The detection theorem

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1 θ_j in the Adams-Novikov spectral sequence

θ_j in the Adams-Novikov spectral sequence

Browder's theorem says that θ_j is detected in the classical Adams spectral sequence by

$$h_j^2 \in \operatorname{Ext}_A^{2,2^{j+1}}(\mathbf{Z}/2,\mathbf{Z}/2).$$

This element is known to be the only one in its bidegree.

It is more convenient for us to work with the Adams-Novikov spectral sequence, which maps to the Adams spectral sequence. It has a family of elements in filtration 2, namely

$$\beta_{i/j} \in \operatorname{Ext}^{2,6i-2j}_{MU_*(MU)}(MU_*,MU_*)$$

for certain values of of i and j. When $j=1$, it is customary to omit it from the notation. 1.2

θ_j in the Adams-Novikov spectral sequence (continued)

Here are the first few of these in the relevant bidegrees.

θ_4 :	$\beta_{8/8}$ and $\beta_{6/2}$
θ_5 :	$\beta_{16/16}$, $\beta_{12/4}$ and β_{11}
θ_6 :	$\beta_{32/32}$, $\beta_{24/8}$ and $\beta_{22/2}$
θ_7 :	$\beta_{64/64}$, $\beta_{48/16}$, $\beta_{44/4}$ and β_{43}

and so on. In the bidegree of θ_j, only $\beta_{2^{j-1}/2^{j-1}}$ has a nontrivial image (namely h_j^2) in the Adams spectral sequence. There is an additional element in this bidegree, namely $\alpha_1\alpha_{2^j-1}$.

We need to show that any element mapping to h_j^2 in the classical Adams spectral sequence has nontrivial image the Adams-Novikov spectral sequence for Ω. 1.3

θ_j in the Adams-Novikov spectral sequence (continued)

Detection Theorem. *Let $x \in \operatorname{Ext}^{2,2^{j+1}}_{MU_*(MU)}(MU_*,MU_*)$ be any element whose image in $\operatorname{Ext}_A^{2,2^{j+1}}(\mathbf{Z}/2,\mathbf{Z}/2)$ is h_j^2 with $j\geq 6$. (Here A denotes the mod 2 Steenrod algebra.) Then the image of x in $H^{2,2^{j+1}}(C_8;\pi_*(\tilde{\Omega}))$ is nonzero.*

We will prove this by showing the same is true after we map the latter to a simpler object involving another algebraic tool, the theory of formal A-modules, where A is the ring of integers in a suitable field. 1.4

2 Formal A-modules

Formal A-modules

Recall the a formal group law over a ring R is a power series

$$F(x,y) = x + y + \sum_{i,j>0} a_{i,j} x^i y^j \in R[[x,y]]$$

with certain properties.

For positive integers m one has power series $[m](x) \in R[[x]]$ defined recursively by $[1](x) = x$ and

$$[m](x) = F(x, [m-1](x)).$$

These satisfy

$$[m+n](x) = F([m](x), [n](x)) \text{ and } [m]([n](x)) = [mn](x).$$

With these properties we can define $[m](x)$ uniquely for all integers m, and we get a homomorphism τ from $\mathbf{Z}$ to $End(F)$, the endomorphism ring of F. 1.5

Formal A-modules (continued)

If the ground ring R is an algebra over the p-local integers $\mathbf{Z}_{(p)}$ or the p-adic integers $\mathbf{Z}_p$, then we can make sense of $[m](x)$ for m in $\mathbf{Z}_{(p)}$ or $\mathbf{Z}_p$.

Now suppose R is an algebra over a larger ring A, such as the ring of integers in a number field or a finite extension of the p-adic numbers. We say that the formal group law F is a formal A-module if the homomorphism τ extends to A in such a way that

$$[a](x) \equiv ax \bmod (x^2) \text{ for } a \in A.$$

The theory of formal A-modules is well developed. Lubin-Tate used them to do local class field theory. 1.6

Formal A-modules (continued)

The example of interest to us is $A = \mathbf{Z}_2[x]/(x^4+1) = \mathbf{Z}_2[\zeta_8]$, where ζ_8 is a primitive 8th root of unity. The maximal ideal of A is generated by $\pi = \zeta_8 - 1$, and π^4 is a unit multiple of 2. There is a formal A-module G over $R_* = A[w^{\pm 1}]$ (with $|w| = 2$) satisfying

$$\log_G(G(x,y)) = \log_G(x) + \log_G(y)$$

where

$$\log_G(x) = \sum_{n \ge 0} \frac{w^{2^n - 1} x^{2^n}}{\pi^n}.$$

The classifying map $\lambda : MU_* \to R_*$ for G factors through BP_*, where the logarithm is

$$\log_F(x) = \sum_{n \ge 0} \ell_n x^{2^n}.$$

1.7

Formal A-modules (continued)

Recall that $BP_* = \mathbf{Z}_{(2)}[v_1, v_2, \dots]$ with $|v_n| = 2(2^n - 1)$. The v_n and the ℓ_n are related by Hazewinkel's formula,

$$\begin{aligned}
\ell_1 &= \frac{v_1}{2} \\
\ell_2 &= \frac{v_2}{2} + \frac{v_1^3}{4} \\
\ell_3 &= \frac{v_3}{2} + \frac{v_1 v_2^2 + v_2 v_1^4}{4} + \frac{v_1^7}{8} \\
\ell_4 &= \frac{v_4}{2} + \frac{v_1 v_3^2 + v_2^5 + v_3 v_1^8}{4} + \frac{v_1^3 v_2^4 + v_1^9 v_2^2 + v_2 v_1^{12}}{8} + \frac{v_1^{15}}{16} \\
&\vdots
\end{aligned}$$

1.8

3 $\pi_*(MU^{(4)})$ and R_*

The relation between $MU^{(4)}$ and formal A-modules

What does all this have to do with our spectrum $\tilde{\Omega} = D^{-1}MU^{(4)}$? Recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$. We saw earlier that inverting a product of this sort is needed to get the Periodicity Theorem, but we did not explain the choice of subscripts of $\overline{\Delta}$. They are the smallest ones that satisfy the second part of the following.

Lemma. *The classifying homomorphism $\lambda : \pi_*(MU) \to R_*$ for G factors through $\pi_*(MU^{(4)})$ in such a way that*

- *the homomorphism $\lambda^{(4)} : \pi_*(MU^{(4)}) \to R_*$ is equivariant, where C_8 acts on $\pi_*(MU^{(4)})$ as before, it acts trivially on A and $\gamma w = \zeta_8 w$ for a generator γ of C_8.*
- *The element $D \in \pi_*(MU^{(4)})$ that we invert to get $\tilde{\Omega}$ goes to a unit in R_*.*

We will prove this later.

1.9

4 The proof of the Detection Theorem

The proof of the Detection Theorem

It follows that we have a map

$$H^*(C_8; \pi_*(D^{-1}MU^{(4)})) = H^*(C_8; \pi_*(\tilde{\Omega})) \to H^*(C_8; R_*).$$

The source here is the E_2-term of the homotopy fixed point spectral sequence for $\pi_*(\Omega)$, and the target is easy to calculate. We will use it to prove the Detection Theorem, namely

Detection Theorem. *Let $x \in \mathrm{Ext}^{2,2^{j+1}}_{MU_*(MU)}(MU_*, MU_*)$ be any element whose image in $\mathrm{Ext}^{2,2^{j+1}}_A(\mathbf{Z}/2, \mathbf{Z}/2)$ is h_j^2 with $j \geq 6$. (Here A denotes the mod 2 Steenrod algebra.) Then the image of x in $H^{2,2^{j+1}}(C_8; \pi_*(\tilde{\Omega}))$ is nonzero.*

We will prove this by showing that the image of x in $H^{2,2^{j+1}}(C_8; R_*)$ is nonzero. 1.10

The proof of the Detection Theorem (continued)

We will calculate with BP-theory. Recall that

$$BP_*(BP) = BP_*[t_1, t_2, \dots] \quad \text{where } |t_n| = 2(2^n - 1).$$

We will abbreviate $\mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*, BP_*)$ by $\mathrm{Ext}^{s,t}$. For a $BP_*(BP)$-comodule M (such as $BP_*(X)$), we will abbreviate $\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*)$ by $\mathrm{Ext}(M)$.

There is a map from this Hopf algebroid to one associated with $H^*(C_8;R_*)$ in which t_n maps to an R_*-valued function on C_8 (regarded as the group of 8th roots of unity) determined by

$$[\zeta](x) = \sum_{n\geq 0}^{F} \langle t_n,\, \zeta\rangle x^{2^n}.$$

An easy calculation shows that the function t_1 sends a primitive root in C_8 to a unit in R_*. 1.11

The proof of the Detection Theorem (continued)

Let

$$b_{1,j-1} = \frac{1}{2}\sum_{0<i<2^j}\binom{2^j}{i}\left[t_1^i | t_1^{2^j-i}\right] \in \mathrm{Ext}^{2,2^{j+1}}$$

It is is known to be cohomologous to $\beta_{2^{j-1}/2^{j-1}}$ and to have order 2. We will show that its image in $H^{2,2^{j+1}}(C_8;R_*)$ is nontrivial for $j\geq 2$.

$H^*(C_8;R_*)$ is the cohomology of the cochain complex

$$R_*[C_8] \xrightarrow{\gamma-1} R_*[C_8] \xrightarrow{\text{Trace}} R_*[C_8] \xrightarrow{\gamma-1} \dots$$

where Trace is multiplication by $1+\gamma+\cdots+\gamma^7$. 1.12

The proof of the Detection Theorem (continued)

The cohomology groups $H^s(C_8;R_*)$ for $s>0$ are periodic in s with period 2. We have

$$\begin{aligned}
H^1(C_8;R_{2m}) &= \ker(1+\zeta_8^m+\cdots+\zeta_8^{7m})/\mathrm{im}\,(\zeta_8^m-1)\\
&= \begin{cases} w^mA/(\pi) & \text{for } m \text{ odd}\\ w^mA/(\pi^2) & \text{for } m\equiv 2 \bmod 4\\ w^mA/(2) & \text{for } m\equiv 4 \bmod 8\\ 0 & \text{for } m\equiv 0 \bmod 8\end{cases}\\
H^2(C_8;R_{2m}) &= \ker(\zeta_8^m-1)/\mathrm{im}\,(1+\zeta_8^m+\cdots+\zeta_8^{7m})\\
&= \begin{cases} w^mA/(8) & \text{for } m\equiv 0 \bmod 8\\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

An easy calculation shows that $b_{1,j-1}$ maps to $4w^{2^j}$, which is the element of order 2 in $H^2(C_8;R_{2^{j+1}})$. 1.13

Sidebar on chromatic fractions

It is common to write $\beta_{i/j}$ as a chromatic fraction $\frac{v_2^i}{2v_1^j}$. What does this mean? For suitable i and j, v_2^i is an element of $\mathrm{Ext}^{0,6i}\left(BP_*/(2,v_1^j)\right)$ and there are short exact sequences

$$0 \longrightarrow BP_*/(2) \xrightarrow{v_1^j} \Sigma^{-2j}BP_*/(2) \longrightarrow \Sigma^{-2j}BP_*/(2,v_1^j) \longrightarrow 0$$

and

$$0 \longrightarrow BP_* \xrightarrow{2} BP_* \longrightarrow BP_*/(2) \longrightarrow 0$$

leading to connecting homomorphisms

$$\begin{array}{ccccc}
\mathrm{Ext}^{0,6i}\left(BP_*/(2,v_1^j)\right) & \to & \mathrm{Ext}^{1,6i-2j}\left(BP_*/(2)\right) & \to & \mathrm{Ext}^{2,6i-2j}\left(BP_*\right)\\
v_2^i & \longmapsto & \frac{v_2^i}{v_1^j} & \longmapsto & \frac{v_2^i}{2v_1^j}.
\end{array}$$

1.14

The proof of the Detection Theorem (continued)

To finish the proof we need to show that the other βs in the same bidegree as $\beta_{2^{j-1}/2^{j-1}} = \beta_{c(j,0)/2^{j-1}}$ map to zero. We will do this for $j \geq 6$. The set of these is

$$\left\{\beta_{c(j,k)/2^{j-1-2k}} : 0 < k < j/2\right\}$$

where $c(j,k) = 2^{j-1-2k}(1+2^{2k+1})/3$.

We will see in the proof of the Lemma below that v_1 and v_2 map to unit multiples of $\pi^3 w$ and $\pi^2 w^3$ respectively. This means we can define a valuation on chromatic fractions compatible with the one on A in which $||2|| = 1$, $||\pi|| = 1/4$, $||v_1|| = 3/4$ and $||v_2|| = 1/2$. We extend the valuation on A to R_* by setting $||w|| = 0$. 1.15

The proof of the Detection Theorem (continued)

Hence for $k \geq 1$ and $j \geq 6$ we have

$$\begin{aligned} ||\beta_{c(j,k)/2^{j-1-2k}}|| &= \left|\left|\frac{v_2^{c(j,k)}}{2v_1^{2^{j-1-2k}}}\right|\right| \\ &= \frac{c(j,k)}{2} - \frac{3\cdot 2^{j-1-2k}}{4} - 1 \\ &= \frac{2^j + 2^{j-1-2k}}{6} - \frac{3\cdot 2^{j-1-2k}}{4} - 1 \\ &= (2^{j-1} - 7\cdot 2^{j-3-2k})/3 - 1 \ \geq\ 5. \end{aligned}$$

This means $\beta_{c(j,k)/2^{j-1-2k}}$ maps to an element that is divisible by 8 and therefore zero, since the homomorphism cannot lower this valuation. 1.16

The proof of the Detection Theorem (continued)

We have to make a similar computation with the element $\alpha_1\alpha_{2^j-1}$. We have

$$\begin{aligned} ||\alpha_{2^j-1}|| &= \left|\left|\frac{v_1^{2^j-1}}{2}\right|\right| \\ &= \frac{3(2^j-1)}{4} - 1 \\ &\geq \frac{21}{4} - 1 \geq 4 \quad \text{for } j \geq 3. \end{aligned}$$

This completes the proof of the Detection Theorem modulo the Lemma. 1.17

5 The proof of the Lemma

The proof of the Lemma

Here it is again.

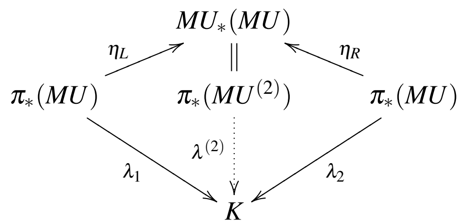
Lemma. *The classifying homomorphism $\lambda : \pi_*(MU) \to R_*$ for G factors through $\pi_*(MU^{(4)})$ in such a way that*

- *the homomorphism $\lambda^{(4)} : \pi_*(MU^{(4)}) \to R_*$ is equivariant, where C_8 acts on $\pi_*(MU^{(4)})$ as before, it acts trivially on A and $\gamma w = \zeta_8 w$ for a generator γ of C_8.*
- *The element $D \in \pi_*(MU^{(4)})$ that we invert to get $\tilde{\Omega}$ goes to a unit in R_*.*

1.18

The proof of the Lemma (continued)

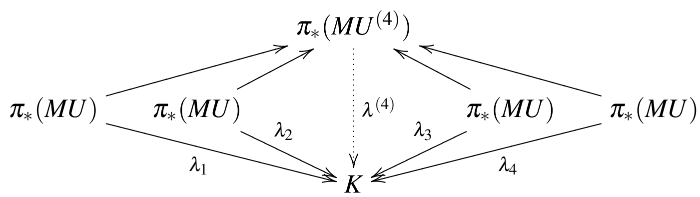
To prove the first part, consider the following diagram for an arbitrary ring K.



The maps λ_1 and λ_2 classify two formal group laws F_1 and F_2 over K. The Hopf algebroid $MU_*(MU)$ represents strict isomorphisms between formal group laws. Hence the existence of $\lambda^{(2)}$ is equivalent to that of a compatible strict isomorphism between F_1 and F_2. 1.19

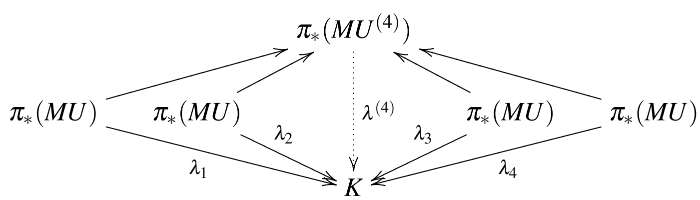
The proof of the Lemma (continued)

Similarly consider the diagram



The existence of $\lambda^{(4)}$ is equivalent to that of compatible strict isomorphisms between the four formal group laws F_j classified by the λ_j. 1.20

The proof of the Lemma (continued)



Now suppose further that K has a C_8-action and that $\lambda^{(4)}$ is equivariant with respect to the previously defined C_8-action on $MU^{(4)}$. Then the isomorphism induced by the fourth power of a generator $\gamma \in C_8$ is the isomorphism sending x to its formal inverse on each of the F_j.

This means that the existence of an equivariant $\lambda^{(4)}$ is equivalent to that of a formal $\mathbf{Z}[\zeta_8]$-module structure on each of the F_j, which are all isomorphic. Setting $K = R_*$ proves the first part of the Lemma. 1.21

The proof of the Lemma (continued)

Here is the Lemma again.

Lemma. *The classifying homomorphism $\lambda : \pi_*(MU) \to R_*$ for G factors through $\pi_*(MU^{(4)})$ in such a way that*

- *the homomorphism $\lambda^{(4)} : \pi_*(MU^{(4)}) \to R_*$ is equivariant, where C_8 acts on $\pi_*(MU^{(4)})$ as before, it acts trivially on A and $\gamma w = \zeta_8 w$ for a generator γ of C_8.*
- *The element $D \in \pi_*(MU^{(4)})$ that we invert to get $\tilde{\Omega}$ goes to a unit in R_*.*

1.22

The proof of the Lemma (continued)

For the second part, recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$, where

$$\overline{\Delta}_k^{(g)} = \begin{cases} x_{2^k-1} & \text{for } g = 2 \\ N_4^g(r_{2^k-1}) & \text{otherwise.} \end{cases}$$

Since our formal A-module is 2-typical we can do the calculations using BP in place of MU. Hence we can replace $x_{2^k-1} \in \pi_* MU$ by $v_k \in \pi_* BP$ and $r_{2^k-1} \in \pi_* MU \wedge MU$ by $t_k \in \pi_* BP \wedge BP$. We have $\overline{\Delta}_k^{(2)} = v_k$. Using Hazewinkel's formula we find that

$$\begin{aligned} v_1 &\mapsto (-\pi^3 - 4\pi^2 - 6\pi - 4)w \;=\; \text{unit} \cdot \pi^3 w \\ v_2 &\mapsto (4\pi^3 + 11\pi^2 + 6\pi - 6)w^3 \;=\; \text{unit} \cdot \pi^2 w^3 \\ v_3 &\mapsto (40\pi^3 + 166\pi^2 + 237\pi + 100)w^7 \;=\; \text{unit} \cdot \pi w^7 \\ v_4 &\mapsto (-15754\pi^3 - 56631\pi^2 - 63495\pi - 9707)w^{15}. \end{aligned}$$

(where each unit is in A) so v_4 (but not v_n for $n < 4$) and therefore $N_2^8(\overline{\Delta}_4^{(2)})$ maps to a unit in R_*. 1.23

The proof of the Lemma (continued)

We have $\overline{\Delta}_k^{(2)} = t_k$. We consider the equivariant composite

$$BP_*^{(2)} \to BP_*^{(4)} \to R_*$$

under which

$$\eta_R(\ell_n) \mapsto \frac{\zeta_8^2 w^{2^n-1}}{\pi^n}.$$

Using the right unit formula we find that

$$\begin{aligned} t_1 &\mapsto (\pi + 2)w \;=\; \text{unit} \cdot \pi w \\ t_2 &\mapsto (\pi^3 + 5\pi^2 + 9\pi + 5)w^3. \end{aligned}$$

This means t_2 (but not t_1) and therefore $N_4^8(\overline{\Delta}_2^{(4)})$ maps to a unit in R_*. 1.24

The proof of the Lemma (continued)

Finally, we have $\overline{\Delta}_n^{(8)} = t_n(1) \in BP_*^{(4)}$, where $t_n(1)$ is the analog of $r_{2^n-1}(1)$. Then we find

$$\begin{aligned} \ell_n(1) &\mapsto \frac{w^{2^n-1}}{\pi^n} \\ \ell_n(2) &\mapsto \frac{(\zeta_8 w)^{2^n-1}}{\pi^n}. \end{aligned}$$

This implies

$$\overline{\Delta}_1^{(8)} = \ell_1(2) - \ell_1(1) \mapsto \frac{\zeta_8 w - w}{\pi} = w.$$

Thus we have shown that each factor of

$$D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$$

and hence D itself maps to a unit in R_*, thus proving the lemma. 1.25