Morava K–theories of Classifying Spaces and Generalized Characters for Finite Groups

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This paper is intended to be an informal introduction to [HKR], where we study the Morava K–theories (which will be partially defined below) of the classifying space of a finite group G and related matters. It will be long on exposition and short on proofs, in the spirit of the third author's lecture at the conference.

In Section 1 we will recall Atiyah's theorem relating the complex K–theory of BG to the complex representation ring $R(G)$. We will also define classical characters on G. In Section 2 we will introduce the Morava K–theories $K(n)^*$ and the related theories $E(n)^*$. In Section 3 we will state our main results and conjectures. Most of the former are generalizations of the classical results stated in Section 1.

In the remaining five sections we will outline the proofs of our results. Section 4 is purely group–theoretic, i.e., it makes no use of any topology. In it we will prove our formula for the number $\chi_{n,p}(G)$ of conjugacy classes of commuting n–tuples of elements of prime power order in a finite group G. We have discussed this material with several prominent group theorists, but we have yet to find one who admits to ever having considered this question. In Section 5 we equate this number with the Euler characteristic of $K(n)^*(BG)$. Our generalized characters are functions on the set of such conjugacy classes with values in certain p–adic fields.

In Section 6 we recall the Lubin–Tate construction from local algebraic number theory. It uses formal group laws to construct abelian extensions of finite extensions of the field of p–adic numbers. We need it in Section 7 where we describe the connection between $E(n)^*(BG)$ and our generalized characters.

In Section 8, we prove a theorem about wreath products. A corollary of this is that our main conjecture (3.5) about $K(n)^*(BG)$ holds for all the symmetric

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Contents

1 Atiyah's theorem and other classical results

Atiyah determined the complex K–theory of the classifying space BG in [Ati61]. A complex representation ρ of a finite group G is a homomorphism $G \to U$ (where U denotes the stable unitary group) and hence gives a map of classifying spaces $BG \to BU$, which defines a vector bundle over BG and an element in $K^*(BG)$. A virtual representation leads to a virtual vector bundle, and we still get an element in $K^*(BG)$. It follows that there is a natural ring homomorphism

$$R(G) \longrightarrow K^*(BG),$$

where $R(G)$ denotes the complex representation ring of G.

This map is not an isomorphism, but it is close to being one. The representation ring $R(G)$ has an augmentation ideal I, the ideal of all virtual representations of degree 0. Let $R(G)\hat{}$ denote the I–adic completion of $R(G)$.

Theorem 1.1 (Atiyah) *For a finite group G,*

$$K^i(BG) \cong \begin{cases} R(G)\hat{} & \textit{if } i \textit{ is even} \\ 0 & \textit{if } i \textit{ is odd.} \end{cases}$$

When G is a p–group, the I–adic completion is roughly the same as the p–adic completion. More precisely, we have

$$R(G)\hat{} \cong \mathbf{Z} \oplus (\mathbf{Z}_p \otimes I),$$

where $\mathbf{Z}_p$ denotes the p–adic integers. For more general finite groups G, I–adic completion is more drastic. The map $R(G) \to R(G)\hat{}$ has a nontrivial kernel, as can be seen by studying the case $G = \mathbf{Z}/(6)$.

In view of 1.1, it is useful to recall some classical facts about the representation ring $R(G)$. A good reference is Serre's book [Ser67b]. Additively, $R(G)$ is a free abelian group whose rank is the number of conjugacy classes of elements in G.

The mod p analog of 1.1 is the following, which was proved in [Kuh87].

Proposition 1.2 *For a finite group G and a prime number p,*

$$K^i(BG; \mathbf{Z}/(p)) \cong \begin{cases} R(G)\hat{} \otimes \mathbf{Z}/(p) & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd,} \end{cases}$$

and the rank of this vector space is the number of conjugacy classes of elements whose order is some power of the prime p.

A representation ρ induces a complex valued function χ, called a *character*, on the set of conjugacy classes as follows. Pick a basis for the complex vector space V on which G acts via ρ. Then for $G \in G$ we define $\chi(g)$ to be the trace of the matrix $\rho(g)$. This turns out to be invariant under conjugation and to be independent of the choice of basis. It leads to a ring isomorphism

$$R(G) \otimes \mathbf{C} \xrightarrow{\chi_G} \mathrm{Cl}(G), \tag{1.3}$$

where $\mathrm{Cl}(G)$ denotes the ring of complex valued functions on the conjugacy classes of elements in G.

It should be noted here that this isomorphism does *not* have an analog in characteristic p, namely $K^*(BG; \mathbf{Z}/(p)) \otimes \bar{\mathbf{F}}_p$ is not naturally isomorphic to the ring of $\bar{\mathbf{F}}_p$–valued functions on conjugacy classes of prime power order, where $\bar{\mathbf{F}}_p$ denote the algebraic closure of the field $\mathbf{Z}/(p)$.

We will illustrate this by studying the representation ring for the quaternion group Q_8. Using the usual notation for quaternions, its elements are ± 1, $\pm i$, $\pm j$, and $\pm k$. There are five conjugacy classes, namely $\{1\}$, $\{-1\}$, $\{\pm i\}$, $\{\pm j\}$, and $\{\pm k\}$. The center is the subgroup of order 2 generated by -1, and there is a group extension

$$1 \longrightarrow \mathbf{Z}/(2) \longrightarrow Q_8 \longrightarrow \mathbf{Z}/(2) \oplus \mathbf{Z}/(2) \longrightarrow 1$$

There are four irreducible representations of Q_8 of degree 1 induced by those on the quotient group. We denote these by 1, α, β and γ. There is a fifth irreducible representation of degree 2, which we denote by σ, obtained by the usual action of Q_8 on the space $\mathbf{H} = \mathbf{C}^2$ of quaternions.

The corresponding characters are displayed in Table 1.4. This is a *character table* for Q_8. There is a column for each irreducible representation and a row for each conjugacy class. The numbers shown are that various values of $\chi(g)$.

Table 1.4: Character table for Q_8

	1	α	β	γ	σ
1	1	1	1	1	2
-1	1	1	1	1	-2
$\pm i$	1	1	-1	-1	0
$\pm j$	1	-1	1	-1	0
$\pm k$	1	-1	-1	1	0

From this table we can read off the multiplicative structure of $R(Q_8)$. We see that $\alpha\beta\gamma = \alpha^2 = \beta^2 = \gamma^2 = 1$, $\sigma\alpha = \sigma\beta = \sigma\gamma = \sigma$, and $\sigma^2 = 1 + \alpha + \beta + \gamma$.

It follows that there is a unique homomorphism from $R(Q_8)$ to a field k of characteristic 2, and it sends α, β and γ to 1 and σ to 0. On the other hand, the ring of k–valued functions on the conjugacy classes of Q_8 admits at least five such homomorphisms, namely evaluation on each of the five conjugacy classes.

In view of (1.3), one can ask for a characterization of $R(G)$ as a subring of $\mathrm{Cl}(G)$. This seems to be a very delicate business, but one can recover its rational form $R(G) \otimes \mathbf{Q}$ in the following way.

First, one does not need all of $\mathbf{C}$ to get the isomorphism (1.3). The trace of $\rho(g)$ is a linear combination of eigenvalues of g, i.e., it lies in a cyclotomic extension of $\mathbf{Q}$. To fix notation let L denote the union of all such extensions, i.e., the maximal abelian extension of $\mathbf{Q}$. Then (1.3) can be replaced by an isomorphism

$$R(G) \otimes L \longrightarrow \mathrm{Cl}(G, L),$$

where $\mathrm{Cl}(G, L)$ denotes the ring of L–valued class functions.

Now the Galois group $\mathrm{Gal}(L{:}\,\mathbf{Q})$ is known to be isomorphic to the group of units in the profinite integers. Each automorphism of L raises each root of unity to a certain power, depending on the order of the root. This group also acts on the set of conjugacy classes in G in a similar way. Elements in $\mathrm{Cl}(G, L)$ coming from representations are *equivariant* with respect to these two Galois actions.

Proposition 1.5 *The subring of* $\mathrm{Cl}(G, L)$ *consisting of Galois equivariant class functions is isomorphic to* $R(G) \otimes \mathbf{Q}$.

Finally, we must mention a theorem of Artin about $R(G)$. Given a subgroup H of G, one has a restriction homomorphism $R(G) \to R(H)$. Given a category $\mathcal{C}$ of subgroups (such as that of all abelian subgroups or all cyclic subgroups) one has a map

$$R(G) \longrightarrow \lim_{\mathcal{C}} R(H) \tag{1.6}$$

and one can ask under what circumstances it is an isomorphism. The limit here is the categorical inverse limit. It is defined to be the subset of the product

$$\prod_{H \in \mathrm{Ob}(\mathcal{C})} R(H)$$

consisting of points in which the coordinates are compatible under all morphisms in $\mathcal{C}$. This means that for any morphism

$$H' \xrightarrow{h} H'',$$

the coordinates $x_{H'} \in R(H')$ and $x_{H''} \in R(H'')$ must satisfy

$$x_{H'} = f^*(x_{H''}).$$

We are assuming that the objects in $\mathcal{C}$ are closed under subgroups, i.e., if a subgroup H is an object of $\mathcal{C}$, then so is any subgroup of H. We all assume that the morphisms in $\mathcal{C}$ are generated by inclusions and conjugations.

We will illustrate the categorical limit with the quaternion group Q_8 (whose character table is shown in 1.4) and the category of abelian subgroups. It has five abelian subgroups: the trivial subgroup, the subgroup of order 2, and three subgroups of order 4 (A_1, A_2 and A_3). An element in the limit

$$\lim_{\mathcal{C}} R(A)$$

is a 5–tuple of representations (one for each abelian subgroup) which are compatible under inclusions and conjugations. This means three things:

(i) All five must have the same degree, since they must have the same restriction to the trivial subgroup. It follows that the limit is spanned by the trivial representation of degree 1 and by elements in which each coordinate is a virtual representation of degree 0.

(ii) The representations of the three subgroups of order 4 must the same restriction to the subgroup of order 2. Let α denote the nontrivial representation of $\mathbf{Z}/(2)$ of degree 1.

(iii) The representations of the subgroups of order 4 must each be invariant under the nontrivial involution of that group, since it is induced by a conjugation in Q_8. If λ denotes a representation of $\mathbf{Z}/(4)$ of degree 1 with eigenvalue i, then the coordinate in the 5–tuple must be a linear combination of 1, λ^2 and $\lambda + \lambda^3$.

It follows that the limit is spanned by five generators whose restrictions to the five subgroups are as shown in Table 1.7. In it there is a row for each subgroup and a column for each generator.

Comparing this with Table 1.4, we see that the map of (1.6) sends α to $C + D + 1$, β to $B + D + 1$, γ to $B + C + 1$ and σ to $A + 2$. It follows that the map is one–to–one and has a cokernel of order 2.

Artin's theorem, which is proved in [Ser67b, Chapter 9], says that in general this map is one–to–one, and the order of its cokernel is a product of primes dividing the order of G.

Table 1.7: Generators of $\lim R(A)$ for Q_8

	A	B	C	D	1
A_1	$\lambda + \lambda^3 - 2$	$\lambda^2 - 1$	0	0	1
A_2	$\lambda + \lambda^3 - 2$	0	$\lambda^2 - 1$	0	1
A_3	$\lambda + \lambda^3 - 2$	0	0	$\lambda^2 - 1$	1
$\mathbf{Z}/(2)$	$2\alpha - 2$	0	0	0	1
$\{e\}$	0	0	0	0	1

Theorem 1.8 (Artin) *Let $\mathcal{C}(G)$ denote the category whose objects are cyclic subgroups of G, with morphisms generated by inclusions and conjugations. Then the natural map*

$$R(G) \otimes \mathbf{Z}[|G|^{-1}] \longrightarrow \lim_{C \in \mathcal{C}(G)} R(C) \otimes \mathbf{Z}[|G|^{-1}]$$

is an isomorphism. The same is true if we replace $\mathcal{C}(G)$ by $\mathcal{A}(G)$, the category of abelian subgroups of G.

Corollary 1.9 *The natural maps*

$$K^*(BG) \otimes \mathbf{Z}[|G|^{-1}] \longrightarrow \lim_{C \in \mathcal{C}(G)} K^*(BC) \otimes \mathbf{Z}[|G|^{-1}]$$

and

$$K^*(BG) \otimes \mathbf{Z}[|G|^{-1}] \longrightarrow \lim_{A \in \mathcal{A}(G)} K^*(BA) \otimes \mathbf{Z}[|G|^{-1}]$$

are isomorphisms for every finite group G.

Note that this is trivially true if we replace K–theory by mod p cohomology. In that case the source and target of the map are both trivial.

2 Morava K–theories

We would like to generalize 1.1 and 1.2 to some other cohomology theories related to BP–theory. Recall that BP is a ring spectrum, which is a minimal wedge summand of the Thom spectrum MU localized at p, with

$$BP_* = \pi_*(BP) = \mathbf{Z}_{(p)}[v_1, v_2, \cdots],$$

where the dimension of v_n is $2p^n - 2$. There are BP–module spectra $E(n)$ and $K(n)$ (the n^{th} Morava K–theory at the prime p) for each $n \geq 0$ with

$$\begin{aligned} E(0)_* = K(0)_* &= \mathbf{Q} \\ K(n)_* &= \mathbf{Z}/(p)[v_n, v_n^{-1}] \\ E(n)_* &= \mathbf{Z}_{(p)}[v_1, v_2, \cdots v_n, v_n^{-1}] \\ &\quad \text{for } n > 0. \end{aligned}$$

We will not elaborate here on the construction of these spectra. The interested reader can find more information in [Rav84] and [Wil82].

Some additional properties of these spectra should be noted. $E(0) = K(0)$ is the rational Eilenberg–Mac Lane spectrum. $E(1)$ is one of $p-1$ isomorphic summands of the complex K–theory spectrum localized at p, and $K(1)$ is its mod p analog. $E(2)$ is related in a similar way to elliptic cohomology; see [Bak89].

Hence 1.1 and 1.2 can easily be translated into descriptions of $E(1)^*(BG)$ and $K(1)^*(BG)$. We will give a partial generalization to $E(n)^*(BG)$ and $K(n)^*(BG)$.

The coefficient ring $K(n)_*$ is a graded field in the sense that every graded module over it is free. This makes it very convenient for computations. In particular there is a Künneth isomorphism

$$K(n)^*(X \times Y) = K(n)^*(X) \otimes K(n)^*(Y). \tag{2.1}$$

A consequence of the Nilpotence Theorem of [DHS88] is that the Morava K–theories, along with ordinary mod p cohomology, are essentially the *only* cohomology theories with Künneth isomorphisms.

For a finite complex X, we know that the rank of $K(n)^*(X)$ is finite, grows monotonically with n, and is bounded above by the rank of $H^*(X; \mathbf{Z}/(p))$. In [Rav82] it was shown that $K(n)^*(BG)$ for finite G also has finite rank. Our results indicate that this rank grows *exponentially* with n.

All of these theories are *complex oriented*. This means that they behave in the expected way on $\mathbf{C}P^\infty$, namely

$$K(n)^*(\mathbf{C}P^\infty) = K(n)^*(\mathrm{pt.})[[x]]$$

with $x \in K(n)^2(\mathbf{C}P^\infty)$, and similarly for $E(n)^*(\mathbf{C}P^\infty)$.

We also have

$$K(n)^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) = K(n)^*(\mathrm{pt.})[[x \otimes 1, 1 \otimes x]].$$

The H–space multiplication

$$\mathbf{C}P^\infty \times \mathbf{C}P^\infty \xrightarrow{m} \mathbf{C}P^\infty$$

induces a homomorphism in $K(n)$–cohomology determined by its behavior on the class x, and we can write

$$m^*(x) = F(x \otimes 1, 1 \otimes x).$$

The power series F is a *formal group law*, that is it satisfies

(i) $F(u, 0) = F(0, u) = u$,

(ii) $F(u, v) = F(v, u)$ and

(iii) $F(F(u, v), w) = F(u, F(v, w))$.

This particular formal group law is characterized by its *p–series*, i.e., by the image of x under the p^{th} power map on $\mathbf{C}P^\infty$. This is given by

$$[p](x) = v_n x^{p^n}. \tag{2.2}$$

This formula enables us to compute $K(n)^*(B\mathbf{Z}/(p^i))$ in the following way. The space $B\mathbf{Z}/(p^i)$ is an S^1–bundle over $\mathbf{C}P^\infty$. As such, its $K(n)$–cohomology can be computed with a Gysin sequence similar to the one for ordinary cohomology. One needs to know the Euler class of the bundle, and this can be derived easily from (2.2). The result is

Proposition 2.3

$$K(n)^*(B\mathbf{Z}/(p^i)) = K(n)^*(\text{pt.})[x]/(x^{p^{ni}}).$$

Using, (2.1), this can easily be extended to finite abelian groups. Notice that for a finite abelian p–group A, the rank of $K(n)^*(BA)$ is the n^{th} power of the order of A.

The reduction map $r : \mathbf{Z}/(p^{m+1}) \to \mathbf{Z}/(p^m)$ induces a homomorphism

$$K(n)^*(B\mathbf{Z}/(p^m)) \xrightarrow{r^*} K(n)^*(B\mathbf{Z}/(p^{m+1}))$$

which sends x to $v_n x^{p^n}$ and is therefore one–to–one. On the other hand, the inclusion map $i : \mathbf{Z}/(p^m) \to \mathbf{Z}/(p^{m+1})$ induces a surjection. From these we can easily deduce

Proposition 2.4 *Let $h : A \to A'$ be a homomorphism of finite abelian groups. Then $K(n)^*(h)$ is onto if h is one–to–one and $K(n)^*(h)$ is one–to–one if h is onto. If A and A' are also p–groups, the converse statements are also true.*

Notice that nothing like this holds for ordinary cohomology.

One can make similar computations with $E(n)$–theory. Again one has a formal group law F with a p–series satisfying

$$\begin{array}{lll} [p](x) & \equiv & px \qquad \bmod (x^2) \quad \text{and} \\ [p](x) & \equiv & v_n x^{p^n} \quad \bmod (p, v_1, \cdots v_{n-1}). \end{array} \tag{2.5}$$

It follows that

$$\begin{array}{lll} [p^i](x) & \equiv & p^i x \qquad \bmod (x^2) \quad \text{and} \\ [p^i](x) & \equiv & v_n^{(p^{ni}-1)/(p^n-1)} x^{p^{ni}} \quad \bmod (p, v_1, \cdots v_{n-1}). \end{array}$$

Furthermore, we have

$$E(n)^*(B\mathbf{Z}/(p^i)) = E(n)^*[[x]]/([p^i](x)). \tag{2.6}$$

After a certain completion, this is a free $E(n)^*$–module of rank p^{ni}, generated by $\{x^j : 0 \leq j < p^{ni}\}$. The analogue of 2.4 holds for $E(n)$–theory.

3 Main results and conjectures

Now we can state our main results, which are partial generalizations of 1.1, 1.2, and 1.9. We will give the second of these first.

1.2 identifies says $K(1)^*(BG)$ is concentrated in even dimensions and its rank is the number of conjugacy classes of elements in G of prime power order. We cannot prove that $K(n)^*(BG)$ is concentrated in even dimensions; this is Conjecture 3.5. We know it is true for finite abelian groups by 2.3. We will discuss this matter further at the end of this Section.

We know that $K(n)^*(BG)$ has finite rank, so we can speak of its Euler characteristic, i.e., the difference between the rank of its even dimensional part and that of its odd dimensional part. We denote this number by $\chi_{n,p}(G)$. From 2.3 we know that for a finite abelian group A,

$$\chi_{n,p}(A) = |A_{(p)}|^n, \tag{3.1}$$

the n^{th} power of the order of the p–component of A.

Like $\chi_{1,p}(G)$ of 1.2, $\chi_{n,p}(G)$ can be described in terms of conjugacy classes. It is the number of *conjugacy classes of of commuting n–tuples* of elements of prime power order in G. More precisely, let

$$G_n = \{(g_1, g_2, \cdots g_n) \colon [g_i, g_j] = e \text{ for } 1 \leq i < j \leq n\},$$

i.e., the coordinates g_i lie in an abelian subgroup of G. We define $G_{n,p}$ similarly with the additional condition that the order of each g_i is some power of p. G acts on both G_n and $G_{n,p}$ by coordinate–wise conjugation.

Theorem 3.2 *Let $\chi_{n,p}(G)$ be the Euler characteristic of $K(n)^*(BG)$, i.e., the difference in ranks between the even and odd–dimensional components of $K(n)^*(BG)$. It is equal to the number of G–orbits in $G_{n,p}$.*

We have a formula for this number, which will be described and proven below in Section 4. We will outline the proof of Theorem 3.2 in Section 5.

In view of this result it is natural to consider characters as functions defined on orbits of $G_{n,p}$, generalizing the classical case of $n = 1$. A more sophisticated way of viewing $G_{n,p}$ is as follows. Let $\mathbf{Z}_p$ denote the p–adic integers. Observe that, as G–sets,

$$G_{n,p} = \mathrm{Hom}((\mathbf{Z}_p)^n, G);$$

Let $\mathrm{Cl}_{n,p}(G)$ be the ring of $\overline{\mathbf{Q}}_p$ valued conjugacy class functions on $G_{n,p}$. Then

$$\mathrm{Cl}_{n,p}(G) = \mathrm{Map}_G(\mathrm{Hom}(\mathbf{Z}_p^n, G), \overline{\mathbf{Q}}_p).$$

We can now state our main theorem, which is a partial generalization of Theorem 1.1.

Theorem 3.3 *Let A be the ring of integers in a finite extension K of the p–adic numbers, with maximal ideal (π). Given a ring homomorphism*

$$\varphi : E(n)^*(\text{ pt. }) \longrightarrow A \subset \overline{\mathbf{Q}}_p$$

such that $\varphi(I_n) \subset (\pi)$, there exist natural isomorphisms, for all finite groups G,

$$E(n)^*(BG) \otimes_{E(n)^*} \overline{\mathbf{Q}}_p \xrightarrow{\chi_G} \mathrm{Cl}_{n,p}(G),$$

which is an isomorphism after a suitable completion of the source.

This is our analogue of classical character theory:

$$\chi_G : R(G) \otimes \mathbf{C} \xrightarrow{\sim} \mathrm{Cl}(G),$$

where $\mathrm{Cl}(G)$ is the ring of complex valued class functions on G.
Our generalization of 1.9 is

Theorem 3.4 *For every finite group G and for each positive integer n and each prime p, the natural map*

$$E(n)^*(BG) \otimes \mathbf{Z}[|G|^{-1}] \longrightarrow \lim_{A \in \mathcal{A}(G)} E(n)^*(BA) \otimes \mathbf{Z}[|G|^{-1}]$$

is an isomorphism. The same is true if we replace $\mathcal{A}(G)$ by the category of abelian subgroups generated by at most n elements.

This theorem is actually true in much greater generality; $E(n)^*$ can be replaced by *any* complex oriented cohomology theory, provided we state it in terms of the category of $\mathcal{A}(G)$. This is proved in Section 2 of [HKR].
Our main conjecture about $K(n)^*(BG)$ is the following.

Conjecture 3.5 *$K(n)^*(BG)$ is concentrated in even dimensions.*

This is true for $n = 1$ by Atiyah's theorem.

Proposition 3.6 *3.5 is true for G if it is true for a p–Sylow subgroup $H \subset G$.*

Proof. The composite stable map

$$BG \xrightarrow{\mathrm{Tr}} BH \longrightarrow BG$$

induces multiplication by the index of H in G in ordinary homology. Since this index is prime to p, it follows that the map is an equivalence after localizing at p. This means that $BG_{(p)}$ is a retract of $BH_{(p)}$. The result follows. ∎

The conjecture holds for abelian p–groups by 2.3. Tezuka–Yagita [TY89] have verified it for the nonabelian groups of order p^3. Theorem 5.4 of [Kuh89] implies that for any stable summand eBG of BG, $K(n)_*(BG)$ has positive Euler characteristic for sufficiently large n.
We can show that it holds for certain wreath products. In order to state this result, we need a preliminary definition.

Definition 3.7 *(a) For a finite group G, an element $x \in K(n)^*(BG)$ is* **good** *if it is a transferred Euler class of a complex subrepresentation of G, i.e., a class of the form $\mathrm{Tr}^*(e(\rho))$ where ρ is a complex representation of a subgroup $H < G$, $e(\rho) \in K(n)^*(BH)$ is its Euler class (i.e., its top Chern class, this being defined since $K(n)^*$ is a complex oriented theory), and $\mathrm{Tr} : BG \to BH$ is the transfer map.*

(b) G is **good** *if $K(n)^*(BG)$ is spanned by good elements as a $K(n)^*$–module.*

Recall that a representation ρ of a subgroup $H \subset G$ leads to an induced representation $\mathrm{Ind}_H^G(\rho)$ of G. Its degree is that of ρ times the index of H in G. One also has a stable transfer map

$$BG \xrightarrow{\mathrm{Tr}} BH.$$

The induced map K–theory sends the element corresponding (under Atiyah's isomorphism) to ρ to that corresponding to $\mathrm{Ind}_H^G(\rho)$.

However, $\mathrm{Tr}^*(e(\rho)) \neq e(\mathrm{Ind}_H^G(\rho))$ in general. For good G, $K(n)^*(BG)$ cannot be described only in terms of representations of G itself. If G is good then $K(n)^*(BG)$ is of course concentrated in even dimensions. We know of no groups which are not good in this sense, so we could strengthen the conjecture by saying that all finite groups are good.

With this definition we have

Theorem 3.8 *If a finite group G is good, then so is the wreath product $W = \mathbf{Z}/(p) \wr G$.*

A variant of this has been proved independently by Hunton [Hun90].

Corollary 3.9 *Let Σ_k denote the symmetric group on k letters. Then*

$$K(n)^*(B\Sigma_k)$$

is concentrated in even dimensions for all k.

Proof. The p–Sylow subgroup of Σ_k is a direct sum of iterated wreath products of $\mathbf{Z}/(p)$ with itself, and is therefore good by 3.8. The result follows by transfer arguments. ∎

Our conjecture is closely related to the one that says that $E(n)^*(BG)$ is torsion free and even–dimensional. Note that then the map of Theorem 1.8 would be monic *before* inverting $|G|$.

Part of Atiyah's proof goes as follows. We want to show that $K(1)_*(BG)$ is concentrated in even dimensions. It suffices to do this for p–groups. Each p–group G has a normal subgroup H of index p. We can assume inductively that $K(1)^*(BH)$ is concentrated in even dimensions. We can study the spectral sequence associated with the fibration

$$BH \longrightarrow BG \longrightarrow B\mathbf{Z}/(p)$$

converging to $K(1)^*(BG)$ with

$$E_2 = H^*(\mathbf{Z}/(p);\, K(1)^*(BH)),$$

where this cohomology has twisted coefficients based on the action of $\mathbf{Z}/(p)$ on H by conjugation. One wants to show that this spectral sequence collapses.

We know inductively that $K(1)^*(BH)$ has a basis corresponding to certain *irreducible* characters of H, i.e., characters associated with irreducible representations. These characters are clearly permuted by outer automorphisms of H, so it follows that $\mathbf{Z}/(p)$ acts on $K(1)^*(BH)$ by permutations. From this it follows easily that the E_2–term is concentrated in even dimensions and the spectral sequence collapses as desired.

The difficulty in generalizing this argument to $n > 1$ is that $\mathbf{Z}/(p)$ need not act on $K(n)^*(BH)$ via a permutation representation so we cannot say that the spectral sequence collapses.

4 Counting the orbits in $G_{n,p}$

In this section we will derive our formula for the number of G–orbits in $G_{n,p}$, i.e. the number of conjugacy classes of commuting n–tuples of elements whose order is a power of the prime p. This is an exercise in elementary group theory. Before proving it we will illustrate the formula with some examples. In this section no mention will be made of Morava K–theory. The notation $\chi_{n,p}(G)$ is used here only for convenience to denote the number of G–orbits in $G_{n,p}$. The connection of this number with Morava K–theory will be discussed in Section 5.

The formula for $\chi_{n,p}(G)$ is

Proposition 4.1 *The number of G–orbits in $G_{n,p}$ is*

$$\chi_{n,p}(G) = \sum_{A<G} \frac{|A|}{|G|} \mu_G(A) \chi_{n,p}(A)$$

where the sum is over all abelian subgroups $A < G$ and μ_G is a Möbius function defined recursively by

$$\sum_{A<A'} \mu_G(A') = 1$$

where the sum is over all abelian subgroups $A' < G$ which contain A. (In particular, $\mu_G(A) = 1$ when A is maximal.)

The Möbius function μ can be defined as above for any partially ordered set in which each element is dominated by only finitely many elements in the set. The classical arithmetic Möbius function is obtained (up to a factor of -1) in this way from the poset of proper subgroups of the integers.

If G is an abelian group, then $\mu_G(G) = 1$ and μ_G vanishes on all proper subgroups, since each is contained in precisely one maximal abelian subgroup. Thus the sum has but one terms and the formula is tautologous in this case.

Now we will consider the quaternion group Q_8 again. The lattice of abelian subgroups is

$$0 \longrightarrow \mathbf{Z}/(2) \longrightarrow 3(\mathbf{Z}/(4))$$

It follows that $\mu_{Q_8}(\mathbf{Z}/(2)) = -2$, while μ_{Q_8} vanishes on the trivial subgroup. (In general it can be shown that μ_G vanishes on any abelian subgroup not containing the center of G.)

Thus the formula gives

$$\begin{aligned} \chi_{n,2}(Q_8) &= \frac{3 \cdot 4 \cdot 4^n - 2 \cdot 2 \cdot 2^n}{8} \\ &= \frac{3 \cdot 4^n - 2^n}{2}. \end{aligned}$$

Note that when $n = 1$ this gives 5, the number of conjugacy classes in Q_8, and that for all n it gives an integer.

If we replace Q_8 by the dihedral group D_8 of order 8, the lattice of abelian subgroups has the same structure, but with two of the $\mathbf{Z}/(4)$s replaced by $(\mathbf{Z}/(2))^2$. This change will not alter any of the numbers in the formula, so we have

$$\chi_{n,2}(D_8) = \chi_{n,2}(Q_8).$$

Now we consider the symmetric group on three letters Σ_3. (It should be noted that the formula for $\chi_{n,p}(G)$ requires us to sum over *all* abelian subgroups, not just those which are p–groups.) Here there are four maximal abelian subgroups, three of order 2 and one of order 3. The intersection of any pair of them is the trivial subgroup. It follows that the value of μ_{Σ_3} on the trivial subgroup is -3. Hence the formula gives

$$\begin{aligned} \chi_{n,p}(\Sigma_3) &= \frac{3 \cdot 2\chi_{n,p}(\mathbf{Z}/(2)) + 3\chi_{n,p}(\mathbf{Z}/(3)) - 3}{6} \\ \chi_{n,2}(\Sigma_3) &= \frac{3 \cdot 2 \cdot 2^n + 3 - 3}{6} \\ &= 2^n \\ &= \chi_{n,2}(\mathbf{Z}/(2)) \\ \chi_{n,3}(\Sigma_3) &= \frac{3 \cdot 2 + 3 \cdot 3^n - 3}{6} \\ &= \frac{3^n + 1}{2} \end{aligned}$$

Recall that $B\Sigma_3$ localized at the prime 3 is one of two stable summands of $B\mathbf{Z}/(3)$, and we see that $\chi_{n,3}(\Sigma_3)$ is roughly half of $\chi_{n,3}(\mathbf{Z}/(3))$.

Next we consider the alternating group A_4. It has five maximal abelian subgroups, four of order three and one isomorphic to $(\mathbf{Z}/(2))^2$. The latter has three subgroups of order 2. Thus we have

$$\begin{array}{ccccc} 0 & \longrightarrow & 3(\mathbf{Z}/(2)) & \longrightarrow & (\mathbf{Z}/(2))^2 \\ \downarrow & & & & \\ 4(\mathbf{Z}/(3)) & & & & \end{array}$$

The Möbius function μ_{A_4} vanishes on each subgroup of order 2, and its value on the trivial subgroup is -4. Thus we have

$$\begin{aligned}
\chi_{n,p}(A_4) &= \frac{4\cdot 3\chi_{n,p}(\mathbf{Z}/(3)) + 4\chi_{n,p}(\mathbf{Z}/(2))^2 - 4}{12} \\
\chi_{n,2}(A_4) &= \frac{4\cdot 3 + 4\cdot 4^n - 4}{12} \\
&= \frac{4^n+2}{3} \\
\chi_{n,3}(A_4) &= \frac{4\cdot 3\cdot 3^n + 4 - 4}{12} \\
&= 3^n \\
&= \chi_{n,3}(\mathbf{Z}/(3)).
\end{aligned}$$

Finally, we look at the symmetric group on 4 letters, Σ_4. It has three cyclic subgroups of order 4. In each case the subgroup of order 2 is contained in A_4. There are six additional subgroups of order 2, each generated by a transposition. There are three more noncyclic subgroups of order 4. In each of them one of the three subgroups of order 2 is contained in A_4 and the other two are not.

Thus the diagram of abelian subgroups is

$$\begin{array}{ccccc}
6(\mathbf{Z}/(2)) & & 3((\mathbf{Z}/(2))^2) & & \\
\uparrow & & \uparrow & & \\
0 & \longrightarrow & 3(\mathbf{Z}/(2)) & \longrightarrow & (\mathbf{Z}/(2))^2 \\
\downarrow & & \downarrow & & \\
4(\mathbf{Z}/(3)) & & 3(\mathbf{Z}/(4)) & &
\end{array}$$

It follows that μ_{Σ_4} vanishes on the six subgroups of order 2 not contained in $_4$ and takes the value -2 on the three that are. Its value on the trivial subgroup is -4. Thus the formula gives

$$\begin{aligned}
\chi_{n,p}(\Sigma_4) &= \frac{4\cdot 3\chi_{n,p}(\mathbf{Z}/(3)) + 4\cdot 4\chi_{n,p}(\mathbf{Z}/(2))^2 + 3\cdot 4\chi_{n,p}(\mathbf{Z}/(4))}{24} \\
&\quad - \frac{6\cdot 2\chi_{n,p}(\mathbf{Z}/(2)) + 4}{24} \\
\chi_{n,2}(\Sigma_4) &= \frac{4\cdot 3 + 4\cdot 4\cdot 4^n + 3\cdot 4\cdot 4^n - 6\cdot 2\cdot 2^n - 4}{24} \\
&= \frac{7\cdot 4^n - 3\cdot 2^n + 2}{6} \\
\chi_{n,3}(\Sigma_4) &= \frac{4\cdot 3\cdot 3^n + 4\cdot 4 + 3\cdot 4 - 6\cdot 2 - 4}{24} \\
&= \frac{3^n+1}{2} \\
&= \chi_{n,3}(\Sigma_3)
\end{aligned}$$

Proof of 4.1. We first treat the case $n = 0$. In this case we are counting conjugacy classes of 0–tuples, so the answer should be one. In particular $\chi_{0,p}(A) = 1$

for any abelian group A. If we multiply the right hand side by $|G|$ we get

$$\sum_{A<G} |A|\mu_G(A) = \sum_{A<G} \sum_{g \in A} \mu_G(A).$$

since $|A|$ is the number of elements in A. If we change the order of summation we get

$$\sum_{g \in G} \sum_{A \ni g} \mu_G(A). \tag{4.2}$$

where the second sum is over all abelian subgroups containing the element g. These groups also contain the cyclic subgroups generated by g, so the inner sum is 1 by the definition of μ_G. Hence the value of the expression in (4.2) is $|G|$ so $\chi_{0,p}(G) = 1$ for all G as expected.

For $n > 0$ we can write

$$\sum_{A<G} \frac{|A|}{|G|} \mu_G(A) \chi_{n,p}(A) = \sum_{A<G} \frac{1}{|G|} \sum_{g_0, g_1, \cdots, g_n \in A} \mu_G(A)$$

where the elements $g_1, \cdots, g_n \in A$ all have order a power of p, since $\chi_{n,p}(A) = |A_{(p)}|^n$. Changing the order of summation gives

$$\sum_{A<G} \frac{|A|}{|G|} \mu_G(A) \chi_{n,p}(A) = \sum_{g_0, g_1, \cdots, g_n \in G} \frac{1}{|G|} \sum_{A \ni g_0, g_1, \cdots, g_n} \mu_G(A)$$

where the outer sum is over all $(n+1)$–tuples (g_i) of commuting elements with all but g_0 having order a power of p. Again the inner sum is one because it is taken over all the abelian subgroups containing the one generated by the g_i, so we have

$$\sum_{A<G} \frac{|A|}{|G|} \mu_G(A) \chi_{n,p}(A) = \sum_{g_0, g_1, \cdots, g_n \in G} \frac{1}{|G|}.$$

Now G acts on the n–tuple $(g_1, \cdots, g_n)$ by coordinate–wise conjugation and the isotropy group is precisely the set of elements g_0 which commute with each of $g_1, \cdots, g_n$. We can write

$$\sum_{g_0, g_1, \cdots, g_n \in G} \frac{1}{|G|} = \sum_{(g_1, \cdots, g_n) \in G_{n,p}} \sum_{g_0} \frac{1}{|G|},$$

and the inner sum is one over the number n–tuples in $G_{n,p}$ conjugate to $(g_1, \cdots, g_n)$. It follows that our expression is the number of G–orbits in $G_{n,p}$ as claimed. ■

5 The Euler characteristic of $K(n)^*(BG)$

In this section we will outline the proof of Theorem 3.2. Thus our aim is to prove the formula of 4.1 with the understanding that $\chi_{n,p}(G)$ denotes the Euler characteristic of $K(n)^*(BG)$. We already know $\chi_{n,p}(A)$ for finite abelian groups A by 2.3. The outline given here will be more pedestrian than the proof given in [HKR], more along the lines in which we first thought of it.

Let $\rho : G \to U(m)$ be a faithful unitary representation of G, so that G acts freely on $U(m)$. This gives a G–action on the flag manifold

$$F(m) = U(m)/T^n$$

(where T^n denotes the maximal torus in $U(m)$, i.e., the group of diagonal matrices) in which every isotropy group is abelian. The Borel construction

$$X = F(m) \times_G EG$$

(where EG is a free contractible G–space) is an $F(m)$–bundle over BG. It is easy to show that its $K(n)$–theoretic Euler characteristic, which we will denote by $\chi_{n,p}(X)$, is $m!$ times that of BG since $\chi_{n,p}(F(m)) = m!$.

For each abelian subgroup $A \subset G$, let $F(m)^{\langle A\rangle}$ denote the subspace of $F(m)$ consisting of points where the isotropy group is precisely A. Then $F(m)^{\langle A\rangle}$ is an open dense subset of $F(m)^A$, the subspace fixed by A. In fact we have

$$F(m)^A = \bigcup_{A'\supset A} F(m)^{\langle A'\rangle}.$$

We can describe the subspaces $F(m)^A$ explicitly as follows. Each $A \subset G$ determines an eigenspace decomposition of the vector space $V = \mathbf{C}^m$ on which G acts. Recall that a point in the flag manifold $F(m)$ is a decomposition of V into one dimensional subspaces. If each of these subspaces is contained in an eigenspace of A, then the flag is fixed by A. If the eigenspace decomposition has the form

$$V = V_1 \oplus V_2 \oplus \cdots V_k$$

where V_i has rank d_i, then $F(m)^A$ is a finite disjoint union of copies of submanifolds of the form

$$F(d_1) \times F(d_2) \times \cdots F(d_k),$$

the number of copies being

$$\frac{m!}{d_1!\, d_2! \cdots d_k!}.$$

(The fact that $F(m)^A$ has the same Euler characteristic as $F(m)$ is a special case of the Lefschetz Fixed Point Theorem.)

Each subspace $F(m)^A \subset F(m)$ is a complex submanifold equipped with a complex normal bundle. The compliment of the zero section in this bundle has trivial Euler characteristic, so we have

$$\chi_{n,p}(F(m)) = \chi_{n,p}(F(m)^A) + \chi_{n,p}(F(m) - F(m)^A),$$

from which we deduce that for each $A \subset G$

$$m! = \chi_{n,p}(F(m)^A) = \sum_{A \subset A'} \chi_{n,p}(F(m)^{\langle A' \rangle}),$$

the sum being over all abelian subgroups A' containing A.

We can solve these equations for $\chi_{n,p}(F(m)^{\langle A \rangle})$, and get

$$\chi_{n,p}(F(m)^{\langle A \rangle}) = m!\, \mu_G(A)$$

where μ_G is the Möbius function defined in Section 4.

The subspace $F(m)^{\langle A \rangle} \subset F(m)$ is not G–invariant because $g \in G$ sends $F(m)^{\langle A \rangle}$ to $F(m)^{\langle gAg^{-1} \rangle}$. Let

$$F(m)^{(A)} = \bigcup_{g \in G} F(m)^{\langle gAg^{-1} \rangle}.$$

It is invariant and each orbit in it is isomorphic to G/A.

It follows that

$$F(m)^{(A)} \times_G EG \simeq F(m)^{(A)}/G \times BA$$

and

$$\chi_{n,p}(F(m)^{(A)} \times_G EG) = \frac{|A|}{|G|} \chi_{n,p}(F(m)^{(A)}) \chi_{n,p}(A).$$

Summing over all conjugacy classes (A) of abelian subgroups $A \subset G$, we get

$$\begin{aligned} m!\, \chi_{n,p}(G) &= \chi_{n,p}(F(m) \times_G EG) \\ &= \sum_{(A)} \chi_{n,p}(F(m)^{(A)} \times_G EG) \\ &= \sum_{A \subset G} \frac{|A|}{|G|} \chi_{n,p}(F(m)^{\langle A \rangle}) \chi_{n,p}(A) \\ &= \sum_{A \subset G} \frac{|A|}{|G|} m!\, \mu_G(A) \chi_{n,p}(A), \end{aligned}$$

which proves Theorem 3.2.

6 The Lubin–Tate construction

In this section we will describe the Lubin–Tate construction [LT65], which is a description of the maximal totally ramified abelian extension of a local field K of characteristic 0 (i.e., of a finite extension of the p–adic numbers $\mathbf{Q}_p$) using formal group laws. Accounts can also be found in [Ser67a] and [Haz78, Section 32]. We need these fields because they are the natural targets for our generalized characters, to be described in the next section.

Recall that the maximal abelian extension of the rationals $\mathbf{Q}$ can be obtained by adjoining all of the roots of unity in $\mathbf{C}$. These are the elements of finite order, or torsion points in the multiplicative group of the complex numbers.

Now suppose we are given a formal group law F defined over the ring of integers A of some finite extension K of the p–adic numbers $\mathbf{Q}_p$. This is power series $F(x, y)$ with certain properties described above. If x and y are elements in the maximal ideal of A, then this power series will converge, since A is complete. The same is true if x and y lie in the maximal ideal $\mathbf{m}$ of the completion of the algebraic closure $\overline{\mathbf{Q}}_p$ of K. In this way we obtain a group structure on $\mathbf{m}$. We will denote this group by $\mathbf{m}_F$.

When F is the additive formal group law $x + y$, then we get the usual additive group structure on $\mathbf{m}$. In this case there are no nontrivial elements of finite order.

When F is the multiplicative formal group law, we get the usual multiplicative groups structure on $1 + \mathbf{m}$. The elements of finite order are roots of unity congruent to 1 modulo $\mathbf{m}$, i.e., the $(p^i)^{\text{th}}$ roots of unity for various i. The field obtained by adjoining all of these elements to $\mathbf{Q}_p$ is the maximal totally ramified abelian extension.

The main result of [LT65] is that with a suitable choice of the formal group law F, the field obtained by adjoining all of the elements of finite order in $\mathbf{m}_F$ is the maximal totally ramified abelian extension of the given field K. In order to specify this choice, we need the notion of a *formal A–module*, which is a formal group law over an A–algebra R with certain additional structure. (Recall that A is the ring of integers in K, a finite extension of $\mathbf{Q}_p$.).

Associated with any formal group law there are power series $[n](x)$ for integers n satisfying

(i) $[1](x) = x$,

(ii) $F([m](x), [n](x)) = [m + n](x)$ and

(iii) $[m]([n](x)) = [mn](x)$.

When F is defined over a $\mathbf{Z}_p$–algebra, one can use continuity to extend this definition to $[u](x)$ for any $u \in \mathbf{Z}_p$. *F is a formal A–module if we can define $[a](x)$ with similar properties for all $a \in A$.* An account of theory of formal A–modules can be found in [Haz78, Section 21].

Theorem 6.1 (Lubin–Tate) *Let A be the ring of integers in a finite extension K of the p–adic numbers $\mathbf{Q}_p$. Let $\pi \in A$ be a generator of the maximal ideal and let q be the cardinality of the residue field $A/(\pi)$. Let $f(x) \in A[[x]]$ be a power series with*

$$f(x) \equiv \pi x \bmod (x)^2 \quad \textit{and} \quad f(x) \equiv u x^q \bmod (\pi),$$

where u is a unit in A.

Then

(i) *There is a unique formal A–module F over A for which $[\pi](x) = f(x)$.*

(ii) *The field obtained by adjoining the elements of finite order in $\mathbf{m}_F$ is the maximal totally ramified abelian extension L of K. These elements are the roots of the iterates of f. Let L_i denote the field obtained by adjoining the elements of order dividing $(q-1)q^{i-1}$ in $\mathbf{m}_F$, i.e., by the roots of $f^{\circ i}$, the i^{th} iterate of f.*

(iii) *The Galois group $\mathrm{Gal}(L : K)$ is isomorphic to the group of units $A^\times$ in A. For x an element of finite order in $\mathbf{m}_F$, the image of x under the automorphism corresponding to $a \in A^\times$ is $[a](x)$. If x is a root of $f^{\circ i}$, then it it fixed by this automorphism if a is congruent to 1 modulo $(\pi)^i$.*

When $A = \mathbf{Z}_p$, we can take

$$f(x) = (1+x)^p - 1 = \sum_{1 \le k \le p} \begin{pmatrix} p \\ k \end{pmatrix} x^k$$

and the statements in the 6.1 can be readily verified. In this case F is the multiplicative formal group law $x + y + xy$ and the roots of the of $f^{\circ i}$ are elements of the form $\zeta - 1$, where ζ is a $(p^i)^{\text{th}}$ root of unity.

More generally, when f is a polynomial of degree q (the simplest example is $\pi x + x^q$), then $g_1(x) = f(x)/x$ is irreducible by Eisenstein's criterion. The same is true of the polynomials $g_i(x)$ defined inductively for $i > 1$ by $g_i(x) = g_{i-1}(f(x))$. L_i is the splitting field for $g_i(x)$.

7 Generalized characters and $E(n)^*(\mathbf{BG})$

In this section we will describe a homomorphism

$$E(n)^*(BG) \xrightarrow{\chi_G} \mathrm{Cl}_{n,p}(G), \tag{7.1}$$

the target being the ring of $\overline{\mathbf{Q}}_p$ valued conjugacy class functions on $G_{n,p}$, the set of commuting n–tuples of elements of prime power order in G.

As remarked in Section 3, an element $\gamma \in G_{n,p}$ is the same thing as a homomorphism

$$(\mathbf{Z}_p)^n \longrightarrow G.$$

Since G is finite, this factors through $(\mathbf{Z}/(p^i))^n$ for sufficiently large i. Thus we get a map

$$E(n)^*(BG) \xrightarrow{\gamma_i^*} E(n)^*(B(\mathbf{Z}/(p^i))^n).$$

Letting i go to ∞, we get a map

$$E(n)^*(BG) \xrightarrow{\gamma^*} \varinjlim_i E(n)^*(B(\mathbf{Z}/(p^i))^n).$$

Note that this direct limit is *not* the same as

$$E(n)^*(\lim_{\substack{\leftarrow \\ i}} B(\mathbf{Z}/(p^i))^n),$$

which is far less interesting in this context.

We could define χ_G as in (7.1) if we had a suitable map

$$\lim_{\substack{\rightarrow \\ i}} E(n)^*(B(\mathbf{Z}/(p^i))^n) \longrightarrow \overline{\mathbf{Q}}_p \tag{7.2}$$

This is where the Lubin–Tate construction comes into the picture.

We will illustrate with an easy example. Let the field K be the unramified extension of $\mathbf{Q}_p$ of degree n, and let π (the generator of its maximal ideal) be p. The cardinality q of the residue field $A/(p)$ is p^n. The homomorphism φ as in Theorem 3.3 must send v_n to a unit in A since v_n is invertible in $E(n)^*$. Then the conditions on $f(x)$ in Theorem 6.1 are identical to those on $\varphi([p](x))$ in (2.5).

This means that (2.6) translates into

$$E(n)^*(B\mathbf{Z}/(p^i)) \otimes K = K[[x]]/(f^{\circ i}(x))$$

and we can extend φ to a homomorphism

$$E(n)^*(B\mathbf{Z}/(p^i)) \xrightarrow{\varphi} L_i$$

by sending x to a root of g_i. We can do this compatibly for all i and get a map

$$\lim_{\substack{\rightarrow \\ i}} E(n)^*(B\mathbf{Z}/(p^i)) \xrightarrow{\varphi} L. \tag{7.3}$$

We will extend this further as in (7.2) in such a way that the characters given by (7.1) will be Galois equivariant in the sense of 1.5. The construction of φ in (7.3) depends on a choice of roots $r_i \in L$ of g_i satisfying

$$r_i = f(r_{i+1}).$$

Recall that the Galois group $\mathrm{Gal}(L : K)$ is isomorphic to the group of units $A^\times$ in A. We can regard $(\mathbf{Z}/(p^i))^n$ as $A/(p^i)$. Then we can define

$$\lim_{\substack{\rightarrow \\ i}} E(n)^*(BA/(p^i)) \xrightarrow{\varphi} L$$

to be the equivariant extension of the φ of (7.3).

We can make this more explicit as follows. We have

$$E(n)^*(BA/(p^i)) = E(n)^*[[x_0, x_1, \cdots x_{n-1}]]/([p^i](x_j)).$$

Pick a $\mathbf{Z}_p$–basis of A of the form $\{a_0 = 1, a_1, \cdots a_{n-1}\}$. Then extend φ from $E(n)^*(B\mathbf{Z}/(p^i))$ to $E(n)^*(BA/(p^i))$ by defining

$$\varphi(x_j) = [a_j](r_i). \tag{7.4}$$

This map φ enables us to define

$$E(n)^*(BG) \otimes_{E(n)^*} K \xrightarrow{\chi_G} \mathrm{Cl}_{n,p}(G, L)^{A^\times}, \tag{7.5}$$

where $\mathrm{Cl}_{n,p}(G, L)^{A^\times}$ denotes the ring of Galois equivariant L–valued conjugacy class functions on $G_{n,p}$.

The Galois equivariance of these characters is not mentioned in [HKR]. If we replace K by L in the source of (7.5), we can drop the equivariant condition on the target. In the theorem stated in [HKR], L is replaced by $\overline{\mathbf{Q}}_p$.

We will now give an indication of why the map of (7.5) is an isomorphism after a suitable completion of the source. It is not hard to show that the target behaves well with respect to abelian subgroups, i.e. that it satisfies an analogue of Artin's Theorem (1.8). (This is Lemma 4.11 of [HKR].) Thus Theorem 3.4 can be used to reduce to the case of abelian groups. Then a routine argument reduces it further to the case of finite cyclic p–groups.

Therefore we will examine the case $G = \mathbf{Z}/(p^i)$ in more detail. In this case the target of χ_G is a free K–module of rank p^{ni}. The same is true of the source after suitable completion, with the generators being the powers of the orientation class $x \in E(n)^2(BG)$. Thus χ_G can be represented by a $(p^{ni} \times p^{ni})$ matrix M_G over K, which must be shown to be nonsingular.

We will use the notation of (7.4). Recall that

$$G_{n,p} = \mathrm{Hom}(A, G).$$

Given $\theta \in \mathrm{Hom}(A, G)$, we need to compute

$$\chi_G(x)((\theta)) \quad = \quad \varphi(\theta^*(x)) \in L,$$

which must be a root of $f^{\circ i}$. Let θ_j for $0 \le j \le p^{ni} - 1$ denote the p^{ni} elements of $G_{n,p}$ and let

$$\lambda_j = \chi_G(x)(\theta_j) \in L.$$

These λ_j are the p^{ni} distinct roots of $f^{\circ i}$, and we have

$$\chi_G(x^k)(\theta_j) = \lambda_j^k \in L.$$

It follows that M_G is a Vandermonde matrix with entries

$$m_{j,k} = \lambda_j^k.$$

Therefore it is nonsingular, completing our outline of the proof of Theorem 3.3.

8 The wreath product theorem

In this section we will prove Theorem 3.8. Recall the definition of good groups given in 3.7. The following properties of such groups are immediate.

Proposition 8.1 *(a) G is good if its p–Sylow subgroup is good.*

(b) If G_1 and G_2 are good then so is their product $G_1 \times G_2$.

(c) Every finite abelian group is good. (In fact, if G is abelian then $K(n)^*(BG)$ is generated by Euler classes of representations of G itself.)

Before specializing to the wreath product situation we note

Lemma 8.2 *If $f : H \to G$ is a homomorphism and $x \in K(n)^*(BG)$ is good, then $f^*(x)$ is a linear combination of good elements in $K(n)^*(BH)$.*

Proof. Suppose $x = \mathrm{Tr}(e(\rho))$ where ρ is a representation of $K < G$. Then there is a pullback diagram of spaces of the form

$$\begin{array}{ccc} \coprod BH_\alpha & \xrightarrow{\coprod f_\alpha} & BK \\ \downarrow & & \downarrow \\ BH & \xrightarrow{f} & BG \end{array}$$

where each H_α is a subgroup of H. This happens because the pullback is a covering of BH whose degree is the index k of K in G. Therefore its higher homotopy groups vanish and it must be a disjoint union of the indicated form. The sum of the indices of the H_α must be k.

By naturality of the transfer,

$$f^*(x) = \sum_\alpha \mathrm{Tr}^* e(f_\alpha^*(\rho)). \qquad \blacksquare$$

Corollary 8.3 *If x and y are good elements of $K(n)^*(BG)$ then their cup product xy is a sum of good elements.*

Proof. $x \times y \in K(n)^*(BG \times G)$ is good and $xy = \Delta^*(x \times y)$ where $\Delta : G \to G \times G$ is the diagonal map. $\blacksquare$

To prove the Theorem 3.8, we study the extension

$$G^p \longrightarrow W \longrightarrow \mathbf{Z}/(p)$$

and the associated spectral sequence

$$E_2 = H^*(\mathbf{Z}/(p); K(n)^*(BG^p)) \Rightarrow K(n)^*(BW). \tag{8.4}$$

$\mathbf{Z}/(p)$ acts on $K(n)^*(BG^p)$ by permuting the factors. Thus, as a module over $\mathbf{Z}/(p)$, we have

$$K(n)^*(BG^p) = F \oplus T$$

where F is a free $\mathbf{Z}/(p)$–module and T has trivial $\mathbf{Z}/(p)$–action. If $\{x_i\}$ is a basis of $K(n)^*(BG)$, then F has basis

$$\{x_{i_1} \otimes x_{i_2} \cdots \otimes x_{i_p}\}$$

where the subscripts $i_1, \dots i_p$ are not all the same, and T has basis $\{Px_i\}$ where $Px_i = x_i \otimes x_i \cdots x_i$. Moreover

$$H^i(\mathbf{Z}/(p); F) = \begin{cases} F^{\mathbf{Z}/(p)} & \text{for } i = 0 \\ 0 & \text{for } i > 0 \end{cases}$$

and

$$H^*(\mathbf{Z}/(p); T) = T \otimes H^*(B\mathbf{Z}/(p))$$

where

$$H^*(B\mathbf{Z}/(p)) = E(u) \otimes P(x)$$

with $u \in H^1$ and $x \in H^2$.

We need to show that each element in

$$E_2^{0,*} = H^0(\mathbf{Z}/(p); T \oplus F)$$

is a permanent cycle. Once we have this, then we know that the spectral sequence has a differential of the form

$$d_{2p^n-1}(u) = v_n x^{p^n}$$

because this happens in the case when G is trivial, i.e., in the Atiyah–Hirzebruch spectral sequence for $K(n)^*(B\mathbf{Z}/(p))$. It would then follow that

$$E_{2p^n} = E_\infty = H^0(\mathbf{Z}/(p); F) \oplus (H^0(\mathbf{Z}/(p); T) \otimes K(n)^*(B\mathbf{Z}/(p)))$$

which is concentrated in even dimensions. We will also show that W is good. Thus we will need the following two lemmas.

Lemma 8.5 *Each element in $H^0(\mathbf{Z}/(p); F)$ is a permanent cycle in the spectral sequence (8.4).*

Lemma 8.6 *Each element in $H^0(\mathbf{Z}/(p); T)$ is a permanent cycle in the spectral sequence (8.4).*

Proof of Lemma 8.5. Let $x = x_{i_1} \otimes \cdots x_{i_p}$ be a basis element of F and let $\sigma(x)$ denote the sum of x and all of its conjugates under the action of $\mathbf{Z}/(p)$. Then $H^0(\mathbf{Z}/(p); F)$ is spanned by these $\sigma(x)$. Moreover $\sigma(x)$ is the image of x under the composite

$$K(n)^*(BG^p) \xrightarrow{\mathrm{Tr}^*} K(n)^*(BW) \xrightarrow{\mathrm{Res}^*} K(n)^*(BG^p).$$

(Res^* here denotes the restriction map induced by the inclusion $G^p \to W$.) An element in $E_2^{0,*} \subset K(n)^*(BG^p)$ is a permanent cycle iff it is the restriction of an element in $K(n)^*(BW)$. Hence this is true of each element of $H^0(\mathbf{Z}/(p); F) = F^{\mathbf{Z}/(p)}$. ■

Proof of 8.6. As in the proof above we need to show that for each basis element $x_i \in K(n)^*(BG)$, $Px_i \in K(n)^*(BG^p)$ is the restriction of an element $y \in K(n)^*(BW)$.

We can assume that x_i is a transferred Euler class $\mathrm{Tr}^*(e(\rho))$ for ρ a complex representation of some subgroup $H < G$. The representation $\rho \oplus \rho \oplus \cdots \rho$ of H^p extends to a representation $\hat{\rho}$ of $\mathbf{Z}/(p) \wr H$ and $e(\hat{\rho})$ restricts to $P(e(\rho))$.

The following diagram commutes.

$$\begin{array}{ccc} K(n)^*(BH^p) & \stackrel{\mathrm{Res}^*}{\longleftarrow} & K(n)^*(B(\mathbf{Z}/(p) \wr H)) \\ \downarrow \mathrm{Tr}^* & & \downarrow \mathrm{Tr}^* \\ K(n)^*(BG^p) & \stackrel{\mathrm{Res}^*}{\longleftarrow} & K(n)^*(BW) \end{array}$$

Hence we have

$$\begin{aligned} \mathrm{Res}^*\mathrm{Tr}^*(e(\hat{\rho})) &= \mathrm{Tr}^*\mathrm{Res}^* e(\hat{\rho}) \\ &= \mathrm{Tr}^*(Pe(\rho)) \\ &= P\mathrm{Tr}^*(e(\rho)) \\ &= Px_i \end{aligned}$$

so we can take $y = \mathrm{Tr}^*(e(\hat{\rho}))$. ■

Proof of Theorem 3.8. We have shown that each element in $E_2^{0,*}$ is a permanent cycle, so the spectral sequence has only one differential. It remains to show that W is good. We have

$$K(n)^*(BW) = H^0(\mathbf{Z}/(p); F) \oplus (H^0(\mathbf{Z}/(p); T) \otimes K(n)^*(B\mathbf{Z}/(p))).$$

$H^0(\mathbf{Z}/(p); F)$ is in the image of the transfer by Lemma 8.5, and Lemma 8.6 shows that $H^0(\mathbf{Z}/(p); T)$ is generated by elements of the form $\mathrm{Tr}^*(e(\hat{\rho}))$ where $\hat{\rho}$ is a representation of some subgroup of W. Recall that

$$K(n)^*(B\mathbf{Z}/(p)) = K(n)^*[x]/(x^{p^n})$$

where $x = e(\lambda)$, λ being a one–dimensional representation of of $\mathbf{Z}/(p)$. It follows from 8.3 that $\mathrm{Tr}^* e(\hat{\rho}) x^i$ is a sum of transferred Euler classes, completing the proof. ■

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