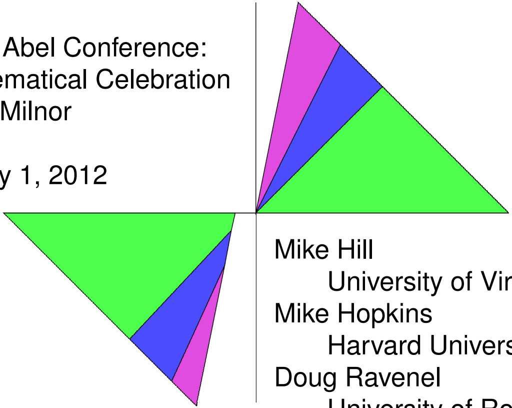
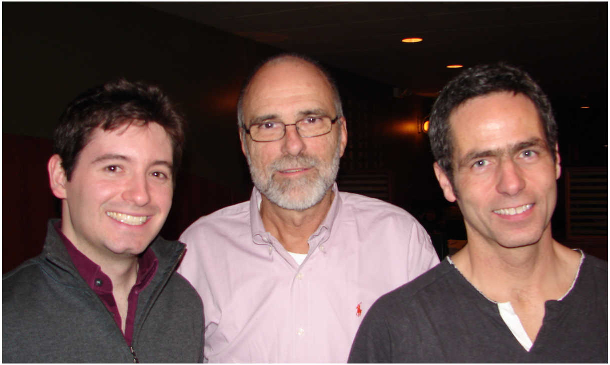
A solution to the Arf-Kervaire invariant problem

Second Abel Conference:
A Mathematical Celebration of John Milnor

February 1, 2012



Mike Hill
University of Virginia
Mike Hopkins
Harvard University
Doug Ravenel
University of Rochester



Mike Hill, myself and Mike Hopkins
Photo taken by Bill Browder
February 11, 2010

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The Kervaire-Milnor classification of exotic spheres

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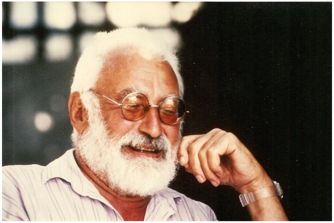
- 

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  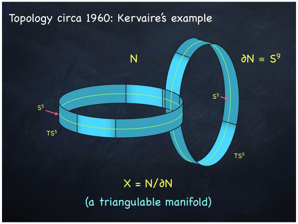

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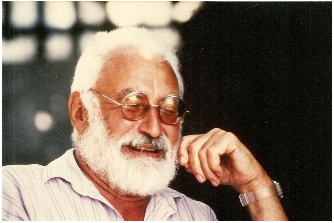
- 

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  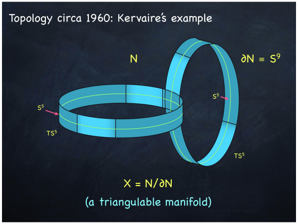

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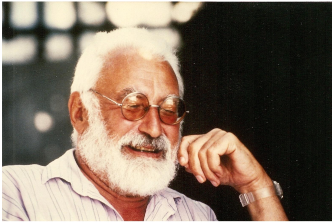
- 

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  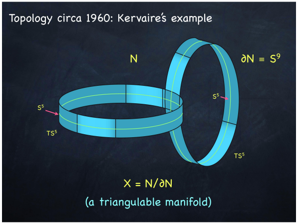

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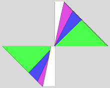

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For example, for $n = 1, 2, 3, \cdots, 18$, it will be shown that the order of the group Θ_n is respectively:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
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(ii) There was an ambiguous factor of two in dimensions congruent to 1 mod 4. The solution to that problem is the subject of this talk.

Pontryagin's early work on homotopy groups of spheres

Back to the 1930s

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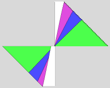

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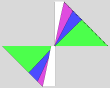

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- Pick a regular value $y \in S^k$. Its inverse image will be a smooth n-manifold M in S^{n+k}.
- By studying such manifolds, Pontryagin was able to deduce things about maps between spheres.

Pontryagin's early work (continued)

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 & & S^{n+k} & \xrightarrow{\quad f \quad} & S^k \\
 & & \uparrow & & \uparrow \\
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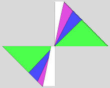
Two maps $f_1, f_2 : S^{n+k} \to S^k$ are homotopic if there is a continuous map $h : S^{n+k} \times [0, 1] \to S^k$ (called a homotopy between f_1 and f_2) such that



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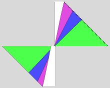

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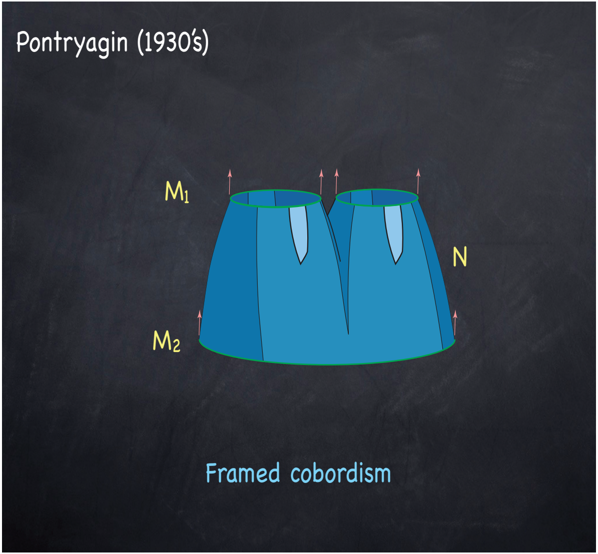
Pontryagin's early work (continued)

Here is an example of a framed cobordism for $n = k = 1$.



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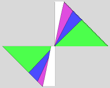
The above homomorphism is an isomorphism in all cases.



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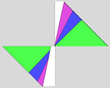
Two framings of an exotic sphere $\Sigma^n \subset S^{n+k}$ differ by a map to the special orthogonal group $SO(k)$, and this map does not depend on the differentiable structure on Σ^n.



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Heinz Hopf
1894-1971

$$\pi_n SO(k) \xrightarrow{J} \pi_{n+k} S^k$$

George Whitehead
1918-2004

called the Hopf-Whitehead *J*-homomorphism.

Exotic spheres as framed manifolds (continued)

Two framings of an exotic sphere $\Sigma^n \subset S^{n+k}$ differ by a map to the special orthogonal group $SO(k)$, and this map does not depend on the differentiable structure on Σ^n. Varying the framing on the standard sphere S^n leads to a homomorphism

Heinz Hopf
1894-1971

$$\pi_n SO(k) \xrightarrow{J} \pi_{n+k} S^k$$

George Whitehead
1918-2004

called the Hopf-Whitehead J-homomorphism. It is well understood by homotopy theorists.

Mike Hill
Mike Hopkins
Doug Ravenel

Exotic spheres as framed manifolds (continued)

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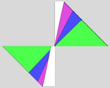

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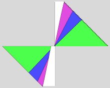

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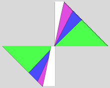

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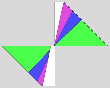

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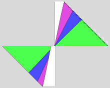

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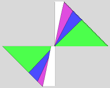

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We now know the value of bP_{4m+2} in every case except $m = 31$.

Mike Hill
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Exotic spheres as framed manifolds (continued)

In other words have a 4-term exact sequence

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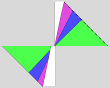

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To say more about this we need to define the Kervaire invariant of a framed manifold.

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Back to the 1940s

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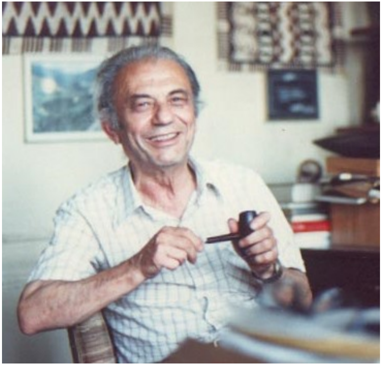

Cahit Arf 1910-1997

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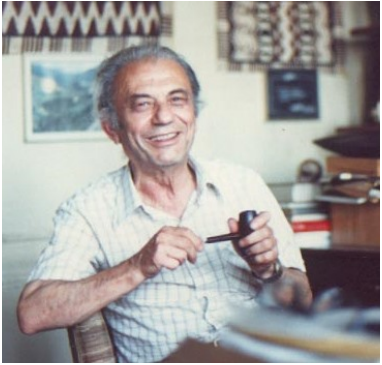

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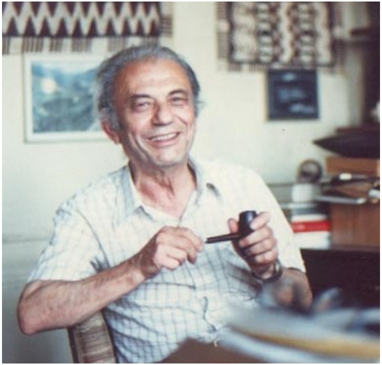

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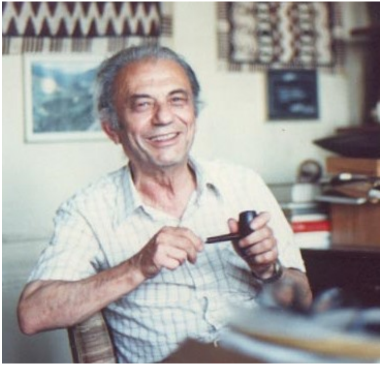

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$$\lambda(a_i, a_{i'}) = 0 \qquad \lambda(b_j, b_{j'}) = 0 \qquad \text{and} \qquad \lambda(a_i, b_j) = \delta_{i,j}.$$

The Arf invariant of a quadratic form in characteristic 2 (continued)

In other words, $\overline{H}$ has a basis for which the bilinear form's matrix has the symplectic form

$$\begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}.$$

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In 1941 Arf proved that this invariant (along with the number n) determines the isomorphism type of q.

From my stamp collection

From my stamp collection

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Money talks: Arf’s definition republished in 2009

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The Kervaire invariant of a framed $(4m+2)$-manifold

Into the 60s
a third time

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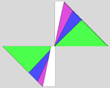
The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

For $M = T^2 \subset S^3$ and $x \in H_1(T^2; \mathbf{Z}/2)$, $q(x)$ is the number of full twists in a cylinder V neighboring a curve representing x.



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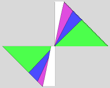

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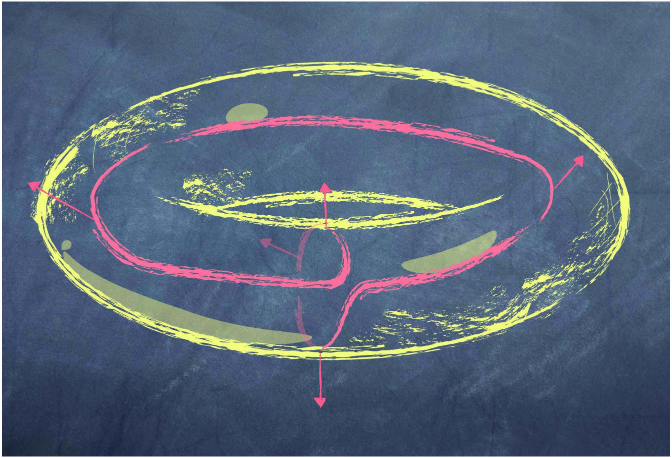

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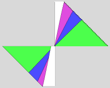

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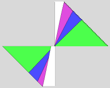

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Kervaire-Milnor Theorem (1963)

$bP_{4m+2} = 0$ iff there is a smooth framed $(4m+2)$-manifold M with $\Phi(M)$ nontrivial.

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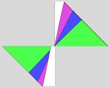
What can we say about $\Phi(M)$?



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What can we say about $\Phi(M)$?

For $m = 0$ there is a framing on the torus $S^1 \times S^1 \subset \mathbf{R}^4$ with nontrivial Kervaire invariant.

The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

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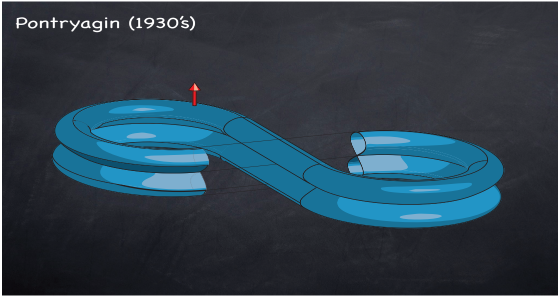

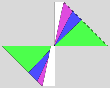

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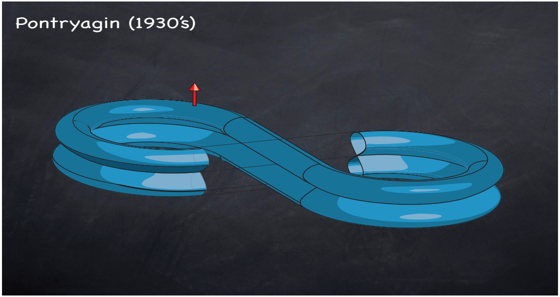

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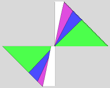

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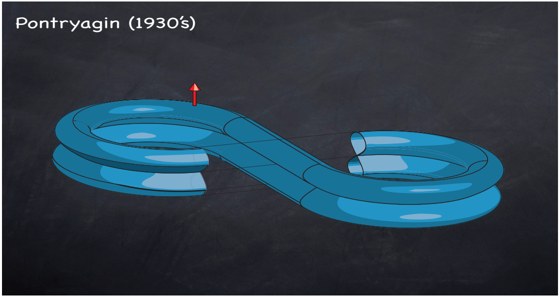

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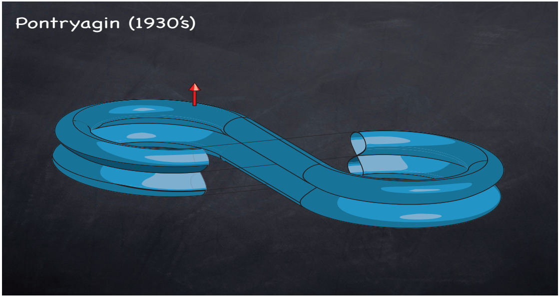

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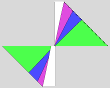

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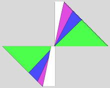
Kervaire (1960) showed it must vanish when $m = 2$, so $bP_{10} = \mathbf{Z}/2$.

Mike Hill
Mike Hopkins
Doug Ravenel



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Kervaire (1960) showed it must vanish when $m = 2$, so $bP_{10} = \mathbf{Z}/2$. This enabled him to construct the first example of a topological manifold (of dimension 10) without a smooth structure.

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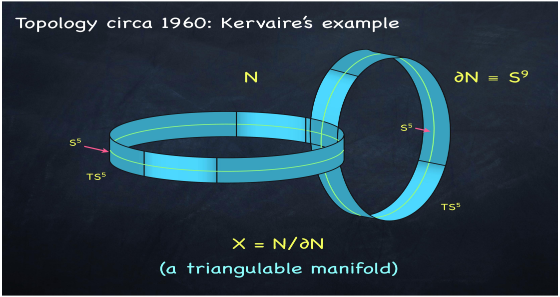

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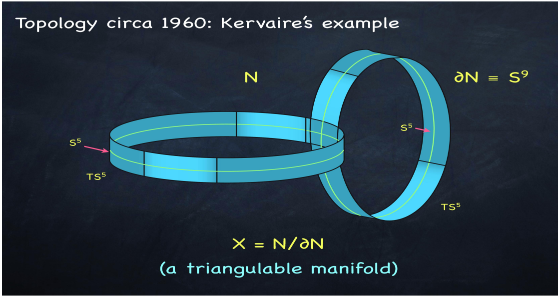

This construction generalizes to higher m, but Kervaire's proof that the boundary is exotic does not.

The Kervaire invariant of a framed $(4m + 2)$-manifold (continued)

More of what we can say about $\Phi(M)$.

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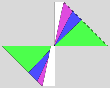

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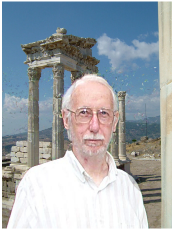

Ed Brown

Frank Peterson
1930-2000

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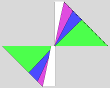

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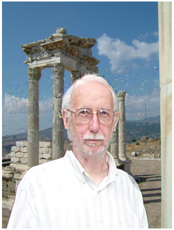

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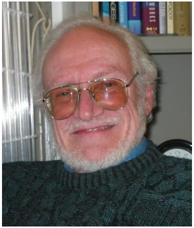
- 

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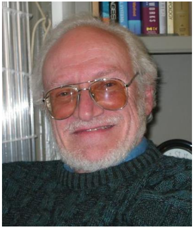
- 

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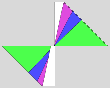

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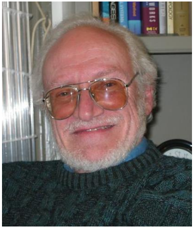
- 

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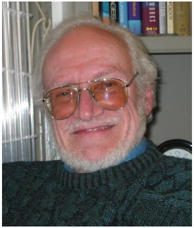
- 

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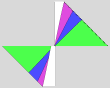

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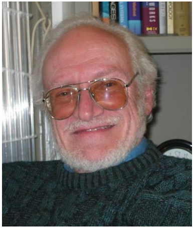
- 

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- θ_j is known to exist for $1 \leq j \leq 5$, i.e., in dimensions 2, 6, 14, 30 and 62.

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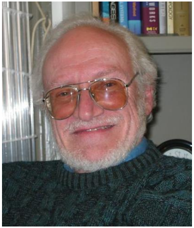
- 

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- θ_j is known to exist for $1 \leq j \leq 5$, i.e., in dimensions 2, 6, 14, 30 and 62. In other words, bP_2, bP_6, bP_{14}, bP_{30} and bP_{62} are all trivial.

And then . . .

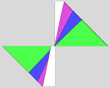
And then . . . the problem went viral!

A wildly popular dance craze

Drawing by Carolyn Snaith 1981
London, Ontario

Speculations about θ_j after Browder's theorem

In the decade following Browder's theorem, many topologists tried without success to construct framed manifolds with nontrivial Kervaire invariant in all such dimensions, i.e., to show that $bP_{2^{j+1}-2} = 0$ for all $j > 0$.



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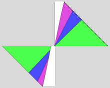

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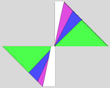

Mark Mahowald's sailboat

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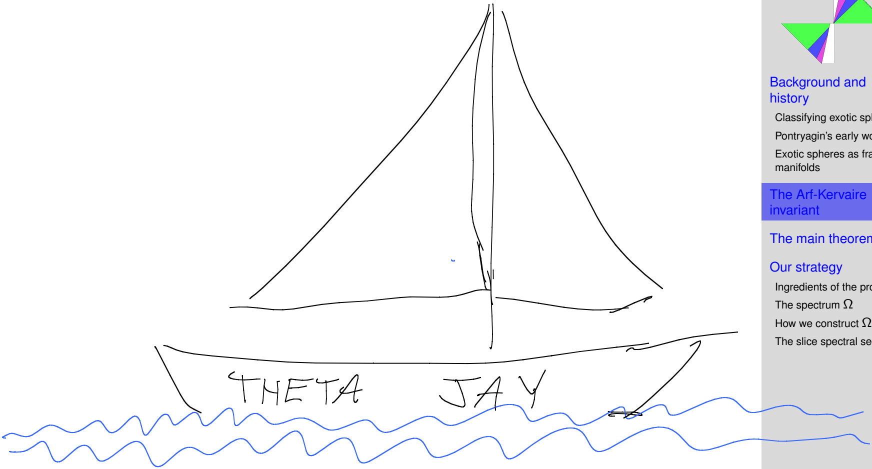

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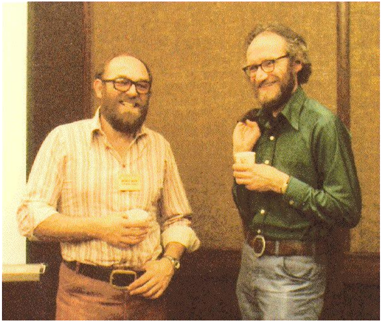

Vic Snaith and Bill Browder in 1981
Photo by Clarence Wilkerson

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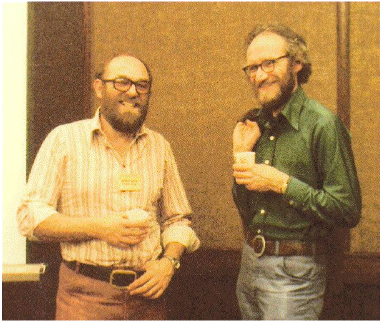

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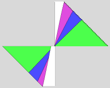

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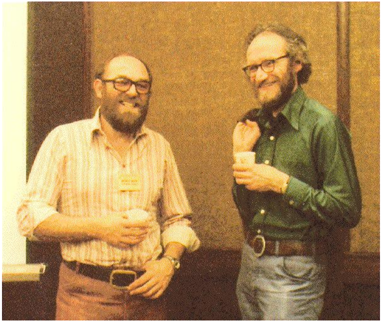

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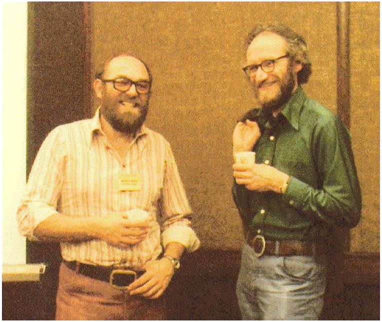

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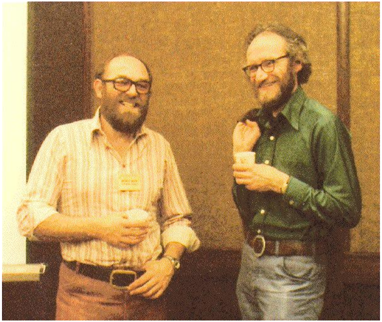

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to 2009

Snaith's book

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Stable Homotopy Around the Arf-Kervaire Invariant, published in early 2009,

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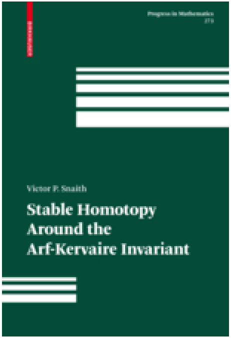
Snaith's book

Fast forward to 2009

Stable Homotopy Around the Arf-Kervaire Invariant, published in early 2009, just before we proved our theorem.

"As ideas for progress on a particular mathematics problem atrophy it can disappear. Accordingly I wrote this book to stem the tide of oblivion."

Snaith’s book (continued)



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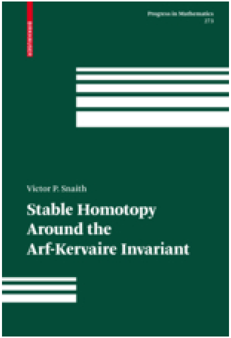

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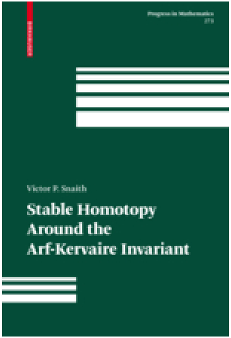

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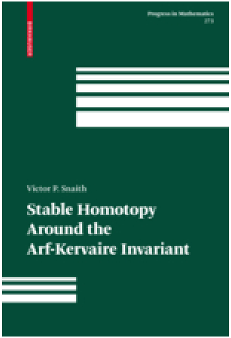

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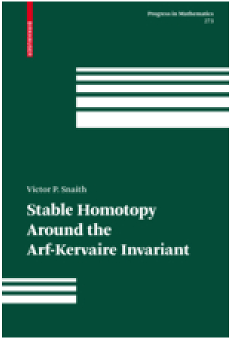

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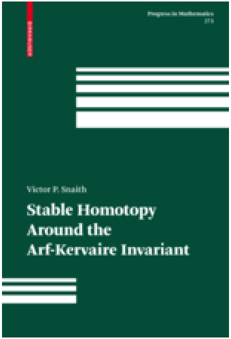

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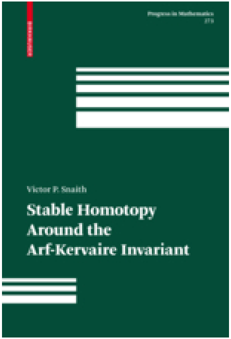

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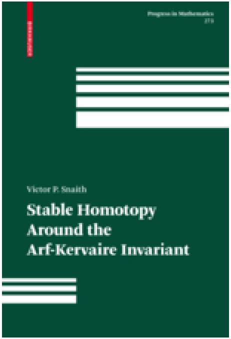

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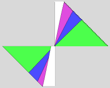

Our main result

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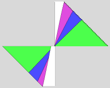

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Mike Hill
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There were several unsuccessful attempts in the 1970s to prove the opposite of what we have proved, namely that $bP_{2^{j+1}-2} = 0$ for all $j > 0$.

Mike Hill
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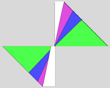
Here is the stable homotopy theoretic formulation.



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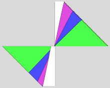

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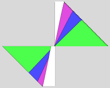

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Questions raised by our theorem

Mike Hill
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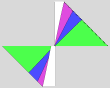

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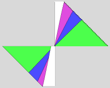

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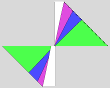

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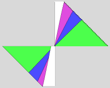
Ingredients of the proof

Our proof has several ingredients.



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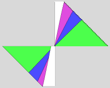

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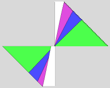

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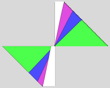

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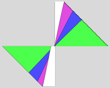

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John Milnor

Sergei Novikov

Dan Quillen
1940–2011

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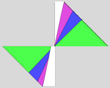

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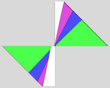

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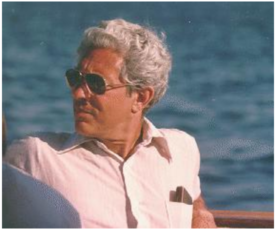

Peter May

John Greenlees

Gaunce Lewis
1949-2006

The spectrum Ω

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Mike Hill
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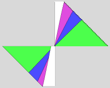

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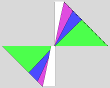

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The spectrum Ω (continued)

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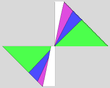
Mike Hill
Mike Hopkins
Doug Ravenel



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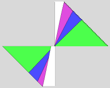

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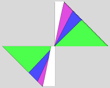

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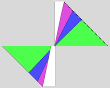
If $\theta_7 \in \pi_{254}(S^0)$ exists, (i) implies it has a nontrivial image in this group, so it cannot exist. The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$.

Mike Hill
Mike Hopkins
Doug Ravenel

How we construct Ω



How we construct Ω

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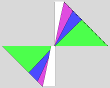

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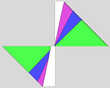

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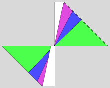

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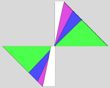

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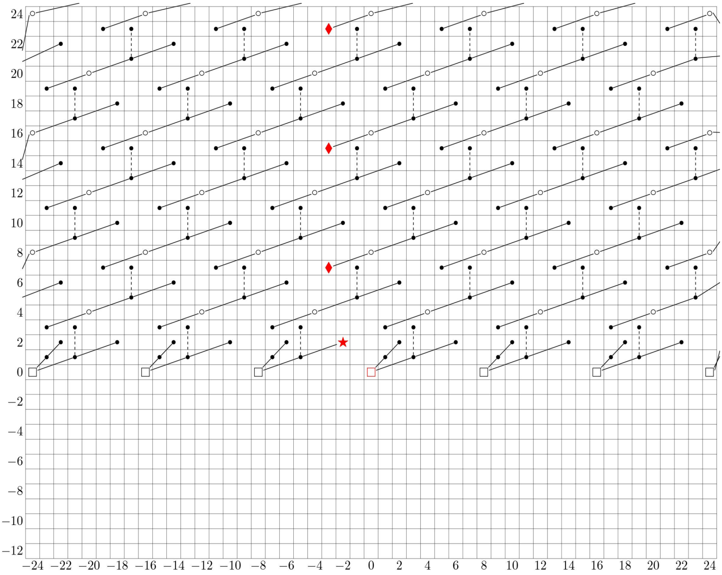
This spectrum is not periodic, but it has a close relative $\tilde{\Omega}$ which is.

A homotopy fixed point spectral sequence



The corresponding slice spectral sequence