WHAT IS AN ∞-CATEGORY?

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This is an expository paper on ∞-categories. It was originally given as a talk at the Regensburg conference of August, 2023. I thank Sadok Kallel for inviting me to publish it here and Siddharth Gurumurthy for some useful discussions.

The main references for this topic are two remarkable books by Jacob Lurie:

- *Higher Topos Theory* published in 2009 (949 pages), which we denote by [HTT].
- *Higher Algebra* last edited in 2017 (1553 pages), which we denote by [HA].

We will adhere to the following color convention:

- Ordinary categories will be written in green.
- ∞-categories (that are not ordinary categories) will be written in purple.

1. Introduction

Before defining ∞-categories (see Definition 1 below), we note some of their general features.

An ∞-category is a generalization of an ordinary category, also known as a 1-category. Like an ordinary category, it has objects and morphisms (also known as 1-morphisms), but composition of morphisms is not well defined. It also has higher structures called k-morphisms for $k > 1$, to be spelled out later. We will describe these explicitly for the ∞-category of topological spaces in Sections 5 to 8.

∞-categories provide a convenient setting for doing homotopy theory.

There is nothing easy about ∞-categories. Most concepts and results from ordinary category theory have ∞-categorical analogs, but the definitions are less obvious and the proofs are harder. For example, the definition of a symmetric monoidal ∞-category $\mathcal{C}$ requires far more than a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with the expected properties. See the discussion at the beginning of [HA, Chapter 2].

For objects W, X and Y in an ordinary category C, one has a morphism sets $C(X,Y)$, $C(W,Y)$ and $C(W,X)$, with a composition map

$$\begin{array}{ccc} C(X,Y) \times C(W,X) & \longrightarrow & C(W,Y) \\ (g,f) & \longmapsto & gf. \end{array}$$

In an ∞-category $\mathcal{C}$, these three sets are topological spaces or simplicial sets, specifically Kan complexes. Given morphisms $f : W \to X$ and $g : X \to Y$, instead of a well defined composite $gf \in \mathcal{C}(W,Y)$, we get a contractible subspace of $\mathcal{C}(W,Y)$. All morphisms in this subspace are homotopic to each other, meaning that they all lie in the same path component.

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Many definitions involve weak equivalences of morphism spaces rather than isomorphisms of morphism sets. For example, an initial object X in $\mathcal{C}$ is one for which $\mathcal{C}(X,Y)$ is contractible (rather than a one point set) for all Y.

In an ∞-category, homotopy limits and colimits are the same as ordinary limits and colimits when they exist. We will see a simple example of this in Section 9.

In an ∞-category one need not worry about a model structure, but concepts of model category theory are needed to develop the theory of ∞-categories.

An ∞-category is a certain kind of simplicial set (but not generally a Kan complex), so it is sort of like a topological space. There is a model structure on the category of simplicial sets due to Joyal in which the fibrant objects are the ∞-categories, see [HTT, Theorem 2.4.6.1]. Hence one can speak of limits of ∞-categories, and certain functors between them are Joyal fibrations, also known as *inner fibrations* [HTT, Definition 2.0.0.3].

2. Review of simplicial sets

The simplicial category Δ is that of finite ordered sets and order preserving maps. For each integer $n \geq 0$, let $[n]$ denote the ordered set $\{0, 1, \ldots, n\}$.

A *simplicial set* X is a contravariant $\mathcal{S}\mathrm{et}$ valued functor on Δ. Its value on $[k]$, its set of k-simplices, is denoted by X_k. X comes equipped with families of maps $X_k \to X_{k-1}$ (called face maps) and $X_k \to X_{k+1}$ (degeneracies), each indexed by i for $0 \leq i \leq k$. The ith such maps are induced respectively by

- the order preserving monomorphism $[k-1] \to [k]$ whose image does not contain i and
- the order preserving epimorphism $[k+1] \to [k]$ sending both i and $i+1$ to i.

A simplex is *degenerate* if it is in the image of a degeneracy map. Otherwise it is *nondegenerate*.

The simplicial set Δ^n, the *standard n-simplex*, is defined by

$$(\Delta^n)_k = \Delta([k],[n]).$$

In its *boundary* $\partial\Delta^n$, the set of k-simplices is the set of such morphisms in Δ which are not surjective.

In its *ith face*, the set of k-simplices is the set of such morphisms whose image does not contain i.

In the *ith horn* $\Lambda^n_i \subseteq \partial\Delta^n$ for $0 \leq i \leq n$, the set of k-simplices is the set of nonsurjective morphisms whose image does contain i.

The *inner faces and horns* are those for which $0 < i < n$. The other two are *outer*

Here are the three horns of a 2-simplex. Only the middle one is inner.

$$\begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \\ & \Lambda^2_0 & \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \\ & \Lambda^2_1 & \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[rr] & & 2 \\ & \Lambda^2_2 & \end{tikzcd} \tag{1}$$

In the ith horn, the missing face is the one opposite the ith vertex.

A *Kan complex* is a simplicial set X for which every map from a horn $\Lambda^n_i \to X$ extends to Δ^n.

The *topological n-simplex* Δ^n_{top} is the space

$$\left\{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0 \text{ and } \sum x_i = 1\right\}.$$

The *geometric realization* $|X|$ of a simplicial set X is the colimit of the Top-valued functor

$$[k] \mapsto X_k \times \Delta^k_{\text{top}}.$$

This space turns out to be the union of geometric realizations of the nondegenerate topological simplices of X, meaning ones not in the image of any degeneracy map. The data given by the face maps determine how they are glued together. In particular, $|\Delta^n| = \Delta^n_{\text{top}} \approx D^n$, $|\partial\Delta^n| \approx S^{n-1}$, and $|\Lambda^n_i| \approx D^{n-1}$.

Given simplicial sets X and Y, one can define a simplicial set $X \times Y$ in which

$$(X \times Y)_n = \coprod_{0 \leq i \leq n} X_i \times Y_{n-i} \quad \text{and} \quad |X \times Y| = |X| \times |Y|.$$

The category of simplicial sets is denoted by $\mathcal{S}\mathrm{et}_\Delta$.

A simplicial map $X \to Y$ is a natural transformation of contravariant functors on Δ. The set of such maps is $\mathcal{S}\mathrm{et}_\Delta(X, Y)$. This can be thickened up to a simplicial set $\underline{\mathcal{S}\mathrm{et}_\Delta(X, Y)}$ in which the set of k-simplices is $\mathcal{S}\mathrm{et}_\Delta(X \times \Delta^k, Y)$.

Hence $\mathcal{S}\mathrm{et}_\Delta$ is enriched over itself.

3. Of all the nerve!

The nerve NC of a small category C is the simplicial set in which the set of n-simplices NC_n is the set of diagrams

$$X_0 \to X_1 \to \cdots \to X_n$$

in C. Face and degeneracy maps are defined by composing adjacent morphisms and inserting identity maps. Equivalently we can regard $[n]$ as the category

$$0 \to 1 \to \cdots \to n$$

and define NC_n to be the set of functors from $[n]$ to C.

This simplicial set has the following property: *Any simplicial map $\Lambda^n_i \to NC$ for $0 < i < n$ extends uniquely to Δ^n.*

The following is an illustration for $n = 2$. The X_i are objects in C. The three diagrams with the dotted arrows removed indicate C-valued functors from the three diagrams of (1), that is maps from the three horns of a 2-simplex to NC. Extending these maps to all of Δ^2 means identifying the dotted arrow. There is a unique way to do this for the inner horn Λ^2_1, but there may or may not be such an arrow for the two outer horns.

$$\begin{tikzcd} & X_1 \arrow[dr, "g"] & \\ X_0 \arrow[ur, dotted, "?"] \arrow[rr, "h"'] & & X_2 \end{tikzcd} \qquad \begin{tikzcd} & X_1 \arrow[dr, "g"] & \\ X_0 \arrow[ur, "f"] \arrow[rr, dotted, "gf"'] & & X_2 \end{tikzcd} \qquad \begin{tikzcd} & X_1 \arrow[dr, dotted, "?"] & \\ X_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & X_2 \end{tikzcd}$$

$$\Lambda^2_0 \qquad\qquad \Lambda^2_1 \qquad\qquad \Lambda^2_2$$

It is known that the category C is determined by its nerve, and that any simplicial set with the property above is the nerve of some small category.

A small category is thus defined by a simplicial set (its nerve) in which each map from an inner horn Λ_i^n extends uniquely to a map from Δ^n. *An ∞-category is defined to be a simplicial set in which this uniqueness condition is dropped.*

4. The main definition

Definition 1. *An* ∞-category *(also called a quasicategory) $\mathcal{C}$ is a simplicial set for which each simplicial map $\Lambda_i^n \to \mathcal{C}$ for $0 < i < n$ extends to a map $\Delta^n \to \mathcal{C}$. A functor $F : \mathcal{C} \to \mathcal{C}'$ from one ∞-category to another is a simplicial map.*

There are some other equivalent definitions of an ∞-category in the literature, but this is the one used by Lurie. There are several features of it worth noting.

- We are not requiring extensions of maps from Λ_0^n and Λ_n^n (known as the left and right outer horns) as in the definition of a Kan complex. Boardman and Vogt [BV73, Definition 4.8] called this *the restricted Kan condition*.
- The extension of each map from an inner horn is not required to be unique, as it is in the nerve of an ordinary category. This means that this notion is more general than that of an ordinary category as seen through its nerve. Hence an ordinary category is a special case of an ∞-category.
- Given such a simplicial set $\mathcal{C}$, we can think of elements of the sets $\mathcal{C}_0$ and $\mathcal{C}_1$ as objects and morphisms. The two face maps $\mathcal{C}_1 \rightrightarrows \mathcal{C}_0$ define the source and target (aka domain and codomain) of each morphism. Elements in the sets $\mathcal{C}_k$ for $k > 1$ can be thought of as *higher morphisms* in $\mathcal{C}$.
- A diagram

$$\begin{tikzcd} & X_1 \arrow[dr, "f_{1,2}"] & \\ X_0 \arrow[ur, "f_{0,1}"] \arrow[rr, dashed, "f_{0,2}"'] & & X_2 \end{tikzcd}$$

  without the dashed arrow is equivalent to a map $\Lambda_1^2 \to \mathcal{C}$. Choosing a dashed arrow (in which the diagram is *not* required to commute) is equivalent to extending this map to $\partial\Delta^2$. Choosing a homotopy between $f_{1,2}f_{0,1}$ and $f_{0,2}$ is equivalent to extending this map to all of Δ^2. Such an extension is guaranteed to exist, but it is not unique. In the nerve of an ordinary category this extension is unique and identifies the composite $f_{1,2}f_{0,1}$. In an ∞-category this extension is only unique up to homotopy, so *composition of morphisms in an ∞-category is not well defined.*
- The simplicial set $\mathcal{S}\mathrm{et}_\Delta(K, \mathcal{D})$ of simplicial maps from a simplicial set K to an ∞-category $\mathcal{D}$ is itself an ∞-category.
- K above could be an ∞-category $\mathcal{C}$, in particular it could be NC for an ordinary category C. In other words, the collection of functors $\mathcal{C} \to \mathcal{D}$ is an ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

To a topological space X we can associate an ∞-category X (also known as $\mathrm{Sing}\, X$, the singular simplicial set of X) in which X_n is the set of continuous maps $|\Delta^n| \to X$. X is also a Kan complex since a map $|\Lambda_i^n| \to X$, for any horn Λ_i^n, extends to $|\Delta^n|$ using a retraction $|\Delta^n| \to |\Lambda_i^n|$.

Such an ∞-category is called an ∞-*groupoid* because all morphisms, i.e., paths in X, are invertible up to homotopy.

5. The ∞-category of topological spaces

Let Top denote the category of compactly generated weak Hausdorff spaces with cardinality less than κ, where κ is a sufficiently large regular cardinal. This version of the category of topological spaces is small, so we can consider its nerve.

There is another construction called the *homotopy coherent nerve* whose definition [HTT, Definition 1.1.5.5] baffled me for several years. Rather than giving it here, I will describe the ∞-category $\mathcal{S}$ (Lurie's notation of [HTT, Definition 1.2.16.1]) one gets by applying it to Top. This is the ∞-category of topological spaces. *We will see that it comes equipped with a large collection of higher morphisms not present in the ordinary category of topological spaces.*

Lurie's $\mathcal{S}$ is actually the homotopy coherent nerve of the category $\mathcal{K}$an of Kan complexes, which is equivalent to the category of CW-complexes. The distinction between CW-complexes and more general spaces does not matter in what follows.

As in Definition 1, $\mathcal{S}$ is a simplicial set. Its vertices and edges are objects and morphisms in Top, meaning spaces and continuous maps.

The set of 2-simplices is more interesting. In the subcategory NTop (the ordinary nerve), it is the set of commutative diagrams of the form

$$\begin{tikzcd} & X_1 \arrow[dr, "f_{1,2}"] & \\ X_0 \arrow[ur, "f_{0,1}"] \arrow[rr, "f_{1,2}f_{0,1}"'] & & X_2. \end{tikzcd}$$

The top two edges can be viewed as a map $\Lambda^1_2 \to N\text{Top}$, with the full diagram being its unique extension to Δ^2.

The set of 2-simplices $\mathcal{S}_2$ consists of similar diagrams in which the bottom arrow is replaced by any map $f_{0,2}$ homotopic to $f_{1,2}f_{0,1}$, with the homotopy $h_{0,2}$ being part of the datum. Thus we have a diagram

$$\begin{tikzcd} & X_1 \arrow[dr, "f_{1,2}"] \arrow[d, phantom, "\Downarrow h_{0,2}"] & \\ X_0 \arrow[ur, "f_{0,1}"] \arrow[rr, "f_{0,2}"'] & {} & X_2. \end{tikzcd} \tag{2}$$

The homotopy is a map

$$I \times X_0 \xrightarrow{h_{0,2}} X_2$$

with certain properties. It is adjoint to a path

$$\begin{array}{ccc} I & \xrightarrow{\widehat{h}_{0,2}} & \text{Top}(X_0, X_2) \\ 0 & \longmapsto & f_{1,2}f_{0,1} \\ 1 & \longmapsto & f_{0,2} \end{array}$$

Here $\text{Top}(X_0, X_2)$, the set of continuous maps from X_0 to X_2, is given the compact-open topology and the map $\widehat{h}_{0,2}$ is required to be continuous.

As in the ordinary case, the top two edges of the diagram (2) can be viewed as a map $\Lambda^2_1 \to \mathcal{S}$. Now there is an extension of it to Δ^2 for each path $\widehat{h}_{0,2}$ in

$\mathrm{Top}(X_0, X_2)$ starting at the point $f_{1,2}f_{0,1}$. *The space of such paths is contractible*, as is the space of paths starting at a given point in any topological space.

6. The set of 3-simplices in $\mathcal{S}$

The following diagram shows four 2-simplices with their homotopies.

$$
\begin{tikzcd}
& & X_0 \arrow[dl, "f_{0,1}"'] \arrow[dr, "f_{0,3}"] \arrow[d, phantom, "\overset{h^2_{0,3}}{\Rightarrow}"] & & \\
& X_1 \arrow[rr, "f_{1,3}"] \arrow[dr, "f_{1,2}"] \arrow[d, phantom, "h_{0,2}\Downarrow"] & {} & X_3 \arrow[dl, phantom, "\Uparrow h_{1,3}"] \arrow[d, phantom, "\overset{h^1_{0,3}}{\Rightarrow}"] & \\
X_0 \arrow[ur, "f_{0,1}"] \arrow[rr, "f_{0,2}"'] & {} & X_2 \arrow[ur, "f_{2,3}"'] & {} & X_0 \arrow[ul, "f_{0,3}"'] \arrow[ll, "f_{0,2}"]
\end{tikzcd}
\tag{3}
$$

Our convention for labeling homotopies is as follows. The subscripts correspond to the first and third vertices of the triangle while the super script corresponds to the second one. The later is omitted when it is uniquely determined by the subscripts.

These four 2-simplices form the boundary of a 3-simplex in $\mathcal{S}$ iff there is a certain double homotopy adjoint to a map $\widehat{h}_{0,3} : I^2 \to \mathrm{Top}(X_0, X_3)$ of the following form.

$$
\begin{tikzcd}
f_{2,3}f_{1,2}f_{0,1} \arrow[r, "\mathrm{Top}(X_0, f_{2,3})\widehat{h}_{0,2}"] \arrow[d, "\mathrm{Top}(f_{0,1}, X_3)\widehat{h}_{1,3}"'] & f_{2,3}f_{0,2} \arrow[d, "\widehat{h}^1_{0,3}"] \\
f_{1,3}f_{0,1} \arrow[r, "\widehat{h}^2_{0,3}"'] & f_{0,3}
\end{tikzcd}
\tag{4}
$$

This is a picture rather than a diagram. Each vertex of the square is not an object but a point in $\mathrm{Top}(X_0, X_3)$, while the upper and left edges are not morphisms but the indicated paths. The other edges are paths adjoint to the homotopies shown in (3).

A comment is in order about the maps

$$\mathrm{Top}(X_0, f_{2,3}) : \mathrm{Top}(X_0, X_2) \to \mathrm{Top}(X_0, X_3)$$

and

$$\mathrm{Top}(f_{0,1}, X_3) : \mathrm{Top}(X_1, X_3) \to \mathrm{Top}(X_0, X_3)$$

appearing in (4).

For a category $\mathcal{C}$ with an object X and a morphism $f : Y \to Y'$, we can compose any morphism $X \to Y$ with f to get a morphism $X \to Y'$. This means that we can use X to define a $\mathcal{S}\mathrm{et}$-valued functor on $\mathcal{C}$,

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathcal{C}(X,-)"] & \mathcal{S}\mathrm{et} \\
Y \arrow[r, mapsto] & \mathcal{C}(X, Y) \\
(f : Y \to Y') \arrow[r, mapsto] & (f_* : \mathcal{C}(X, Y) \to \mathcal{C}(X, Y'))\,.
\end{tikzcd}
$$

Dually we can precompose any morphism $W \to X$ with $g : W' \to W$ to get a morphism $W' \to X$. This leads to a contravariant functor,

$$\begin{tikzcd} C \arrow[r, "C(-,X)"] & \mathcal{S}\mathrm{et} \\ W \arrow[r, mapsto] & C(W,X) \\ (g : W' \to W) \arrow[r, mapsto] & (g^* : C(W,X) \to C(W',X)) . \end{tikzcd}$$

If the morphism sets of C come equipped with natural topologies, then these functors are Top-valued. Thus in the case $C = \mathrm{Top}$, they are *endofunctors*.

7. The set of 4-simplices in $\mathcal{S}$

For each 4-simplex, the additional datum is a map $\widehat{h}_{0,4} : I^3 \to \mathrm{Top}(X_0, X_4)$ of the form

$$\begin{tikzcd} f_{3,4}f_{2,3}f_{1,2}f_{0,1} \arrow[rr, "F_0(f_{3,4}f_{2,3})\widehat{h}_{0,2}"] \arrow[dr, "F_0(f_{3,4})F^3(f_{0,1})\widehat{h}_{1,3}"] \arrow[dd, "F^4(f_{1,2}f_{0,1})\widehat{h}_{2,4}"] \arrow[drrr, phantom, "F_0(f_{3,4})\widehat{h}_{0,3}"] \arrow[dddr, phantom, "F^4(f_{0,1})\widehat{h}_{1,4}"] & & f_{3,4}f_{2,3}f_{0,2} \arrow[dr, "F_0(f_{3,4})\widehat{h}^1_{0,3}"] \arrow[dd, "F^4(f_{0,2})\widehat{h}_{2,4}"] \arrow[dddr, phantom, "\widehat{h}^1_{0,4}"] & \\ & f_{3,4}f_{1,3}f_{0,1} \arrow[rr, "F_0(f_{3,4})\widehat{h}^2_{0,3}"] \arrow[dd, "F^4(f_{0,1})\widehat{h}^2_{1,4}"] \arrow[ddrr, phantom, "\widehat{h}^2_{0,4}"] & & f_{3,4}f_{0,3} \arrow[dd, "\widehat{h}^{1,2}_{0,4}"] \\ f_{2,4}f_{1,2}f_{0,1} \arrow[rr, "F_0(f_{2,4})\widehat{h}_{0,2}"] \arrow[dr, "F^4(f_{0,1})\widehat{h}^3_{1,4}"] \arrow[drrr, phantom, "\widehat{h}^3_{0,4}"] & & f_{2,4}f_{0,2} \arrow[dr, "\widehat{h}^{1,3}_{0,4}"] & \\ & f_{1,4}f_{0,1} \arrow[rr, "\widehat{h}^{2,3}_{0,4}"] & & f_{0,4} \end{tikzcd}$$

where F_i and F^i denote the endofunctors $\mathrm{Top}(X_i, -)$ and $\mathrm{Top}(-, X_i)$.

The restriction of $\widehat{h}_{0,4}$ to the left and top faces are the composite double homotopies indicated in green. The restrictions to the three faces abuting $f_{0,4}$ (the front lower right corner) are adjoint to the double homotopies $h^i_{0,4}$ indicated in blue.

The restriction of $\widehat{h}_{0,4}$ to the back face (not labeled) is the composite

$$\begin{tikzcd} I \times I \arrow[r, "\widehat{h}_{2,4} \times \widehat{h}_{0,2}"] & \mathrm{Top}(X_2, X_4) \times \mathrm{Top}(X_0, X_2) \arrow[d, "\mathrm{comp}"] \\ & \mathrm{Top}(X_0, X_4). \end{tikzcd}$$

The five labeled faces of the cube are associated with the five 3-dimensional faces of the corresponding 4-simplex in $\mathcal{S}$. These five tetrahedra fit together in a 3-dimensional analog of (3), with the central tetrahedron corresponding to the front face of the cube, on which the map restricts to $\widehat{h}^2_{0,4}$.

8. The set $\mathcal{S}_{n+1}$ for $n > 3$

For each $(n+1)$-simplex there is a sequence of spaces and continuous maps

$$X_0 \xrightarrow{f_{0,1}} X_1 \xrightarrow{f_{1,2}} \cdots \xrightarrow{f_{n,n+1}} X_{n+1}$$

and a map

$$\begin{aligned} I^n &\xrightarrow{\widehat{h}_{0,n}} \mathbf{Top}(X_0, X_{n+1}) \\ (0,\ldots,0) &\longmapsto f_{n,n+1}\cdots f_{0,1} \\ (1,\ldots,1) &\longmapsto f_{0,n+1} \end{aligned}$$

We refer to these two points as the left and right vertices of the n-cube, and the n faces meeting each of them as the left and right faces.

The $n+2$ faces of the associated $(n+1)$-simplex correspond to the n right faces of this cube, along with the two left faces

$$\{(t_1,\ldots,t_{n-1},0)\} \qquad \text{and} \qquad \{(0,t_2,\ldots,t_n)\}\,.$$

To sum up, the ∞-category $\mathcal{S}$ of topological spaces is a simplicial set in which

- there is a vertex for each topological space in $\mathbf{Top}$,
- there is an edge for each continuous map, and
- for $n > 0$, there is an $(n+1)$-simplex for each sequence of spaces and continuous maps

  $$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} X_{n+1}$$

  and each map $\widehat{h}_{0,n} : I^n \to \mathbf{Top}(X_0, X_{n+1})$ meeting certain boundary conditions described above.

To repeat, there is an $(n+1)$-simplex *for every suitable datum. This construction does not involve any choices.*

9. A colimit in $\mathcal{S}$

A pleasant feature of ∞-categories is the fact that limits and colimits are the same as homotopy limits and colimits. The "connective tissue" needed to pass from an ordinary colimit to a homotopy colimit is "built into" an ∞-category.

We will illustrate this with an elementary example taken from the highly recommended paper of Dwyer and Spalinski [DS95], a very friendly introduction to model categories. Consider the following pushout diagrams in $\mathbf{Top}$.

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \\ D^n & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} S^{n-1} & \longrightarrow & * \\ \downarrow & & \\ *, & & \end{array} \tag{5}$$

where the maps in the left diagram are each the inclusion of the boundary of the n-dimensional disk. The two diagrams are homotopy equivalent but have distinct pushouts, namely S^n and $*$. What to do?

One solution is to define a model structure on the category of pushout diagrams in $\mathbf{Top}$, in which equivalences and fibrations are levelwise equivalences and fibrations, and cofibrations are defined in terms of lifting properties. This is described in [DS95]. It turns out that the left diagram in (5) is cofibrant, but the right one is not. The evident map from the left to the right is a cofibrant approximation. The colimit functor on such diagrams is homotopy invariant on cofibrant objects *but not in general.*

Another solution is to develop the theory of homotopy limits and colimits as Bousfield and Kan did in the "yellow monster" [BK72]. It turns out that the homotopy colimit of each diagram in (5) is S^n.

In an ordinary category C, the colimit of a diagram p is an initial object in the category of objects equipped with compatible maps from all the objects in p, which we denote by $C_{p/}$, *the category of objects under* p. If p is a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f'"'] & B \\ B', \end{tikzcd}$$

then an object in $C_{p/}$ is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f'"'] & B \arrow[d] \\ B' \arrow[r] & X. \end{tikzcd}$$

Now suppose we have an ∞-category $\mathcal{C}$ and a simplicial map $\widetilde{p} : K \to \mathcal{C}$ for a simplicial set K. Then we can define $\mathcal{C}_{\widetilde{p}/}$, *the* ∞*-category of objects under* $\widetilde{p}$, and we can look for an initial object in it.

In the case at hand, K is the nerve of the pushout category

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet \end{tikzcd}$$

Let p be the diagram on the right of (5), and choose a map $\widetilde{p} : K \to \mathcal{S}$ that does the right thing on the three vertices and two nondegenerate edges of K. *There are many such maps, and any one of them wi (ll do.* (To see that such maps exist, note that $|K|$ is contractible.) An object in $\mathcal{S}_{/\widetilde{p}}$ leads a diagram of the form

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] \arrow[dr, "f"] & * \arrow[d] \arrow[dl, phantom, "h_1 \swarrow"] \\ * \arrow[r] \arrow[ur, phantom, "h_2 \nearrow"] & Y, \end{tikzcd}$$

(compare with (2)) for some space Y. This is a pair of 2-simplices in $\mathcal{S}$ sharing a common edge. It amounts to a map $f : S^{n-1} \to Y$ *equipped with a pair of null homotopies* h_1 and h_2 that are determined by the choice of $\widetilde{p}$. These define extensions of f to the northern and southern hemispheres of S^n, meaning the diagram has the same information as a map $S^n \to Y$. It follows that S^n, which is the homotopy colimit of p in Top, *is the ordinary colimit of* $\widetilde{p}$ *(for any choice of* $\widetilde{p}$*!) in* $\mathcal{S}$.

More details can be found in [HTT, 4.2.4].

10. Bousfield localization in ∞-categories

Bousfield localization may be the best construction in model category theory. One starts with a model category $\mathcal{M}$, and tries to alter the model structure in the following way. We enlarge the class of weak equivalences in some way without altering the class of cofibrations. This means there are more trivial cofibrations (cofibrations which are also weak equivalences) and hence fewer fibrations, since they must have the right lifting property with respect to all trivial cofibrations. However there are just as many trivial fibrations as before, since they must have the right lifting property with respect to all cofibrations. See [HTT, A.3.7].

The hard part of this is verifying that the proposed new model structure (with more weak equivalences but fewer fibrations) satisfies the factorization axiom saying that each map can be factored as a trivial cofibration followed by a fibration. There is a theorem saying this can be done under mild hypotheses on $\mathcal{M}$, but no assumptions are needed about how we enlarge the class of weak equivalences. Thus we get a new model structure *with a much more interesting fibrant replacement functor L.*

For example, when we enlarge the class of weak equivalences in the category of spaces or spectra to those maps inducing an isomorphism of homotopy groups in dimensions up to a chosen integer m, but not necessarily in higher dimensions, *the resulting fibrant replacement functor is the mth Postnikov section.* This means attaching cells in dimensions above $m+1$ so as to kill of all the higher homotopy groups. The fibrant objects are those spaces or spectra with trivial homotopy groups above dimension m.

When we enlarge the class of weak equivalences in the category of spaces or spectra to all maps inducing an isomorphism in the nth Morava E-theory (or the nth Morava K-theory) for a fixed prime p and height n, *the resulting fibrant replacement functor is the L_n (or $L_{K(n)}$) of chromatic homotopy theory.* In this case there is no easy description of the fibrant objects.

[HTT, Proposition 5.5.4.15] is statement about an analog of Bousfield localization. The input is a presentable ∞-category $\mathcal{C}$ with a set of morphisms S that are meant to be made into weak equivalences. *Presentable* means that $\mathcal{C}$ has small colimits and every object is a colimit of small objects. An object is *small* if the mapping space from it to each filtered colimit is equivalent to the colimit of the mapping spaces.

In [HTT, Definition 5.5.4.1] an object Z is said to be *S-local* if each morphism $s : X \to Y$ in S induces a weak equivalence $\mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$. A morphism $s : A \to B$ is an *S-equivalence* if it induces a weak equivalence $\mathcal{C}(B,Z) \to \mathcal{C}(A,Z)$ for each S-local object Z.

Let $\overline{S}$ be the set of all S-equivalences. It can be explicitly constructed from S. Let $\mathcal{C}'$ be the full subcategory of S-local objects. Then

(i) For each object $X \in \mathcal{C}$, there exists an S-equivalence $s : X \to X'$ where X' is S-local.
(ii) The ∞-category $\mathcal{C}'$ is presentable.
(iii) The inclusion functor $\mathcal{C}' \to \mathcal{C}$ has a left adjoint L. This is the analog of Bousfield's fibrant replacement functor in model category theory.

11. The ∞-category of spectra

The passage from $\mathcal{S}$, the ∞-category of spaces, to Sp, the ∞-category of spectra, is described by Lurie in [HA, 1.4]. We need to do the following.

- Pass to $\mathcal{S}_*$, the ∞-category of pointed spaces. This is straightforward. $\mathcal{S}_*$ is the homotopy coherent nerve of the ordinary category of pointed spaces (or Kan complexes). An ∞-category $\mathcal{C}$ is pointed if it has a zero object 0 which is both initial and final, meaning that the spaces $\mathcal{C}(X,0)$ and $\mathcal{C}(0,Y)$ are contractible in all cases. This object need not be unique.
- $\mathcal{S}_*$ has a loop functor Ω, leading to a tower
$$\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*$$
of ∞-categories and functors.
- Then Sp is the homotopy limit of this tower, which is the same as the limit in the ∞-category of ∞-categories.

To unpack this definition, note that a vertex in this homotopy limit (meaning an object in the ∞-category Sp) consists of a sequence of vertices (i.e., pointed spaces) $X_0, X_1, X_2, \ldots$, along with weak equivalences $X_i \to \Omega X_{i+1}$ in $\mathcal{S}_*$. *This coincides with the original definition of an Ω-spectrum.*

The ∞-category Sp satisfies the following, which is [HA, Definition 1.1.1.9].

Definition 2. *An ∞-category $\mathcal{C}$ is* stable *if*

(1) It is pointed.

(2) For each morphism $f : X \to Y$ there are pullback and pushout diagrams
$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0' \arrow[r] & Z, \end{tikzcd}$$
the fiber *and* cofiber *sequences of f.*

(3) A diagram of the above form is a pushout if and only if it is a pullback, i.e., fiber sequences and cofiber sequences are the same.

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