=1$	$\widehat{u}$	$\widehat{h}_{1,1}\widehat{u}$	$\widehat{b}_{1,1}\widehat{u}$	$\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{u}$
$t=0$	$*$	$\widehat{\beta}$	$\widehat{h}_{1,1}\widehat{\beta}$	$\widehat{b}_{1,1}\widehat{\beta}$
	$s=0$	$s=1$	$s=2$	$s=3$

Small descent spectral sequence for $j=1$

$t=4$	$\widehat{b}_{2,0}\widehat{u}$	$\widehat{h}_{1,1}\widehat{b}_{2,0}\widehat{u}$	$\widehat{b}_{1,1}\widehat{b}_{2,0}\widehat{u}$	$\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{b}_{2,0}\widehat{u}$
$t=3$	$\widehat{h}_{2,0}\widehat{u}$	$\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{u}$	$\widehat{b}_{1,1}\widehat{h}_{2,0}\widehat{u}$	$\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{h}_{2,0}\widehat{u}$
$t=2$	$\widehat{u}$	$\widehat{h}_{1,1}\widehat{u}$	$\widehat{b}_{1,1}\widehat{u}$	$\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{u}$
	$\widehat{\beta}$	$\widehat{h}_{1,1}\widehat{\beta}$	$\widehat{b}_{1,1}\widehat{\beta}$	$\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{\beta}$
$t=1$	$*$			
	$s=0$	$s=1$	$s=2$	$s=3$

Similarly, we can also see that $\widehat{b}_{1,1}^{k+1}\widehat{\beta}_{i+1/2}$ is renamed $\widehat{v}_2^i\widehat{h}_{1,1}\widehat{b}_{1,1}^k\widehat{\gamma}_1$. For example, in the chromatic spectral sequence we have

$$d_e(v_2^{-1}\widehat{h}_{1,1}\widehat{u}_{p-1}) = d_e\left(\widehat{t}_1^p\left(\frac{v_2^{-1}\widehat{v}_2^{p-1}\widehat{v}_3}{pv_1} - \frac{\widehat{v}_2^{2p-1}}{pv_1^{p+1}}\right)\right) = \frac{\widehat{v}_2^{p-1}\widehat{v}_3\widehat{t}_1^p}{pv_1v_2} = \widehat{h}_{1,1}\widehat{v}_2^{p-1}\widehat{\gamma}_1$$

and $\quad d_i(v_2^{-1}\widehat{h}_{1,1}\widehat{u}_{p-1}) = -\widehat{t}_1^p \otimes \dfrac{\widehat{v}_2^p\widehat{t}_1^{p^2-p}}{pv_1^2} + \cdots = \widehat{b}_{1,1}\widehat{\beta}_{p/2}.$

The following result concerns higher Cartan-Eilenberg differentials, and we will prove it in the next section.

Theorem 10.10. *Assume that $m>0$. The Cartan-Eilenberg spectral sequence of Table 1 for $j=1$ has the following differentials (along with those of Lemma 10.7) and no others in our range of dimensions:*

(i) $\tilde{d}_3(\widehat{h}_{2,0}\widehat{u}_i) = v_2\widehat{b}_{1,1}\widehat{\beta}'_{i+1}$ *for* $i \not\equiv 0$ *mod* p.

(ii) $\tilde{d}_3(\widehat{h}_{2,0}^{\varepsilon}\widehat{b}_{2,0}^k\widehat{u}_i) = v_2\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{h}_{2,0}^{\epsilon}\widehat{b}_{2,0}^{k-1}\widehat{u}_{i-1}$ *for* $i \not\equiv 0$ *mod* p, $k \ge 1$ *and* $\varepsilon = 0$ *or* 1.

(iii) $\tilde{d}_{2k+3}(\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{b}_{2,0}^k\widehat{u}_i) = v_2^{k+1}\widehat{h}_{1,1}\widehat{b}_{1,1}^{k+1}\widehat{\beta}'_{i+1/k+1}$ *for* $i \equiv -1$ *mod* p *and* $0 \le k < p-1$.

(iv) $\tilde{d}_{2k+2}(\widehat{h}_{1,1}\widehat{b}_{2,0}^k\widehat{u}_i) = v_2^{k+1}\widehat{b}_{1,1}^{k+1}\widehat{\beta}_{i+1/k+2}$ *for* $i \equiv -1$ *mod* p *and* $1 \le k < p-1$ *(the case $k=0$ is Lemma 10.7(ii)).*

(v) $\tilde{d}_{2p-1}(\widehat{h}_{1,1}\widehat{b}_{2,0}^{p-1}\widehat{u}_i) = v_2^{p-1}\widehat{b}_{1,1}^p\widehat{u}_{i-p+1}$ *for* $i \equiv -1$ *mod* p.

All differentials commute with multiplication by $\widehat{b}_{1,1}$.

We will prove Theorem 10.10 in the next section.

Since each source of the stated differentials lies in $\tilde{E}_r^{0,*}$ or $\tilde{E}_r^{1,*}$, it cannot be the target of another differential. Moreover, each differential has maximal length for the bidegree of its source. Thus, the source should be a permanent cycle if a differential is trivial.

Remark 10.11. We can define a decreasing filtration on B_{m+1} and U_{m+1} by

$$||\widehat{\beta}'_{i/j}|| = i - j - 1, \quad ||\widehat{u}_i|| = i + [i/p], \quad \text{and} \quad ||p|| = ||v_1|| = ||v_2|| = 1.$$

In (10.8) all elements along the same diagonal (e.g., $\widehat{\beta}_2$, $\widehat{\beta}'_{3/2}$, $\widehat{\beta}_{3/3}$ and $v_2^{-1}\widehat{u}_1$ in filtration 0) have the same filtration, and the source and target of each differential listed in Theorem 10.10 have the same filtration. A similar filtration for $m = 0$ is discussed in [Rav04, 7.4.6].

Remark 10.12. Again, we obtained the differentials of the form $d_r(x) = v_2^t y$ $(t \geq 1)$, each of which doesn't kill y but makes y into a v_2^t-torsion element, as we have already seen in Remark 10.9. For example, the differential in (i) means that $\widehat{b}_{1,1}\widehat{\beta}'_{j+1}$ is killed by v_2 in the Ext group. In the chromatic cobar complex we have

$$d(v_2^{-1}\widehat{h}_{2,0}\widehat{u}_i) = -\widehat{b}_{1,1}\widehat{\beta}'_{i+1} \pm \widehat{v}_2^i\widehat{h}_{2,0}\widehat{\gamma}_1,$$

so $\pm\widehat{v}_2^i\widehat{h}_{2,0}\widehat{\gamma}_1$ is the new name for $\widehat{b}_{1,1}\widehat{\beta}'_{i+1}$. Similarly, $\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{h}_{2,0}^{\varepsilon}\widehat{b}_{2,0}^{k-1}\widehat{u}_{i-1}$ is renamed $\widehat{v}_2^i\widehat{h}_{2,0}^{\varepsilon}\widehat{b}_{2,0}^k\widehat{\gamma}_1$ by (ii), and $\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{\beta}'_p$ is renamed $\widehat{v}_2^{p-1}\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{\gamma}_1$ by (iii).

We will now demonstate the ramifications of these differentials for $p = 3$. There is a family of them associated with each nonmaximal component of (10.8). For example, for $\widehat{\beta}_2$ at $p = 3$ we have

$$\begin{tikzcd}
\boxed{\widehat{\beta}_2} & \widehat{h}_{1,1}\widehat{\beta}_2 & \underline{\widehat{b}_{1,1}\widehat{\beta}_2} & \widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{\beta}_2 & \underline{\widehat{b}_{1,1}^2\widehat{\beta}_2} & \widehat{h}_{1,1}\widehat{b}_{1,1}^2\widehat{\beta}_2 \\
\widehat{\beta}'_{3/2} \arrow[ur, "d_1"] & \boxed{\widehat{h}_{1,1}\widehat{\beta}'_{3/2}} & \widehat{b}_{1,1}\widehat{\beta}'_{3/2} \arrow[ur, "d_1"] & \boxed{\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{\beta}'_{3/2}} & \widehat{b}_{1,1}^2\widehat{\beta}'_{3/2} \arrow[ur, "d_1"] & \underline{\widehat{h}_{1,1}\widehat{b}_{1,1}^2\widehat{\beta}'_{3/2}} \\
 & x_1 \arrow[uur, "\tilde{d}_3"] & \boxed{\widehat{h}_{1,1}x_1} & \widehat{b}_{1,1}x_1 \arrow[uur, "\tilde{d}_3"] & \underline{\widehat{h}_{1,1}\widehat{b}_{1,1}x_1} & \\
 & & & x_2 \arrow[ur, "\tilde{d}_3"] & \widehat{h}_{1,1}x_2 \arrow[uur, "\tilde{d}_5"'] &
\end{tikzcd}$$

where $x_k = v_2^{-k}\widehat{h}_{2,0}\widehat{b}_{2,0}^{k-1}\widehat{u}_k$, and the boxed elements are elements not killed by differentials. The underlined elements also survive, however, each of these changes into v_2-torsion element (cf. Remark 10.9 and 10.12)

By Remark 10.4, $\widehat{h}_{1,1}\widehat{\beta}'_{3/2}$ corresponds to $\mu_2(\widehat{\beta}_2) = \langle\widehat{h}_{1,1}, \widehat{h}_{1,1}, \widehat{\beta}_2\rangle$. Here $\tilde{d}_r$ for $r \geq 2$ denotes a differential in the Cartan-Eilenberg spectral sequence for $j = 1$, while d_1 is related to the action $\widehat{r}_p$. Thus we have

$$\widehat{h}_{1,1}\widehat{\beta}'_{3/2} = \mu_2(\widehat{\beta}_2), \quad \widehat{b}_{1,1}\widehat{\beta}_2 = \mu_1(\mu_2(\widehat{\beta}_2)) \quad \text{and} \quad \widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{\beta}'_{3/2} = \mu_2(\mu_1(\mu_2(\widehat{\beta}_2))).$$

Note that $\widehat{b}_{1,1}\widehat{\beta}_2$ is renamed $\widehat{v}_2\widehat{h}_{2,0}\widehat{\gamma}_1$ by Remark 10.12. Similarly, for $\widehat{\beta}_{3/3}$ we have

$$\begin{tikzcd}
\boxed{\widehat{\beta}_{3/3}} & \underline{\widehat{h}_{1,1}\widehat{\beta}_{3/3}} & \boxed{\widehat{b}_{1,1}\widehat{\beta}_{3/3}} & \underline{\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{\beta}_{3/3}} & \underline{\widehat{b}^2_{1,1}\widehat{\beta}_{3/3}} \\
y_1 \arrow[ur, "d_1"] & \boxed{\widehat{h}_{1,1}y_1} & \widehat{b}_{1,1}y_1 \arrow[ur, "d_1"] & \underline{\widehat{h}_{1,1}\widehat{b}_{1,1}y_1} & \\
 & & y_2 \arrow[ur, "\tilde{d}_3"] & \widehat{h}_{1,1}y_2 \arrow[uur, "\tilde{d}_4"'] &
\end{tikzcd}$$

where $y_k = v_2^{-k}\widehat{b}_{2,0}^{k-1}\widehat{u}_k$. Note that $\widehat{h}_{1,1}\widehat{\beta}_{3/3}$ is renamed $\widehat{v}_2\widehat{\gamma}_1$ by Remark 10.12. We also have

$$\begin{tikzcd}
\boxed{\widehat{\beta}_{3/2}} & \widehat{h}_{1,1}\widehat{\beta}_{3/2} & \underline{\widehat{b}_{1,1}\widehat{\beta}_{3/2}} \\
\widehat{\beta}_{4/3} \arrow[ur, "\widehat{r}_3"] & \underline{\widehat{h}_{1,1}\widehat{\beta}_{4/3}} & \\
z \arrow[ur, "d_2"] & \widehat{h}_{1,1}z \arrow[uur, "d_2"'] &
\end{tikzcd}$$

where $z = v_2^{-1}\widehat{u}_2$, and we have $\widehat{h}_{1,1}\widehat{\beta}_{4/3} = \mu_2(\widehat{\beta}_{3/2})$.

Theorem 10.13. *Assume that $m > 0$. Below dimension $p|\widehat{v}_3|$ the Cartan-Eilenberg $\tilde{E}_\infty$-term of Table 1 for $j = 1$ is the direct sum of the followings:*

(i) *the $A(m+1)/I_2 \otimes P(\widehat{v}_2^p)$-module generated by*

$$\begin{gathered}
\left\{\widehat{\beta}'_1, \widehat{\beta}'_2, \dots, \widehat{\beta}'_p \,;\, \widehat{\beta}_{p/1}, \widehat{\beta}_{p/2} \,;\, \widehat{h}_{1,1}\widehat{\beta}'_p\right\} \\
\oplus \\
P(\widehat{b}_{1,1}) \otimes \left\{\widehat{h}_{1,1}\widehat{\beta}'_{p/p-i+1}, \widehat{\beta}_{p/j} \mid 2 \le i \le p-1,\ 3 \le j \le p\right\} \\
\oplus \\
E(\widehat{h}_{2,0}) \otimes P(\widehat{b}_{2,0}) \otimes \left(\begin{array}{c} P(\widehat{b}_{1,1}) \otimes \{\widehat{u}_0\} \\ \oplus \\ \left\{\widehat{h}_{1,1}\widehat{u}_i \mid 0 \le i \le p-2\right\} \end{array}\right);
\end{gathered}$$

(ii) *the $A(m+1)/I_3 \otimes P(\widehat{v}_2^p)$-module generated by*

$$E(\widehat{h}_{2,0}) \otimes P(\widehat{b}_{1,1}, \widehat{b}_{2,0}) \otimes \left(\begin{array}{c} E(\widehat{h}_{1,1}) \otimes \{\widehat{v}_2^{p-1}\widehat{\gamma}_1\} \\ \oplus \\ \{\widehat{v}_2^i\widehat{\gamma}_1 \mid 2 \le i \le p-2\} \\ \oplus \\ \{\widehat{v}_2\widehat{\gamma}_1\} \end{array}\right) \Bigg/ \left(\begin{array}{c} \widehat{v}_2^{p-1}\widehat{h}_{1,1}\widehat{b}_{2,0}\widehat{\gamma}_1, \\ \widehat{v}_2^{p-1}\widehat{b}_{1,1}\widehat{b}_{2,0}\widehat{\gamma}_1, \\ \widehat{v}_2^i\widehat{b}_{1,1}\widehat{b}_{2,0}\widehat{\gamma}_1 \end{array}\right)$$

where the second summand is only for $p \ge 5$;

(iii) *the $A(m+1)/I_2$-module generated by*

$$\begin{gathered}
\left\{\widehat{\beta}_{p^2/k} \mid 1 \le k \le p^2 - p + 1\right\} \\
\oplus \\
P(\widehat{b}_{1,1}) \otimes \left(\begin{array}{c} \left\{\widehat{\beta}_{p^2/p^2-\ell}, \widehat{h}_{1,1}\widehat{\beta}_{p^2+\ell/p^2} \mid 0 \le \ell \le p-2\right\} \\ \oplus \\ E(\widehat{h}_{1,1}, \widehat{h}_{2,0}) \otimes P(\widehat{b}_{2,0}) \otimes \left\{\widehat{u}_{p/k} \mid 2 \le k \le p\right\} \end{array}\right); \text{ and}
\end{gathered}$$

(iv) *the $A(m+2)/I_3$-module generated by*

$$E(\widehat{h}_{1,1},\widehat{h}_{2,0})\otimes P(\widehat{b}_{1,1},\widehat{b}_{2,0})\otimes\Big\{\widehat{\gamma}_\ell\mid \ell\geq 2\Big\}.$$

Remark 10.14. By Theorem 10.10 (iii) and (iv) we know that some elements in the second summand of Theorem 10.13 (i) have higher v_2-torsion. They should be renamed chromatically so as to be understood explicitly that they are v_2-torsion.

Now we have computed $\mathrm{Ext}_{\Gamma(m+1)}(T_m^{(1)}\otimes E_{m+1}^2)$ and obtain

$$\mathrm{Ext}^n_{\Gamma(m+1)}(T_m^{(1)})\cong \mathrm{Ext}^n_{BP_*(BP)}(BP_*,BP_*(T(m)_{(1)}))$$

for $n\geq 2$ (Proposition 5.3). There are no Adams-Novikov differentials in this range because there are no elements in filtration $\geq 2p+1$, the first such element being $\widehat{v}_2\widehat{b}_{1,1}^{p-1}\widehat{\gamma}_1$, which is out of our range. Thus, Theorem 10.13 gives us the homotopy groups of $T(m)_{(1)}$ as desired.

The elements for $(p,m)=(3,1)$ are listed in Figure 1 and depicted in Figure 2.

$t-s$	Element
46	$\widehat{\beta}_1$
98	$\widehat{\beta}_2$
142	$\widehat{\beta}_{3/3}$
146	$\widehat{\beta}_{3/2}$
150	$\widehat{\beta}_3$ $\widehat{\beta}'_3$
154	$\widehat{u}_0$
189	$\widehat{v}_2\widehat{\gamma}_1$
193	$\widehat{h}_{1,1}\widehat{\beta}'_{3/2}$
197	$\widehat{h}_{1,1}\widehat{\beta}'_3$
201	$\widehat{h}_{1,1}\widehat{u}_0$
202	$\widehat{\beta}_4$
205	$\widehat{h}_{2,0}\widehat{u}_0$
240	$\widehat{v}_2\widehat{h}_{2,0}\widehat{\gamma}_1$
241	$\widehat{v}_2^2\widehat{\gamma}_1$
252	$\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{u}_0$
253	$\widehat{h}_{1,1}\widehat{u}_1$
254	$\widehat{\beta}_5$
284	$\widehat{b}_{1,1}\widehat{\beta}_{3/3}$
288	$\widehat{v}_2^2\widehat{h}_{1,1}\widehat{\gamma}_1$
292	$\widehat{v}_2^2\widehat{h}_{2,0}\widehat{\gamma}_1$

$t-s$	Element
296	$\widehat{b}_{1,1}\widehat{u}_0$
297	$\widehat{\gamma}_2$
298	$\widehat{\beta}_{6/3}$
302	$\widehat{\beta}_{6/2}$
304	$\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{u}_1$
306	$\widehat{\beta}'_6$ $\widehat{\beta}_6$
308	$\widehat{b}_{2,0}\widehat{u}_0$
310	$\widehat{u}_3$
331	$\widehat{v}_2\widehat{b}_{1,1}\widehat{\gamma}_1$
335	$\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{\beta}'_{3/2}$
339	$\widehat{v}_2^2\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{\gamma}_1$
343	$\widehat{v}_2\widehat{b}_{2,0}\widehat{\gamma}_1$
344	$\widehat{h}_{1,1}\widehat{\gamma}_2$
345	$\widehat{v}_2^4\widehat{\gamma}_1$
347	$\widehat{b}_{1,1}\widehat{h}_{2,0}\widehat{u}_0$
348	$\widehat{h}_{2,0}\widehat{\gamma}_2$
349	$\widehat{h}_{1,1}\widehat{\beta}'_{6/2}$ $\widehat{v}_2\widehat{\gamma}_2$
353	$\widehat{h}_{1,1}\widehat{\beta}'_6$

$t-s$	Element
355	$\widehat{h}_{1,1}\widehat{b}_{2,0}\widehat{u}_0$
357	$\widehat{h}_{1,1}\widehat{u}_3$
358	$\widehat{\beta}_7$
359	$\widehat{h}_{2,0}\widehat{b}_{2,0}\widehat{u}_0$
361	$\widehat{h}_{2,0}\widehat{u}_3$
382	$\widehat{v}_2\widehat{b}_{1,1}\widehat{h}_{2,0}\widehat{\gamma}_1$
383	$\widehat{v}_2^2\widehat{b}_{1,1}\widehat{\gamma}_1$
394	$\widehat{v}_2\widehat{h}_{2,0}\widehat{b}_{2,0}\widehat{\gamma}_1$
395	$\widehat{v}_2^2\widehat{b}_{2,0}\widehat{\gamma}_1$ $\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{\gamma}_2$
396	$\widehat{v}_2^4\widehat{h}_{2,0}\widehat{\gamma}_1$ $\widehat{v}_2\widehat{h}_{1,1}\widehat{\gamma}_2$
397	$\widehat{v}_2^5\widehat{\gamma}_1$
400	$\widehat{v}_2\widehat{h}_{2,0}\widehat{\gamma}_2$
401	$\widehat{v}_2^2\widehat{\gamma}_2$
406	$\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{b}_{2,0}\widehat{u}_0$
407	$\widehat{h}_{1,1}\widehat{b}_{2,0}\widehat{u}_1$
408	$\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{u}_3$
409	$\widehat{h}_{1,1}\widehat{u}_4$
410	$\widehat{\beta}_8$

FIGURE 1. The elements of $\mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*(T(1)_{(1)}))$ for $p=3$, and $t-s\leq 426$.

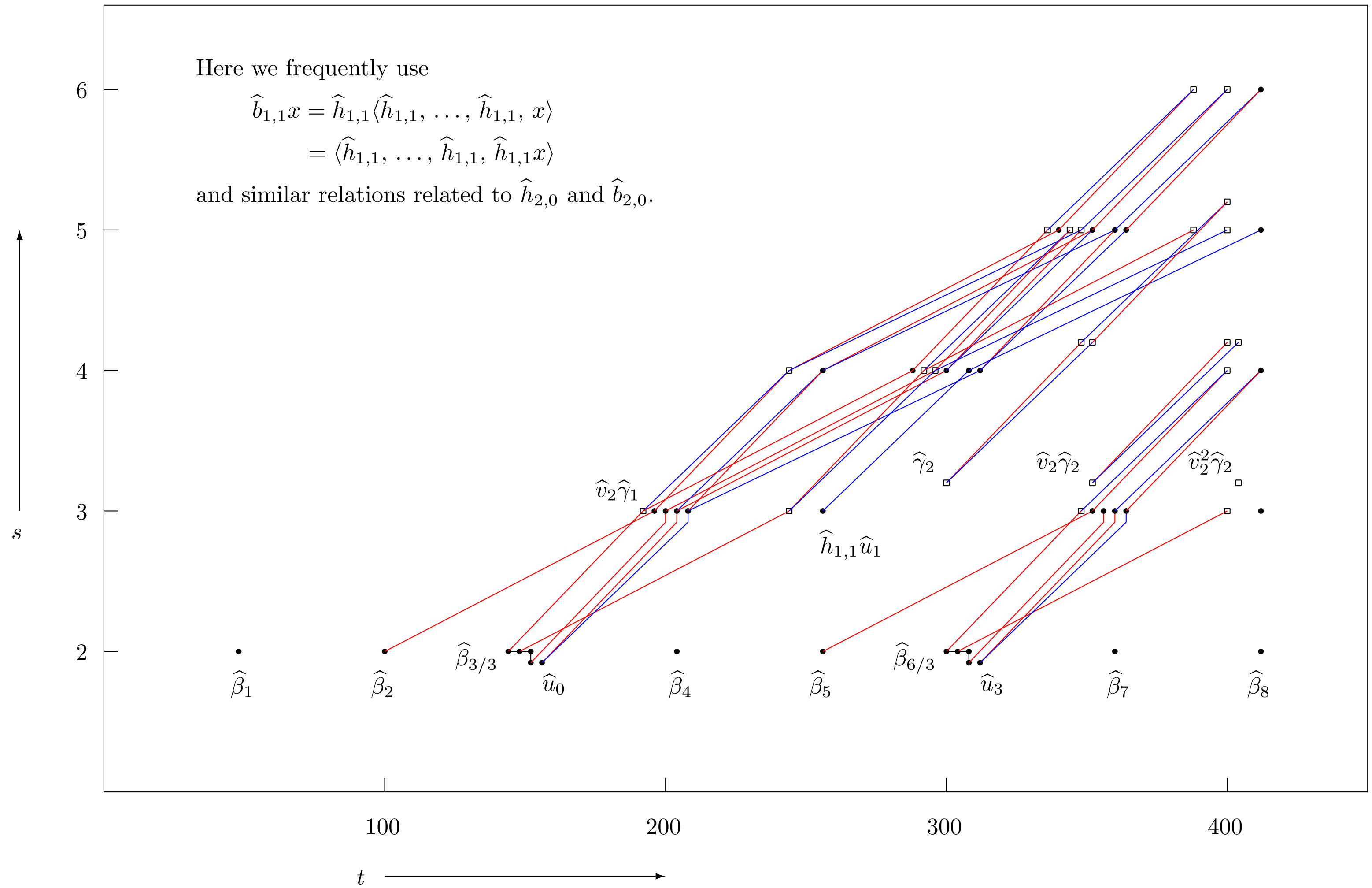

FIGURE 2. $\mathrm{Ext}_{BP_*(BP)}(BP_*(T(1)_{(1)}))$ for $p = 3$ in dimensions up to 426 dimension. There are no Adams-Novikov differentials in this range.

- Solid dots indicate v_2-torsion free elements, and squares indicate elements killed by v_2.
- Short vertical and horizontal lines indicate multiplication by p and v_1.
- Red lines (resp. blue lines) indicate multiplication by $h_{2,1}$ (resp. $h_{3,0}$) and the Massey product operation $\langle h_{2,1}, h_{2,1}, -\rangle$ (resp. $\langle h_{3,0}, h_{3,0}, -\rangle$). The composite of the two is multiplication by $b_{2,1}$ (resp. $b_{3,0}$).

11. The proof of Theorem 10.10

In this section we will give a detailed proof of Theorem 10.10 for $m > 0$[6]. Recall that the reduced Cartan-Eilenberg spectral sequence is a quotient of the Cartan-Eilenberg spectral sequence as is characterized in Lemma 1.4, and it is enough to prove each differential by computing in $C_{\Gamma(m+1)}(T_m^{(1)} \otimes N^2)$.

As shorthand, we will also use the symbols $\widehat{b}_{1,1}$ and $\widehat{b}_{2,0}$ for their cobar representatives, namely

$$\begin{aligned} \widehat{b}_{1,j} &= p^{-1} d\left(\widehat{t}_1^{p^{j+1}}\right) = - \sum_{0<\ell<p^{j+1}} p^{-1} \binom{p^{j+1}}{\ell} \widehat{t}_1^{\ell} \otimes \widehat{t}_1^{p^{j+1}-\ell} \\ \text{and} \qquad \widehat{b}_{2,0} &\equiv p^{-1}\left(\widehat{t}_2^p \otimes 1 + 1 \otimes \widehat{t}_2^p - (\widehat{t}_2 \otimes 1 + 1 \otimes \widehat{t}_2)^p\right) \\ &\equiv - \sum_{0<\ell<p} p^{-1} \binom{p}{\ell} \widehat{t}_2^{\ell} \otimes \widehat{t}_2^{p-\ell} \qquad \bmod \left(\widehat{t}_1\right). \end{aligned}$$

Lemma 11.1. *For $m > 0$, we have a cocycle*

$$\widehat{b}'_{2,0} = p^{-1}(v_1^p \widehat{b}_{1,1} + d(\widehat{t}_2^p))$$

in $C_{\Gamma(m+1)}$, which projects to $\widehat{b}_{2,0}$ in $C_{\Gamma(m+2)}$.

Proof. If follows from $d(\widehat{t}_2^p) = (\widehat{t}_2^p \otimes 1 + 1 \otimes \widehat{t}_2^p - (\widehat{t}_2 \otimes 1 + v_1 \widehat{b}_{1,0} + 1 \otimes \widehat{t}_2)^p)$. □

It also follows that $\widehat{b}_{2,0} \otimes x$ is permanent for a permanent cycle x.

Lemma 11.2. *Let $\overline{\widehat{t}_3}$ be the conjugation of $\widehat{t}_3$. Then we have*

$$\Delta(\overline{\widehat{t}_3}) = \overline{\widehat{t}_3} \otimes 1 + 1 \otimes \overline{\widehat{t}_3} - v_1 \widehat{b}_{2,0} - v_2 \widehat{b}_{1,1} + \begin{cases} \widehat{t}_1^{p^2} \otimes \widehat{t}_1 & \text{for } m = 1 \\ 0 & \text{for } m \geq 2. \end{cases}$$

The difference between $\overline{\widehat{t}_3}$ and $-\widehat{t}_3$ has trivial image in $\Gamma(m+2)$.

Proof. By definition, $\overline{\widehat{t}_3} = -\widehat{t}_3 + \widehat{t}_1^{1+p^2}$ for $m = 1$ and $\overline{\widehat{t}_3} = -\widehat{t}_3$ for $m \geq 2$. Since

$$\Delta(\widehat{t}_3) = \widehat{t}_3 \otimes 1 + 1 \otimes \widehat{t}_3 + v_1 \widehat{b}_{2,0} + v_2 \widehat{b}_{1,1} + \begin{cases} \widehat{t}_1 \otimes \widehat{t}_1^{p^2} & \text{for } m = 1 \\ 0 & \text{for } m \geq 2. \end{cases}$$

we have the result. □

[6]The case $m = 0$ is very different and was treated in [Rav04, §7.4].

Proof of Theorem 10.10 (i). Notice that we may use $\dfrac{\widehat{v}_2^i\widehat{v}_3}{pv_1}$ instead of $\widehat{u}_i$ because these have the same $\delta^1\delta^0$-image (8.3) into U^2_{m+1}. For $i>0$, we have

$$\begin{aligned}
d\left(\widehat{t}_2\otimes 1\otimes \frac{\widehat{v}_2^i\widehat{v}_3}{pv_1}\right) &= \widehat{t}_2\otimes (v_2\widehat{t}_1^{p^2}-v_2^{p^{m+1}}\widehat{t}_1)\otimes 1\otimes \frac{\widehat{v}_2^i}{pv_1}\\
d\left(\widehat{t}_2\otimes \widehat{t}_1\otimes \frac{v_2^{p^{m+1}}\widehat{v}_2^i}{pv_1}\right) &= \widehat{t}_2\otimes \widehat{t}_1\otimes 1\otimes \frac{v_2^{p^{m+1}}\widehat{v}_2^i}{pv_1}\\
d\left(\widehat{t}_2\widehat{t}_1^{p^2}\otimes 1\otimes \frac{v_2\widehat{v}_2^i}{pv_1}\right) &= -\left(\widehat{t}_2\otimes \widehat{t}_1^{p^2}+\widehat{t}_1^{p^2}\otimes \widehat{t}_2\right)\otimes 1\otimes \frac{v_2\widehat{v}_2^i}{pv_1}\\
d\left(\widehat{t}_1^{p^2}\otimes 1\otimes \frac{v_2\widehat{v}_2^{i+1}}{(i+1)p^2v_1}\right) &= \widehat{t}_1^{p^2}\otimes \widehat{t}_2\otimes 1\otimes \frac{v_2\widehat{v}_2^i}{pv_1}+\widehat{b}_{1,1}\otimes 1\otimes \frac{v_2\widehat{v}_2^{i+1}}{(i+1)pv_1}.
\end{aligned}$$

The sum of the preimages on the left is a cochain representing $\widehat{h}_{2,0}\widehat{u}_i$; summing on the right gives the desired result. □

Proof of Theorem 10.10 (ii). We will prove this for $k=1$ and $\varepsilon=1$. The general case follows because we can replace $\widehat{b}_{2,0}$ by $\widehat{b}'_{2,0}$ (Lemma 11.1) and tensor all equations on the left with the cocycle $(\widehat{b}'_{2,0})^{k-1}$.

We have

$$\eta_R(\widehat{v}_2)\equiv \widehat{v}_2+z \qquad \bmod I^{p^{m+1}},$$

where $I=(p,v_1,\ldots)$ and $z=v_1\widehat{t}_1^p+p\widehat{t}_2$. By this and Lemma 11.2 we have

$$\begin{aligned}
d(\widehat{b}_{2,0}\otimes 1\otimes \widehat{u}_i) &= \widehat{b}_{2,0}\otimes d\,(1\otimes \widehat{u}_i)\\
&= -\widehat{b}_{2,0}\otimes v_2\sum_{0<k<p}\binom{i+p}{k}z^k\otimes 1\otimes \frac{\widehat{v}_2^{i+p-k}}{\binom{i+p}{p}pv_1^{p+1}}\\
&= -\widehat{b}_{2,0}\otimes v_2\left((i+p)\widehat{t}_1^p\otimes 1\otimes \frac{\widehat{v}_2^{i+p-1}}{\binom{i+p}{p}pv_1^p}+\cdots\right),
\end{aligned}$$

$$\begin{aligned}
d&\left(-\widehat{t}_3\otimes v_2\left(-(i+p)\widehat{t}_1^p\otimes 1\otimes \frac{\widehat{v}_2^{i+p-1}}{\binom{i+p}{p}pv_1^{p+1}}+\cdots\right)\right)\\
&= -(v_1\widehat{b}_{2,0}+v_2\widehat{b}_{1,1})\otimes v_2\left(-(i+p)\widehat{t}_1^p\otimes 1\otimes \frac{\widehat{v}_2^{i+p-1}}{\binom{i+p}{p}pv_1^{p+1}}+\cdots\right)\\
&\qquad -\widehat{t}_3\otimes -v_2(i+p)\widehat{t}_1^p\otimes \binom{i+p-1}{p}\widehat{t}_1^{p^2}\otimes 1\otimes \frac{\widehat{v}_2^{i-1}}{\binom{i+p}{p}pv_1}\\
&= -\widehat{b}_{2,0}\otimes v_2\left(-(i+p)\widehat{t}_1^p\otimes 1\otimes \frac{\widehat{v}_2^{i+p-1}}{\binom{i+p}{p}pv_1^p}+\cdots\right)\\
&\qquad -v_2\widehat{b}_{1,1}\otimes v_2\left(-(i+p)\widehat{t}_1^p\otimes 1\otimes \frac{\widehat{v}_2^{i+p-1}}{\binom{i+p}{p}pv_1^{p+1}}+\cdots\right)\\
&\qquad +iv_2\widehat{t}_3\otimes \widehat{t}_1^p\otimes \widehat{t}_1^{p^2}\otimes 1\otimes \frac{\widehat{v}_2^{i-1}}{pv_1},
\end{aligned}$$

and

$$d\left(-i\widehat{t}_3 \otimes \widehat{t}_1^p \otimes 1 \otimes \frac{\widehat{v}_2^{i-1}\widehat{v}_3}{pv_1}\right)$$
$$= -iv_2\widehat{b}_{1,1} \otimes \widehat{t}_1^p \otimes 1 \otimes \frac{\widehat{v}_2^{i-1}\widehat{v}_3}{pv_1} - i\widehat{t}_3 \otimes \widehat{t}_1^p \otimes v_2\widehat{t}_1^{p^2} \otimes 1 \otimes \frac{\widehat{v}_2^{i-1}}{pv_1}.$$

The sum of the terms on the left represents $d(\widehat{b}_{2,0}\widehat{u}_i)$, and the terms on the right add up to

$$\widehat{b}_{1,1} \otimes \widehat{t}_1^p \otimes 1 \otimes \left(-\frac{iv_2\widehat{v}_2^{i-1}\widehat{v}_3}{pv_1} + \frac{(i+p)v_2^2\widehat{v}_2^{i+p-1}}{\binom{i+p}{p}pv_1^{p+1}}\right) + \cdots$$
$$= -iv_2\widehat{b}_{1,1} \otimes \widehat{t}_1^p \otimes 1 \otimes \left(\frac{\widehat{v}_2^{i-1}\widehat{v}_3}{pv_1} - \frac{\widehat{v}_2^{i+p-1}}{\binom{i+p-1}{p}pv_1^{p+1}}\right) + \cdots$$

The inspection of $\tilde{E}_2^{3,1}$ as described in Corollary 10.2 shows that the element represents $-iv_2\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{u}_{i-1}$ as claimed. $\square$

We will derive (iii), (iv) and (i) from (i) and (ii) using Massey product arguments.

Figure 3 illustrates these products and others like them for $p = 5$ and $m > 0$. The arrows labeled $\tilde{d}_3$ are related to the differential of (ii). For example, the differential $\tilde{d}_3(\widehat{b}_{2,0}\widehat{u}_4) = v_2\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{u}_3$ is denoted

$$\widehat{b}_{2,0}\widehat{u}_4 \xrightarrow{\tilde{d}_3} v_2\widehat{b}_{1,1}\widehat{u}_3.$$

Proof of Theorem 10.10 (iii). For $k = 0$ this is a direct consequence of (i) via multiplication by $\widehat{h}_{1,1}$. We will illustrate with the case $i = p-1$ and $k \le 2$ (the other cases are similarly shown). For $k = 1$, we have the sequence analogous to that of Remark 10.4:

$$\widehat{b}_{2,0}\widehat{u}_{p-1} \xrightarrow{\tilde{d}_3} v_2\widehat{b}_{1,1}\widehat{u}_{p-2} \xrightarrow{\tilde{d}_2} v_2^2\widehat{b}_{1,1}\widehat{\beta}_{2p-3/p} \xrightarrow{\widehat{r}_p} \cdots \xrightarrow{\widehat{r}_p} v_2^2\widehat{b}_{1,1}\widehat{\beta}_{p/3}.$$

This allows us to identify $v_2\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{u}_{p-2}$ with the Massey product $\mu_{p-1}(v_2^2\widehat{b}_{1,1}\widehat{\beta}_{p/3})$ up to unit scalar. It follows that the differential on $\widehat{h}_{2,0}\widehat{h}_{1,1}(\widehat{b}_{2,0}\widehat{u}_{p-1})$ is the value of $\widehat{h}_{2,0}\widehat{h}_{1,1}\mu_{p-1}(v_2^2\widehat{b}_{1,1}\widehat{\beta}_{p/3})$. Now $\widehat{h}_{2,0}\widehat{h}_{1,1}$ (resp. $\widehat{b}_{1,1}$) is the image of $\widehat{\beta}_2$ (resp. $\widehat{\beta}_{p/p}$) under a suitable reduction map, so we have

$$\begin{aligned} v_2^2\widehat{h}_{2,0}\widehat{h}_{1,1}\widehat{b}_{1,1}\mu_{p-1}(\widehat{\beta}_{p/3}) &= v_2^2\widehat{b}_{1,1}\widehat{\beta}_2\mu_{p-1}(\widehat{\beta}_{p/3}) \\ &= v_2^2\widehat{b}_{1,1}\mu_{p-1}(\widehat{\beta}_2)\widehat{\beta}_{p/3} \qquad \text{by Lemma A.8} \\ &= v_2^2\widehat{b}_{1,1}\mu_{p-1}(\widehat{\beta}_2)v_1^{p-3}\widehat{\beta}_{p/p} = v_1^{p-3}v_2^2\widehat{b}_{1,1}^2\mu_{p-1}(\widehat{\beta}_2). \end{aligned}$$

By Example A.9 we have $v_1^{p-3}\mu_{p-1}(\widehat{\beta}_2) = \mu_2(\widehat{\beta}_{p-1})$, so

$$\tilde{d}_5(\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{b}_{2,0}\widehat{u}_{p-1}) = v_2^2\widehat{b}_{1,1}^2\mu_2(\widehat{\beta}_{p-1}) = v_2^2\widehat{b}_{1,1}^2\widehat{h}_{1,1}\widehat{\beta}'_{p/2}$$

as claimed. For $k = 2$, consider the sequence[7]

$$\widehat{b}_{2,0}^2\widehat{u}_{p-1} \xrightarrow{\tilde{d}_3} v_2\widehat{b}_{1,1}\widehat{b}_{2,0}\widehat{u}_{p-2} \xrightarrow{\tilde{d}_3} v_2^2\widehat{b}_{1,1}^2\widehat{u}_{p-3} \xrightarrow{\tilde{d}_2} v_2^3\widehat{b}_{1,1}^2\widehat{\beta}_{2p-4/p} \xrightarrow{\widehat{r}_p} \cdots \xrightarrow{\widehat{r}_p} v_2^3\widehat{b}_{1,1}^2\widehat{\beta}_{p/4}$$

[7]Note that we may assume that $p \ge 5$ since $0 \le k < p-1$.

$$
\begin{tikzcd}
\widehat{\beta}_5 & \widehat{\beta}_6 & \widehat{\beta}_7 & \widehat{\beta}_8 & \widehat{\beta}_9 & \widehat{\beta}'_{10} \\
v_2\widehat{\beta}_{5/2} & \widehat{\beta}_{6/2} \arrow[ul, "\widehat{r}_3"'] & \widehat{\beta}_{7/2} \arrow[ul, "\widehat{r}_3"'] & \widehat{\beta}_{8/2} \arrow[ul, "\widehat{r}_3"'] & \widehat{\beta}_{9/2} \arrow[ul, "\widehat{r}_3"'] & \widehat{\beta}'_{10/2} \arrow[ul, "\widehat{r}_3"'] \\
v_2^2\widehat{b}_{1,1}\widehat{\beta}_{5/3} & v_2\widehat{\beta}_{6/3} \arrow[ul, "\widehat{r}_3"'] & \widehat{\beta}_{7/3} \arrow[ul, "\widehat{r}_3"'] & \widehat{\beta}_{8/3} \arrow[ul, "\widehat{r}_3"'] & \widehat{\beta}_{9/3} \arrow[ul, "\widehat{r}_3"'] & \widehat{\beta}'_{10/3} \arrow[ul, "\widehat{r}_3"'] \\
v_2^3\widehat{b}_{1,1}^2\widehat{\beta}_{5/4} & v_2^2\widehat{b}_{1,1}\widehat{\beta}_{6/4} \arrow[ul, "\widehat{r}_3"'] & v_2\widehat{\beta}_{7/4} \arrow[ul, "\widehat{r}_3"'] & \widehat{\beta}_{8/4} \arrow[ul, "\widehat{r}_3"'] & \widehat{\beta}_{9/4} \arrow[ul, "\widehat{r}_3"'] & \widehat{\beta}'_{10/4} \arrow[ul, "\widehat{r}_3"'] \\
v_2^4\widehat{b}_{1,1}^3\widehat{\beta}_{5/5} & v_2^3\widehat{b}_{1,1}^2\widehat{\beta}_{6/5} \arrow[ul, "\widehat{r}_3"'] & v_2^2\widehat{b}_{1,1}\widehat{\beta}_{7/5} \arrow[ul, "\widehat{r}_3"'] & v_2\widehat{\beta}_{8/5} \arrow[ul, "\widehat{r}_3"'] & \widehat{\beta}_{9/5} \arrow[ul, "\widehat{r}_3"'] & \widehat{\beta}'_{10/5} \arrow[ul, "\widehat{r}_3"'] \\
v_2^4\widehat{b}_{1,1}^4\widehat{u}_0 & v_2^3\widehat{b}_{1,1}^3\widehat{u}_1 \arrow[ul, "\tilde{d}_2"'] & v_2^2\widehat{b}_{1,1}^2\widehat{u}_2 \arrow[ul, "\tilde{d}_2"'] & v_2\widehat{b}_{1,1}\widehat{u}_3 \arrow[ul, "\tilde{d}_2"'] & \widehat{u}_4 \arrow[ul, "\tilde{d}_2"'] & \\
& v_2^3\widehat{b}_{1,1}^3\widehat{b}_{2,0}\widehat{u}_1 \arrow[ul, "\tilde{d}_3"'] & v_2^2\widehat{b}_{1,1}^2\widehat{b}_{2,0}\widehat{u}_2 \arrow[ul, "\tilde{d}_3"'] & v_2\widehat{b}_{1,1}\widehat{b}_{2,0}\widehat{u}_3 \arrow[ul, "\tilde{d}_3"'] & \widehat{b}_{2,0}\widehat{u}_4 \arrow[ul, "\tilde{d}_3"'] & \\
& & v_2^2\widehat{b}_{1,1}^2\widehat{b}_{2,0}^2\widehat{u}_2 \arrow[ul, "\tilde{d}_3"'] & v_2\widehat{b}_{1,1}\widehat{b}_{2,0}^2\widehat{u}_3 \arrow[ul, "\tilde{d}_3"'] & \widehat{b}_{2,0}^2\widehat{u}_4 \arrow[ul, "\tilde{d}_3"'] & \\
& & & v_2\widehat{b}_{1,1}\widehat{b}_{2,0}^3\widehat{u}_3 \arrow[ul, "\tilde{d}_3"'] & \widehat{b}_{2,0}^3\widehat{u}_4 \arrow[ul, "\tilde{d}_3"'] & \\
& & & & \widehat{b}_{2,0}^4\widehat{u}_4 \arrow[ul, "\tilde{d}_3"'] &
\end{tikzcd}
$$

FIGURE 3. The case $p = 5$ and $m > 0$.

which allows us to identify $v_2\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{b}_{2,0}\widehat{u}_{p-2}$ with $\mu_{p-1}(v_2^3\widehat{b}_{1,1}^2\widehat{\beta}_{p/4})$. It follows that the differential on $\widehat{h}_{1,1}\widehat{h}_{2,0}(\widehat{b}_{2,0}^2\widehat{u}_{p-1})$ is the value of $\widehat{h}_{1,1}\widehat{h}_{2,0}(\mu_{p-1}(v_2^3\widehat{b}_{1,1}^2\widehat{\beta}_{p/4}))$. By Example A.9 we have $v_1^{p-4}\mu'_{p-1}(\widehat{\beta}_2) = \mu'_3(\widehat{\beta}_{p-2})$, so

$$
\begin{aligned}
\tilde{d}_7(\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{b}_{2,0}^2\widehat{u}_{p-1}) &= \widehat{h}_{2,0}\widehat{h}_{1,1}\mu_{p-1}(v_2^3\widehat{b}_{1,1}^2\widehat{\beta}_{p/4}) = \mu'_{p-1}(\widehat{h}_{2,0}\widehat{h}_{1,1})v_2^3\widehat{b}_{1,1}^2\widehat{\beta}_{p/4} \\
&= v_2^3\widehat{b}_{1,1}^2\mu'_{p-1}(\widehat{\beta}_2)\widehat{\beta}_{p/4} = v_1^{p-4}v_2^3\widehat{b}_{1,1}^2\mu'_{p-1}(\widehat{\beta}_2)\widehat{\beta}_{p/p} \\
&= v_2^3\widehat{b}_{1,1}^3\mu'_3(\widehat{\beta}_{p-2}) = v_2^3\widehat{b}_{1,1}^3\widehat{h}_{1,1}\widehat{\beta}'_{p/3}
\end{aligned}
$$

as claimed. □

Proof of Theorem 10.10 (iv) and (v). Look at the elements $\widehat{b}_{2,0}^k\widehat{u}_{p-1}$ in the last column of Figure 3.

We have the sequence

$$
\widehat{b}_{2,0}^k\widehat{u}_{p-1} \xrightarrow{\tilde{d}_3} \cdots \xrightarrow{\tilde{d}_3} v_2^k\widehat{b}_{1,1}^k\widehat{u}_{p-1-k} \xrightarrow{\tilde{d}_2} v_2^{k+1}\widehat{b}_{1,1}^k\widehat{\beta}_{2p-2-k/p} \xrightarrow{\widehat{r}_p} \cdots \xrightarrow{\widehat{r}_p} v_2^{k+1}\widehat{b}_{1,1}^k\widehat{\beta}_{p/k+2}
$$

for $k < p - 1$, and

$$
\widehat{b}_{2,0}^{p-1}\widehat{u}_{p-1} \xrightarrow{\tilde{d}_3} \cdots \xrightarrow{\tilde{d}_3} v_2^{p-1}\widehat{b}_{1,1}^{p-1}\widehat{u}_0
$$

for $k = p-1$. Thus we have

$$\tilde{d}_r(\widehat{b}_{2,0}^k \widehat{u}_{p-1}) = \begin{cases} \mu_{p-1}(v_2^{k+1}\widehat{b}_{1,1}^k \widehat{\beta}_{p/k+2}) & \text{for } k < p-1 \\ \mu_{p-1}(v_2^{p-1}\widehat{b}_{1,1}^{p-1}\widehat{u}_0) & \text{for } k = p-1 \end{cases}$$

up to unit scalar multiplication. Since $\widehat{h}_{1,1}\mu_{p-1}(x) = \widehat{b}_{1,1}x$ we have

$$\tilde{d}_r(\widehat{h}_{1,1}\widehat{b}_{2,0}^k \widehat{u}_{p-1}) = \begin{cases} v_2^{k+1}\widehat{b}_{1,1}^{k+1}\widehat{\beta}_{p/k+2} & \text{for } k < p-1 \\ v_2^{p-1}\widehat{b}_{1,1}^{p}\widehat{u}_0 & \text{for } k = p-1 \end{cases}$$

as claimed. □

Appendix A. Massey products

Here we recall the definition of Massey products very briefly (cf. [Rav86, A1.4]) and prove some results used in this paper. Let C be a differential graded algebra, which makes $H^*(C)$ a graded algebra. For $x \in C$ or $x \in H^*(C)$, let $\overline{x} = (-1)^{1+\deg(x)}x$, where $\deg(x)$ denotes the sum of its internal and cohomogical degrees of x. We have $d(\overline{x}) = -\overline{d(x)}$, $\overline{(xy)} = -\overline{x}\,\overline{y}$, and $d(xy) = d(x)y - \overline{x}d(y)$.

Let $\alpha_k \in H^*(C)$ $(k = 1, 2, \dots)$ be a finite collection of elements and with representative cocycles $a_{k-1,k} \in C$. When $\overline{\alpha_1}\alpha_2 = 0$ and $\overline{\alpha_2}\alpha_3 = 0$, there are cochains $a_{0,2}$ and $a_{1,3}$ such that $d(a_{0,2}) = \overline{a_{0,1}}a_{1,2}$ and $d(a_{1,3}) = \overline{a_{1,2}}a_{2,3}$, and we have a cocycle $b_{0,3} = \overline{a_{0,2}}a_{2,3} + \overline{a_{0,1}}a_{1,3}$. The corresponding class in $H^*(C)$ represents the Massey product $\langle \alpha_1,\, \alpha_2,\, \alpha_3 \rangle$, which is the coset comprising all cohomology classes represented by such $b_{0,3}$ for all possible choices of $a_{i,j}$. Two choices of $a_{0,2}$ or $a_{1,3}$ differ by a cocycle. The **indeterminacy** of $\langle \alpha_1,\, \alpha_2,\, \alpha_3 \rangle$ is the set

$$\alpha_1 H^{|\alpha_2\alpha_3|}(C) + H^{|\alpha_1\alpha_2|}(C)\alpha_3.$$

If the triple product contains zero, then one such choice yields a $b_{0,3}$ which is the coboundary of a cochain $a_{0,3}$.

If we have two 3-fold Massey products $\langle \alpha_1,\, \alpha_2,\, \alpha_3 \rangle$ and $\langle \alpha_2,\, \alpha_3,\, \alpha_4 \rangle$ containing zero, then the $a_{i-1,i}$ and $a_{i-2,i}$ can be chosen so that here are cochains $a_{0,3}$ and $a_{1,4}$ with $d(a_{0,3}) = b_{0,3}$ and $d(a_{1,4}) = b_{1,4}$, and the 4-fold Massey product $\langle \alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4 \rangle$. represented by the cocycle $b_{0,4} = \overline{a_{0,3}}a_{3,4} + \overline{a_{0,2}}a_{2,4} + \overline{a_{0,1}}a_{1,4}$. More generally, if we have the series of cocycles $b_{j,k}$ and cochains $a_{j,k}$ satisfying

$$b_{j,k} = \sum_{j<\ell<k} \overline{a_{j,\ell}}a_{\ell,k} \quad \text{for } i \le j < k \le i+n \tag{A.1}$$

n-fold Massey products $\langle \alpha_{i+1}, \dots, \alpha_{i+n} \rangle$ represented by $b_{i,i+n}$. The cochains $a_{j,k}$ chosen above are called the **defining system** for the Massey product (cf. [Rav86, A1.4.3]).

If two products $\langle \alpha_1,\, \dots,\, \alpha_{n-1} \rangle$ and $\langle \alpha_2,\, \dots,\, \alpha_n \rangle$ are strictly defined (meaning all the lower order products in sight have trivial indeterminacy), then we have

$$\alpha_1\langle \alpha_2,\, \dots,\, \alpha_n \rangle = \langle \overline{\alpha}_1,\, \dots,\, \overline{\alpha_{n-1}} \rangle \alpha_n$$

(cf. [Rav86, A1.4.6(c)]). In fact, we can relax the hypothesis of strict definition in the following way.

Lemma A.2. *Suppose that $\langle \alpha_1,\, \dots,\, \alpha_{n-1} \rangle$ and $\langle \alpha_2,\, \dots,\, \alpha_n \rangle$ are defined and have representatives x and y respectively with the common defining system $a_{i,j}$ $(0 < i < j < n)$. Then, the cocycle $\overline{x}a_{n-1,n}$ is cohomologous to $a_{0,1}y$.*

Proof. If both x and y contain zero, then we would have cochains $a_{1,n}$ and $a_{0,n-1}$ satisfying $d(a_{0,n-1}) = x$ and $d(a_{1,n}) = y$. Hence we could define the cocycle $b_{0,n}$ (A.1). In that case we would have

$$\begin{aligned} d(b_{0,n}) &= d(\overline{a_{0,1}}a_{1,n}) + d(\overline{a_{0,n-1}}a_{n-1,n}) + d(\tilde{b}_{0,n}) \\ &= -a_{0,1}y + \overline{x}a_{n-1,n} + d(\tilde{b}_{0,n}) = 0 \end{aligned}$$

where

$$\tilde{b}_{0,n} = \sum_{1<i<n-1} \overline{a_{0,i}}a_{i,n}.$$

Even if x and y do not contain zero, so we don't have cochains $a_{1,n}$ and $a_{0,n-1}$, we can still define $\tilde{b}_{0,n}$. A routine calculation gives the desired value of $d(\tilde{b}_{0,n})$. $\square$

We also have Massey products in the spectral sequence associated with a filtered differential graded algebra or a filtered differential graded module over a filtered differential graded algebra. Our Cartan-Eilenberg spectral sequence is not associated with such a filtration, but we can get around this as follows. Let $T_m^* = \bigoplus_{i\geq 0} T_m^i$ be a bigraded comodule algebra with i being the second grading. Then the algebra structure of T_m^* is given by the pairings $T_m^i \otimes T_m^j \to T_m^{i+j}$.

Recall that for a Hopf algebroid (A, Γ) and a comodule algebra M the cup product in the cobar complex $C = C_\Gamma(M)$ is given by

$$\begin{aligned} &(\gamma_1 \otimes \cdots \otimes \gamma_s \otimes m_1) \cup (\gamma_{s+1} \otimes \cdots \otimes \gamma_{s+t} \otimes m_2) \\ &\qquad = \gamma_1 \otimes \cdots \otimes \gamma_s \otimes m_1^{(1)}\gamma_{s+1} \otimes \cdots \otimes m_1^{(t)}\gamma_{s+t} \otimes m_1^{(t+1)}m_2 \end{aligned}$$

(cf. [Rav86, A1.2.15]) where $\gamma_i \in \Gamma(m+1)$ and $m_j \in M$, and $m_1^{(1)} \otimes \cdots \otimes m_1^{(t+1)}$ is the iterated coproduct on m_1. The coboundary operator is a derivation with respect to this product and C is a filtered differential graded algebra, i.e., we have

$$d(x \cup y) = d(x) \cup y + (-1)^{\deg(x)} x \cup d(y).$$

Now we have two quadrigraded Cartan-Eilenberg spectral sequences:

$$\mathrm{Ext}_{G(m+1)}(\mathrm{Ext}_{\Gamma(m+2)}(T_m^*)) \implies \mathrm{Ext}_{\Gamma(m+1)}(T_m^*), \tag{A.3}$$

which is associated with a filtration on $C = C_{\Gamma(m+1)}(T_m^*)$, and

$$\mathrm{Ext}_{G(m+1)}(\mathrm{Ext}_{\Gamma(m+2)}(T_m^* \otimes E_{m+1}^1)) \implies \mathrm{Ext}_{\Gamma(m+1)}(T_m^* \otimes E_{m+1}^1), \tag{A.4}$$

which is associated with a filtration on $C' = C_{\Gamma(m+1)}(T_m^* \otimes E_{m+1}^1)$. We may regard the Cartan-Eilenberg spectral sequence of (6.1) as a quotient of the degree $p^i - 1$ component of (A.4).

Since C' is a left differential module over C, (A.4) is a module over (A.3). x Then we can make a similar product $\langle \alpha_1, \ldots, \alpha_j \rangle$ with $\alpha_i \in H^*(C)$ $(1 \leq i < j)$ and $\alpha_j \in H^*(C')$. In particular, we will be interested in Massey products of the form

$$\mu_k(y) = \langle \widehat{h}_{1,1}, \ldots, \widehat{h}_{1,1}, y \rangle \qquad \text{and} \qquad \mu'_k(x) = \langle x, \widehat{h}_{1,1}, \ldots, \widehat{h}_{1,1} \rangle \tag{A.5}$$

with k factors $\widehat{h}_{1,1}$. Specially, $\mu_1(y)$ is the ordinary product $\widehat{h}_{1,1}y$. For $1 < k < p$, $\mu_k(y)$ is defined only if $0 \in \mu_{k-1}(y)$. If $\mu_k(\mu_{p-k}(y))$ is defined for some k, then it contains $\widehat{b}_{1,1}y$.

Remark A.6. $\widehat{h}_{1,1} \in \mathrm{Ext}^1_{\Gamma(m+1)}(T_m^{(1)})$ is represented in the cobar complex by

$$x = -d(\widehat{t}_1^p) = \left(\widehat{t}_1 \otimes 1 + 1 \otimes \widehat{t}_1\right)^p - 1 \otimes \widehat{t}_1^p \equiv \widehat{t}_1^p \otimes 1 \qquad \mathrm{mod}\ (p),$$

which means that $\widehat{h}_{1,1}$ becomes trivial when we pass to $\mathrm{Ext}^1_{\Gamma(m+1)}(T_m^p)$. Similarly, we have

$$x \cup x = d\left(x \cup \widehat{t}_1^p\right) = d\left(\sum_{i>0} \binom{p}{i} \widehat{t}_1^i \otimes \widehat{t}_1^{2p-i}\right).$$

Thus $\widehat{h}_{1,1} \cup \widehat{h}_{1,1} \in \mathrm{Ext}^2_{\Gamma(m+1)}(T_m^{2p-2})$ maps trivially to $\mathrm{Ext}^2_{\Gamma(m+1)}(T_m^{2p-1})$.

Lemma A.7. *Let $x_1 = x$ as above and define x_i inductively on i by*

$$x_i = \left(x_{i-1} \cup \widehat{t}_1^p - \widehat{t}_1^p \cup x_{i-1}\right)/i \qquad (1 < i < p).$$

Then x_i is in $C_{\Gamma(m+1)}(T_m^{(i-1)(p-1)})$ and it satisfies

$$x_i \equiv (-1)^{i+1} \widehat{t}_1^{ip} \otimes 1/i! \quad \mathrm{mod}\ (p) \qquad \text{and} \qquad d(x_i) = \sum_{0<j<i} x_j \cup x_{i-j}.$$

Proof. We prove these statements by induction. For the first statement, assume that $x_i \in C_{\Gamma(m+1)}(T_m^{(i-1)(p-1)})$. This means that it has the form $c\widehat{t}_1^{i+p-1} \otimes \widehat{t}_1^{(i-1)(p-1)}$ modulo $C_{\Gamma(m+1)}(T_m^{(i-1)(p-1)-1})$ for some scalar c, and so we have

$$\begin{aligned} x_{i+1} &= (x_i \cup \widehat{t}_1^p - \widehat{t}_1^p \cup x_i)/(i+1) \\ &\equiv c(\widehat{t}_1^{i+p-1} \otimes \widehat{t}_1^{(i-1)(p-1)+p} - \widehat{t}_1^{i+p-1} \otimes \widehat{t}_1^{(i-1)(p-1)+p})/(i+1) \equiv 0 \end{aligned}$$

modulo $C_{\Gamma(m+1)}(T_m^{i(p-1)})$. For the congruence, we see that

$$\begin{aligned} (i+1)!x_{i+1} &= i!(x_i \cup \widehat{t}_1^p - \widehat{t}_1^p \cup x_i) \equiv (-1)^{i+1}\left((\widehat{t}_1^{ip} \otimes 1) \cup \widehat{t}_1^p - \widehat{t}_1^p \cup (\widehat{t}_1^{ip} \otimes 1)\right) \\ &= (-1)^{i+1}\left(\widehat{t}_1^{ip} \otimes \widehat{t}_1^p - \widehat{t}_1^{(i+1)p} \otimes 1 - \widehat{t}_1^{ip} \otimes \widehat{t}_1^p\right) = (-1)^{i+2}\widehat{t}_1^{(i+1)p} \otimes 1. \end{aligned}$$

For the derivation formula, we see that

$$\begin{aligned} &(i+1)d(x_{i+1}) - x_i \cup x_1 - x_1 \cup x_i \\ &\qquad = d(x_i) \cup \widehat{t}_1^p - \widehat{t}_1^p \cup d(x_i) \\ &\qquad = \left(\sum_{0<j<i} x_j \cup x_{i-j}\right) \cup \widehat{t}_1^p - \widehat{t}_1^p \cup \left(\sum_{0<j<i} x_j \cup x_{i-j}\right) \\ &\qquad = \sum_{0<j<i} \left(x_j \cup (x_{i-j} \cup \widehat{t}_1^p - \widehat{t}_1^p \cup x_{i-j}) + (x_j \cup \widehat{t}_1^p - \widehat{t}_1^p \cup x_j) \cup x_{i-j}\right) \\ &\qquad = \sum_{0<j<i} \left((i+1-j)x_j \cup x_{i+1-j} + (j+1)x_{j+1} \cup x_{i-j}\right) \\ &\qquad = (i+1) \sum_{1<j<i} x_j \cup x_{i+1-j}. \end{aligned}$$

□

We can use these elements to define the Massey products that we need. The following result follows easily from Lemma A.7.

Lemma A.8. *Suppose that $\alpha, \beta \in \operatorname{Ext}_{\Gamma(m+1)}(T_m^h \otimes E_{m+1}^2)$ are represented by cocycles a_1 and b_1, and that there are cochains*

$$a_i,\, b_i \in C_{\Gamma(m+1)}(T_m^{h+(i-1)(p-1)} \otimes E_{m+1}^2) \quad \text{for } 1 < i \leq k$$

satisfying

$$d(a_i) = \sum_{0<j<i} a_{i-j} \cup x_j \quad \text{and} \quad d(b_i) = \sum_{0<j<i} x_j \cup b_{i-j},$$

where x_j are as in Lemma A.7. Then the Massey products

$$\mu'_k(\alpha),\, \mu_k(\beta) \in \operatorname{Ext}_{\Gamma(m+1)}(T_m^{h+k(p-1)} \otimes E_{m+1}^2)$$

are defined and are represented by the cocycles

$$\sum_{0<i<k+1} a_{k+1-i} \cup x_i \quad \text{and} \quad \sum_{0<i<k+1} x_i \cup b_{k+1-i}.$$

Moreover, we have $\alpha\mu_k(\beta) = \mu'_k(\overline{\alpha})\beta$ using these representatives.

Here are two examples of such products.

Example A.9. For $0 < k < p$ and $\ell > 0$, the Massey product $\mu_k(\widehat{\beta}'_{p\ell-k+1})$ is defined and it is represented by

$$\sum_{0<i<k+1} x_i \cup (-1)^{k-i}\frac{(p\ell-k)!}{(p\ell-i)!}\widehat{\beta}'_{p\ell+1-i/k+1-i}.$$

We have an equality $v_1\mu_k(\widehat{\beta}'_{p\ell+1-k}) = \mu_{k-1}(\widehat{\beta}'_{p\ell+2-k})/(k-1-p\ell)$ for $k > 1$.

Example A.10. For $0 < k < p$ and $\ell > 0$, the Massey product $\mu_k(\widehat{\beta}_{p\ell/p+2-k})$ is defined and it is represented by

$$x_1 \cup v_2^{-1}\widehat{u}_{p\ell+k-1-p} + \sum_{1<i<k+1} x_i \cup (-1)^{i+1}\frac{(p\ell+k)!}{\ell(p\ell+k-i)!}\widehat{\beta}_{p\ell+k-i/p+2-i}.$$

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