A guided tour of the EHP sequence

Happy Birthday Joe!

Doug Ravenel
University of Rochester
November 20, 2005

The EHP sequence for $p = 2$

The EHP sequence is a recursive method for computing the homotopy groups of spheres. It has a p-primary version for each prime p. It is easiest to describe at $p = 2$, but we will concentrate later on the case $p = 3$.

At $p = 2$, for each $n > 0$ there is a 2-local fiber sequence

$$S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}$$

leading to long exact sequences of homotopy groups

$$\cdots \longrightarrow \pi_{n+k}(S^n) \xrightarrow{E} \pi_{n+k+1}(S^{n+1}) \xrightarrow{H} \pi_{n+k+1}(S^{2n+1}) \xrightarrow{P} \pi_{n+k-1}(S^n) \longrightarrow \cdots$$

These can be assembled in to an exact couple, which leads to a spectral sequence that we will say more about later.

Results of James, Serre and Toda

James defined the *kth reduced product* (or James construction) J_kX for a pointed space X to be the quotient of the k-fold Cartesian product of X obtained by allowing two adjacent coordinates to be interchanged if one of them is the base point. Thus we get maps

$$X = J_1X \to J_2X \to J_3X \to \cdots$$

and we can define $J_\infty X$ to be the direct limit.

He then constructed a map $J_\infty X \to \Omega\Sigma X$ and proved that it is a weak equivalence. He also showed that there is a splitting

$$\Sigma J_kX \simeq \bigvee_{1\le i\le k} \Sigma X^{(i)}$$

where $X^{(i)}$ denote the i-fold smash power of X.

For $X = S^n$ this means that

$$\Omega S^{n+1} \simeq J_\infty S^n$$

and the splitting leads to projection maps

$$\Sigma\Omega S^{n+1} \to S^{kn+1} \qquad \text{for each } k \ge 0$$

which are adjoint to maps

$$\Omega S^{n+1} \xrightarrow{H_k} \Omega S^{kn+1}$$

known as the *James-Hopf maps*. The construction of the EHP sequence depends on determining the fibers of these maps in certain cases.

Theorem 1. (i) *The fiber of the James-Hopf map*

$$\Omega S^{2n} \xrightarrow{H_2} \Omega S^{4n-1}$$

is S^{2n-1}, and there is an odd primary equivalence (due to Serre)

$$\Omega S^{2n} \simeq S^{2n-1} \times \Omega S^{4n-1}.$$

(ii) *The p-local fiber of*

$$\Omega S^{2n+1} \xrightarrow{H_p} \Omega S^{2pn+1}$$

is $J_{p-1}S^{2n}$.

For $p = 2$ this gives us the fiber sequences mentioned at the start of the talk.

For p odd, (i) says that the homotopy groups of an even dimensional sphere can be expressed in terms of those of odd dimensional spheres, so even dimensional spheres are *uninteresting*. It is useful to replace S^{2n} by

$$\widehat{S}^{2n} := J_{p-1}S^{2n}.$$

Then (ii) gives us a p-local fiber sequence

$$\widehat{S}^{2n} \xrightarrow{E} \Omega S^{2n+1} \xrightarrow{H_p} \Omega S^{2pn+1}$$

The odd primary replacement for the fibration

$$S^{2n-1} \xrightarrow{E} \Omega S^{2n} \xrightarrow{H_p} \Omega S^{4n-1}$$

is the fibration

$$\Omega S^{2n-1} \xrightarrow{E} \Omega^2 \widehat{S}^{2n} \longrightarrow \Omega^2 S^{2pn-1}$$

constructed by Toda.

Mahowald's master diagram at $p = 2$

The 2-primary fibration can be rewritten as

$$\Omega^n S^n \longrightarrow \Omega^{n+1} S^{n+1} \longrightarrow \Omega^{n+1} S^{2n+1}$$

and this fits into the following commutative diagram.

$$\begin{array}{ccccc}
\mathbf{R}P^{n-1} & \longrightarrow & \mathbf{R}P^n & \longrightarrow & S^n \\
\downarrow \rho & & \downarrow \rho & & \| \\
O(n) & \longrightarrow & O(n+1) & \longrightarrow & S^n \\
\downarrow j & & \downarrow j & & \downarrow \\
\Omega^n S^n & \longrightarrow & \Omega^{n+1} S^{n+1} & \longrightarrow & \Omega^{n+1} S^{2n+1} \\
\downarrow s & & \downarrow s & & \downarrow \\
Q\mathbf{R}P^{n-1} & \longrightarrow & Q\mathbf{R}P^n & \longrightarrow & QS^n \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^\infty(\mathbf{R}P^{n-1} \wedge J) & \longrightarrow & \Omega^\infty(\mathbf{R}P^n \wedge J) & \longrightarrow & \Omega^\infty(S^n \wedge J)
\end{array}$$

The top row is a cofiber sequence, and every other row is a fiber sequence. The vertical maps ρ, j and s are the reflection map, the map inducing the J-homoorphism, and the Snaith map respectively.

Each of the bottom four rows leads to a spectral sequence of homotopy groups. The ones for the third and fourth rows are called the *EHP spectral sequence* and *stable EHP spectral sequence* respectively, with the latter converging to the stable homotopy of $\mathbf{R}P^\infty$. The one for the bottom row is explicitly known, and this gives us a lot of information about the EHP sequence.

The odd primary master diagram

For odd primes there is no known analog of the second row, and we need to replace the real projective spaces above by suitable skeleta of the p-localization of $B\Sigma_p$, the classifying space of the symmetric group on p letters. We will denote this space simply by B. It has one cell in each dimension congruent to 0 and -1 modulo $q = 2p - 2$. There are two diagrams depending on the parity of the dimension of the sphere. They are

$$\begin{array}{ccccc}
B^{qn-q} & \longrightarrow & B^{qn-1} & \longrightarrow & S^{qn-1} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{2n-1}S^{2n-1} & \longrightarrow & \Omega^{2n}\widehat{S}^{2n} & \longrightarrow & \Omega^{2n}S^{2pn-1} \\
{\scriptstyle s}\downarrow & & {\scriptstyle s}\downarrow & & \downarrow \\
QB^{qn-q} & \longrightarrow & QB^{qn-1} & \longrightarrow & QS^{qn-1} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\infty}(B^{qn-q}\wedge J) & \longrightarrow & \Omega^{\infty}(B^{qn-1}\wedge J) & \longrightarrow & \Omega^{\infty}(S^{qn-1}\wedge J)
\end{array}$$

and

$$\begin{array}{ccccc}
B^{qn-1} & \longrightarrow & B^{qn} & \longrightarrow & S^{qn} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{2n}\widehat{S}^{2n} & \longrightarrow & \Omega^{2n+1}S^{2n+1} & \longrightarrow & \Omega^{2n+1}S^{2pn+1} \\
{\scriptstyle s}\downarrow & & {\scriptstyle s}\downarrow & & \downarrow \\
QB^{qn-1} & \longrightarrow & QB^{qn} & \longrightarrow & QS^{qn} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\infty}(B^{qn-1}\wedge J) & \longrightarrow & \Omega^{\infty}(B^{qn}\wedge J) & \longrightarrow & \Omega^{\infty}(S^{qn}\wedge J)
\end{array}$$

The EHP spectral sequence for $p = 3$

Theorem 2. *For each prime p there is a spectral sequence converging the the the homotopy of the p-local sphere spectrum with*

$$E_1^{k,2m+1} = \pi_{k+2m+1}(S^{2pm+1})$$

and

$$E_1^{k,2m} = \pi_{k+2m}(S^{2pm-1})$$

with

$$d_r : E_r^{k,n} \to E_r^{k-1,n-r}$$

there is a similar spectral sequence converging to $\pi_*(S^n_{(p)})$ if n is odd and $\pi_*(\widehat{S}^n_{(p)})$ if n is even, with $E_1^{k,j} = 0$ for $j > n$.

Here is a picture of the E_1-term modulo torsion in low dimensions.

	k	0	1	2	3	4	5	6	7	8	9	10	11	12
$[S^1]$	$n=1$	$\mathbf{Z}$												
$[S^5]$	$n=2$				$\mathbf{Z}$									
$[S^7]$	$n=3$					$\mathbf{Z}$								
$[S^{11}]$	$n=4$								$\mathbf{Z}$					
$[S^{13}]$	$n=5$									$\mathbf{Z}$				
$[S^{17}]$	$n=6$												$\mathbf{Z}$	
$[S^{19}]$	$n=7$													$\mathbf{Z}$

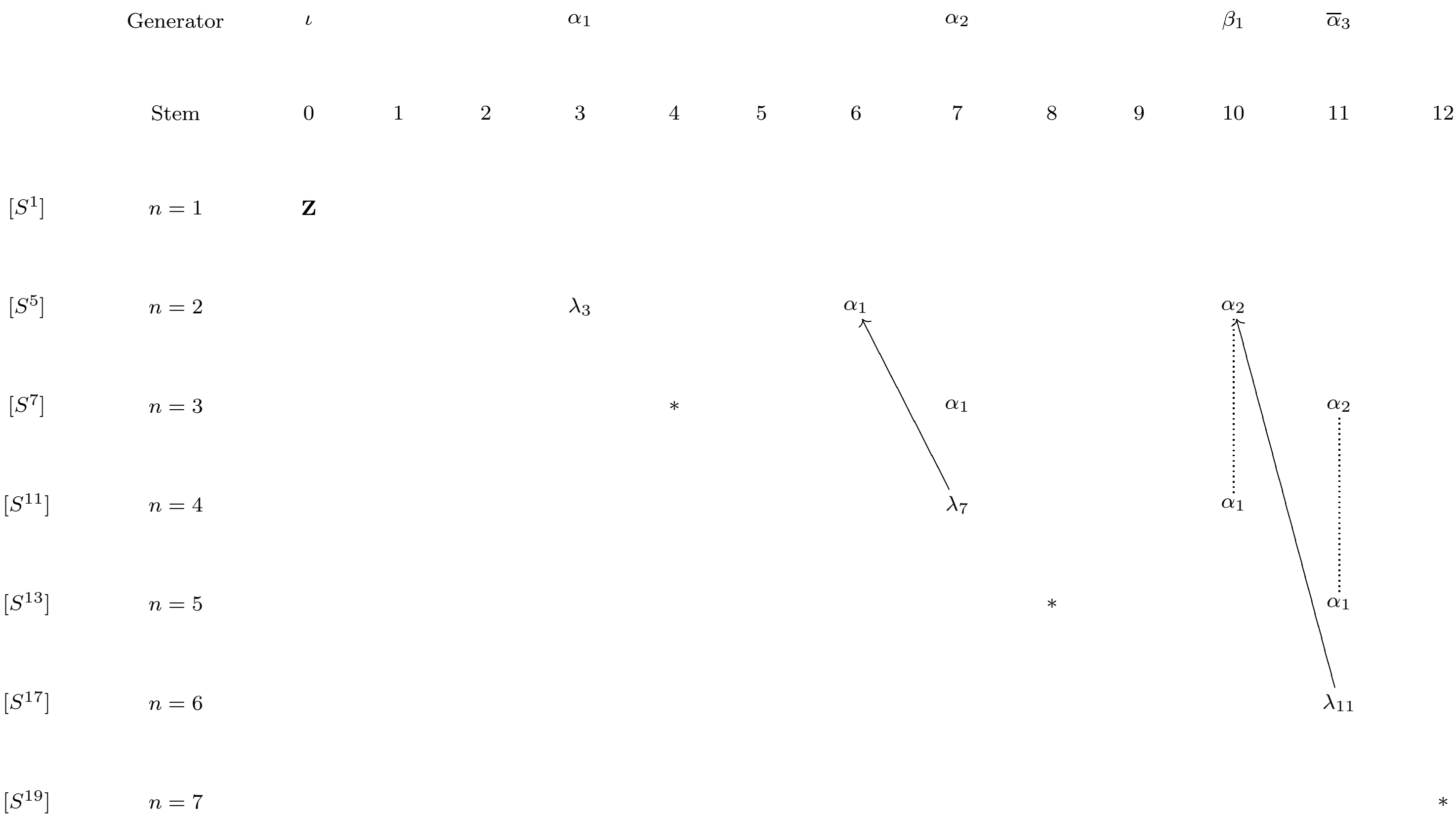
Here are some simple observations.

- In the row for $n = 1$, we have $E_1^{k,1} = 0$ for $k > 0$ since we know all of $\pi_*(S^1)$.
- $E_1^{k,n} = 0$ when k is small in relation to n due to the connectivity of the sphere indicated on the far left of each row.
- The differential
$$d_1 : E_1^{qm,2m+1} \longrightarrow E_1^{qm-1,2m}$$
is multiplication by p.

From these we can deduce the following groups in the 3-stem:

$$\begin{aligned} \pi_4(S^1) &= 0 \\ \pi_5(\widehat{S}^2) &= \mathbf{Z} \\ \pi_{2n+2}(S^{2n-1}) = \pi_{2n+3}(\widehat{S}^{2n}) &= \mathbf{Z}/(3) \qquad \text{for } n > 1. \end{aligned}$$

The surviving generator is α_1, and it follows that it appears in each row of the E_1-term for $n > 1$.



The following higher differentials and group extensions occur in this range. They can all be inferred from the bottom row of the master diagram.

$$
\begin{aligned}
d_2(\lambda_7) &= \lambda_3\alpha_1 \\
d_4(\lambda_{11}) &= \lambda_3\alpha_2 \\
3\cdot\lambda_7\alpha_1 &= \lambda_3\alpha_2 \\
3\cdot\lambda_8\alpha_1 &= \lambda_4\alpha_2
\end{aligned}
$$

University of Rochester, Rochester, NY 14627