chi-l A solution to the Arf-Kervaire invariant problem II: Overview of the proof

Unni Namboodiri Lectures
University of Chicago

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Mike Hill
University of Virginia
Mike Hopkins
Harvard University 2.1
Doug Ravenel
University of Rochester

1 Our strategy

Our strategy

Main Theorem. *The Arf-Kervaire elements $\theta_j \in \pi_{2^{j+1}-2}(S^0)$ do not exist for $j \geq 7$.*

We will prove it by producing a map $S^0 \to \Omega$, where Ω is a nonconnective E_∞-ring spectrum with the following properties. 2.2

(i) Detection Theorem. It has an Adams-Novikov spectral sequence in which the image of each θ_j is nontrivial. This means that if θ_j exists, we will see its image in $\pi_*(\Omega)$.
(ii) Periodicity Theorem. It is 256-periodic, meaning that $\pi_k(\Omega)$ depends only on the reduction of k modulo 256.
(iii) Gap Theorem. $\pi_k(\Omega) = 0$ for $-4 < k < 0$. This property is our zinger. Its proof involves a new tool we call the slice spectral sequence. It will be the subject of tomorrow's talk.

How the theorem follows from the existence of Ω

Here again are the properties of Ω:

(i) Detection Theorem. If θ_j exists, it has nontrivial image in $\pi_*(\Omega)$.
(ii) Periodicity Theorem. $\pi_k(\Omega)$ depends only on the reduction of k modulo 256.
(iii) Gap Theorem. $\pi_{-2}(\Omega) = 0$.

(ii) and (iii) imply that $\pi_{254}(\Omega) = 0$.

If $\theta_7 \in \pi_{254}(S^0)$ exists, (i) implies it has a nontrivial image in this group, so it cannot exist. The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$. 2.3

2 The construction of Ω

How we construct the spectrum Ω

The construction of Ω requires the use of equivariant stable homotopy theory.

Roughly speaking, an equivariant G-spectrum is a spectrum X with an action of the group G. For us the group of interest will be C_8. This leads to a fixed point spectrum X^G and a homotopy fixed point spectrum X^{hG}, with a map $X^G \to X^{hG}$.

For a G-space X, X^G is the subspace fixed by all of G, which is the same as the space of equivariant maps from a point to X, $\mathrm{Map}_G(*,X)$. To get X^{hG}, we replace the point here by an free contractible G-space EG.

The homotopy type of $X^{hG} = \mathrm{Map}_G(EG,X)$ is known to be independent of the choice of EG. The unique map $EG \to *$ leads to the map $X^G \to X^{hG}$. 2.4

How we construct the spectrum Ω (continued)

We construct a C_8-spectrum $\tilde{\Omega}$ and show that

- $\tilde{\Omega}^{hC_8}$ satisfies the detection and periodicity theorems.
- $\tilde{\Omega}^{C_8}$ satisfies the gap theorem.

Hence our proof depends on a fourth property:

(iv) Fixed Point Theorem The map $\tilde{\Omega}^{C_8} \to \tilde{\Omega}^{hC_8}$ is an equivalence.

We will come back to the definition of $\tilde{\Omega}$ below. 2.5

3 MU and its equivariant relatives

MU and its equivariant relatives

The starting point for the construction of $\tilde{\Omega}$ is the action of C_2 on the complex cobordism spectrum MU given by complex conjugation. The resulting C_2-equivariant spectrum is denoted by $MU_{\mathbf{R}}$ and is called real cobordism theory. This terminology follows Atiyah's definition of real K-theory, by which he meant complex K-theory equipped with complex conjugation.

Next we use a formal tool we call the norm N_H^G for inducing up from an H-spectrum to a G-spectrum when H is a subgroup of G. 2.6

MU and its equivariant relatives (continued)

For an H-space X, we have a G-space

$$\mathrm{Map}_H(G,X),$$

where H acts on G by right multiplication and G acts on the mapping space via right multiplication in G. The underlying space here is the Cartesian product $X^{|G/H|}$. G permutes the factors the same way it permutes cosets, and each factor is invariant under H. The norm functor is an analogous construction in the stable category.

The case of interest to us is $X = MU_{\mathbf{R}}$, $H = C_2$ and $G = C_{2^{n+1}}$. This means that the underlying spectrum of $N_H^G X$ is $MU^{(2^n)}$, the 2^n-fold smash power of MU. 2.7

MU and its equivariant relatives (continued)

In order to proceed further we need to introduce $RO(G)$-graded homotopy, where $RO(G)$ denotes the orthogonal representation ring of G. Let S^V denote the one point compactification of orthogonal representation V, and for a G-space or spectrum X define

$$\pi_V^G X = \mathrm{Map}_G(S^V, X).$$

Note that when the action of G on V is trivial, an equivariant map $S^V = S^{\dim V} \to X$ must land in the fixed point set X^G, so

$$\pi_n^G X = \pi_n X^G.$$

In the stable category we can make sense of this for virtual as well as actual representations, so we get homotopy groups indexed by $RO(G)$, which we denote collectively by $\pi_\star^G X$. 2.8

MU and its equivariant relatives (continued)

Recall that

$$\pi_*(MU) = \mathbf{Z}[x_1, x_2, \ldots] \qquad \text{with } |x_i| = 2i.$$

It turns out that any choice of generator $x_i : S^{2i} \to MU$ is the image under the forgetful functor of a map

$$S^{i\rho} \xrightarrow{\bar{x}_i} MU_{\mathbf{R}}.$$

Here ρ denotes the regular real representation of C_2, which is the same thing as the complex numbers $\mathbf{C}$ acted on by conjugation. 2.9

MU and its equivariant relatives (continued)

For $G = C_{2^{n+1}}$, the G-spectrum $N_{C_2}^G MU_{\mathbf{R}}$ is underlain by $MU^{(2^n)}$. $\pi_* MU^{(2^n)}$ is a graded polynomial algebra over $\mathbf{Z}$ where

- there are 2^n generators in each positive even dimension $2i$.
- they are acted on transitively by G.

For a group generator $\gamma \in G$ and polynomial generator $r_i \in \pi_{2i}$, the set

$$\left\{ \gamma^j r_i \colon 0 \le j < 2^n \right\}$$

is algebraically independent, and $\gamma^{2^n} r_i = (-1)^i r_i$. 2.10

4 The slice filtration

The slice filtration

Now we introduce our main technical tool, the slice filtration.

First we need to recall some things about the classical Postnikov tower. The mth Postnikov section $P^m X$ of a space or spectrum X is obtained by killing all homotopy groups of X above dimension m by attaching cells. The fiber of the map $X \to P^m X$ is $P_m X$, the m-connected cover of X.

These two functors have some universal properties. Let $\mathscr{S}$ and $\mathscr{S}_{>m}$ denote the categories of spectra and m-connected spectra. The functor P^m is Dror nullification with respect to the subcategory $\mathscr{S}_{>m}$. This means

- For all spectra X, $P_m X \in \mathscr{S}_{>m}$.
- For all $A \in \mathscr{S}_{>m}$ and $X \in \mathscr{S}$, map of function spectra $\mathscr{S}(A, P_m X) \to \mathscr{S}(A, X)$ is a weak equivalence.

In other words, the map $P_m X \to X$ is universal among maps from m-connected spectra to X. 2.11

More about the Postnikov tower

Similarly the map $X \to P^m X$ is universal among maps from X to spectra which are $\mathscr{S}_{>m}$-null in the sense that all maps to them from m-connected spectra are null. In other words,

- The spectrum $P^m X$ is $\mathscr{S}_{>m}$-null.
- For any $\mathscr{S}_{>m}$-null spectrum Z, the map $\mathscr{S}(P^m X, Z) \to \mathscr{S}(X, Z)$ is an equivalence.

Since $\mathscr{S}_{>m} \subset \mathscr{S}_{>m-1}$, there is a natural transformation $P^m \to P^{m-1}$, whose fiber is denoted by $P_m^m X$.

Thus we get a the Postnikov tower

$$\begin{tikzcd}
\cdots \arrow[r] & P^{m+1}X \arrow[r] & P^m X \arrow[r] & P^{m-1}X \arrow[r] & \cdots \\
 & P_{m+1}^{m+1}X \arrow[u] & P_m^m X \arrow[u] & P_{m-1}^{m-1}X \arrow[u] &
\end{tikzcd}$$

in which the homotopy limit is X and the homotopy colimit is contractible. The mth fiber $P_m^m X$ is $H\pi_m X$, the Eilenberg-Mac Lane spectrum for the mth homotopy group of X. 2.12

An equivariant Postnikov tower

In what follows G will be an arbitrary finite cyclic 2-group, and $g = |G|$. Let $\mathscr{S}^G$ denote the category of G-equivariant spectra. We need an equivariant analog of $\mathscr{S}_{>m}$. Our choice for this is somewhat novel.

Recall that $\mathscr{S}_{>m}$ is the category of spectra built up out of spheres of dimension $> m$ using arbitrary wedges and mapping cones.

For a subgroup H of G with $|H| = h$ and an integer k, let

$$\widehat{S}(k\rho_H) = G_+ \wedge_H S^{k\rho_H}$$

where ρ_H denotes the regular real representation of H. Its underlying spectrum is a wedge of g/h spheres of dimension kh which are permuted by elements of G and are invariant under H.

2.13

An equivariant Postnikov tower (continued)

We will replace the set of sphere spectra by

$$\mathscr{A} = \left\{ \widehat{S}(k\rho_H), \Sigma^{-1}\widehat{S}(k\rho_H) \colon H \subset G, k \in \mathbf{Z} \right\}.$$

We will refer to the elements in this set as slice cells. Note that $\Sigma^{-2}\widehat{S}(k\rho_H)$ (and larger desuspensions) are not slice cells. A free slice cell is one of the form $\widehat{S}(k\rho_{\{e\}})$, a wedge of g k-spheres permuted by G. Note that

$$\Sigma^{-1}\widehat{S}(k\rho_{\{e\}}) = \widehat{S}((k-1)\rho_{\{e\}}).$$

Nonfree slice cells are said to be isotropic.

In order to define $\mathscr{S}^G_{>m}$, we need to assign a dimension to each element in $\mathscr{A}$. We do this in terms of the underlying wedge summands, namely

$$\dim \widehat{S}(k\rho_H) = kh \quad \text{and} \quad \dim \Sigma^{-1}\widehat{S}(k\rho_H) = kh - 1.$$

2.14

An equivariant Postnikov tower (continued)

Then $\mathscr{S}^G_{>m}$ is the category built up out of elements in $\mathscr{A}$ of dimension $> m$ using arbitrary wedges, mapping cones and smash products with equivariant suspension spectra.

With this definition it is possible to construct functors P^G_m and P^m_G with the same formal properties as in the classical case.

Thus we get an equivariant analog of the Postnikov tower

$$\begin{tikzcd}
\cdots \arrow[r] & P^{m+1}_G X \arrow[r] & P^m_G X \arrow[r] & P^{m-1}_G X \arrow[r] & \cdots \\
& {}^G P^{m+1}_{m+1} X \arrow[u] & {}^G P^m_m X \arrow[u] & {}^G P^{m-1}_{m-1} X \arrow[u] &
\end{tikzcd}$$

in which the homotopy limit is X and the homotopy colimit is contractible.

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5 The slice spectral sequence

The slice spectral sequence

$$\begin{tikzcd}
\cdots \arrow[r] & P_G^{m+1}X \arrow[r] & P_G^{m}X \arrow[r] & P_G^{m-1}X \arrow[r] & \cdots \\
& {}^GP_{m+1}^{m+1}X \arrow[u] & {}^GP_m^mX \arrow[u] & {}^GP_{m-1}^{m-1}X \arrow[u] &
\end{tikzcd}$$

We call this the slice tower. ${}^GP_m^mX$ is the mth slice and the decreasing sequence of subgroups of $\pi_*(X)$ is the slice filtration. We also get slice filtrations of the $RO(G)$-graded homotopy $\pi_\star^G(X)$ and the homotopy groups of fixed point sets $\pi_*(X^H)$.

There is an important difference between this tower and the classical one. In the classical case the map $X \to P^mX$ does not change homotopy groups in dimensions $\leq m$. This is not true in the equivariant case. 2.16

The slice spectral sequence (continued)

Equivalently, in the classical case, P_m^mX is an Eilenberg-Mac Lane spectrum whose mth homotopy group is that of X. In our case, $\pi_*({}^GP_m^mX)$ need not be concentrated in dimension m.

This means the slice filtration leads to a (possibly noncollapsing) slice spectral sequence converging to $\pi_*(X)$ and its variants.

One variant has the form

$$E_2^{s,t} = \pi_{t-s}^G({}^GP_t^tX) \implies \pi_{t-s}^G(X).$$

Recall that $\pi_*^G(X)$ is by definition $\pi_*(X^G)$, the homotopy of the fixed point set.

This is the spectral sequence we will use to study $MU_{\mathbf{R}}^{(4)}$ and its relatives. 2.17

The slice spectral sequence (continued)

A large portion of our paper is devoted to proving that the slice spectral sequence has the desired properties. From now on we will drop the symbol G from the functors P^m, P_m and P_m^m.

Slice Theorem . *In the slice tower for $N_{C_2}^G MU_{\mathbf{R}}$, every odd slice is contractible and $P_{2m}^{2m} = \widehat{W}_m \wedge H\mathbf{Z}$, where $\widehat{W}_m$ is a certain wedge of $2m$-dimensional slice cells (to be discussed in tomorrow's talk) and $H\mathbf{Z}$ is the integer Eilenberg-Mac Lane spectrum. $\widehat{W}_m$ never has any free summands.* 2.18

Our G-spectrum $\tilde{\Omega}$ (where $G = C_8$) is obtained from the E_∞-ring spectrum $N_{C_2}^G MU_{\mathbf{R}}$ by inverting a certain element $D \in \pi_{19\rho_G}^G N_{C_2}^G MU_{\mathbf{R}}$. The choices of G and D are the simplest ones leading to a homotopy fixed point set with the detection property. The slice tower for $\tilde{\Omega}$ has similar properties to that of $N_{C_2}^G MU_{\mathbf{R}}$.

6 The gap theorem

Proving the gap theorem

The gap theorem follows from the fact that π_{-2}^G vanishes for each isotropic slice, i.e., for each one of the form

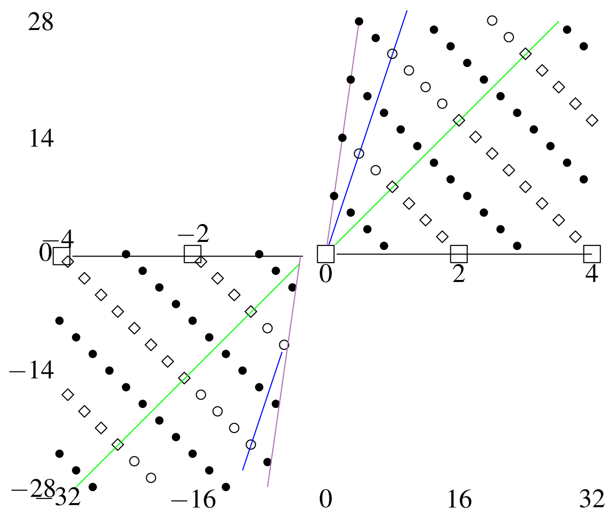
$$\widehat{S}(k\rho_H) \wedge H\mathbf{Z}$$

for any nontrivial subgroup H and any integer k. This will be the subject of tomorrow's talk.

In order to give a feel for these calculations we offer the following picture of $\pi_*^G S^{k\rho_G} \wedge H\mathbf{Z}$ for $G = C_8$ and various integers k. 2.19

Some C_8 slices

A picture of $\pi_*^G S^{k\rho_G} \wedge H\mathbf{Z}$ for $G = C_8$ and various integers k.



2.20

7 The periodicity theorem

Proving the Periodicity Theorem

We now outline the proof of the Periodicity Theorem, assuming the Slice Theorem.

We establish some differentials in the slice spectral sequence and show that certain elements become permanent cycles after inverting a certain $D \in \pi_{19\rho_G}^G N_{C_2}^G MU_{\mathbf{R}}$. This leads to an equivariant self map

$$\Sigma^{256}\tilde{\Omega} \to \tilde{\Omega}.$$

It is an ordinary homotopy equivalence, and this is known to imply that it induces an equivalence on homotopy fixed point sets. 2.21

Digression on Geometric Fixed Points

The key tool for studying differentials in the slice spectral sequence is the geometric fixed point spectrum, denoted by $\Phi^G X$ for a G-spectrum X. It has much nicer properties than the usual fixed point spectrum X^G, which fails to commute with smash products and infinite suspensions.

In order to define it we need the isotropy separation sequence, which in the case of a finite cyclic 2-group G is the cofiber sequence

$$EC_{2+} \to S^0 \to \tilde{E}C_2.$$

Here EC_2 is a G-space via the projection $G \to C_2$ and S^0 has the trivial action, so $\tilde{E}C_2$ is also a G-space. 2.22

Geometric Fixed Points (continued)

$$EC_{2+} \to S^0 \to \tilde{E}C_2.$$

Under this action EC_2^G is empty while for any proper subgroup H of G, $EC_2^H = EC_2$, which is contractible. For an arbitrary finite group G it is possible to construct a G-space with the similar properties.

Definition. *For a finite cyclic 2-group G and G-spectrum X, the geometric fixed point spectrum is*

$$\Phi^G X = (X \wedge \tilde{E}C_2)^G.$$

2.23

Geometric Fixed Points (continued)

$$\Phi^G X = (X \wedge \tilde{E}C_2)^G.$$

This functor has the following properties:

- For G-spectra X and Y, $\Phi^G(X \wedge Y) = \Phi^G X \wedge \Phi^G Y$.
- For a G-space X, $\Phi^G \Sigma^\infty X = \Sigma^\infty (X^G)$.
- A map $f : X \to Y$ is a G-equivalence iff $\Phi^H f$ is an ordinary equivalence for each subgroup $H \subset G$.

From the suspension property we can deduce that

$$\Phi^{C_8} MU^{(4)} = MO,$$

the unoriented cobordism spectrum. Its homotopy type has been well understood since Thom's work in the 50s.

Moreover there is a theorem saying that inverting a certain element in the slice spectral sequence converging to $\pi_* X^G$ gives one converging to $\pi_* \Phi^G X$. 2.24

Back to the Periodicity Theorem

Our knowledge of the slice spectral sequence converging to

$$\pi_* \Phi^G MU^{(4)} = \pi_* MO$$

gives us a very good handle on the one converging to $\pi_*(MU^{(4)})^G$. This enables us to prove the Periodicity Theorem.

Typically one proves theorems about differentials in such spectral sequences by means of some sort of extended power construction. In our case, all of the necessary geometry is encoded in the relation between $\pi_* MU$ and $\pi_* MO$! 2.25

8 The detection theorem

Proving the Detection Theorem

The proof of the detection theorem is a calculation with the Adams-Novikov spectral sequence. It is the one part of our proof that could have been done 30 years ago.

Similar methods were used in the 70s to prove an odd primary analog (for $p > 3$) of our theorem. In that proof a key tool is a homomorphism

$$\left\{ \begin{array}{c} \text{Adams-Novikov} \\ E_2\text{-term for } S^0 \end{array} \right\} \to H^* \left(C_p; \begin{array}{c} \text{certain} \\ \text{coefficients} \end{array} \right)$$

It is based on the fact that a formal group law of height $p-1$ can have nontrivial automorphisms of order p. 2.26

Proving the Detection Theorem (continued)

For us it is a composite homomorphism

$$\left\{ \begin{array}{c} E_2\text{-term} \\ \text{for } S^0 \end{array} \right\} \to \left\{ \begin{array}{c} E_2\text{-term} \\ \text{for } \Omega \end{array} \right\} \to H^* \left(C_8; \begin{array}{c} \text{something} \\ \text{easy} \end{array} \right)$$

in which each θ_j has a nontrivial image.

It is based on the fact that at the prime 2 a formal group law of height 4 can have nontrivial automorphisms of order 8. 2.27

9 The slice and reduction theorems

Proving the Slice Theorem

Recall that a pivotal step in our proof is the Slice Theorem, which identifies the layers in the slice tower for $MU_{\mathbf{R}}$ and its relatives.

For each cyclic 2-group $G = C_{2^{n+1}}$ there is a equivariant (noncommutative) ring spectrum A which is a certain wedge of slice cells. It maps to $N_{C_2}^{G} MU_{\mathbf{R}}$ in such a way that the underlying wedge of spheres hits all of the underlying homotopy of $MU^{(2^n)}$. Thus both $N_{C_2}^{G} MU_{\mathbf{R}}$ and S^0 are A-modules.

Reduction Theorem. *The A-smash product $N_{C_2}^{G} MU_{\mathbf{R}} \wedge_A S^0$ is equivariantly equivalent to the integer Eilenberg-Mac Lane spectrum $H\mathbf{Z}$.* 2.28

The proof of this is the hardest calculation in our paper. Deriving the Slice Theorem from it is a formality.