

Drawing by Carolyn Snaith, 1981

THE ARF-KERVAIRE INVARIANT PROBLEM IN ALGEBRAIC TOPOLOGY

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ABSTRACT. This paper gives the history and background of one of the oldest problems in algebraic topology, along with an outline of our solution to it. A rigorous account can be found in our preprint [HHR]. The third author has a website with numerous links to related papers and talks we have given on the subject since announcing our result in April, 2009.

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Main Theorem. The Arf-Kervaire elements $\theta_j \in \pi_{2^{j+1}-2}^S$ do not exist for $j \ge 7$.

Here π_k^S denotes the *k*th stable homotopy group of spheres, which will be defined shortly.

The kth (for a positive integer k) homotopy group of the topological space X, denoted by $\pi_k(X)$, is the set of continuous maps to X from the k-sphere S^k , up to continuous deformation. For technical reasons we require that each map send a specified point in S^k (called a *base point*) to a specified point $x_0 \in X$. When X is path connected the choice of these two points is irrelevant, so it is usually omitted from the notation. When X is not path connected, we get different collections of maps depending on the path connected component of the base point.

This set has a natural group structure, which is abelian for k > 1. The word *natural* here means that a continuous base point preserving map $f : X \to Y$ induces a homomorphism $f_* : \pi_k(X) \to \pi_k(Y)$, sometimes denoted by $\pi_k(f)$.

It is known that the group $\pi_{n+k}(S^n)$ is independent of n for n > k. There is a homomorphism $E : \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ defined as follows. $S^{n+1}[S^{n+k+1}]$ can be

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obtained from $S^n [S^{n+k}]$ by a double cone construction known as *suspension*. The cone over S^n is an (n+1)-dimensional ball, and gluing two such balls together along there common boundary gives and (n+1)-dimensional sphere. A map $f : S^{n+k} \to S^n$ can be canonically extended (by suspending both its source and target) to a map $Ef : S^{n+k+1} \to S^{n+1}$, and this leads to the suspension homomorphism E. The Freudenthal Suspension Theorem [Fre38], proved in 1938, says that it is onto for k = n and an isomorphism for n > k. For this reason the group $\pi_{n+k}(S^n)$ is said to be *stable* when n > k, and it is denoted by π_k^S and called the *stable k-stem*.

The Main Theorem above concerns the case $k = 2^{j+1} - 2$. The θ_j in the theorem is a hypothetical element related a geometric invariant of certain manifolds studied originally by Pontryagin starting in the 1930s, [Pon38], [Pon50] and [Pon55]. The problem came into its present form with a theorem of Browder [Bro69] published in 1969. There were several unsuccessful attempts to solve it in the 1970s. They were all aimed at proving the opposite of what we have proved, namely that ll of the θ_j exist.

The θ_j in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial. Browder's theorem says that such things can exist only in dimensions that are 2 less than a power of 2.

Some homotopy theorists, most notably Mahowald, speculated about what would happen if θ_j existed for all j. They derived numerous consequences about homotopy groups of spheres. The possible nonexistence of the θ_j for large j was known as the DOOMSDAY HYPOTHESIS.

After 1980, the problem faded into the background because it was thought to be too hard. In 2009, just a few weeks before we announced our theorm, Snaith published a book [Sna09] on the problem "to stem the tide of oblivion." On the difficulty of the problem, he wrote

In the light of ... the failure over fifty years to construct framed manifolds of Arf-Kervaire invariant one this might turn out to be a book about things which do not exist. This [is] why the quotations which preface each chapter contain a preponderance of utterances from the pen of Lewis Carroll.

Our proof is two giant steps away from anything that was attempted in the 70s. We now know that the world of homotopy theory is very different from what they had envisioned then.

1. BACKGROUND AND HISTORY

1.1. **Pontryagin's early work on homotopy groups of spheres.** The Arf-Kervaire invariant problem has its origins in Pontryagin's early work on a geometric approach to the homotopy groups of spheres, [Pon38], [Pon50] and [Pon55].

Pontryagin's approach to maps $f: S^{n+k} \to S^n$ is to assume that f is smooth and that the base point y_0 of the target is a regular value. (Any continuous f can be continuously deformed to a map with this property.) This means that $f^{-1}(y_0)$ is a closed smooth kmanifold M in S^{n+k} . Let D^n be the closure of an open ball around y_0 . If it is sufficiently small, then $V^{n+k} = f^{-1}(D^n) \subset S^{n+k}$ is an (n+k)-manifold homeomorphic to $M \times D^n$ with boundary homeomorphic to $M \times S^{n-1}$. It is also a tubular neighborhood of M^k and comes equipped with a map $p: V^{n+k} \to M^k$ sending each point to the nearest point in M. For each $x \in M$, $p^{-1}(x)$ is homeomorphic to a closed n-ball B^n . The pair $(p, f|V^{n+k})$ defines an explicit homeomorphism

$$V^{n+k} \xrightarrow{(p,f|V^{n+k})} \approx M^k \times D^n.$$

This structure on M^k is called a *framing*, and M is said to be *framed in* \mathbb{R}^{n+k} . A choice of basis of the tangent space at $y_0 \in S^n$ pulls back to a set of linearly independent normal vector fields on $M \subset \mathbb{R}^{n+k}$. These will be indicated in Figures ??-?? and ?? below.

Conversely, suppose we have a closed sub-k-manifold $M \subset \mathbb{R}^{n+k}$ with a closed tubular neighborhood V and a homeomorphism h to $M \times D^n$ as above. This is called a *framed* sub-k-manifold of \mathbb{R}^{n+k} . Some remarks are in order here.

- The existence of a framing puts some restrictions on the topology of M. All of its characteristic classes must vanish. In particular it must be orientable.
- A framing can be twisted by a map g : M → SO(n), where SO(n) denotes the group of orthogonal n × n matrices with determinant 1. Such matrices act on Dⁿ in an obvious way. The twisted framing is the composite

$$V \xrightarrow{h} M^k \times D^n \xrightarrow{} M^k \times D^n$$
$$(m, x) \longmapsto (m, g(m)(x))$$

We will say more about this later.

• If we drop the assumption that M is framed, then the tubular neighborhood V is a (possibly nontrivial) disk bundle over M. The map $M \rightarrow y_0$ needs to be replaced by a map to the classifying space for such bundles, BO(n). This leads to unoriented bordism theory, which was analyzed by Thom in [Tho54]. Two helpful references for this material are the books by Milnor-Stasheff[MS74] and Stong[Sto68].

Pontryagin constructs a map $P(M,h) : S^{n+k} \to S^n$ as follows. We regard S^{n+k} as the one point compactification of \mathbb{R}^{n+k} and S^n as the quotient $D^n/\partial D^n$. This leads to a diagram

The map P(M,h) is the extension of p_2h obtained by sending the compliment of V in S^{n+k} to the point at infinity in S^n . For n > k, the choice of the embedding (but not the choice of framing) of M into the Euclidean space is irrelevant. Any two embeddings (with suitably chosen framings) lead to the same map P(M,h) up to continuous deformation.

To proceed further, we need to be more precise about what we mean by continuous deformation. Two maps $f_1, f_2 : X \to Y$ are *homotopic* if there is a continuous map $h : X \times [0, 1] \to Y$ (called a *homotopy between* f_1 and f_2) such that

$$h(x,0) = f_1(x)$$
 and $h(x,1) = f_2(x)$.

Now suppose $X = S^{n+k}$, $Y = S^n$, and the map h (and hence f_1 and f_2) is smooth with y_0 as a regular value. Then $h^{-1}(y_0)$ is a framed (k+1)-manifold N whose boundary is the disjoint union of $M_1 = f^{-1}(y_0)$ and $M_2 = g^{-1}(y_0)$. This N is called a *framed cobordism* between M_1 and M_2 , and when it exists the two closed manifolds are said to be *framed cobordant*.

Sidebar 1 The Hopf-Whitehead *J*-homomorphism

Suppose our framed manifold is S^k with a framing that extends to a D^{k+1} . This will lead to the trivial element in $\pi_{n+k}(S^n)$, but twisting the framing can lead to nontrivial elements. The twist is determined up to homotopy by an element in $\pi_k(SO(n))$. Pontryagin's construction thus leads to the homomorphism

$$\pi_k(SO(n)) \xrightarrow{J} \pi_{n+k}(S^n)$$

introduced by Hopf [Hop35] and Whitehead [Whi42]. Both source and target known to be independent of n for n > k + 1. In this case the source group for each k (denoted simply by $\pi_k(SO)$ since n is irrelevant) was determined by Bott [Bot59] in his remarkable periodicity theorem. He showed

$$\pi_k(SO) = \begin{cases} \mathbf{Z} & \text{for } k \equiv 3 \text{ or } 7 \text{ mod } 8\\ \mathbf{Z}/2 & \text{for } k \equiv 0 \text{ or } 1 \text{ mod } 8\\ 0 & \text{otherwise.} \end{cases}$$

Here is a table showing these groups for $k \leq 10$.

k	1	2	3	4	5	6	7	8	9	10
$\pi_k(SO)$	$\mathbf{Z}/2$	0	Ζ	0	0	0	Ζ	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0

In each case where the group is nontrivial, its the image under J of its generator is known to generate a direct summand. In the *j*th case we denote this image by β_j and its dimension by $\phi(j)$, which is roughly 2j. The first three of these are the Hopf maps $\eta \in \pi_1^S$, $\nu \in \pi_3^S$ and $\sigma \in \pi_7^S$. After that we have $\beta_4 \in \pi_8^S$, $\beta_5 \in \pi_9^S$, $\beta_6 \in \pi_{11}^S$ and so on.

For the case $\pi_{4m-1}(SO) = \mathbf{Z}$, the image under J is known to be a cyclic group whose order a_m is the denominator of $B_m/4m$, where B_m is the *m*th Bernoulli number. Details can be found in [Ada66] and [MS74]. Here is a table showing these values for $m \leq 10$.

m	1	2	3	4	5	6	7	8	9	10
a_m	24	240	504	480	264	65,520	24	16,320	28,728	13,200

Let $\Omega_{k,n}^{\text{fr}}$ denote the cobordism group of framed k-manifolds in \mathbb{R}^{n+k} . The above construction leads to Pontryagin's isomorphism

$$\Omega_{k,n}^{\mathrm{fr}} \xrightarrow{\approx} \pi_{n+k}(S^n).$$

First consider the case k = 0. Here the 0-dimensional manifold M is a finite set of points in \mathbb{R}^n . Each comes with a framing which can be obtained from a standard one by an element in the orthogonal group O(n). We attach a sign to each point corresponding to the sign of the associated determinant. With these signs we can count the points algebraically and get an integer called the *degree of f*. Two framed 0-manifolds are cobordant iff they have the same degree.

Now consider the case k = 1. M is a closed 1-manifold, i.e., a disjoint union of circles. Two framings on a single circle differ by a map from S^1 to the group SO(n), and it is known that

$$\pi_1(SO(n)) = \begin{cases} 0 & \text{for } n = 1 \\ \mathbf{Z} & \text{for } n = 2 \\ \mathbf{Z}/2 & \text{for } n > 2. \end{cases}$$

Figure ?? illustrates the two different framings on S^1 for n = 2. It turns about that any disjoint union of framed circles is cobordant to a single framed circle. This can be used to show that

$$\pi_{n+1}(S^n) = \begin{cases} 0 & \text{for } n = 1\\ \mathbf{Z} & \text{for } n = 2\\ \mathbf{Z}/2 & \text{for } n > 2. \end{cases}$$

The case k = 2 is more subtle. As in the 1-dimensional case we have a complete classification of closed 2-manifolds, and it is only necessary to consider path connected ones. The existence of a framing implies that the surface is orientable, so it is characterized by its genus.

If the genus is zero, namely if $M = S^2$, then there is a framing which extends to a 3-dimensional ball. This makes M cobordant to the empty set, which means that the map is *null homotopic* (or, more briefly, *null*), meaning that it is homotopic to a constant map. Any two framings on S^2 differ by an element in $\pi_2(SO(n))$. This group is known to vanish, so any two framings on S^2 are equivalent, and the map $f: S^{n+2} \to S^n$ is null.

Now suppose the genus is one Suppose we can find an embedded arc on which the framing extends to a disk. Then there is a cobordism which effectively cuts along the arc and attaches two disks as shown. This process is called *framed surgery*. If we can do this, then we have converted the torus to a 2-sphere and shown that the map $f: S^{n+2} \to S^n$ is null.

When can we find such a closed curve in M? It must represent a generator of $H_1(M)$ and carry a trivial framing. This leads to a map

(1)
$$\varphi: H_1(M; \mathbf{Z}/2) \to \mathbf{Z}/2$$

defined as follows. Each class in H_1 can be represented by a closed curve which is framed either trivially or nontrivially. It can be shown that homologous curves have the same framing invariant, so φ is well defined. At this point Pontryagin made a famous mistake which went undedected for over a decade: he assumed that φ was a homomorphism. We now know this is not the case, and we will say more about it below in §1.3.

On that basis he argued that φ must have a nontrivial kernel, since the source group is $(\mathbb{Z}/2)^2$. Therefore there is a closed curve along which we can do the surgery. It follows that M can be surgered into a 2-sphere, leading to the erroneous conclusion that $\pi_{n+2}(S^n) = 0$ for all n. Freudenthal [Fre38] and later George Whitehead [Whi50] both proved that it is $\mathbb{Z}/2$ for $n \geq 2$. Pontryagin corrected his mistake in [Pon50], and in [Pon55] he gave a complete account of the relation between framed cobordism and homotopy groups of spheres.

1.2. **Our main result.** Our main theorem can be stated in three different but equivalent ways:

- *Manifold formulation:* It says that a certain geometrically defined invariant $\Phi(M)$ (the Arf-Kervaire invariant, to be defined later) on certain manifolds M is always zero.
- *Stable homotopy theoretic formulation:* It says that certain long sought hypothetical maps between high dimensional spheres do not exist.

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• *Unstable homotopy theoretic formulation:* It says something about the EHP sequence (to be defined below), which has to do with unstable homotopy groups of spheres.

The problem solved by our theorem is nearly 50 years old. There were several unsuccessful attempts to solve it in the 1970s. They were all aimed at proving the opposite of what we have proved.

Here again is the stable homotopy theoretic formulation.

Main Theorem. The Arf-Kervaire elements $\theta_j \in \pi_{2^{j+1}-2}^S$ do not exist for $j \ge 7$.

1.3. The manifold formulation. Let λ be a nonsingular anti-symmetric bilinear form on a free abelian group H of rank 2n with mod 2 reduction \overline{H} . It is known that \overline{H} has a basis of the form $\{a_i, b_i : 1 \le i \le n\}$ with

$$\lambda(a_i, a_{i'}) = 0 \qquad \lambda(b_j, b_{j'}) = 0 \qquad \text{and} \qquad \lambda(a_i, b_j) = \delta_{i,j}.$$

In other words, \overline{H} has a basis for which the bilinear form's matrix has the symplectic form

A quadratic refinement of λ is a map $q: \overline{H} \to \mathbb{Z}/2$ satisfying

$$q(x+y) = q(x) + q(y) + \lambda(x,y)$$

Its Arf invariant is

$$\operatorname{Arf}(q) = \sum_{i=1}^{n} q(a_i)q(b_i) \in \mathbf{Z}/2$$

In 1941 Arf [Arf41] proved that this invariant (along with the number n) determines the isomorphism type of q.

An equivalent definition is the "democratic invariant" of Browder [Bro69]. The elements of \overline{H} "vote" for either 0 or 1 by the function q. The winner of the election (which is never a tie) is $\operatorname{Arf}(q)$. Here is a table illustrating this for three possible refinements q, q'and q'' when \overline{H} has rank 2.

x	0	a	b	a+b	Arf invariant
q(x)	0	0	0	1	0
q'(x)	0	1	1	1	1
$q^{\prime\prime}(x)$	0	1	0	0	0

The value each refinement on a + b is determined by those on a and b, and q'' is isomorphic to q. Thus the vote is three to one in each case. When \overline{H} has rank 4, it is 10 to 6.

Let M be a 2m-connected smooth closed manifold of dimension 4m + 2 with a framed embedding in \mathbb{R}^{4m+2+n} . We saw above that this leads to a map $f: S^{n+4m+2} \to S^n$ and hence an element in $\pi_{n+4m+2}(S^n)$. Let $H = H_{2m+1}(M; \mathbb{Z})$, the homology group in the middle dimension. Each $x \in H$ is represented by an immersion $i_x : S^{2m+1} \hookrightarrow M$ with a stably trivialized normal bundle. Hhas an antisymmetric bilinear form λ defined in terms of intersection numbers.

In 1960 Kervaire [Ker60] defined a quadratic refinement q on its mod 2 reduction in terms of the trivialization of each sphere's normal bundle. The *Kervaire invariant* $\Phi(M)$ is defined to be the Arf invariant of q. In the case m = 0, when the dimension of the manifold is 2, Kervaire's q is Pontryagin's map φ of (1).

What can we say about $\Phi(M)$?

- Kervaire [Ker60] showed it must vanish when k = 2. This enabled him to construct the first example of a topological manifold (of dimension 10) without a smooth structure. Let N be a smooth 10-manifold with boundary given as the union of two copies of the tangent disk bundle of S⁵, glued together along a common copy of D⁵ × D⁵ where the fibers in one copy are parallel to the base in the other. The boundary is homeomorphic to S⁹. Thus we can get a closed topological manifold X by gluing on a 10-ball along its common boundary with n, or equivalently collapsing ∂N to a point. X then has nontrivial Kervaire invariant. On the other hand, Kervaire proved that any smooth framed manifold X cannot have a smooth structure. Equivalently, the boundary ∂N cannot be diffeomorphic to S⁹. It must be an exotic 9-sphere.
- For k = 0 there is a framing on the torus S¹ × S¹ ⊂ R⁴ with nontrivial Kervaire invariant. Pontryagin used it in [Pon50] (after some false starts in the 30s) to show π_{n+2}(Sⁿ) = Z/2 for all n ≥ 2.
- There are similar constructions for k = 1 and k = 3, where the framed manifolds are $S^3 \times S^3$ and $S^7 \times S^7$ respectively. Like S^1 , S^3 and S^7 are both parallelizable, meaning that their trivial tangent bundles are trivial. The framings can be twisted in such a way as to yield a nontrivial Kervaire invariant.
- Brown-Peterson [BP66b] showed that it vanishes for all positive even k. This means that apart from the 2-dimensioanl case, any smooth framed manifold with nontrivail Kervaire invariant must a dimension congruent to 6 modulo 8.
- Browder [Bro69] showed that it can be nontrivial only if $k = 2^{j-1} 1$ for some positive integer j. This happens iff the element h_j^2 is a permanent cycle in the Adams spectral sequence, which was originally introduced in [Ada58]. (More information about it can be found below in §??) in [Rav86] and[Rav04].) The corresponding element in $\pi_{n+2^{j+1}-2}^S$ is θ_j , the subject of our theorem. This is the stable homotopy theoretic formulation of the problem.
- θ_j is known to exist for $1 \le j \le 3$, i.e., in dimensions 2, 6, and 14. In these cases the relevant framed manifold is $S^{2^j-1} \times S^{2^j-1}$ with a twisted framing as discussed above. The framings on S^{2^j-1} represent the elements h_j in the Adams spectral sequence. The Hopf invariant one theorem of Adams [Ada60] says that for j > 3, h_j is not a permanent cycle in the Adams spectral sequence because it supports a nontrivial differential. (His original proof was not written in this language, but had to do with secondary cohomlogy operations.) This means that for j > 3, a smooth framed manifold representing θ_j (i.e., having a nontrivial Kervaire invariant) cannot have the form $S^{2^j-1} \times S^{2^j-1}$.
- θ_j is also known to exist for j = 4 and j = 5, i.e., in dimensions 30 and 62. In both cases the existence was first established by purely homotopy theoretic means, without constructing a suitable framed manifold. For j = 4 this was

done by Barratt, Mahowald and Tangora in [MT67] and [BMT70]. A framed 30manifold with nontrivial Kervaire invariant was later constructed by Jones [Jon78]. For j = 5 the homotopy theory was done in 1985 by Barratt-Jones-Mahowald in [BJM84].

• Our theorem says θ_j does *not* exist for $j \ge 7$. The case j = 6 is still open.

Kervaire's 10-manifold with boundary described above can be generalized to a (4k+2)manifold with boundary constructed in a similar way. In all cases except k = 0, 1 or 3, any framing of this manifold will do because the tangent bundle of S^{2k+1} is nontrivial and leads to a nontrivial invariant. The boundary is homeomorphic to S^{4k+1} , but may or amy not be diffeomorphic to the standard sphere. If it is, then attaching a (4k + 2)-disk to it will produce a smooth framed manifold with nontrivial Kervaire invariant. If it is not, then we have an exotic (4k + 1)-sphere bounding a framed manifold and hence not detected by framed cobordism.

1.4. The unstable formulation. Assume all spaces in sight are localized and the prime 2. For each n > 0 there is a fiber sequence due to James, [Jam55], [Jam56a], [Jam56b] and [Jam57]

$$S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}.$$

Here $\Omega X = \Omega^1 X$ where $\Omega^k X$ denotes the space of continuous base point preserving maps to X from the k-sphere S^k , known as the kth loop space of X. This leads to a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_{m+n}(S^n) \xrightarrow{E} \pi_{m+n+1}(S^{n+1}) \xrightarrow{H} \pi_{m+n+1}(S^{2n+1}) \xrightarrow{P} \pi_{m+n-1}(S^n) \longrightarrow \cdots$$

Here

- *E* stands for *E*inhängung, the German word for suspension.
- *H* stands for *H*opf invariant.
- *P* stands forWhitehead *p*roduct.

Assembling these for fixed m and various n leads to a diagram

where

- Sequences of arrows labeled H, P, E, H, P (or any subset thereof) in that order are exact.
- The groups in the top and bottom rows are inductively known, and we can compute those in the middle row by induction on *n*.
- The groups in the top and bottom rows vanish for large n, making E an isomorphism.
- An element in the middle row has trivial suspension (is killed by E) iff it is in the image of P.

• It desusupends (is in the image of E) iff its Hopf invariant (image under H) is trivial.

When m = n - 1 this diagram is

$$\pi_{2n+1}(S^{n+1})$$

$$\downarrow H$$

$$\pi_{2n}(S^{2n-1}) \qquad \mathbf{Z} \qquad 0$$

$$\downarrow P \qquad \qquad \downarrow P \qquad \qquad \downarrow P$$

$$\downarrow P \qquad \qquad \downarrow P \qquad \qquad \downarrow P$$

$$\downarrow P \qquad \qquad \downarrow P \qquad \qquad \downarrow P$$

$$\downarrow H \qquad \qquad \qquad \downarrow H \qquad \qquad \qquad \downarrow H$$

$$\pi_{2n-2}(S^{2n-3}) \qquad \mathbf{Z} \qquad 0$$

The image under P of the generator of the upper Z is denoted by $w_n \in \pi_{2n-1}(S^n)$ and is called the *Whitehead square*.

- When n is even, $H(w_n) = 2$ and w_n has infinite order.
- w_n is trivial for n = 1, 3 and 7. In those cases the generator of the upper Z is the Hopf invariant (image under H) of one of the three Hopf maps in $\pi_{2n+1}(S^{n+1})$,

$$S^3 \xrightarrow{\eta} S^2$$
, $S^7 \xrightarrow{\nu} S^4$ and $S^{15} \xrightarrow{\sigma} S^8$.

- For other odd values of n, twice the generator of the upper Z is $H(w_{n+1})$, so w_n has order 2.
- It turns out that w_n is divisible by 2 iff $n = 2^{j+1} 1$ and θ_j exists, in which case $w_n = 2\theta_j$.
- Each Whitehead square w_{2n+1} ∈ π_{4n+1}(S²ⁿ⁺¹) (except the cases n = 0, 1 and 3) desuspends to a lower sphere until we get an element with a nontrivial Hopf invariant, which is always some β_i (see Sidebar 1). More precisely we have

$$H(w_{(2s+1)2^{j}-1}) = \beta_{j}$$

for each j > 0 and $s \ge 0$. This result is essentially Adams' 1962 solution to the vector field problem [Ada62].

Recall the EHP sequence

$$\cdots \longrightarrow \pi_{m+n}(S^n) \xrightarrow{E} \pi_{m+n+1}(S^{n+1}) \xrightarrow{H} \pi_{m+n+1}(S^{2n+1}) \xrightarrow{P} \pi_{m+n-1}(S^n) \longrightarrow \cdots$$

Given some $\beta_j \in \pi_{\phi(j)+2n+1}(S^{2n+1})$ for $\phi(j) < 2n$, one can ask about the Hopf invariant of its image under P, which vanishes when β_j is in the image of H. In most cases the answer is known and is due to Mahowald, [Mah67] and [Mah82]. The remaining cases have to do with θ_j . The answer that he had hoped for is the following, which can be found in [Mah67]. (To our knowledge, Mahowald never referred to this as the World Without End Hypothesis. We chose that term to emphasize its contrast with the Doomsday Hypothesis.)

World Without End Hypothesis (Mahowald 1967).

- The Arf-Kervaire element $\theta_j \in \pi_{2^{j+1}-2}^S$ exists for all j > 0.
- It desuspends to $S^{2^{j+1}-1-\phi(j)}$ and its Hopf invariant is β_j .
- Let j, s > 0 and suppose that $m = 2^{j+2}(s+1) 4 \phi(j)$ and $n = 2^{j+1}(s+1) 2 \phi(j)$. Then $P(\beta_j)$ has Hopf invariant θ_j .

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This describes the systematic behavior in the EHP sequence of elements related to the image of J, and the θ_j are an essential part of the picture. Because of our theorem, we now know that this hypothesis is incorrect.

1.5. Questions raised by our theorem.

EHP sequence formulation. The World Without End Hypothesis was the nicest possible statement of its kind given all that was known prior to our theorem. Now we know it cannot be true since θ_j does not exist for $j \ge 7$. This means the behavior of the indicated elements $P(\beta_j)$ for $j \ge 7$ is a mystery.

Adams spectral sequence formulation. We now know that the h_j^2 for $j \ge 7$ are not permanent cycles, so they have to support nontrivial differentials. We have no idea what their targets are.

Manifold formulation. Here our result does *not* lead to any obvious new questions. It appears rather to be the final page in the story.

Our method of proof offers a new tool for studying the stable homotopy groups of spheres. We look forward to learning more with it in the future.

2. OUR STRATEGY

- 2.1. Ingredients of the proof. Our proof has several ingredients.
 - It uses methods of stable homotopy theory, which means it uses spectra instead of topological spaces. Recall that a space X has a homotopy group π_k(X) for each positive integer k. A spectrum X has an abelian homotopy group π_k(X) defined for every integer k. For the sphere spectrum S⁰, π_k(S⁰) is the stable k-stem homotopy group π_k^S. The hypothetical θ_j is an element of this group for k = 2^{j+1} 2.
 - It uses complex cobordism theory, the subject of Sidebar 2. This is a branch of algebraic topology having deep connections with algebraic geometry and number theory. It includes some highly developed computational techniques that began with work by Milnor [Mil60], Novikov ([Nov60], [Nov62] and [Nov67]) and Quillen [Qui69] in the 60s. A pivotal tool in the subject is the theory of formal group laws, the subject of Sidebar 3.
 - It also makes use of newer less familiar methods from equivariant stable homotopy theory. A helpful introduction to this subject is the paper of Greenlees-May [GM95]. This means there is a finite group G (a cyclic 2-group) acting on all spaces in sight, and all maps are required to commute with these actions. When we pass to spectra, we get homotopy groups indexed not just by the integers Z, *but by* RO(G), *the orthogonal representation ring of* G. Our calculations make use of this richer structure.

2.2. The spectrum Ω . We will produce a map $S^0 \to \Omega$, where Ω is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.

 (i) Detection Theorem. It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each θ_j is nontrivial. This means that if θ_j exists, we will see its image in π_{*}(Ω).

- (ii) *Periodicity Theorem.* It is 256-periodic, meaning that $\pi_k(\Omega)$ depends only on the reduction of k modulo 256.
- (iii) Gap Theorem. $\pi_k(\Omega) = 0$ for -4 < k < 0. This property is our zinger. Its proof involves a new tool we call the slice spectral sequence.

If $\theta_7 \in \pi_{254}(S^0)$ exists, (i) implies it has a nontrivial image in $\pi_{254}(\Omega)$. On the other hand, (ii) and (iii) imply that $\pi_{254}(\Omega) = 0$, so θ_7 cannot exist. The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \mod 256$ for $j \ge 7$.

2.3. How we construct Ω . The construction of Ω requires the use of *equivariant stable* homotopy theory. It is discussed extensively in [HHR, §2 and Appendices A and B]. Our spectrum Ω will be the fixed point spectrum for the action of C_8 (the cyclic group of order 8) on an equivariant spectrum Ω .

To construct it we start with the complex cobordism spectrum MU. It can be thought of as the set of complex points of an algebraic variety defined over the real numbers. This means that it has an action of C_2 defined by complex conjugation. The resulting C_2 equivariant spectrum is denoted by $MU_{\mathbf{R}}$ and is called *real cobordism theory*. This terminology follows Atiyah's definition of real K-theory [Ati66], by which he meant complex K-theory equipped with complex conjugation.

2.4. The norm. Next we use a formal tool we call the norm N_H^G for inducing up from an H-spectrum to a G-spectrum when H is a subgroup of G. It is treated in [HHR, $\S2.3$]. Since the only groups we consider here are cyclic, we will denote N_H^G by N_h^g where h =|H| and g = |G|.

The analogous construction on the space level is easy to desribe. Let X be a space that is acted on by a subgroup H of G. Let

$$Y = \operatorname{Map}_{H}(G, X),$$

the space of H-equivariant maps from G to X. Here the action of H on G is by right multiplication, and the resulting object has an action of G by left multiplication. As a set, $Y = X^{|G/H|}$, the |G/H|-fold Cartesian power of X. A general element of G permutes these factors, each of which is left invariant by the subgroup H. For an H-spectrum X, the analogous G-spectrum is denoted by $N_H^G X$.

In particular for a finite cyclic 2-group $\vec{G} = C_{2^{n+1}}$ we get a G-spectrum

$$MU^{(G)} = N_2^g M U_{\mathbf{R}}$$

underlain by $MU^{(2^n)}$. We are most interested in the case n = 2. This spectrum is not periodic, but it has a close relative $\tilde{\Omega}$ (to defined below in §5) which is.

2.5. Fixed points and homotopy fixed points. For a G-space X, X^G is the subspace fixed by all of G, which is the same as the space of equivariant maps from a point to X, $\operatorname{Map}_{G}(*, X)$. To get X^{hG} , we replace the point here by an free contractible *G*-space *EG*. The homotopy type of $X^{hG} = \operatorname{Map}_{G}(EG, X)$ is known to be independent of the choice of EG. The unique map $EG \to *$ leads to a map $\varphi : X^G \to X^{hG}$.

We construct a C_8 -spectrum $\Omega_{\mathbb{O}}$ and show that

- Ω₀^{hC₈} satisfies the detection and Periodicity Theorems.
 Ω₀^{C₈} satisfies the Gap Theorem.

Hence our proof depends on a fourth property:

Sidebar 2 Complex cobordism

The complex cobordism spectum MU is a critical tool in modern stable homotpy theory. Its use in computing the stable homotopy groups of spheres is the subject of [Rav86]. Much of the needed background material on vector bundles and Thom spaces can be found in [MS74].

Let U(n) denote the *n*-dimensional unitary group and BU(n) its classifying space. The latter is the space of complex *n*-planes in an infinite dimensional complex vector space. It is the base space of an *n*-dimensional complex vector bundle where the fiber at each point is the set of vectors in the given *n*-plane. We denote the one point compactification of its total space (also known as the Thom space of the vector bundle) by MU(n). The inclusion map $U(n) \rightarrow U(n+1)$ leads to a map $\Sigma^2 MU(n) \rightarrow MU(n+1)$, which is adjoint to $MU(n) \rightarrow \Omega^2 MU(n+1)$.

Such maps can be looped and iterated, allowing us to define

$$MU_k = \lim \Omega^{2n-k} MU(n)$$

with MU_k homeomorphic to ΩMU_{k+1} . The collection $\{MU_k\}$ is the spectrum MU. Its homotopy groups are defined by

$$\pi_i M U = \lim \pi_{2n+i} M U(n).$$

Its homology and cohomology can be similarly defined. The inclusions $U(m) \times U(n) \rightarrow U(m+n)$ lead to maps $MU(m) \wedge MU(n) \rightarrow MU(m+n)$ and $MU \wedge MU \rightarrow MU$. This makes MU into an E_{∞} -ring spectrum. There are generalized homology and cohomology theories MU_* and MU^* defined by

 $MU_i(X) = \pi_i X \wedge MU$ and $MU^i(X) = [X, MU_i],$

with the latter being endowed with cup products.

We know that

 $\pi_*MU = \mathbf{Z}[x_i: i > 0]$ and $H_*MU = \mathbf{Z}[m_i: i > 0]$

where $|x_i| = |m_i| = 2i$. After localization at a prime p, MU splits into a wedge of copies of a spectrum BP (for Brown-Peterson [BP66a]) with

$$\pi_*BP = \mathbf{Z}_{(p)}[v_i: i > 0]$$
 and $H_*BP = \mathbf{Z}_{(p)}[\ell_i: i > 0]$

where $|v_i| = |\ell_i| = 2(p^i - 1)$. This is related to a corresponding fact about formal groups laws described in Sidebar 3.

(iv) *Fixed Point Theorem*. The map $\Omega_{\mathbb{O}}^{C_8} \to \Omega_{\mathbb{O}}^{hC_8}$ is an equivalence. It proof is given in [HHR, §10]. This fixed point set is our spectrum Ω .

We will come back to the definition of $\Omega_{\mathbb{O}}$ below.

2.6. RO(G)-graded homotopy. Let RO(G) denote the orthogonal representation ring of G and let S^V denote the one point compactification of orthogonal representation V. For a G-space or spectrum X define

$$\pi_V^G X = [S^V, X]_G.$$

Note that when the action of G on V is trivial, an equivariant map $S^V = S^{\dim V} \to X$ must land in X^G , so

$$\pi_n^G X = \pi_n X^G.$$

In the stable category we can make sense of this for virtual as well as actual representations, so we get homotopy groups indexed by RO(G), which we denote collectively by $\pi_*^G X$. We denote the ordinary homotopy of the underlying spectrum of X by $\pi_*^u X$.

We use the norm in two ways:

- Externally as a functor from *H*-spectra to *G*-spectra. In particular, for a representation *V* of *H*, $N_h^g S^V = S^{indV}$, the one point compactification of the induced representation of *V*.
- Internally as a homomorphism

$$N_h^g: \pi^H_\star X \to \pi^G_\star X$$

for a G-ring spectrum X. Here we are using the forgetful functor to regard X as an H-spectrum with RO(H)-graded homotopy. The external norm takes us to $\pi_{\star}^{G}N_{h}^{g}X$. Then $N_{h}^{g}X$ is $X^{(g/h)}$ as a G-spectrum, and we use the multiplication on X to get a map $N_{h}^{g}X \to X$.

We will use the same notation for both. Recall that

$$\pi_*(MU) = \mathbf{Z}[x_1, x_2, \dots]$$
 with $|x_i| = 2i$.

It turns out that any choice of generator $x_i : S^{2i} \to MU$ is the image of the forgetful functor of a map

$$S^{i\rho} \xrightarrow{\overline{x}_i} MU_{\mathbf{R}}$$

Here ρ denotes the regular real representation of C_2 , which is the same thing as the complex numbers C acted on by conjugation.

- For $G = C_{2^{n+1}}, \pi^u_* M U^{(G)}$ is a graded polynomial algebra over \mathbf{Z} where
 - there are 2^n generators in each positive even dimension 2i.
 - they are acted on transitively by G.

For a group generator $\gamma \in G$ and polynomial generator $r_i \in \pi_{2i}$, the set

$$\left\{\gamma^j r_i \colon 0 \le j < 2^n\right\}$$

is algebraically independent, and $\gamma^{2^n} r_i = (-1)^i r_i$.

3. The slice filtration

Now we introduce our main technical tool, which is treated in [HHR, §6]. It is an equivariant analog of the Postnikov tower.

3.1. **Postnikov towers.** First we need to recall some things about the classical Postnikov tower. The *m*th Postnikov section $P^m X$ of a space or spectrum X is obtained by killing all homotopy groups of X above dimension m by attaching cells. The fiber of the map $X \to P^m X$ is $P_{m+1}X$, the *m*-connected cover of X.

These two functors have some universal properties. Let S and $S_{>m}$ denote the categories of spectra and *m*-connected spectra.

The functor P^m is Farjoun nullification [Far96] with respect to the subcategory $S_{>m}$. This means the map $X \to P^m X$ is universal among maps from X to spectra which are $S_{>m}$ -null in the sense that all maps to them from m-connected spectra are null. In other words,

- The spectrum $P^m X$ is $S_{>m}$ -null.
- For any $S_{>m}$ -null spectrum Z, the map $S(P^mX, Z) \to S(X, Z)$ is an equivalence.

Since $S_{>m} \subset S_{>m-1}$, there is a natural transformation $P^m \to P^{m-1}$, whose fiber is denoted by $P_m^m X$. It is an Eilenberg-Mac Lane spectrum with homotopy concentrated in dimension m.

In what follows G will be a finite cyclic 2-group, and g = |G|. Let S^G denote the category of G-equivariant spectra. We need an equivariant analog of $S_{>m}$. Our choice for this is somewhat novel.

3.2. Slice cells. Recall that $S_{>m}$ is the category of spectra built up out of spheres of dimension > m using arbitrary wedges and mapping cones.

For a subgroup H of G with |H| = h and an integer k, let

$$S(k\rho_H) = G_+ \wedge_H S^{k\rho_H}$$

where ρ_H denotes the regular real representation of H. Its underlying spectrum is a wedge of g/h spheres of dimension kh which are permuted by elements of G and are invariant under H.

We will replace the set of sphere spectra by

$$\mathcal{A} = \left\{ \widehat{S}(k\rho_H), \, \Sigma^{-1}\widehat{S}(k\rho_H) \colon H \subset G, \, k \in \mathbf{Z} \right\}.$$

We will refer to the elements in this set as *slice cells* or simply as *cells*. Note that $\Sigma^{-2}\widehat{S}(k\rho_H)$ (and larger desuspensions) are *not* cells. A *free cell* is one of the form $\widehat{S}(k\rho_{\{e\}})$, a wedge of g k-spheres permuted by G. Note that

$$\Sigma^{-1}\widehat{S}(k\rho_{\{e\}}) = \widehat{S}((k-1)\rho_{\{e\}}).$$

Nonfree cells are said to be *isotropic*.

In order to define $S^G_{>m}$, we need to assign a dimension to each element in \mathcal{A} . We do this in terms of the underlying wedge summands, namely

dim
$$\widehat{S}(k\rho_H) = kh$$
 and dim $\Sigma^{-1}\widehat{S}(k\rho_H) = kh - 1$.

Then $\mathcal{S}_{>m}^G$ is the category built up out of elements in \mathcal{A} of dimension > m using arbitrary wedges, mapping cones and smash products with equivariant suspension spectra.

3.3. The slice tower. With this definition it is possible to construct functors P_{m+1}^G and P_G^m with the same formal properties as in the classical case. Thus we get a tower

in which the homotopy limit is X and the homotopy colimit is contractible.

We call this the *slice tower*. ${}^{G}P_{m}^{m}X$ is the *nth slice* and the decreasing sequence of subgroups of $\pi_{*}(X)$ is the *slice filtration*. We also get slice filtrations of the RO(G)-graded homotopy $\pi_{*}^{G}(X)$ and the homotopy groups of fixed point sets $\pi_{*}(X^{H})$.

There is an important difference between this tower and the classical one. In the classical case the map $X \to P^m X$ does not change homotopy groups in dimensions $\leq m$. This is not true in this equivariant case.

In the classical case, $P_m^m X$ is an Eilenberg-Mac Lane spectrum whose *n*th homotopy group is that of X. In our case, $\pi_*({}^GP_m^m X)$ need not be concentrated in dimension m.

This means the slice filtration leads to a (possibly noncollapsing) *slice spectral sequence* converging to $\pi_*(X)$ and its variants.

One variant has the form

$$E_2^{s,t} = \pi_{t-s}^G({}^GP_t^tX) \implies \pi_{t-s}^G(X)$$

Recall that $\pi^G_*(X)$ is by definition $\pi_*(X^G)$, the homotopy of the fixed point set. This is the spectral sequence we will use to study MU(G) and its relatives.

3.4. The Slice Theorem. A large portion of our paper is devoted to proving that the slice spectral sequence has the desired properties. From now on we will drop the symbol G from the functors P^m , P_{m+1} and P_m^m .

Slice Theorem. In the slice tower for $MU^{(G)}$, every odd slice is contractible and $P_{2m}^{2m} = \widehat{W}_m^{(G)} \wedge H\mathbf{Z}$, where $\widehat{W}_m^{(G)}$ is a certain wedge of 2*m*-dimensional slice cells (to be named later) and HZ is the integer Eilenberg-Mac Lane spectrum with trivial G action. $\widehat{W}_m^{(G)}$ never has any free summands.

In order to specify $\widehat{W}_m^{(G)}$, we need the following definition.

Definition. Suppose X is a G-spectrum such that its underlying homotopy group $\pi_k^u(X)$ is free abelian. A refinement of $\pi_k^u(X)$ is an equivariant map

$$c: \widehat{W} \to X$$

in which \widehat{W} is a wedge of slice cells of dimension k whose underlying spheres represent a basis of $\pi_k^u(X)$.

Recall that in $\pi^u_*(MU_{\mathbf{R}})$, any monomial in the polynomial generators in dimension 2m is represented by an equivariant map from $S^{m\rho_2}$.

 $\pi^u_*(MU^{(C_8)})$ is a polynomial algebra with 4 generators in every positive even dimension. We will denote the generators in dimension 2i by $r_i(j)$ for $1 \le j \le 4$. The action of a generator $\gamma \in G = C_8$ is given by

$$\gamma(r_i(j)) = \begin{cases} r_i(j+1) & \text{for } 1 \le j \le 3\\ (-1)^i r_i(1) & \text{for } j = 4. \end{cases}$$

We will explain how $\pi^u_*(MU^{(C_8)})$ can be refined.

 $\pi_2^u(MU^{(\hat{C_8})})$ has 4 generators $r_1(j)$ that are permuted up to sign by G. It is refined by an equivariant map

$$\widehat{W}_1^{(C_8)} = \widehat{S}(\rho_2) \to MU^{(C_8)}.$$

Recall that the underlying spectrum of $\widehat{S}(\rho_2)$ is a wedge of 4 copies of S^2 .

In $\pi_4^u(MU^{(C_8)})$ there are 14 monomials that fall into 4 orbits under the action of G, each corresponding to a map from a slice cell.

$$\widehat{S}(2\rho_2) \longleftrightarrow \{r_1(1)^2, r_1(2)^2, r_1(3)^2, r_1(4)^2\}
\widehat{S}(2\rho_2) \longleftrightarrow \{r_1(1)r_1(2), r_1(2)r_1(3), r_1(3)r_1(4), r_1(4)r_1(1)\}
\widehat{S}(2\rho_2) \longleftrightarrow \{r_2(1), r_2(2), r_2(3), r_2(4)\}
\widehat{S}(\rho_4) \longleftrightarrow \{r_1(1)r_1(3), r_1(2)r_1(4)\}$$

(Recall that $\widehat{S}(\rho_4)$ is underlain by $S^4 \vee S^4$.) It follows that $\pi_4^u(MU^{(C_8)})$ is refined by an equivariant map from

$$\widehat{W}_2^{(C_8)} = \widehat{S}(2\rho_2) \vee \widehat{S}(2\rho_2) \vee \widehat{S}(\rho_4) \vee \widehat{S}(2\rho_2).$$

3.5. **Refining** $\pi^u_* M U^{(G)}$. More generally, we can proceed as follows. The $\pi^u_* M U_{\mathbf{R}}$ is refined by a map from a wedge of $S^{k\rho_2}$ s. More specifically, for each i > 0 we have a map

$$A_i = \bigvee_{m \ge 0} S^{mi\rho_2} \to MU_{\mathbf{R}}$$

representing the powers of the generator $r_i \in \pi_{2i}^u M U_{\mathbf{R}}$. Using the multiplication on $MU_{\mathbf{R}}$, we get a map

$$A^{(C_2)} = \bigwedge_{i>0} A_i \to MU_{\mathbf{R}}.$$

The spectrum $A^{(C_2)}$ is a wedge of $S^{m\rho_2}$ s representing each monomial in the r_i s.

Now we apply the norm functor N_2^g to this map and get a map

(2)
$$A^{(G)} = N_2^g A^{(C_2)} \to M U^{(G)}$$

which refines the underlying homotopy of the target. $A^{(G)}$ is a wedge of even dimensional slice cells, and the $\widehat{W}_m^{(G)}$ in the Slice Theorem is the wedge of slice cells in dimension 2m.

3.6. The definition of $\Omega_{\mathbb{O}}$. Our G-spectrum $\Omega_{\mathbb{O}}$ (where $G = C_8$) is obtained from the E_{∞} -ring spectrum $MU^{(G)}$ by inverting a certain element $D \in \pi_{19\rho_G}^G$. The choices of G and D are the simplest ones leading to a homotospy fixed point set with the detection property, as show in [HHR, §11]. The slice tower for $\Omega_{\mathbb{O}}$ has similar properties to that of $MU^{(G)}$.

4. The Gap Theorem

4.1. Derivation from the Slice Theorem. Assuming the Slice Theorem, the Gap Theorem follows from the fact that π^G_{-2} vanishes for each isotropic slice, i.e., for each one of the form

$$\widehat{S}(k\rho_H) \wedge H\mathbf{Z}$$

for nontrivial H. Proving this amounts to computing the homology of a certain chain complex. Details can be found in [HHR, §3.2].

In order to give a feel for these calculations, a picture of $\pi^G_* S^{k\rho_G} \wedge H\mathbf{Z}$ for $G = C_8$ and various integers k isd shown in Figure 1.

In order to derive the Gap Theorem from the Slice Theorem we need to find the groups

$$\pi^G_*(W(m\rho_h) \wedge H\mathbf{Z}) = \pi^H_*(S^{m\rho_h} \wedge H\mathbf{Z}).$$

We need this for all integers m because eventually we will obtain the spectrum $\Omega_{\mathbb{O}}$ by inverting a certain element in $\pi_{19\rho_8}^G(MU^{(C_8)})$. Here is what we will learn.

Vanishing Theorem.

- For m ≥ 0, π_k^H(S^{mρ_h} ∧ HZ) = 0 unless m ≤ k ≤ mh.
 For m < 0 and h > 1, π_k^H(S^{mρ_h} ∧ HZ) = 0 unless mh ≤ k ≤ m 2. The upper bound can be improved to m 3 except in the case (h, m) = (2, -2) when $\pi^{H}_{-4}(S^{-2\rho_2} \wedge H\mathbf{Z}) = \mathbf{Z}.$

Gap Corollary. For h > 1 and all integers $m, \pi_k^H(S^{m\rho_h} \wedge H\mathbf{Z}) = 0$ for -4 < k < 0.

This means a similar statement must hold for $\pi_*^{C_8}(\Omega_{\mathbb{O}}) = \pi_*(\Omega)$, which gives the Gap Theorem.

Assuming the Slice Theorem, the Gap Theorem (the statement that $\pi_{-2}(\Omega) = 0$) follows immediately from the Gap Corollary above, as does the following.



FIGURE 1. The homotopy of the slices $S^{k\rho_8} \wedge H\mathbf{Z}$ for small k. The horizontal and vertical axes are t-s and s in the slice spectral sequence. The numbers along the (t-s)-axis are even values of k. The groups $\pi_*^G S^{k\rho_8} \wedge H\mathbf{Z}$ lie along diagonals where t = 8k. Squares, bullets, circles and diamaonds stand for \mathbf{Z} , $\mathbf{Z}/2$, $\mathbf{Z}/4$ and $\mathbf{Z}/8$ respectively. Lines of slopes 1, 3 and 7 separate regions with various types of torsion for even k. The gap can be seen in this chart. The vanishing regions seens here also occur in the slice spectral sequence for $\pi_*^G MU^{(G)}$.

Vanishing Line Corollary. The slice spectral sequence for $\pi^G_*MU^{(G)}$ is confined to the first quadrant and

$$E_2^{s,t} = 0$$
 for $s > (g-1)(t-s)$,

where g = |G|. If we invert any element in $\pi^G_{k\pi_G} MU^{(G)}$, the spectral sequence is confined to the first and third quadrants, and in the latter

$$E_2^{s,t} = 0$$
 for $s < \min(0, (g-1)(t-s+4)+4-g)$.

The proofs of the Vanishing Theorem and Gap Corollary are surprisingly easy.

4.2. An easy calculation. We begin by constructing an equivariant cellular chain complex $C(m\rho_g)_*$ for $S^{m\rho_g}$, where $m \ge 0$. In it the cells are permuted by the action of G. It is a complex of $\mathbb{Z}[G]$ -modules and is uniquely determined by fixed point data of $S^{m\rho_g}$.

For $H \subset G$ we have

$$(S^{m\rho_g})^H = S^{mg/h}$$

This means there is a G-CW-complex with one cell in dimension m, two cells in each dimension from m + 1 to 2m, four cells in each dimension from 2m + 1 to 4m, and so on.

In other words,

$$C(m\rho_g)_k = \begin{cases} 0 & \text{unless } m \le k \le gm \\ \mathbf{Z} & \text{for } k = m \\ \mathbf{Z}[G/H] & \text{for } mg/2h < k \le mg/h \text{ and } h < g. \end{cases}$$

Each of these is a cyclic $\mathbb{Z}[G]$ -module. The boundary operator is uniquely determined by the fact that $H_*(C(m\rho_q)) = H_*(S^{gm})$.

Then we have

$$\pi^G_*(S^{m\rho_g} \wedge H\mathbf{Z}) = H_*(\operatorname{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g))).$$

These groups are nontrivial only for $m \le k \le gm$, which gives the Vanishing Theorem for $m \ge 0$.

We will look at the bottom three groups in the complex $\operatorname{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g)_*)$. Since $C(m\rho_g)_k$ is a cyclic $\mathbf{Z}[G]$ -module, the Hom group is always \mathbf{Z} .

For m > 1 our chain complex $C(m\rho_g)$ has the form

Applying Hom_{**Z**[G]}(**Z**, \cdot) to this gives (in dimensions $\leq 2m$ for m > 4)

$$\mathbf{Z} \stackrel{2}{\leftarrow} \mathbf{Z} \stackrel{0}{\leftarrow} \mathbf{Z} \stackrel{2}{\leftarrow} \mathbf{Z} \stackrel{0}{\leftarrow} \mathbf{Z} \stackrel{2}{\leftarrow} \mathbf{Z} \stackrel{0}{\leftarrow} \mathbf{Z} \stackrel{0}{\leftarrow} \mathbf{Z} \stackrel{0}{\leftarrow} \mathbf{Z} \stackrel{0}{\leftarrow} \cdots$$

$$m \quad m+1 \quad m+2 \quad m+3 \quad m+4$$

It follows that for $m \leq k < 2m$,

$$\pi_k^G(S^{m\rho_g} \wedge H\mathbf{Z}) = \begin{cases} \mathbf{Z}/2 & k \equiv m \text{ mod } 2\\ 0 & \text{otherwise.} \end{cases}$$

For negative multiples of ρ_g , $S^{-m\rho_g}$ (with m > 0) is the equivariant Spanier-Whitehead dual of $S^{m\rho_g}$. This means that

$$\pi^G_*(S^{-m\rho_g} \wedge H\mathbf{Z}) = H^*(\operatorname{Hom}_{\mathbf{Z}[G]}(C(m\rho_g), \mathbf{Z}))$$

Applying the functor $\operatorname{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$ to our chain complex $C(m\rho_q)$

$$\mathbf{Z} \xleftarrow{\epsilon} \mathbf{Z}[C_2] \xleftarrow{1-\gamma}{\mathbf{Z}[C_2]} \xleftarrow{1+\gamma}{\mathbf{Z}[C_2 \text{ or } C_4]} \xleftarrow{1-\gamma}{\mathbf{Z}[C_2 \text{ or } C_4]} \xleftarrow{1-\gamma}{\mathbf{Z}[C_4 \text{ or } C_$$

gives a cochain complex beginning with

$$\mathbf{Z} \xrightarrow{1} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{2} \cdots$$
$$-m - m - 1 \qquad -m - 2 \qquad -m - 3 \qquad -m - 4$$

Here is a diagram showing both functors in the case $m \ge 4$.

$$m \qquad m+1 \qquad m+2 \qquad m+3 \qquad m+4$$

$$\mathbf{Z} \underbrace{\leftarrow}_{2} \qquad \mathbf{Z} \underbrace{\leftarrow}_{0} \qquad \mathbf{Z} \underbrace{\leftarrow}_{2} \qquad \mathbf{Z} \underbrace{\leftarrow}_{0} \qquad$$

Note the difference in behavior of the map $\epsilon : \mathbf{Z}[C_2] \to \mathbf{Z}$ under the functors $\operatorname{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$ and $\operatorname{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$. They convert it to maps of degrees 2 and 1 respectively. This difference is responsible for the Gap.

5. The Periodicity Theorem

We now outline the proof of the Periodicity Theorem (assuming the Slice Theorem), which is treated in [HHR, §9].

We establish some differentials in the slice spectral sequence and show that certain elements become permanent cycles after inverting a certain $D \in \pi^G_{19\rho_G} MU^{(G)}$, where $G = C_8$. This lead to an equivariant self map

$$\Sigma^{256}\Omega_{\mathbb{O}} \to \Omega_{\mathbb{O}}.$$

It is an ordinary homotopy equivalence, and we will see that this implies formally that it induces an equivalence on homotopy fixed point sets.

5.1. Geometric fixed points. The key tool for studying differentials in the slice spectral sequence is the *geometric fixed point spectrum*, denoted by $\Phi^G X$ for a *G*-spectrum *X*. A detailed account and references can be found in [HHR, §2.5]. It has much nicer properties than the usual fixed point spectrum X^G , which is awkward for two reasons:

- it fails to commute with smash products and
- it fails to commute with infinite suspensions.

The geometric fixed set $\Phi^G X$ is a convenient substitute that avoids these difficulties. In order to define it we need the *isotropy separation sequence*, which in the case of a finite cyclic 2-group G is the cofiber sequence

$$EC_{2+} \to S^0 \to \tilde{E}C_2$$

Here EC_2 is a *G*-space via the projection $G \to C_2$ and S^0 has the trivial action, so $\tilde{E}C_2$ is also a *G*-space. Under this action EC_2^G is empty while for any proper subgroup *H* of *G*, $EC_2^H = EC_2$, which is contractible. For an arbitrary finite group *G* it is possible to construct a *G*-space with the similar properties.

This functor has the following properties:

- For G-spectra X and Y, $\Phi^G(X \wedge Y) = \Phi^G X \wedge \Phi^G Y$.
- For a G-space X, $\Phi^G \Sigma^\infty X = \Sigma^\infty (X^G)$.
- A map $f: X \to Y$ is a G-equivalence iff $\Phi^H f$ is an ordinary equivalence for each subgroup $H \subset G$.

From the suspension property we can deduce that for any finite cyclic 2-group G,

$$\Phi^G M U^{(G)} = M O$$

the unoriented cobordism spectrum. Its homotopy type has been well understood since Thom's work in the 50s.

Geometric Fixed Point Theorem. Let σ denote the sign representation. Then for any *G*-spectrum X, $\pi_{\star}(\tilde{E}C_2 \wedge X) = a_{\sigma}^{-1}\pi_{\star}(X)$, where $a_{\sigma} \in \pi_{-\sigma}X$ is induced by the inclusion of the fixed point set $S^0 \to S^{\sigma}$.

In [HHR, §5.4] we define specific polynomial generators

$$\overline{r}_i^G \in \pi_{i\rho_2}^{C_2} M U^{(G)}$$

that are convenient for our purposes. We denote the underlying homotopy classes by $r_i \in \pi_{2i}^u MU^{(G)}$.

Recall that $\pi_*(MO) = \mathbb{Z}/2[y_i : i > 0, i \neq 2^k - 1]$ where $|y_i| = i$. It is not hard to show that

$$\pi_*(MU^{(C_8)}) = \mathbf{Z}[r_i, \gamma(r_i), \gamma^2(r_i), \gamma^3(r_i) : i > 0]$$

where $|r_i| = 2i$, γ is a generator of G and $\gamma^4(r_i) = (-1)^i r_i$. In $\pi_{i\rho_8}(MU^{(4)})$ we have the element

$$Nr_i = r_i \gamma(r_i) \gamma^2(r_i) \gamma^3(r_i)$$

Applying the functor Φ^G to the map $N_2^8 r_i : S^{i\rho_8} \to MU^{(C_8)}$ gives a map $S^i \to MO$.

Lemma 1. The generators r_i and y_i satisfy

$$\Phi^{G} N_{2}^{g} r_{i} = \begin{cases} 0 & \text{for } i = 2^{k} - 1 \\ y_{i} & \text{otherwise.} \end{cases}$$

5.2. Some slice differentials. We know that the slice spectral sequence for $MU^{(G)}$ has a vanishing line of slope g - 1. We will describe the subring of elements lying on it.

Let $f_i \in \pi_i(MU^{(G)})$ be the composite

$$S^i \xrightarrow{a_{i\rho_g}} S^{i\rho_g} \xrightarrow{Nr_i} MU^{(G)},$$

where $a_{i\rho_g}$ is the inclusion of the fixed point set. The following facts about f_i are easy to prove.

- It appears in the slice spectral sequence in $E_2^{(g-1)i,gi}$, which is on the vanishing line.
- The subring of elements on the vanishing line is the polynomial algebra on the f_i .
- Under the map

$$\pi_*(MU^{(G)}) \to \pi_*(\Phi^G MU^{(G)}) = \pi_*(MO)$$

we have

$$f_i \mapsto \begin{cases} 0 & \text{for } i = 2^k - 1\\ y_i & \text{otherwise} \end{cases}$$

• Any differential landing on the vanishing line must have a target in the ideal $(f_1, f_3, f_7, ...)$. A similar statement can be made after smashing with $S^{2^k\sigma}$.

For an oriented representation V there is a map $u_V : S^{|V|} \to \Sigma^V H \mathbf{Z}$, which lies in $\pi_{V-|V|}(H\mathbf{Z})$. It satisfies $u_{V+W} = u_V u_W$, so $u_{2^k\sigma} = u_{2\sigma}^{2^{k-1}}$.

Slice Differentials Theorem. In the slice spectral sequence for $\Sigma^{2^k\sigma}MU^{(G)}$ for k > 0, we have $d_r(u_{2^k\sigma}) = 0$ for $r < 1 + g(2^k - 1)$, and

$$d_{1+g(2^k-1)}(u_{2^k\sigma}) = a_{\sigma}^{2^k} f_{2^k-1}.$$

Sketch of proof: Inverting a_{σ} in the slice spectral sequence will make it converge to $\pi_*(MO)$. This means each power of $u_{2\sigma}$ has to support a nontrivial differential. The only way this can happen is as indicated in the theorem.

Typically one proves theorems about differentials in such spectral sequences by means of some sort of extended power construction. In our case, all of the necessary geometry is encoded in the relation between $\pi^u_* MU^{(G)}$ and $\pi_* MO$!

5.3. Some RO(G)-graded calculations. For a cyclic 2-group G let

$$\overline{\mathfrak{d}}_k^G = N_2^g \overline{r}_{2^k - 1} = \overline{r}_{2^k - 1} \gamma(\overline{r}_{2^k - 1}) \dots \gamma^{g/2 - 1}(\overline{r}_{2^k - 1})$$

$$\in \pi_{(2^k - 1)\rho_a}(MU^{(G)})$$

We want to invert this element and study the resulting slice spectral sequence. As explained previously, for $G = C_8$ it is confined to the first and third quadrants with vanishing lines of slopes 0 and 7.

The differential d_r on $u_{2\sigma}^{2^k}$ described in the theorem is the last one possible since its target, $a_{\sigma}^{2^{k+1}} f_{2^{k+1}-1}$, lies on the vanishing line. If we can show that this target is killed by an earlier differential after inverting $\overline{\mathfrak{d}}_k^G$, then $u_{2\sigma}^{2^k}$ will be a permanent cycle.

We have

$$\begin{split} f_{2^{k+1}-1}\overline{\mathfrak{d}}_{k}^{G} &= (a_{\rho_{g}}^{2^{k+1}-1}N_{2}^{g}\overline{r}_{2^{k+1}-1})(N_{2}^{g}\overline{r}_{2^{k}-1}) \\ &= a_{\rho_{g}}^{2^{k}}N_{2}^{g}\overline{r}_{2^{k+1}-1}(a_{\rho_{g}}^{2^{k}-1}N_{2}^{g}\overline{r}_{2^{k}-1}) \\ &= a_{\rho_{g}}^{2^{k}}\overline{\mathfrak{d}}_{k+1}^{G}f_{2^{k}-1} \\ &= a_{V}^{2^{k}}\overline{\mathfrak{d}}_{k+1}^{G}a_{\sigma}^{2^{k}}f_{2^{k}-1} \quad \text{where } V = \rho_{g} - \sigma \\ &= a_{V}^{2^{k}}p\overline{\mathfrak{d}}_{k+1}^{G}d_{1+8(2^{k}-1)}(u_{2^{k}\sigma}). \end{split}$$

Corollary. In the RO(G)-graded slice spectral sequence for $(\overline{\mathfrak{d}}_k^G)^{-1} MU^{(G)}$, the class $u_{2^{k+1}\sigma} = u_{2\sigma}^{2^k}$ is a permanent cycle.

5.4. An even trickier RO(G)-graded calculation. The corollary shows that inverting a certain element makes a power of $u_{2\sigma}$ a permanent cycle. We need to invert something to make a power of $u_{2\rho_8}$ a permanent cycle.

We will get this by using the norm property of u. It says that if V is an oriented representation of a subgroup $H \subset G$ with $V^H = 0$ and V' is the induced representation of V, then the norm functor N_h^g from H-spectra to G-spectra satisfies $N_h^g(u_V)u_{V''} = u_{V'}$, where V'' is the induced representation of the trivial representation of degree |V|.

From this we can deduce that $u_{2\rho_8} = u_{8\sigma_8} N_4^8(u_{4\sigma_4}) N_2^8(u_{2\sigma_2})$, where σ_g denotes the sign representation on C_g .

We have $u_{2\rho_8} = u_{8\sigma_8} N_4^8(u_{4\sigma_4}) N_2^8(u_{2\sigma_2})$.

By the Corollary we can make a power of each factor a permanent cycle by inverting some $\overline{\mathfrak{d}}_{k_m}^{(2^m)}$ for $1 \le m \le 3$. If we make k_m too small we will lose the detection property, that is we will get a spectrum that does not detect the θ_j . It turns out that k_m must be chosen so that $8|2^mk_m$.

- Inverting δ^{C₂}/₄ makes u_{32σ₂} a permanent cycle.
 Inverting δ^{C₄}/₂ makes u_{8σ4} a permanent cycle.
- Inverting $\overline{\mathfrak{d}}_1^{C_8}$ makes $u_{4\sigma_8}$ a permanent cycle.
- Inverting the product D of the norms of all three makes $u_{32\rho_8} = u_{2\rho_8}^{16}$ a permanent cycle.

Let

$$D = \overline{\mathfrak{d}}_1^{C_8} N_4^8 (\overline{\mathfrak{d}}_2^{C_4}) N_2^8 (\overline{\mathfrak{d}}_4^{C_2}) = N_2^8 (\overline{r}_1^{C_8} \overline{r}_3^{C_4} \overline{r}_{15}^{C_2}) \in \pi_{19\rho_8} (MU^{(C_8)}).$$

The we define $\Omega_{\mathbb{O}} = D^{-1}MU^{(C_8)}$ and $\Omega = \Omega_{\mathbb{O}}^{hC_8}$. Since the inverted element is represented by a map from $S^{m\rho_8}$, the slice spectral sequence for $\pi_*(\Omega) = \pi_*^{C_8}(\Omega_{\mathbb{Q}})$ has the usual properties:

- It is concentrated in the first and third quadrants and confined by vanishing lines of slopes 0 and 7.
- It has the gap property, i.e., no homotopy between dimensions -4 and 0.

5.5. The proof of the Periodicity Theorem.

Preperiodicity Theorem. Let
$$\Delta_1^{(8)} = u_{2\rho_8} \left(\overline{\mathfrak{d}}_1^{(8)}\right)^2 \in E_2^{16,0}(D^{-1}MU^{(C_8)}) = E_2^{16,0}(\Omega_{\mathbb{O}}).$$

Then $\left(\Delta_1^{(8)}\right)^{16}$ is a permanent cycle.

To prove this, note that $\left(\Delta_1^{(8)}\right)^{16} = u_{32\rho_8} \left(\overline{\Delta}_1^{(8)}\right)^{32}$. Both $u_{32\rho_8}$ and $\overline{\Delta}_1^{(8)}$ are permanent cycles, so $\left(\Delta_1^{(8)}\right)^{16}$ is also one.

Hence we have an equivariant map $\Pi:\Sigma^{256}\Omega_{\mathbb O}\to\Omega_{\mathbb O}$ where

- $u_{32\rho_8}: S^{256-32\rho_8} \to \Omega_0$ induces to the unit map from S^0 on the underlying ring spectrum and
- $\Delta_1^{(8)}$ is invertible because it is a factor of *D*.

The above imply that the underlying map $i_0\Pi$ of ordinary spectra is a homotopy equivalence. It is known that any such map induces an equivalence of homotopy fixed point sets, so

$$\Sigma^{256}\Omega^{hC_8}_{\mathbb{O}} \xrightarrow{\Pi^{hC_8}} \tilde{\Omega}^{hC_8}$$

Unfortunately the slice spectral sequence tells us nothing about this homotopy fixed point set. We know it detects all of the θ_i , but there is no direct way of showing that it has the gap property.

Fortunately we have a theorem (the subject the next section) stating that in this case the homotopy fixed set is equivalent to the actual fixed point set Ω . The slice spectral sequence tells us that the latter has the gap property. Thus we have proved

Periodicity Theorem. Let $\Omega = (D^{-1}MU^{(C_8)})^{hC_8}$. Then $\Sigma^{256}\Omega$ is equivalent to Ω .

6. THE HOMOTOPY FIXED POINT THEOREM

In order to finish the proof of the main theorem, we need to show that the actual fixed point set of $\Omega_{\mathbb{O}} = D^{-1}MU^{(C_8)}$ is equivalent to its homotopy fixed point set. A detailed account can be found in [HHR, §10]. The slice spectral sequence computes the homotopy of the former while the Hopkins-Miller spectral sequence, also known as the homotopy fixed point spectral sequence (which is known to detect θ_i), computes that of the latter.

Here is a general approach to showing that actual and homotopy fixed points are equivalent for a G-spectrum X.

We have an equivariant map $EG_+ \to S^0$. Mapping both into X gives a map of G-spectra $\varphi : X_+ \to F(EG_+, X_+)$. Passing to fixed points would give a map $X^G \to X^{hG}$, which we would like to be an equivalence. We will prove the stronger statement that φ is a G-equivalence.

The case of interest is $X = \Omega_0$ and $G = C_8$. We will argue by induction on the order of the subgroups H of G, the statement being obvious for the trivial group. We will smash φ with the isotropy separation sequence

$$EG_+ \to S^0 \to \tilde{E}G.$$

This gives us the following diagram in which both rows are cofiber sequences.

$$\begin{split} EG_{+} \wedge \Omega_{\mathbb{O}} & \longrightarrow \tilde{\Omega} & \longrightarrow \tilde{E}G \wedge \Omega_{\mathbb{O}} \\ & \downarrow^{\varphi'} & \downarrow^{\varphi} & \downarrow^{\varphi''} \\ EG_{+} \wedge F(EG_{+}, \Omega_{\mathbb{O}}) & \longrightarrow F(EG_{+}, \Omega_{\mathbb{O}}) & \longrightarrow \tilde{E}G \wedge F(EG_{+}, \Omega_{\mathbb{O}}) \end{split}$$

The map φ' is an equivalence because $\Omega_{\mathbb{O}}$ is nonequivariantly equivalent to $F(EG_+, \Omega_{\mathbb{O}})$, and EG_+ is built up entirely of free G-cells.

Thus it suffices to show that φ'' is an equivalence, which we will do by showing that both its source and target are contractible. Both have the form $\tilde{E}G \wedge X$ where X is a module spectrum over $\Omega_{\mathbb{O}}$, so it suffices to show that $\tilde{E}G \wedge \Omega_{\mathbb{O}}$ is contractible.

We will show that it is *H*-equivariantly contractible by induction on the order of the subgroups *H* of *G*. Over the trivial group $\tilde{E}G$ itself is contractible. Let *H* be a subgroup, $H' \subset H$ the subgroup of index 2 and $H_2 = H/H'$.

We will smash our spectrum with the cofiber sequence

$$EH_{2+} \to S^0 \to \tilde{E}H_2.$$

Then $EH_2 \wedge EG \wedge \Omega_0$ is contractible over H', so it suffices to show that its H-fixed point set is contractible. It is

$$\Phi^H(\tilde{E}G \wedge \Omega_{\mathbb{O}}) = \Phi^H(\tilde{E}G) \wedge \Phi^H(\Omega_{\mathbb{O}}),$$

and $\Phi^H(\Omega_{\mathbb{O}})$ is contractible because $\Phi^H(D) = 0$. Thus it remains to show that $EH_{2+} \wedge \tilde{E}G \wedge \Omega_{\mathbb{O}}$ is *H*-contractible. But this is equivalent to the *H'*-contractibility of $\tilde{E}G \wedge \Omega_{\mathbb{O}}$, which we have by induction.

7. THE DETECTION THEOREM

The Detection Theorem is the subject of [HHR, $\S11$].

7.1. θ_j in the Adams-Novikov spectral sequence. Browder's Theorem says that θ_j is detected in the classical Adams spectral sequence by

$$h_j^2 \in \operatorname{Ext}_{\mathcal{A}}^{2,2^{j+1}}(\mathbf{Z}/2,\mathbf{Z}/2),$$

where A denotes the mod 2 Steenrod algebra. This element is known to be the only one in its bidegree.

It is more convenient for us to work with the Adams-Novikov spectral sequence, which maps to the Adams spectral sequence. It has a family of elements in filtration 2, namely

$$\beta_{i/j} \in \operatorname{Ext}_{MU_*(MU)}^{2,6i-2j}(MU_*, MU_*)$$

for certain values of of i and j. When j = 1, it is customary to omit it from the notation. The definition of these elements can be found in [Rav86, Chapter 5].

Here are the first few of these in the relevant bidegrees.

and so on. In the bidegree of θ_j , only $\beta_{2^{j-1}/2^{j-1}}$ has a nontrivial image (namely h_j^2) in the Adams spectral sequence. There is an additional element in this bidegree, namely $\alpha_1 \alpha_{2^{j-1}}$.

We need to show that any element mapping to h_j^2 in the classical Adams spectral sequence has nontrivial image the Adams-Novikov spectral sequence for Ω .

Detection Theorem. Let $x \in \operatorname{Ext}_{MU_*(MU)}^{2,2^{j+1}}(MU_*, MU_*)$ be any element whose image in $\operatorname{Ext}_{\mathcal{A}}^{2,2^{j+1}}(\mathbb{Z}/2, \mathbb{Z}/2)$ is h_j^2 with $j \ge 6$. Then the image of x in $H^{2,2^{j+1}}(C_8; \pi_*(\Omega_{\mathbb{O}}))$ is nonzero.

We will prove this by showing the same is true after we map the latter to a simpler object involving another algebraic tool, *the theory of formal A-modules*, where A is the ring of integers in a suitable field.

7.2. Formal A-modules. Recall the a formal group law (see Sidebar 3) over a ring R is a power series

$$F(x,y) = x + y + \sum_{i,j>0} a_{i,j} x^i y^j \in R[[x,y]]$$

with certain properties.

For positive integers m one has power series $[m](x) \in R[[x]]$ defined recursively by [1](x) = x and

$$[m](x) = F(x, [m-1](x))$$

These satisfy

$$[m+n](x) = F([m](x), [n](x))$$
 and $[m]([n](x)) = [mn](x)$.

With these properties we can define [m](x) uniquely for all integers m, and we get a homomorphism τ from **Z** to End(F), the endomorphism ring of F.

If the ground ring R is an algebra over the p-local integers $\mathbf{Z}_{(p)}$ or the p-adic integers \mathbf{Z}_p , then we can make sense of [m](x) for m in $\mathbf{Z}_{(p)}$ or \mathbf{Z}_p .

Now suppose R is an algebra over a larger ring A, such as the ring of integers in a number field or a finite extension of the p-adic numbers. We say that the formal group law F is a *formal A-module* if the homomorphism τ extends to A in such a way that

$$[a](x) \equiv ax \mod (x^2)$$
 for $a \in A$

The theory of formal A-modules is well developed. Lubin-Tate [LT65] used them to do local class field theory.

The example of interest to us is $A = \mathbb{Z}_2[\zeta_8]$, where ζ_8 is a primitive 8th root of unity. The maximal ideal of A is generated by $\pi = \zeta_8 - 1$, and π^4 is a unit multiple of 2. There is a formal A-module F over $R_* = A[w^{\pm 1}]$ (with |w| = 2) satisfying

$$\log_F(F(x,y)) = \log_F(x) + \log_F(y)$$

where

$$\log_F(x) = \sum_{n \ge 0} \frac{w^{2^n - 1} x^{2^n}}{\pi^n}$$

The classifying map $\lambda: MU_* \to R_*$ for F factors through BP_* , where the logarithm is

$$\log(x) = \sum_{n \ge 0} \ell_n x^{2^n}.$$

Recall that $BP_* = \mathbf{Z}_{(2)}[v_1, v_2, ...]$ with $|v_n| = 2(2^n - 1)$. The v_n and the ℓ_n are related by Hazewinkel's formula,

(3)
$$2\ell_n = v_n + \sum_{0 < i < n} \ell_i v_{n-i}^{2^i}$$

This can be shown to imply that

$$\lambda(v_n) = \pi^{4-n} w^{2^n - 1} \cdot \text{unit} \quad \text{for} \quad 1 \le n \le 4,$$

where each unit is in A. It follows that there are valuations on BP_* and R_* compatible under λ with

$$||\pi|| = 1/4$$
 and $||w|| = 1$
 $||v_n|| = \max(0, (4-n)/4).$

7.3. The relation between $MU^{(C_8)}$ and formal A-modules. What does this have to do with our spectrum $\Omega_{\mathbb{O}} = D^{-1}MU^{(4)}_{\mathbf{R}}$? Recall that $D = \overline{\Delta}_1^{(8)}N_4^8(\overline{\Delta}_2^{(4)})N_2^8(\overline{\Delta}_4^{(2)})$. We saw earlier that inverting a product of this sort is needed to get the Periodicity Theorem, but we did not explain the choice of subscripts of $\overline{\Delta}$. They are the smallest ones that satisfy the second part of the following.

Lemma 2. The classifying homomorphism $\lambda : \pi_*(MU) \to R_*$ for F factors through $\pi_*(MU_{\mathbf{B}}^{(4)})$ in such a way that

- the homomorphism $\lambda^{(4)} : \pi_*(MU_{\mathbf{R}}^{(4)}) \to R_*$ is equivariant, where C_8 acts on $\pi_*(MU_{\mathbf{R}}^{(4)})$ as before, it acts trivially on A and $\gamma w = \zeta_8 w$ for a generator γ of C_8 .
- The element $D \in \pi_*(MU_{\mathbf{R}}^{(4)})$ that we invert to get $\Omega_{\mathbb{O}}$ goes to a unit in R_* .

We will sketch the proof of this later.

7.4. The proof of the Detection Theorem. It follows that we have a map

$$H^*(C_8; \pi_*(D^{-1}MU_{\mathbf{R}}^{(4)})) = H^*(C_8; \pi_*(\Omega_{\mathbb{O}})) \to H^*(C_8; R_*).$$

The source here is the E_2 -term of the homotopy fixed point spectral sequence for Ω , and the target is easy to calculate. We will use it to prove the Detection Theorem, namely

Detection Theorem. Let $x \in \operatorname{Ext}_{MU_*(MU)}^{2,2^{j+1}}(MU_*, MU_*)$ be any element whose image in $\operatorname{Ext}_A^{2,2^{j+1}}(\mathbb{Z}/2,\mathbb{Z}/2)$ is h_j^2 with $j \ge 6$. (Here A denotes the mod 2 Steenrod algebra.) Then the image of x in $H^{2,2^{j+1}}(C_8; \pi_*(\Omega_{\mathbb{O}}))$ is nonzero.

We will prove this by showing that the image of x in $H^{2,2^{j+1}}(C_8; R_*)$ is nonzero. We will calculate with BP-theory. Recall that

$$BP_*(BP) = BP_*[t_1, t_2, \dots]$$
 where $|t_n| = 2(2^n - 1)$.

We will abbreviate $\operatorname{Ext}_{BP_*(BP)}^{s,t}(BP_*, BP_*)$ by $\operatorname{Ext}^{s,t}$.

There is a map from this Hopf algebroid to one associated with $H^*(C_8; R_*)$ in which t_n maps to an R_* -valued function on C_8 (regarded as the group of 8th roots of unity) determined by

$$[\zeta](x) = \sum_{n \ge 0}^{F} \langle t_n, \zeta \rangle x^{2^n}.$$

An easy calculation shows that the function t_1 sends a primitive root in C_8 to a unit in R_* . Let

$$b_{1,j-1} = \frac{1}{2} \sum_{0 < i < 2^j} {2^j \choose i} \left[t_1^i | t_1^{2^j - i} \right] \in \operatorname{Ext}^{2, 2^{j+1}}$$

It is is known to be cohomologous to $\beta_{2^{j-1}/2^{j-1}}$ and to have order 2. We will show that its image in $H^{2,2^{j+1}}(C_8; R_*)$ is nontrivial for $j \ge 2$.

 $H^*(C_8; R_*)$ is the cohomology of the cochain complex

$$R_*[C_8] \xrightarrow{\gamma-1} R_*[C_8] \xrightarrow{\text{Trace}} R_*[C_8] \xrightarrow{\gamma-1} \cdots$$

where Trace is multiplication by $1 + \gamma + \cdots + \gamma^7$.

The cohomology groups $H^s(C_8; R_*)$ for s > 0 are periodic in s with period 2. The part of it that matters to us is

$$H^2(C_8; R_{2m}) = w^m A/(8)$$
 for $m \equiv 0 \mod 8$

An easy calculation shows that $b_{1,j-1}$ maps to $4w^{2^j}$, which is the element of order 2 in $H^2(C_8; R_{2^{j+1}})$.

To finish the proof we need to show that the other β s in the same bidegree map to zero. We will do this for $j \ge 6$. The set of these is

$$\left\{\beta_{c(j,k)/2^{j-1-2k}} : 0 \le k < j/2\right\}$$

where $c(j,k) = 2^{j-1-2k}(1+2^{2k+1})/3$. Note that $\beta_{c(j,0)/2^{j-1}} = \beta_{2^{j-1}/2^{j-1}}$, so we need to show that the elements with k > 0 map to zero.

Using the valuation on BP_* defined above, we find that for $k \ge 1$ and $j \ge 6$

$$||\beta_{c(j,k)/2^{j-1-2k}}|| = \left|\left|\frac{v_2^{c(j,k)}}{2v_1^{2^{j-1-2k}}}\right|\right| \ge 5.$$

This means $\beta_{c(j,k))/2^{j-1-2k}}$ maps to an element that is divisible by 8 and therefore zero.

A similar computation with the element $\alpha_1 \alpha_{2^j-1}$ shows

$$||\alpha_{2^{j}-1}|| \ge 4$$
 for $j \ge 3$.

This completes the proof of the Detection Theorem modulo the lemma.

7.5. The proof of Lemma 2. To prove the first part, consider the following diagram for an arbitrary ring K.



The maps λ_1 and λ_2 classify two formal group laws F_1 and F_2 over K. The Hopf algebroid $MU_*(MU)$ represents strict isomorphisms between formal group laws. Hence the existence of $\lambda^{(2)}$ is equivalent to that of a compatible strict isomorphism between F_1 and F_2 .

Similarly consider the diagram



The existence of $\lambda^{(4)}$ is equivalent to that of compatible strict isomorphisms between the formal group laws F_j classified by the λ_j .

Now suppose that K has a C_8 -action and that $\lambda^{(4)}$ is equivariant with respect to the previously defined C_8 -action on $MU_{\mathbf{R}}^{(4)}$. Then the isomorphism induced by the fourth power of a generator $\gamma \in C_8$ is the isomorphism sending x to its formal inverse on each of the F_i .

This means that the existence of an equivariant $\lambda^{(4)}$ is equivalent to that of a formal $\mathbb{Z}[\zeta_8]$ -module structure on each of the F_j , which are all isomorphic. This proves the first part of the Lemma.

For the second part of the Lemma, recall that

$$D = N_2^8 (\overline{r}_{15}^{C_2} \overline{r}_3^{C_4} \overline{r}_1^{C_8}).$$

The norm sends products to products, and N(x) is a product of conjugates of x under the action of C_8 . Hence its image in R_* is a unit multiple of that of a power of x, so it suffices to show that each of the three elements $\overline{r}_{15}^{C_2}$, $\overline{r}_3^{C_4}$ and $\overline{r}_1^{C_8}$ maps to a unit in R_* . We refer the reader to [HHR, §11] for the details.

8. THE SLICE AND REDUCTION THEOREMS

Recall that a pivotal step in our proof is the Slice Theorem, which identifies the layers in the slice tower for $MU_{\mathbf{R}}$ and its relatives. On Friday Dugger will explain how it follows from the Reduction Theorem, which will be stated below.

For each cyclic 2-group $G = C_{2^{n+1}}$ there is a certain equivariant (noncommutative) ring spectrum A which is a certain wedge of slice cells. It maps to $N_2^g M U_{\mathbf{R}}$ in such a way that the underlying wedge of spheres hits all of the underlying homotopy of $MU^{(2^n)}$. Thus both $N_2^g M U_{\mathbf{R}}$ and S^0 are A-modules.

Reduction Theorem. The A-smash product $N_2^g M U_{\mathbf{R}} \wedge_A S^0$ is equivariantly equivalent to the integer Eilenberg-Mac Lane spectrum HZ.

The proof of this is the hardest calculation in our paper and is treated in [HHR, $\S7$].

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Sidebar 3 Formal group laws

Formal groups laws are are key tool in complex cobordism theory, which is part of the infrastructure of our proof. The definitive reference is Hazewinkel's book [Haz78]. A much briefer account can be found in [Rav86, Appendix 2]. A *formal group law F over* a ring R is a power series $F(x, y) \in R[[x, y]]$ satisfying three conditions, namely

- (i) COMMUTATIVITY F(y, x) = F(x, y)
- (ii) IDENTITY F(x, 0) = F(0, x) = x
- (iii) ASSOCIATIVITY F(F(x, y), z) = F(x, F(y, z))

Elementary examples include the additive and multiplicative formal group laws, x + yand 1 + x + y + xy respectively. When R is torsion free, F is determined by a power series over $R \otimes \mathbf{Q}$ in one variable called the *logarithm of F* satisfying

$$\log_F(F(x,y)) = \log_F(x) + \log_F(y)$$

For our two elementary examples it is x and $\sum_{n>0}(-1)^n x^{n+1}/(n+1)$.

There is a universal example due to Lazard [Laz55] of a formal group law G over a ring L with the property that for any formal group law F over any ring R, there is a unique ring homomorphism $\theta : L \to R$ sending G to F. It is given by

$$G(x,y) = \sum_{i,j} a_{i,j} x^i y^j$$
 and $L = \mathbf{Z}[a_{i,j}]/I$

where I is the ideal generated by the relations among the coefficients $a_{i,j}$ forced by conditions (i)-(iii) above. To describe L explicitly, it is useful to give it a grading such that G(x, y) is homogeneous of degree 2 if |x| = |y| = 2. This means that $a_{i,j}| = 2(1 - i - j)$. Lazard then shows that

$$L = \mathbf{Z}[x_i : i > 0] \qquad \text{where } |x_i| = -2i.$$

There is a formal group law F defined over the complex cobordism ring MU^* defined as follows. We know that

$$MU^*(\mathbb{C}P^\infty) = MU^*[[x]]$$

where $x \in MU^2 \mathbb{C}P^{\infty}$ is the first Chern class of the canonical complex line bundle. Similarly,

$$MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = MU^*[[x \otimes 1, 1 \otimes x]]$$

There is a map $\mathbb{C}P^{\infty} \times \mathbb{C}P^{\infty} \to \mathbb{C}P^{\infty}$ related to the tensor product of line bundles. It sends x to a power series $F(x \otimes 1, 1 \otimes x)$, which can easily be shown to be a formal group law. Quillen [Qui69] showed that the Lazard classifing map $L \to MU^*$ is an isomorphism.

Cartier [Car67] showed that when R is a torsion free $\mathbf{Z}_{(p)}$ -algebra, F is canonically isomorphic to a formal group law F' which is *p*-typical, meaning that is logarithm has the form

$$\log_{F'}(x) = x + \sum_{n>0} \ell_n x^{p^n} \in R \otimes \mathbf{Q}[[x]].$$

The analog of the Lazard ring for such groups is much smaller than L, having polynomial generators v_n only in each dimension of the form $2(1-p^n)$. The ℓ_n and the v_n are related by Hazewinkel's formula (3).