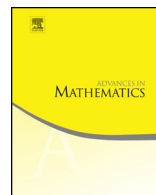




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The homological slice spectral sequence in motivic and Real bordism [☆]

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ABSTRACT

For a motivic spectrum $E \in \mathcal{SH}(k)$, let $\Gamma(E)$ denote the global sections spectrum, where E is viewed as a sheaf of spectra on Sm_k . Voevodsky's slice filtration determines a spectral sequence converging to the homotopy groups of $\Gamma(E)$. In this paper, we introduce a spectral sequence converging instead to the mod 2 homology of $\Gamma(E)$ and study the case $E = BPGL\langle m \rangle$ for $k = \mathbb{R}$ in detail. We show that this spectral sequence contains the \mathcal{A}_* -comodule algebra $\mathcal{A}_* \square_{\mathcal{A}(m)_*} \mathbb{F}_2$ as permanent cycles, and we determine a family of differentials interpolating between $\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2$ and $\mathcal{A}_* \square_{\mathcal{A}(m)_*} \mathbb{F}_2$. Using this, we compute the spectral sequence completely for $m \leq 3$. In the height 2 case, the Betti realization of $BPGL\langle 2 \rangle$ is the C_2 -spectrum $BP_{\mathbb{R}}\langle 2 \rangle$, a form of which was shown by Hill and Meier to be an equivariant model for $\mathrm{tmf}_1(3)$. Our spectral sequence therefore gives a computation of the comodule algebra $H_* \mathrm{tmf}_0(3)$. As a consequence, we deduce a new (2-local) Wood-type splitting

$$\mathrm{tmf} \wedge X \simeq \mathrm{tmf}_0(3)$$

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of tmf -modules predicted by Davis and Mahowald, for X a certain 10-cell complex.

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1. Introduction

The classical truncated Brown–Peterson spectrum $BP\langle m \rangle$ admits an action by the cyclic group of order 2, via the complex conjugation action on complex bordism, MU . The fixed points admit familiar geometric models for small heights m and give higher height analogues of connective real K -theory ko . We give a spectral sequence converging to the homology of these fixed point spectra by lifting the calculation to a simpler one in the \mathbb{R} -motivic stable homotopy category.

1.1. Motivation

The chromatic approach to stable homotopy gives a method to compute the stable homotopy groups of spheres via a step-by-step procedure. This procedure comes from the height filtration on the moduli stack of formal groups, mirrored in stable homotopy by Bousfield localization. The chromatic convergence theorem of Hopkins–Ravenel [34]

states that the p -local sphere spectrum may be recovered as the limit of its chromatic tower

$$S_{(p)}^0 \simeq \varprojlim (\cdots \rightarrow L_n S^0 \rightarrow L_{n-1} S^0 \rightarrow \cdots \rightarrow L_0 S^0)$$

where L_n denotes Bousfield localization at a height n Morava E -theory $E(k, \Gamma)$, for Γ a height n formal group over a perfect field k of characteristic p . The spectrum $L_n S^0$ is built in finitely many steps from the $K(i)$ -local spheres $L_{K(i)} S^0$ for $i \leq n$, via chromatic fracture squares. The spectrum $L_{K(n)} S^0$, in turn, admits a description as a homotopy fixed-point spectrum

$$L_{K(n)} S^0 \simeq E(k, \Gamma)^{h\mathbb{G}_n}$$

by a theorem of Devinatz–Hopkins [11], where $\mathbb{G}_n = \text{Aut}(\Gamma)$ is the Morava stabilizer group. This, in principle, reduces many questions in stable homotopy theory to understanding the \mathbb{G}_n -equivariant spectrum $E(k, \Gamma)$.

However, this is not easy to access in practice, even at small heights. It was an observation of Ravenel [35] and Hopkins–Miller [36] that $L_{K(n)} S^0$ is well approximated by the fixed points of $E(k, \Gamma)$ at *finite* subgroups of \mathbb{G}_n , and that these theories are more computable. These theories

$$EO_n(G) := E(k, \Gamma)^{hG}$$

are known as the Hopkins–Miller higher real K -theories.

The $EO_n(G)$'s have proven very effective at carrying rich information in an accessible way. For instance, studying these theories led to the solution of the Kervaire invariant problem, at primes $p \geq 5$ by Ravenel [35] and at $p = 2$ by Hill–Hopkins–Ravenel [19]. At the prime $p = 3$, they were used by Goerss–Henn–Mahowald–Rezk [13] to produce an explicit finite resolution of the $K(2)$ -local sphere, giving a conceptual framework to the calculation of $\pi_* L_{K(2)} S^0$ by Shimomura–Wang [37]. We refer the reader also to the introduction of [4] for a nice discussion of these theories and their history.

Connective models

Working with the $EO_n(G)$'s in practice is limited by the size of these theories. The $EO_n(G)$'s are non-connective, and their mod p homology vanishes, making it impossible to understand these theories directly from the point of view of the Adams spectral sequence. Moreover, the homotopy groups of the $EO_n(G)$'s are not degreewise finitely generated, so passing to the connective cover does not give a substantial improvement. A key feature of the study of the $EO_n(G)$'s is thus a search for good connective models $eo_n(G)$ with strong finiteness properties.

At heights $h = 2^{n-1}m$ with $p = 2$, the group \mathbb{G}_h contains a subgroup isomorphic to $G = C_{2^n}$, and Beaudry–Hill–Shi–Zeng defined and studied candidates for good connective models of $EO_h(G)$ in this case via Real bordism [5]. These theories—known

as the $BP^{((G))}\langle m \rangle$'s—are particularly accessible as they arise as the fixed points of a G -spectrum whose action comes from geometry, as opposed to the action on $E(k, \Gamma)$, which is defined via obstruction theory. An important manifestation of this is that the $BP^{((G))}\langle m \rangle$'s have a well understood slice filtration, in the sense of Hill–Hopkins–Ravenel [19].

In upcoming work [9], the first and second named authors use the slice filtration to establish the desired finiteness properties of the $BP^{((G))}\langle m \rangle$'s, showing that their fixed points are fp spectra of type $m|G|/2$, in the sense of Mahowald–Rezk [29]. For explicit calculations, however, we would like to know the comodules $H_*BP^{((G))}\langle m \rangle^G$, and the main aim of the present paper is to initiate such homology computations by exploring the case $G = C_2$.

The $G = C_2$ case

The cyclic group of order 2 is often used as a test case for computations in equivariant homotopy in general and for the study of the $EO_n(G)$'s and $BP^{((G))}\langle m \rangle$'s in particular. Quite a lot is known in this case; for instance, the homotopy fixed point spectral sequence computing $\pi_*EO_n(C_2)$ was completely computed at all heights by Hahn–Shi in [16]. Li–Shi–Wang–Xu showed that a large class of elements in the homotopy groups of spheres is detected by the $EO_n(C_2)$'s [27]. The $EO_n(C_2)$'s were studied extensively by Kitchloo–Wilson; they used these theories to prove new nonimmersion results for real projective spaces [25] [24], demonstrating that these theories are useful far beyond their role as a test case.

For $G = C_2$, the connective models $BP^{((G))}\langle m \rangle$ were first defined by Hu–Kriz [22] and are also known as the $BP_{\mathbb{R}}\langle m \rangle$'s, as they model the classical truncated Brown–Peterson spectra $BP\langle m \rangle$ with their C_2 -action via complex conjugation. At small heights m , the $BP_{\mathbb{R}}\langle m \rangle$'s and their fixed points admit familiar geometric models:

m	$BP_{\mathbb{R}}\langle m \rangle$	$BP_{\mathbb{R}}\langle m \rangle^{C_2}$
-1	$H\mathbb{F}_2$	$H\mathbb{F}_2$
0	$H\mathbb{Z}_{(2)}$	$H\mathbb{Z}_{(2)}$
1	$k_{\mathbb{R}}$	ko
2	$\mathrm{tmf}_1(3)$	$\mathrm{tmf}_0(3)$

Here $k_{\mathbb{R}}$ denotes connective Real K -theory in the sense of Atiyah [1], and $\mathrm{tmf}_1(3)$ and $\mathrm{tmf}_0(3)$ are spectra of topological modular forms with level structure (see [20] and [21]).

\mathbb{R} -motivic homotopy

For a real variety X , its set of complex points $X(\mathbb{C})$ comes equipped with the complex analytic topology, with respect to which the action of $C_2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is continuous. This extends to a symmetric monoidal left adjoint

$$\beta : \mathcal{SH}(\mathbb{R}) \rightarrow \mathcal{S}p^{C_2}$$

from the \mathbb{R} -motivic stable homotopy category to the category of genuine C_2 -spectra, called Betti realization. Real bordism and the $BP_{\mathbb{R}}\langle m \rangle$'s lift along β to \mathbb{R} -motivic analogues, known as motivic bordism and the $BPGL\langle m \rangle$'s.

Consequently, we lift the calculation of $H_*BP_{\mathbb{R}}\langle m \rangle^{C_2}$ along β to a simpler and more fundamental calculation in the \mathbb{R} -motivic stable homotopy category, and it is in this setting in which we do our computations. In the motivic setting, the role of fixed points $(-)^{C_2}$ is played by the global sections functor $\Gamma : \mathcal{SH}(k) \rightarrow \mathcal{Sp}$, the right adjoint to the unique colimit-preserving symmetric monoidal functor

$$i_* : \mathcal{Sp} \rightarrow \mathcal{SH}(k)$$

For $E \in \mathcal{SH}(k)$, $\Gamma(E)$ may be computed as the mapping spectrum $F(\mathrm{Spec}(k)_+, E)$ in $\mathcal{SH}(k)$ and thus is the global sections of E , viewed as a sheaf of spectra on Sm_k , the category of smooth schemes over $\mathrm{Spec}(k)$.

The advantage of working motivically is that we may discard the so-called negative cone in our calculations. As we show in Section 6, the presence of the negative cone in the C_2 -equivariant homology of a point results in serious complications. In Section 2, we give a method of comparing C_2 -equivariant calculations of the kind studied here to the corresponding \mathbb{R} -motivic calculations, and we use this to show that the results must agree in non-positive weights. In practice, this means we may discard the negative cone and still recover the correct result in degrees $a + b\sigma$ for $b \leq 0$ and, in particular, in integer degrees.

1.2. Main results

Section 2

We begin by defining the homological slice spectral sequence (HSSS) in Section 2. This spectral sequence arises by smashing the slice tower of $E \in \mathcal{SH}(k)$ with the motivic spectrum $i_*H\mathbb{F}_2$, giving a spectral sequence of \mathcal{A}_* -comodules. We show that when E is slice connective, this spectral sequence has strong vanishing lines and converges to the comodule $H_*\Gamma(E)$ (Proposition 2.6).

For our applications to the $BP_{\mathbb{R}}\langle m \rangle$'s, we compare the global sections functor Γ and the genuine fixed points functor $(-)^{C_2}$, producing a natural transformation

$$\Gamma(E) \rightarrow \beta(E)^{C_2}$$

which we show to be an equivalence in many cases. In particular,

Proposition 1.1 (Corollary 2.20). *For all m , there is an equivalence of spectra*

$$\Gamma(BPGL\langle m \rangle) \rightarrow BP_{\mathbb{R}}\langle m \rangle^{C_2}$$

Section 3

In [6], Behrens–Shah lift the Tate square in C_2 -equivariant homotopy along β to an arithmetic square

$$\begin{array}{ccc} E & \longrightarrow & E[\rho^{-1}] \\ \downarrow & & \downarrow \\ E_{\rho}^{\wedge}[\tau^{-1}] & \longrightarrow & E_{\rho}^{\wedge}[\tau^{-1}][\rho^{-1}] \end{array}$$

for $E \in \mathcal{SH}(\mathbb{R})$. In contrast to the Tate square, this is not usually a pullback, but we show that it becomes a pullback after applying $\Gamma(-)$ in many cases. For $E = i_* H\mathbb{F}_2 \otimes BPGL$, we completely determine the effect of the arithmetic square on bigraded homotopy groups; the result in weight zero is as follows.

Proposition 1.2 (Proposition 3.13). *On homotopy groups, the arithmetic square of $i_* H\mathbb{F}_2 \otimes BPGL$ is given by*

$$\begin{array}{ccc} H_*\Gamma(BPGL) & \longrightarrow & \mathcal{A}_* \\ \downarrow & & \downarrow \varphi \\ \mathbb{F}_2[z, \chi_i | i \geq 1]_z^{\wedge} & \hookrightarrow & \mathbb{F}_2[z, \chi_i | i \geq 1]_z^{\wedge}[z^{-1}] \end{array} \quad \begin{array}{c} \xi_i \\ \downarrow \\ \chi_i z^{2^i-1} + \chi_{i-1} z^{-1} \end{array}$$

with $|z| = -1$ and $|\chi_i| = 2(2^i - 1)$.

This leads to a somewhat explicit description of the \mathcal{A}_* -comodule $H_*\Gamma(BPGL)$ via the associated Mayer–Vietoris sequence (Theorem 3.15). This corrects an error of Hu–Kriz, who gave similar formulas in [23] (see Remark 3.9). The ρ -local part of this arithmetic square is complicated by the following unexpected fact about the reduction map $BPGL \rightarrow M\mathbb{F}_2$ from the motivic Brown–Peterson spectrum to the mod 2 motivic Eilenberg–MacLane spectrum.

Theorem 1.3 (Theorem 3.10). *The composite*

$$H\mathbb{F}_2 \simeq \Gamma(BPGL[\rho^{-1}]) \rightarrow \Gamma(M\mathbb{F}_2[\rho^{-1}]) \simeq \prod_{i \geq 0} \Sigma^i H\mathbb{F}_2$$

has components $Sq^i : H\mathbb{F}_2 \rightarrow \Sigma^i H\mathbb{F}_2$.

Equivariantly, identifying $\Phi^{C_2}(H\mathbb{F}_2)$ with the connective cover of $H\mathbb{F}_2^{tC_2}$, the composite

$$H\mathbb{F}_2 \simeq \Phi^{C_2} BP_{\mathbb{R}} \rightarrow \Phi^{C_2} H\mathbb{F}_2 \simeq \prod_{i \geq 0} \Sigma^i H\mathbb{F}_2$$

has components $Sq^i : H\mathbb{F}_2 \rightarrow \Sigma^i H\mathbb{F}_2$.

We prove Theorem 1.3 as an application of Mahowald's theorem on $H\mathbb{F}_2$. The description of the map appearing here comes via comparison with the Tate-valued Frobenius of Nikolaus–Scholze [31]. We expect this fact to be of independent interest.

Section 4

The HSSS for $BPGL\langle m \rangle$ has a straightforward E_2 -page.

Proposition 1.4 (Corollary 4.5). *The E_2 -page of the HSSS for $BPGL\langle m \rangle$ is isomorphic, as an \mathcal{A}_* -comodule algebra, to*

$$(\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2)[\rho, x_1, \bar{v}_1, \dots, \bar{v}_m]$$

where

$$\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2 \cong H_* H\mathbb{Z} \cong \mathbb{F}_2[\zeta_1^2, \zeta_2, \dots]$$

The tridegrees are given as follows

$$\begin{aligned} |\zeta_i| &= (2^i - 1, 0, 1 - 2^i) \\ |\rho| &= (-1, -1, 1) \\ |x_1| &= (0, -1, 0) \\ |\bar{v}_i| &= (2(2^i - 1), 2^i - 1, 0) \end{aligned}$$

where we use Adams trigrading $(t - s, w, s)$, representing (stem, weight, filtration). The classes ρ and \bar{v}_i are comodule primitives, and

$$\psi(x_1) = 1 \otimes x_1 + \xi_1 \otimes \rho$$

Our main application of the results of Section 3 on the arithmetic square is to determine a large class of permanent cycles in the HSSS for $BPGL\langle m \rangle$. This comes via the edge homomorphism

$$H_* \Gamma(BPGL\langle m \rangle) \rightarrow H_* \Gamma(P^0 BPGL\langle m \rangle) = H_* H\mathbb{Z}$$

Theorem 1.5 (Theorem 4.18). *In weight zero, the image of the edge homomorphism in the HSSS for $BPGL\langle m \rangle$ is precisely*

$$\mathcal{A}_* \square_{\mathcal{A}(m)_*} \mathbb{F}_2 \subset \mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2$$

The classes x_1, ρ , and \bar{v}_i are all permanent cycles, so the subalgebra

$$(\mathcal{A}_* \square_{\mathcal{A}(m)_*} \mathbb{F}_2)[\rho, x_1, \bar{v}_1, \dots, \bar{v}_m]$$

of E_2 consists of permanent cycles. We use the arithmetic square to identify also a class of permanent cycles in negative weights.

Proposition 1.6 (Corollary 3.19). *The classes defined inductively $x_0 = \rho$, x_1 the class above, and*

$$x_n = \sum_{i=0}^{n-1} x_i \zeta_{n-i}^{2^i}$$

are permanent cycles in the HSSS for $BPGL\langle m \rangle$, for all n and m . The tridegree of x_n is $(2^n - 2, -1, 2 - 2^n)$, and the coaction is as follows

$$\psi(x_n) = \sum_{i=0}^n \zeta_i^{2^{n-i}} \otimes x_{n-i}$$

Heuristically, the x_n classes may be thought of as indecomposable elements of the form $[\xi_n \rho]$. We determine a family of differentials in the HSSS for $BPGL\langle m \rangle$, which may be expressed in terms of the x_n 's. The classes x_n appear as Massey products when taking homology with respect to these differentials.

Theorem 1.7 (Theorem 4.16). *In the HSSS for $BPGL\langle m \rangle$, we have the differentials*

$$d_{2^{i+1}-1}(\zeta_j^{2^{i+1-j}}) = \bar{v}_i \rho^{2^i-1} \left(p_{j-1} \left(\frac{x_1}{\rho}, \dots, \frac{x_{j-1}}{\rho} \right) \right)^{2^{i+1-j}}$$

for all $1 \leq i \leq m$ and $1 \leq j \leq i+1$, where p_j is the polynomial

$$\zeta_j = p_j(\xi_1, \dots, \xi_j)$$

given by the inversion formulas in the Hopf algebra \mathcal{A}_ .*

The classes x_n are not divisible by ρ , but all of the ρ exponents appearing are nonnegative when this expression is expanded. Using the heuristic $x_n = [\xi_n \rho]$, this expression may then be read more compactly as

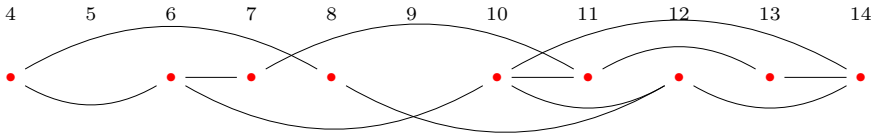
$$d_{2^{i+1}-1}(\zeta_j^{2^{i+1-j}}) = \bar{v}_i [\rho^{2^i-1} \zeta_{j-1}^{2^{i+1-j}}]$$

As each $\zeta_j^{2^{i+1-j}}$ supports a nonzero $d_{2^{i+1}-1}$, its square is a nonzero cycle on the next page. This stops when $i = m+1$, thus interpolating between

$$\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2 = \mathbb{F}_2[\zeta_1^2, \zeta_2, \dots]$$

and

$$\mathcal{A}_* \square_{\mathcal{A}(m)_*} \mathbb{F}_2 = \mathbb{F}_2[\zeta_1^{2^{m+1}}, \zeta_2^{2^m}, \dots, \zeta_{m+1}^2, \zeta_{m+2}, \dots]$$

Fig. 1. The $\mathcal{A}(2)_*$ -comodule \overline{M}_2 .

Section 5

These differentials are explicit but complicated due to the inversion formulas in \mathcal{A}_* , which makes computing homology with respect to these differentials quite difficult, though purely algebraic. This limits our understanding of the behavior of this spectral sequence in general beyond Theorems 1.7 and 1.5. However, we make the following conjecture.

Conjecture 1.8. *All differentials in the HSSS for $BPGL\langle m \rangle$ are generated under the Leibniz rule by those in Theorem 1.7. In particular, the spectral sequence collapses on $E_{2^{m+1}}$.*

Given Conjecture 1.8, computing $H_*\Gamma(BPGL\langle m \rangle)$ becomes the purely algebraic problem of computing homology with respect to these differentials, modulo comodule algebra extension problems. In Section 5, we verify this conjecture for $m \leq 3$, and compute the corresponding spectral sequences completely.

At height 1, this recovers the classical computation of H_*ko , without the use of the Wood cofiber sequence or knowledge of π_*ko . At height 2, it is a theorem of Hill–Meier that the C_2 -spectrum $tmf_1(3)$ is a form of $BP_{\mathbb{R}}\langle 2 \rangle$ [21], so our computation determines the \mathcal{A}_* -comodule algebra $H_*tmf_0(3)$.

Theorem 1.9 (Theorem 5.11). *There is an isomorphism of \mathcal{A}_* -comodule algebras*

$$H_*tmf_0(3) \cong \mathcal{A}_* \square_{\mathcal{A}(2)_*} M_2$$

where M_2 is the $\mathcal{A}(2)_*$ -comodule algebra given by the square zero extension $\mathbb{F}_2\{1\} \oplus \overline{M}_2$, and \overline{M}_2 is the $\mathcal{A}(2)_*$ -comodule is displayed below in Fig. 1.

Prior to the work of Hill–Lawson [20], it was not known how to produce a spectrum $Tmf_0(3)$ (and thereby a connective model $tmf_0(3)$ of the periodic spectrum $TMF_0(3)$ of topological modular forms with level structure) as the global sections of a derived stack. In [10], Davis and Mahowald proposed several hands-on definitions that were suitable for computation. In particular, they construct a certain 10-cell complex X [10, Theorem 2.1] and propose $tmf \wedge X$ as such a connective model.

Our computation in Theorem 1.9 allows us to join this definition with the derived algebro-geometric one of Hill–Lawson. We use the Adams spectral sequence to construct a map $X \rightarrow \mathrm{tmf}_0(3)$, resulting in the following Wood-type splitting.

Theorem 1.10 (Corollary 5.13). *There is a 2-local equivalence of tmf -modules*

$$\mathrm{tmf}_0(3) \simeq \mathrm{tmf} \wedge X$$

Rationally, X has rank 4, corresponding to the fact that the map of stacks $\mathcal{M}_0(3) \rightarrow \mathcal{M}_{ell}$ is a 4-fold cover. Base changing, this also gives the corresponding splittings for $\mathrm{Tmf}_0(3)$ and $\mathrm{TMF}_0(3)$.

At height 3, there is no known familiar geometric model of $BP_{\mathbb{R}}\langle 3 \rangle$, but our computation extends the program of computing $H_*eO_n(G)$ at $p = 2$ to height 3 for the first time, while also demonstrating the staggering complexity of these computations at heights > 2 . We refer the reader to Section 5.3 for a description of the comodule algebra M_3 .

Theorem 1.11 (Theorem 5.22). *The E_{∞} page of the HSSS for $BPGL\langle 3 \rangle$ is isomorphic to the square zero extension*

$$(\mathcal{A}_* \square_{\mathcal{A}(3)_*} M_3) \oplus (\mathcal{A}_* \square_{\mathcal{A}(2)_*} \overline{M}_2)$$

for M_3 an explicit $\mathcal{A}(3)_*$ -comodule algebra of dimension 165, where $\overline{M}_3 \cdot \overline{M}_2 = 0$, and M_2 is the comodule of Theorem 1.9.

We could, in principle, go further to height 4 and beyond; we see no reason our methods should not extend. However, we are unable to give a general description of a comodule M_m at height m , and already at height 3, it is a significant task to give a proper account of the computation.

Section 6

We finish in Section 6 by defining and exploring the HSSS in equivariant stable homotopy. Whereas the slice spectral sequence of Hill–Hopkins–Ravenel [19] has proven an effective tool to compute the homotopy groups of the fixed points E^G of a G -spectrum E , this gives a way to compute the *homology* of E^G .

When $G = C_2$, the HSSS for $BP_{\mathbb{R}}\langle m \rangle$ is closely related to the motivic spectral sequences studied in the body of the paper. However, the presence of the negative cone in the equivariant homology of a point results in some unexpected differences. We show in particular that the $RO(C_2)$ -graded HSSS of $k_{\mathbb{R}}$ has an exotic differential of the form

$$d_5(2u_{2\sigma}^{-1} \cdot \zeta_1^2 \zeta_2) = \bar{v}_1^2$$

This differential leaves a class in the negative cone and kills a class in the positive cone.

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1.4. Notation and conventions

- (1) We work at the prime $p = 2$ in this paper, so homology $H_*(-)$ will always denote mod 2 homology $H_*(-; \mathbb{F}_2)$ and our Steenrod algebras are the 2-primary versions.
- (2) Sm_k denotes the category of smooth schemes over $\mathrm{Spec}(k)$, and $\mathcal{SH}(k)$ denotes the category of k -motivic spectra. $\mathcal{SH}(k)_{\mathrm{cell}}$ denotes the cellular category, the localizing subcategory generated by the bigraded spheres $S^{s,w}$.
- (3) $\pi_{s,w}^{\mathbb{R}}(-)$ denotes bigraded homotopy groups for an \mathbb{R} -motivic spectrum, where s denotes the stem or topological degree, and w denotes the weight.
- (4) $M\mathbb{Z}$ and $M\mathbb{F}_2$ are the integral and mod 2 \mathbb{R} -motivic homology spectra, so that $\pi_{*,*}^{\mathbb{R}}M\mathbb{Z} = \mathbb{Z}[\rho, \tau^2]/(2\rho)$ and $\pi_{*,*}^{\mathbb{R}}M\mathbb{F}_2 = \mathbb{F}_2[\rho, \tau]$ where $|\rho| = (-1, -1)$ and $|\tau| = (0, -1)$.
- (5) \mathcal{A}_* denotes the classical mod 2 dual Steenrod algebra, so that

$$\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \dots]$$

where $|\xi_i| = 2^i - 1$ are the usual Milnor generators, and ζ_i denotes the conjugate of ξ_i . $\mathcal{A}(m)_*$ denotes the quotient Hopf algebra

$$\mathcal{A}(m)_* = \mathbb{F}_2[\xi_1, \dots, \xi_{m+1}]/(\xi_1^{2^{m+1}}, \dots, \xi_{m+1}^2)$$

and

$$\mathcal{A}_* \square_{\mathcal{A}(m)_*} \mathbb{F}_2 \cong \mathbb{F}_2[\zeta_1^{2^{m+1}}, \zeta_2^{2^m}, \dots, \zeta_{m+1}^2, \zeta_{m+2}, \dots]$$

Dually, \mathcal{A} denotes the mod 2 Steenrod algebra, $\mathcal{A}(m)$ the subalgebra generated by Sq^{2^i} for $i \leq m$, and

$$\mathcal{A}/\mathcal{A}(m) = \mathcal{A} \otimes_{\mathcal{A}(m)} \mathbb{F}_2$$

- (6) $\mathcal{A}^{\mathbb{R}}$ denotes the \mathbb{R} -motivic mod 2 Steenrod algebra, and we use the following notation for generators

$$\mathcal{A}_{*,*}^{\mathbb{R}} = \pi_{*,*}^{\mathbb{R}}M\mathbb{F}_2[\bar{t}_1, \bar{t}_2, \dots, \tau_0, \tau_1, \dots]/(\tau_i^2 = \tau \bar{t}_{i+1} + \rho \tau_{i+1} + \rho \tau_0 \bar{t}_{i+1})$$

where $|\bar{\tau}_i| = (2(2^i - 1), 2^i - 1)$ and $|\tau_i| = (2(2^i - 1) + 1, 2^i - 1)$. We use the following notation for the quotient Hopf algebra

$$\mathcal{E}_{*,*}^{\mathbb{R}}(m) = \pi_{*,*}^{\mathbb{R}} M\mathbb{F}_2[\tau_0, \tau_1, \dots, \tau_m] / (\tau_i^2 = \rho\tau_{i+1})$$

Moreover, $c(-)$ denotes Hopf conjugation in $\mathcal{A}_{*,*}^{\mathbb{R}}$.

- (7) $\mathcal{S}p$ denotes the category of spectra, and $\mathcal{S}p^G$ denotes the category of genuine G -spectra. For a G -spectrum X , $\pi_*(X)$ denotes the $RO(G)$ -graded homotopy groups of X .
- (8) We let $i_* : \mathcal{S}p \rightarrow \mathcal{SH}(k)$ denote the unique symmetric monoidal colimit-preserving functor and $\Gamma(-)$ its right adjoint. $\beta : \mathcal{SH}(\mathbb{R}) \rightarrow \mathcal{S}p^{C_2}$ denotes the C_2 -equivariant Betti realization functor.
- (9) Our spectral sequences are Adams trigraded, so that a class in tridegree $(t - s, w, s)$ is in stem $t - s$, weight w , and filtration s . All of the charts drawn in Section 5 are in weight $w = 0$.

2. The slice spectral sequence for generalized homology

In this section we define the HSSS in motivic stable homotopy and establish its basic properties. In Section 2.1, we work at the level of generality of $\mathcal{SH}(k)$, where k is a perfect field, and we use the very effective slice filtration of Spitzweck–Østvær [38]. This allows us to have strong convergence in a wide generality, due to Morel’s connectivity theorem. However, for the spectra we consider - namely, standard quotients of the algebraic cobordism spectrum MGL - the corresponding slice towers for the effective, cellular effective, very effective, and cellular very effective all coincide, as shown by Heard [17].

This filtration determines a tower of motivic spectra, and our spectral sequences arise via smashing these towers pointwise with an ordinary spectrum, using that $\mathcal{SH}(k)$ is tensored over $\mathcal{S}p$. This construction is analogous to one construction of the classical Atiyah–Hirzebruch spectral sequence, where one does the same with the Postnikov tower of a spectrum.

In Section 2.3, we specialize to $\mathcal{SH}_{\text{cell}}(\mathbb{R})$, the cellular subcategory of \mathbb{R} -motivic spectra, where we use the results of Behrens and Shah [6] on Betti realization to compare with C_2 -equivariant stable homotopy.

2.1. General slice towers

We begin with the very effective slice filtration, a modification of Voevodsky’s slice filtration [42], developed by Spitzweck–Østvær [38] and further studied by Bachmann [3]. The reader will notice we double the usual grading conventions, so that our slice towers are run at half speed; we explain this below in Remark 2.2.

Following the discussion in [17, Section 3], we let

$$\mathcal{K}_{2t} = \{\Sigma^{2a,a}\Sigma_+^{\infty} X; a \geq t, X \in \text{Sm}_k\} \subset \mathcal{SH}(k)$$

and we set $\mathcal{K}_{2t+1} = \mathcal{K}_{2t}$. Letting $\mathcal{SH}(k)^{t,\text{veff}}$ denote the full subcategory of $\mathcal{SH}(k)$ generated under colimits and extensions by \mathcal{K}_t , we have a filtration

$$\dots \subset \mathcal{SH}(k)^{t+1,\text{veff}} \subset \mathcal{SH}(k)^{t,\text{veff}} \subset \mathcal{SH}(k)^{t-1,\text{veff}} \subset \dots$$

of $\mathcal{SH}(k)$, and we say E is slice $\geq t$ if $E \in \mathcal{SH}(k)^{t,\text{veff}}$. Associated to any $E \in \mathcal{SH}(k)$, we have a natural cofiber sequence

$$P_{t+1}E \rightarrow E \rightarrow P^tE$$

such that $P_{t+1}E \in \mathcal{K}_{t+1}$, and

$$\text{Map}_{\mathcal{SH}(k)}(K, P^tE) \simeq *$$

for all $K \in \mathcal{SH}(k)^{t+1,\text{veff}}$. We define the fiber P_t^tE of the map

$$P^tE \rightarrow P^{t-1}E$$

to be the t -slice of E . The slice filtration gives a natural tower of spectra under E :

$$\dots \rightarrow P^tE \rightarrow P^{t-1}E \rightarrow \dots$$

The colimit is always contractible and the limit is E .

Now, given any $K \in \mathcal{SH}(k)$, we can smash the slice tower for E with K to produce a new filtered motivic spectrum $K \otimes P^\bullet E$. Applying (bigraded) homotopy groups, this gives a spectral sequence.

Definition 2.1. The K -homology slice spectral sequence for E is the spectral sequence associated to the filtered motivic spectrum $K \otimes P^\bullet E$. We will generically denote this by

$$E_*^{s,w,t}(E; K).$$

In particular, the E_2 -term is given by

$$E_2^{s,w,t}(E; K) = K_{t-s,w}(P_t^tE).$$

We use Adams grading $(t-s, w, s)$ so that the d_r differentials change tridegree by $(-1, 0, r)$.

Remark 2.2. We have chosen to double the gradings to cohere with two closely related spectral sequences when $k = \mathbb{R}$: slice spectral sequences in equivariant homotopy and the classical Atiyah–Hirzebruch spectral sequence. Applying Betti realization to the spectral sequences we consider recovers the analogous slice spectral sequences in C_2 -equivariant

homotopy (see Section 6), and if one base changes to \mathbb{C} and inverts τ , one recovers a classical Atiyah–Hirzebruch spectral sequence. Our choice of gradings recovers the standard grading conventions on these spectral sequences, in both cases.

Example 2.3. When $K = S^{0,0}$ is the sphere spectrum, this is the ordinary slice spectral sequence of E . When $K = S^{0,0}[\rho^{-1}]$, this is the ρ -localized slice spectral sequence of E .

The ordinary slice spectral sequence converges strongly for any E , due to the Morel connectivity theorem [30, Section 5.3]. When we smash with an arbitrary motivic spectrum K , this is no longer guaranteed. For the following result, cf. [38, Proposition 5.11].

Proposition 2.4. *Suppose K is slice bounded below. Then, the K -homology slice spectral sequence of E converges conditionally in the sense that the natural map*

$$K \otimes E \rightarrow \varprojlim (K \otimes P^\bullet E)$$

*induces an isomorphism on bigraded homotopy groups, and $\operatorname{colim}(K \otimes P^\bullet E) \simeq *$.*

If, in addition, E is slice bounded below, the spectral sequence converges strongly in the sense that there are at most finitely many filtrations in which E_2 is non-zero in any given stem.

Proof. The spectrum $\operatorname{colim} P^\bullet E$ is contractible as the categories \mathcal{K}_t form a set of compact generators of $\mathcal{SH}(k)$. Smashing with K , we use that the smash product commutes with colimits.

One has a fiber sequence

$$\varprojlim (K \otimes P_{t+1} E) \rightarrow K \otimes E \rightarrow \varprojlim (K \otimes P^t E)$$

By assumption K is slice $\geq n$ for some fixed n , so $K \otimes P_{t+1} E$ is slice $\geq n + t + 1$ for all t . It follows from Morel's connectivity theorem that if $E \in \mathcal{SH}(k)^{t, \text{veff}}$, then $\pi_{s,w} E = 0$ for $s < t/2 + w$. In a fixed weight w , therefore, $\pi_{s,w}(K \otimes P_{t+1} E)$ vanishes for $s < w + (n + t + 1)/2$. This vanishing range for s is strictly increasing in t , so the bigraded homotopy groups of the limit vanish. If E is slice bounded below, then the same connectivity constraints imply that, in a fixed weight, only finitely many slices contribute to a given stem. \square

Remark 2.5. Without the bound on K , the result is much more delicate, even when E is 0-connective, and the comparison map may fail to be an isomorphism. For example, when $K = EGL(n) = \bar{v}_n^{-1} BPGL\langle n \rangle$ and $E = MGL$, smashing the slice associated graded for E with K gives the zero spectrum.

For our applications, both K and E will be slice connective, and this sharpens our convergence results to give strong vanishing lines.

Proposition 2.6. *If E and K are slice ≥ 0 , the K -homology slice spectral sequence of E in weight zero is a right half-plane spectral sequence concentrated between the lines $y = x$ and $y = -x$. That is, the groups*

$$E_2^{s,0,t}(E; K)$$

vanish if $s > t - s$ or $t < 0$.

Proof. If E is slice ≥ 0 , then $P_t^t E \simeq *$ for $t < 0$, which gives the lower vanishing line. The upper vanishing line follows as above from Morel's connectivity theorem, which implies in particular that since $K \otimes P_t^t E$ is slice $\geq t$, $\pi_{t-s,0}(K \otimes P_t^t E)$ vanishes for $(t-s) < t/2$. This vanishing line is equivalent to $(t-s) < s$. \square

The slice filtration is multiplicative, so we have the following.

Proposition 2.7. *If K and E are both ring spectra, then this is a spectral sequence of algebras.*

In fact, we have more structure that we can consider from naturality in K . Since the spectral sequence arises from a filtration of E that is completely independent of K , all of the structure commutes with K -cooperations.

Proposition 2.8. *If K is a flat homology theory, then the K -homology slice spectral sequence for any E is a spectral sequence of $K_{*,*}K$ -comodules. If E is a ring, this is a spectral sequence of $K_{*,*}K$ -comodule algebras.*

Example 2.9. Let $k = \mathbb{R}$ and $K = M\mathbb{F}_2$, the motivic mod 2 Eilenberg–MacLane spectrum. This gives a spectral sequence of $\mathcal{A}_{*,*}^{\mathbb{R}}$ -comodules converging to the bigraded motivic homology of E with E_2 -page the bigraded motivic homology of the slice associated graded of E . If E is a ring spectrum, this is a spectral sequence of $\mathcal{A}_{*,*}^{\mathbb{R}}$ -comodule algebras.

2.2. The homological slice spectral sequence (HSSS)

2.2.1. Global sections homology

There is an essentially unique colimit-preserving symmetric monoidal functor

$$i_* : \mathcal{S}p \rightarrow \mathcal{SH}(k).$$

This functor admits a right adjoint

$$\Gamma(-) : \mathcal{SH}(k) \rightarrow \mathcal{S}p$$

which sends a motivic spectrum E to the mapping spectrum

$$F(S^{0,0}, E)$$

Viewing E as a sheaf of spectra on Sm_k , since $S^{0,0} = \mathrm{Spec}(k)_+$, we may think of this as taking the global sections of E . We now define our main spectral sequence of interest.

Definition 2.10. For $E \in \mathcal{SH}(k)$, the *homological slice spectral sequence* (HSSS) of E is the spectral sequence $E_*^{s,w,t}(E; i_* H\mathbb{F}_2)$ of Definition 2.1.

That is, we tensor the slice tower of E with $i_* H\mathbb{F}_2$ and take the spectral sequence associated to the resulting tower. We begin with the following lemma, which follows immediately from the adjunction $i_* \dashv \Gamma$.

Lemma 2.11. For $E \in \mathcal{SH}(k)$, we have an isomorphism of abelian groups

$$\pi_{s,w} E \cong \pi_s \Gamma(\Sigma^{0,-w} E)$$

When E is a ring spectrum, this is an isomorphism of bigraded rings.

Since the unit is compact in $\mathcal{SH}(k)$, i_* preserves compact objects and hence $\Gamma(-)$ admits a right adjoint. In particular $\Gamma(-)$ preserves colimits, and we deduce the following projection formula.

Proposition 2.12. For any spectrum K and $E \in \mathcal{SH}(k)$, we have

$$\Gamma(i_* K \otimes E) \simeq K \otimes \Gamma(E)$$

Applying Lemma 2.11 in the case that $K = H\mathbb{F}_2$, we have the following.

Corollary 2.13. For any $E \in \mathcal{SH}(k)$, the homotopy groups of the spectrum $\Gamma(i_* H\mathbb{F}_2 \otimes E)$ are the mod 2 homology groups of $\Gamma(E)$:

$$(i_* H\mathbb{F}_2)_{*,0} E \cong H_* \Gamma(E)$$

Since $H\mathbb{F}_2$ is an E_∞ -ring spectrum, so is $i_* H\mathbb{F}_2$. Since $H\mathbb{F}_2 \otimes H\mathbb{F}_2$ is a tensor product of free associative algebras generated by spheres, the same is true upon taking i_* .

Proposition 2.14. We have an equivalence of associative $i_* H\mathbb{F}_2$ -algebras

$$i_* H\mathbb{F}_2^{\otimes 2} \simeq i_* H\mathbb{F}_2[\zeta_1, \dots],$$

where just as classically, the topological degree of ζ_i is $(2^i - 1)$, and it is in weight 0.

In homotopy, the induced coaction on the ζ_i 's is also the usual one, by functoriality. Put another way, the motivic spectrum $i_* H\mathbb{F}_2 \otimes E$ records all of the information of the mod 2 homology of $\Gamma(E)$, even with the dual Steenrod coaction.

Corollary 2.15. *The HSSS of E is a spectral sequence of bigraded \mathcal{A}_* -comodules that converges to $H_*\Gamma(\Sigma^0, -^*E)$. If E is a ring spectrum, this is a spectral sequence of bigraded \mathcal{A}_* -comodule algebras.*

2.2.2. Global sections homology and motivic homology

The global sections functor is also lax monoidal, so given an E_∞ -monoid R in $\mathcal{SH}(k)$, $\Gamma(R)$ is an E_∞ -ring spectrum. The counit of the adjunction $i_* \dashv \Gamma$ gives an E_∞ -map

$$i_*\Gamma(R) \rightarrow R.$$

It follows from Lemma 2.11 that when $R = M\mathbb{Z}$ or $R = M\mathbb{F}_2$, the motivic Eilenberg–MacLane spectra, the global sections spectra are $H\mathbb{Z}$ and $H\mathbb{F}_2$, respectively. We therefore have canonical E_∞ -maps

$$i_*H\mathbb{Z} \rightarrow M\mathbb{Z} \text{ and } i_*H\mathbb{F}_2 \rightarrow M\mathbb{F}_2$$

We will use these to identify the slice associated graded in many cases.

Applying the functor i_* to ordinary homotopy groups, we have a canonical map of algebras

$$\mathcal{A}_* = \pi_*(H\mathbb{F}_2 \otimes H\mathbb{F}_2) \xrightarrow{i_*} \pi_{*,*}(i_*H\mathbb{F}_2 \otimes i_*H\mathbb{F}_2) \rightarrow \pi_{*,*}(M\mathbb{F}_2 \otimes M\mathbb{F}_2) =: \mathcal{A}_{*,*}^k$$

If the latter dual Steenrod algebra is flat over $\pi_{*,*}(M\mathbb{F}_2)$, we have a Hopf algebroid structure, and our map of algebras is compatible with the coactions in the following sense.

Proposition 2.16. *If $E \in \mathcal{SH}(k)$, then we have a natural map*

$$H_*\Gamma(E) \rightarrow H_{*,*}(E; M\mathbb{F}_2)$$

When $\mathcal{A}_{,*}^k$ is flat, this is a map of $\mathcal{A}_{*,*}^k$ -comodules via the map*

$$\mathcal{A}_* \rightarrow \mathcal{A}_{*,*}^k$$

2.3. Betti realization and the arithmetic square

In this section, we restrict our attention now to $k = \mathbb{R}$ and work in the cellular context. That is, we work in the category $\mathcal{SH}_{\text{cell}}(\mathbb{R})$, the localizing subcategory of $\mathcal{SH}(\mathbb{R})$ generated by the bigraded spheres $S^{s,w}$. Moreover, we implicitly work in the 2-complete context everywhere, so that our $\mathcal{SH}_{\text{cell}}(\mathbb{R})$ stands for $\widehat{\mathcal{SH}_{\text{cell}}(\mathbb{R})}_2$, and our Sp^{C_2} stands for $(Sp^{C_2})_2^\wedge$.

There is a strong relationship between \mathbb{R} -motivic stable homotopy and C_2 -equivariant stable homotopy, which comes via the Betti realization functor

$$\beta : \mathcal{SH}(\mathbb{R}) \rightarrow \mathcal{Sp}^{C_2}$$

If X is a smooth scheme over $\mathrm{Spec}(\mathbb{R})$, its set of complex points $X(\mathbb{C})$ can be equipped with the complex analytic topology, and the action of $C_2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $X(\mathbb{C})$ is continuous. The functor β is obtained by left Kan extension of $X \mapsto \Sigma_+^\infty X(\mathbb{C})$ along the canonical functor $\mathrm{Sm}_{\mathbb{R}} \rightarrow \mathcal{SH}(\mathbb{R})$.

Pushing and pulling computations along the functor β have been incredibly fruitful (see [12], [7], and [15] for example). There has therefore been an effort to make the connection provided by β conceptually precise (see [12], [18], [2], and [6]). Behrens–Shah combined and extended these results to characterize β as a localization functor on $\mathcal{SH}_{\mathrm{cell}}(\mathbb{R})$. They showed, in particular, that after completing at a prime, the functor

$$\mathrm{Sing} : \mathcal{Sp}^{C_2} \rightarrow \mathcal{SH}_{\mathrm{cell}}(\mathbb{R})$$

is a fully faithful right adjoint to β , and we may therefore regard \mathcal{Sp}^{C_2} as a reflective subcategory of $\mathcal{SH}_{\mathrm{cell}}(\mathbb{R})$ with reflection functor given by β [6, Theorem 1.12].

On the other hand, \mathcal{Sp}^{C_2} may be described in terms of pullback squares. For $E \in \mathcal{Sp}^{C_2}$, there is a commutative square

$$\begin{array}{ccc} E & \longrightarrow & \tilde{E}C_2 \otimes E \\ \downarrow & & \downarrow \\ F(EC_{2+}, E) & \longrightarrow & \tilde{E}C_2 \otimes F(EC_{2+}, E) \end{array}$$

known as the *Tate square* of E , which is a pullback. Behrens–Shah [6, Theorem 1.10] provided an extension of this construction to all $E \in \mathcal{SH}_{\mathrm{cell}}(\mathbb{R})$, forming a commutative square

$$\begin{array}{ccc} E & \longrightarrow & E[\rho^{-1}] \\ \downarrow & & \downarrow \\ E_{\rho}^{\wedge}[\tau^{-1}] & \longrightarrow & E_{\rho}^{\wedge}[\tau^{-1}][\rho^{-1}] \end{array}$$

which we call the *arithmetic square* of E . This square is *not* in general a pullback, but the results of Behrens–Shah imply the following.

Proposition 2.17. *$E \in \mathcal{SH}_{\mathrm{cell}}(\mathbb{R})$ is in the essential image of the fully faithful right adjoint*

$$\mathrm{Sing} : \mathcal{Sp}^{C_2} \rightarrow \mathcal{SH}_{\mathrm{cell}}(\mathbb{R})$$

if and only if the arithmetic square of E is a pullback. In particular, the functor $\beta(-)$ may be regarded as the endofunctor of $\mathcal{SH}_{\mathrm{cell}}(\mathbb{R})$ given by the pullback

$$\begin{array}{ccc}
\beta(E) & \longrightarrow & E[\rho^{-1}] \\
\downarrow & & \downarrow \\
E_{\rho}^{\wedge}[\tau^{-1}] & \longrightarrow & E_{\rho}^{\wedge}[\tau^{-1}, \rho^{-1}]
\end{array}$$

This, in turn, describes the genuine fixed points functor $(-)^{C_2} : \mathcal{S}p^{C_2} \rightarrow \mathcal{S}p$ as a special case of the global sections functor of Section 2.2.1.

Proposition 2.18. *The genuine fixed points functor $(-)^{C_2}$ is equivalent to the composite $\Gamma \circ \text{Sing}$.*

The unit map for the reflection functor β gives a natural transformation

$$\eta_E : \Gamma(E) \rightarrow \Gamma(\beta(E))$$

We will show that η_E is an equivalence in many cases of interest; that is, the arithmetic square of E becomes a pullback after applying $\Gamma(-)$ in these cases. This follows by induction up the slice tower of E , given the following base case.

Proposition 2.19. *The map*

$$\Gamma(\Sigma^{s,w} M\mathbb{Z}) \rightarrow \Gamma(\Sigma^{s,w} \beta(M\mathbb{Z}))$$

is an equivalence if $w > -2$. Equivalently, the map

$$\pi_{*,*}^{\mathbb{R}} M\mathbb{Z} \rightarrow \pi_{*,*}^{\mathbb{R}} \beta(M\mathbb{Z})$$

is an iso in weights < 2 .

Proof. On one hand, this is immediate from Lemma 2.11 and the fact that β induces an isomorphism

$$\pi_{*,*}^{\mathbb{R}} M\mathbb{Z} \rightarrow \pi_{*,*}^{C_2} H\mathbb{Z}$$

in weights < 2 (see [12]).

However, we may argue directly. From Voevodsky's computation [41]

$$\pi_{*,*}^{\mathbb{R}} M\mathbb{Z} = \mathbb{Z}[\tau^2, \rho]/(2\rho)$$

where $|\tau| = (0, -2)$ and $|\rho| = (-1, -1)$, one sees that applying bigraded homotopy groups to the arithmetic square of $M\mathbb{Z}$ gives the square

$$\begin{array}{ccc}
\mathbb{Z}[\tau^2, \rho]/(2\rho) & \longrightarrow & \mathbb{F}_2[\tau^2, \rho^\pm] \\
\downarrow & & \downarrow \\
\mathbb{Z}[\tau^{\pm 2}, \rho]/(2\rho) & \longrightarrow & \mathbb{F}_2[\tau^{\pm 2}, \rho^\pm]
\end{array}$$

The resulting Mayer–Vietoris sequence is exact in weights $w < 2$, and the result now follows from Lemma 2.11 and Proposition 2.17. \square

In particular, we also have that η_E is an equivalence when E is of the form $\Sigma^{2t,t}M\mathbb{Z}$ for $t \geq 0$. If E is slice connective with slices of this form, a similar argument to that of Proposition 2.4 gives the following.

Corollary 2.20. *Suppose $E \in \mathcal{SH}_{\text{cell}}(\mathbb{R})$ has the property that, for all $t \geq 0$,*

$$P_{2t}^{2t}E = \bigoplus_{I_t} \Sigma^{2t,t}M\mathbb{Z}$$

and $P_{2t}^{2t}E$ vanishes for $t < 0$. Then

$$\Gamma(\Sigma^{s,w}E) \rightarrow \Gamma(\Sigma^{s,w}\beta(E))$$

is an equivalence for $w > -2$. Equivalently, the induced map

$$\pi_{*,*}^{\mathbb{R}}E \rightarrow \pi_{*,*}^{\mathbb{R}}\beta(E)$$

is an isomorphism in weights $w < 2$.

Corollary 2.21. *Suppose $E \in \mathcal{SH}_{\text{cell}}(\mathbb{R})$ is as in Corollary 2.20. Then*

$$H_*\Gamma(\Sigma^{s,w}E) \rightarrow H_*\Gamma(\Sigma^{s,w}\beta(E))$$

is an isomorphism for $w > -2$. Equivalently, the map

$$\pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes E) \rightarrow \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes \beta(E)) = \pi_{*,*}^{\mathbb{R}}(\beta(i_*H\mathbb{F}_2 \otimes E))$$

is an isomorphism in weights $w < 2$.

In particular, combining Corollary 2.21 with Proposition 2.17, one may compute the bigraded homology of $\Gamma(E)$ in weights $w < 2$ in terms of that of the motivic spectra

$$\Gamma(E[\rho^{-1}]) \quad \Gamma(E_{\rho}^{\wedge}[\tau^{-1}]) \quad \Gamma(E_{\rho}^{\wedge}[\tau^{-1}, \rho^{-1}])$$

by applying the Mayer–Vietoris sequence. We will apply this in Section 3 for $E = BPGL$, the 2-local summand of the algebraic cobordism spectrum.

Recall there is a classifying map $L \rightarrow MGL_{*,*}^{\mathbb{R}}$ that is an isomorphism in bidegrees of the form $(2k, k)$, where $L \cong \mathbb{Z}[a_1, a_2, \dots]$ is the Lazard ring, and $|a_i| = (2i, i)$ (see [17] for this isomorphism and [33] for the description of the Lazard ring). We say an MGL -module is a *standard quotient of MGL* if it is equivalent to the quotient of MGL by some collection of polynomials that are a_i modulo decomposables. Standard quotients of MGL satisfy the hypotheses of Corollary 2.20, by the following theorem of Hopkins–Morel and Levine–Tripathi [26].

Proposition 2.22. *Let $I \subset L$ be the ideal generated by some collection of the a_i ’s. Then the slice associated graded of MGL/I is*

$$M\mathbb{Z}[a_1, a_2, \dots]/I := M\mathbb{Z}[a_i | i \notin I]$$

Working 2-locally, MGL splits as a sum of shifts of the spectrum

$$BPGL := MGL / (a_i | i \neq 2^k - 1)$$

As is standard, we let $\bar{v}_i := a_{2^i - 1}$, and our computations with the HSSS will focus on the truncated versions

$$BPGL\langle m \rangle = BPGL / (\bar{v}_{m+1}, \bar{v}_{m+2}, \dots)$$

2.4. The edge homomorphism

For any $E \in \mathcal{SH}_{\text{cell}}(\mathbb{R})$, we have a natural map of motivic spectra

$$E \rightarrow P^k E,$$

and so we can consider the induced map of slice towers. Since we have a map of filtered spectra, we have an induced map of spectral sequences. The slice spectral sequence for $P^k E$ is exactly the truncation of the slice spectral sequence for E where we throw away all classes in degrees $(t - s, s)$ with $t > k$.

There is a very important case of the truncated slice tower: the case $k = 0$. For standard quotients E of MGL , Proposition 2.22 implies that $P^0 E \simeq M\mathbb{Z}$. We therefore have a map of HSSS’s

$$E_*^{s,w,t}(E; i_* H\mathbb{F}_2) \rightarrow E_*^{s,w,t}(M\mathbb{Z}; i_* H\mathbb{F}_2).$$

The latter is concentrated on the line $t = 0$, but recognizing that this comes from a map of filtered spectra, we see that this gives the edge homomorphism

$$\pi_{*,*}^{\mathbb{R}}(i_* H\mathbb{F}_2 \otimes E) \rightarrow H_{*,*}^{\mathbb{R}}(i_* H\mathbb{F}_2; \mathbb{Z}).$$

Again, this map is a map of $\mathcal{A}_{*,*}^{\mathbb{R}}$ -comodule (algebras). By definition, every class in the image of the edge homomorphism admits a lift along this map, so we have the following.

Proposition 2.23. *The image of the edge homomorphism in the HSSS consists of permanent cycles.*

3. Homology of $\Gamma(BPGL\langle m \rangle)$ via the arithmetic square

We have seen in the previous section that computing the homology groups $H_*\Gamma(E)$ amounts to computing the motivic homotopy groups

$$\pi_{*,0}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes E)$$

and that, when E is as in Proposition 2.22, this \mathcal{A}_* -comodule may be recovered from the arithmetic square of $i_*H\mathbb{F}_2 \otimes E$. In this section, we explore this approach in the case $E = BPGL\langle m \rangle$.

3.1. The arithmetic square of $i_*H\mathbb{F}_2 \otimes BPGL\langle m \rangle$

3.1.1. The τ -local part

In this section, we determine bottom row of the arithmetic square of $i_*H\mathbb{F}_2 \otimes BPGL\langle m \rangle$, namely the map

$$\pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes BPGL\langle m \rangle)_{\rho}^{\wedge}[\tau^{-1}] \rightarrow \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes BPGL\langle m \rangle)_{\rho}^{\wedge}[\tau^{-1}, \rho^{-1}]$$

The main advantage of applying the functor $(-)_{\rho}^{\wedge}[\tau^{-1}]$ is that it converts $i_*H\mathbb{F}_2$ homology to $M\mathbb{F}_2$ -homology.

Proposition 3.1. *The canonical map $i_*H\mathbb{F}_2 \rightarrow M\mathbb{F}_2$ induces an equivalence*

$$(i_*H\mathbb{F}_2 \otimes E)_{\rho}^{\wedge}[\tau^{-1}] \rightarrow (M\mathbb{F}_2 \otimes E)_{\rho}^{\wedge}[\tau^{-1}]$$

for any $E \in \mathcal{SH}_{\text{cell}}(\mathbb{R})$.

Proof. The map in question is obtained by taking the limit of the maps

$$i_*H\mathbb{F}_2 \otimes E \otimes C(\rho^i)[\tau^{-1}] \rightarrow M\mathbb{F}_2 \otimes E \otimes C(\rho^i)[\tau^{-1}]$$

(see [6, Theorem 1.10]). By filtering $C(\rho^i)$, it suffices to show this map is an equivalence when $i = 1$. By [6, Proposition 8.3], the map

$$C(\rho) \rightarrow \text{Spec}(\mathbb{C}_+)$$

becomes an equivalence after applying 2-completion and cellularization. It therefore suffices to show that the map

$$i_* H\mathbb{F}_2 \otimes \mathrm{Spec}(\mathbb{C}_+)[\tau^{-1}] \rightarrow M\mathbb{F}_2 \otimes \mathrm{Spec}(\mathbb{C}_+)[\tau^{-1}]$$

is an equivalence, for which it suffices to base change to \mathbb{C} and show that

$$i_* H\mathbb{F}_2[\tau^{-1}] \rightarrow M\mathbb{F}_2[\tau^{-1}]$$

is an equivalence in $\mathcal{SH}(\mathbb{C})$, by [6, Theorem 1.7]. Finally, by [6, Theorem 1.1], it suffices to apply \mathbb{C} -Betti realization, which gives the identity map of $H\mathbb{F}_2$. \square

The motivic homology of $BPGL\langle m \rangle$ for $-1 \leq m \leq \infty$ is described similarly to the classical case. We have the following computation of Ormsby [32, Theorem 3.8].

Proposition 3.2. *The motivic homology of $BPGL\langle m \rangle$ is*

$$H_{*,*}^{\mathbb{R}} BPGL\langle m \rangle \cong \mathcal{A}_{*,*}^{\mathbb{R}} \square_{\mathcal{E}_{*,*}^{\mathbb{R}}(m)} M\mathbb{F}_{2*,*} = \frac{\mathbb{F}_2[\rho, \tau, \bar{t}_i, c(\tau_j) | i \geq 1, j \geq m+1]}{(c(\tau_j)^2 = c(\tau_{j+1})\rho + c(\bar{t}_{j+1})\tau)}$$

Corollary 3.3. *The map*

$$\pi_{*,*}^{\mathbb{R}}(i_* H\mathbb{F}_2 \otimes BPGL)_{\rho}^{\wedge}[\tau^{-1}] \rightarrow \pi_{*,*}^{\mathbb{R}}(i_* H\mathbb{F}_2 \otimes BPGL)_{\rho}^{\wedge}[\tau^{-1}, \rho^{-1}]$$

is given by the inclusion

$$\frac{\mathbb{F}_2[\rho, \tau^{\pm}, \bar{t}_i, c(\tau_j) | i \geq 1, j \geq m+1]}{(c(\tau_j)^2 = c(\tau_{j+1})\rho + c(\bar{t}_{j+1})\tau)}_{\rho}^{\wedge} \hookrightarrow \frac{\mathbb{F}_2[\rho, \tau^{\pm}, \bar{t}_i, c(\tau_j) | i \geq 1, j \geq m+1]}{(c(\tau_j)^2 = c(\tau_{j+1})\rho + c(\bar{t}_{j+1})\tau)}_{\rho}^{\wedge}[\rho^{-1}]$$

Proof. By Proposition 3.1, we may prove this claim instead for the map

$$\pi_{*,*}^{\mathbb{R}}(M\mathbb{F}_2 \otimes BPGL)_{\rho}^{\wedge}[\tau^{-1}] \rightarrow \pi_{*,*}^{\mathbb{R}}(M\mathbb{F}_2 \otimes BPGL)_{\rho}^{\wedge}[\tau^{-1}, \rho^{-1}]$$

which follows from Proposition 3.2. \square

Remark 3.4. Ormsby's computation may be recovered using the spectral sequence of Section 2.1 with $K = M\mathbb{F}_2$ and $E = BPGL\langle m \rangle$. This computation mirrors exactly the classical computation of $H_*BP\langle m \rangle$ using the Atiyah–Hirzebruch spectral sequence.

3.1.2. The ρ -local part

Corollary 3.3 determines the bottom row of the arithmetic square for $i_* H\mathbb{F}_2 \otimes BPGL\langle m \rangle$. The key input for this calculation was the fact that, working τ -locally, $i_* H\mathbb{F}_2$ homology agrees with $M\mathbb{F}_2$ -homology. On the other hand, working ρ -locally, we have the following [6, Theorem 1.2].

Proposition 3.5. *The functor*

$$\Gamma : \mathcal{SH}(\mathbb{R})_{\text{cell}}[\rho^{-1}] \rightarrow \mathcal{S}p$$

is an equivalence of categories with inverse given by $i_(-)[\rho^{-1}]$.*

The following is easy to obtain by appealing to C_2 -equivariant homotopy and using [6, Theorem 1.5] and [14, Proposition 4.9], or directly by simply adapting the arguments of [14, Proposition 4.9] to the motivic setting.

Lemma 3.6. *Under this equivalence, Γ sends $BPGL[\rho^{-1}]$ to $H\mathbb{F}_2$. Coning off $(\bar{v}_{m+1}, \bar{v}_{m+2}, \dots)$, we have $\Gamma(BPGL\langle m \rangle[\rho^{-1}]) \simeq H\mathbb{F}_2[y^{2^{m+1}}]$, as an $H\mathbb{F}_2$ -module, where $|y| = 1$.*

This tells us that the gluing map in the arithmetic square for $i_*H\mathbb{F}_2 \otimes BPGL$

$$i_*H\mathbb{F}_2 \otimes BPGL[\rho^{-1}] \rightarrow (i_*H\mathbb{F}_2 \otimes BPGL)_{\rho}^{\wedge}[\tau^{-1}, \rho^{-1}]$$

is given on bigraded homotopy by some ring map of the form

$$\mathcal{A}_*[\rho^{\pm}] \rightarrow \mathbb{F}_2[\rho, \tau^{\pm}, \bar{t}_i]_{\rho}^{\wedge}[\rho^{-1}]$$

To determine this gluing map, we can embed both groups into

$$\pi_{*,*}^{\mathbb{R}}(M\mathbb{F}_2 \otimes M\mathbb{F}_2)_{\rho}^{\wedge}[\tau^{-1}, \rho^{-1}]$$

via the Thom reduction map $BPGL \rightarrow M\mathbb{F}_2$ and the canonical map $i_*H\mathbb{F}_2 \rightarrow M\mathbb{F}_2$.

Understanding these embeddings reduces to understanding the composite

$$i_*H\mathbb{F}_2[\rho^{-1}] \simeq BPGL[\rho^{-1}] \rightarrow M\mathbb{F}_2[\rho^{-1}]$$

Much of the subtlety in our computations comes from the observation that this map is not homotopic to the canonical map of 2.2.2

$$i_*H\mathbb{F}_2[\rho^{-1}] \rightarrow M\mathbb{F}_2[\rho^{-1}]$$

The two maps of course have the same effect in homotopy; we need to pass to homology to tell them apart. The following allows us to describe the effect of the canonical map in homology.

Proposition 3.7. *Regarding the classical Milnor generators ξ_i as elements of $\mathcal{A}_{*,*}^{\mathbb{R}}$ via the composite*

$$\mathcal{A}_* = \pi_*(H\mathbb{F}_2 \otimes H\mathbb{F}_2) \xrightarrow{i_*} \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes i_*H\mathbb{F}_2) \rightarrow \pi_{*,*}^{\mathbb{R}}(M\mathbb{F}_2 \otimes M\mathbb{F}_2) = \mathcal{A}_{*,*}^{\mathbb{R}}$$

one has the recursion formulas

$$\begin{aligned}\bar{t}_0 &:= 1 \\ \rho^{2^i} \bar{t}_i &= \xi_{i-1}^2 (\tau_0 \rho + \tau) + \xi_i \rho + \bar{t}_{i-1} \tau^{2^{i-1}}\end{aligned}$$

Proof. We appeal to C_2 -equivariant homotopy, where the corresponding formulas were computed by Hu–Kriz [22, Theorem 6.18]. The result follows \mathbb{R} -motivically by the factorization

$$\begin{array}{ccc} \mathcal{A}_* & \longrightarrow & \mathcal{A}_{*,*}^{\mathbb{R}} \\ & \searrow & \downarrow \\ & & \mathcal{A}_{*,*}^{C_2} \end{array} \quad \square$$

Let can denote the equivalence

$$\text{can} : H\mathbb{F}_2[y] \xrightarrow{\simeq} \Gamma(M\mathbb{F}_2[\rho^{-1}])$$

where $H\mathbb{F}_2[y]$ is the free E_1 - $H\mathbb{F}_2$ -algebra on a class y in degree 1, and can is defined by giving $\Gamma(M\mathbb{F}_2[\rho^{-1}])$ an $H\mathbb{F}_2$ -algebra structure via the canonical map of 2.2.2 and by sending y to $\tau/\rho \in \pi_1 \Gamma(M\mathbb{F}_2[\rho^{-1}])$.

Proposition 3.8. *The map*

$$H\mathbb{F}_2 \simeq \Gamma(BPGL[\rho^{-1}]) \rightarrow \Gamma(M\mathbb{F}_2[\rho^{-1}]) \xrightarrow{\text{can}^{-1}} H\mathbb{F}_2[y]$$

is not homotopic to the unit map $\eta : H\mathbb{F}_2 \rightarrow H\mathbb{F}_2[y]$.

Proof. If it were, the composite

$$i_* H\mathbb{F}_2 \otimes i_* H\mathbb{F}_2[\rho^{-1}] \simeq i_* H\mathbb{F}_2 \otimes BPGL[\rho^{-1}] \rightarrow M\mathbb{F}_2 \otimes M\mathbb{F}_2[\rho^{-1}]$$

would be homotopic to the canonical map

$$i_* H\mathbb{F}_2 \otimes i_* H\mathbb{F}_2[\rho^{-1}] \rightarrow M\mathbb{F}_2 \otimes M\mathbb{F}_2[\rho^{-1}]$$

By Proposition 3.7, this map sends the class ξ_1 to $\rho \bar{t}_1 + \tau_0$. This would imply that τ_0 is in the image of the (injective) map

$$\pi_{*,*}^{\mathbb{R}}(M\mathbb{F}_2 \otimes M\mathbb{Z})_{\rho}^{\wedge}[\tau^{-1}, \rho^{-1}] \rightarrow \pi_{*,*}^{\mathbb{R}}(M\mathbb{F}_2 \otimes M\mathbb{F}_2)_{\rho}^{\wedge}[\tau^{-1}, \rho^{-1}]$$

since the reduction map $BPGL \rightarrow M\mathbb{F}_2$ factors through $M\mathbb{Z}$. Voevodsky's relations

$$\tau_i^2 = \tau \bar{\tau}_{i+1} + \rho \tau_{i+1} + \rho \tau_0 \bar{\tau}_{i+1}$$

would then imply that all τ_i 's are in the image of this map, so that it is an isomorphism, a contradiction. \square

Remark 3.9. In [23], Hu–Kriz compute the homology groups $H_*BP_{\mathbb{R}}^{C_2}$ using the isotropy separation sequence. By Corollary 2.20, this is the same as computing

$$\pi_{*,0}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes BPGL)$$

Our computations differ from theirs, and this is related to what seems to be an error on page 114, where they claim that $\eta_R(u_{\sigma}^{-1})$ is in the image of the reduction map

$$f : \pi_*(\tilde{E}C_2 \otimes F(EC_{2+}, H\underline{\mathbb{F}}_2 \otimes BP_{\mathbb{R}})) \rightarrow \pi_*(\tilde{E}C_2 \otimes F(EC_{2+}, H\underline{\mathbb{F}}_2 \otimes H\underline{\mathbb{F}}_2))$$

In their notation, the claim is that $f(\rho) = \eta_R(\sigma)$. As in the proof of the previous proposition, this would imply that τ_0 is in the image of this map, which leads to contradictions.

The composite in Proposition 3.8 gives *some* ring map

$$H\mathbb{F}_2 \rightarrow H\mathbb{F}_2[y]$$

and we identify it by showing it is structured enough to apply Mahowald's theorem on $H\mathbb{F}_2$ [28].

Theorem 3.10. *The composite*

$$H\mathbb{F}_2 \simeq \Gamma(BPGL[\rho^{-1}]) \rightarrow \Gamma(M\mathbb{F}_2[\rho^{-1}]) \xrightarrow{\text{can}^{-1}} H\mathbb{F}_2[y] \simeq \prod_{i \geq 0} \Sigma^i H\mathbb{F}_2$$

has components $Sq^i : H\mathbb{F}_2 \rightarrow \Sigma^i H\mathbb{F}_2$.

Proof. We note first that the map

$$H\mathbb{F}_2 = \Gamma(i_*H\mathbb{F}_2[\rho^{-1}]) \simeq \Gamma(BPGL[\rho^{-1}]) \rightarrow \Gamma(M\mathbb{F}_2[\rho^{-1}])$$

is an E_{∞} map since it may be factored as

$$\Gamma(i_*H\mathbb{F}_2[\rho^{-1}]) \simeq \Gamma(BPGL[\rho^{-1}]) \rightarrow \Gamma(MGL_{(2)}[\rho^{-1}]) \rightarrow \Gamma(M\mathbb{F}_2[\rho^{-1}])$$

The first map is the unit map $H\mathbb{F}_2 \rightarrow MO$, which is E_{∞} , and the map $MGL_{(2)} \rightarrow M\mathbb{F}_2$ is E_{∞} .

Mahowald's theorem on $H\mathbb{F}_2$ implies that there is a pushout square in E_2 -algebras in Sp

$$\begin{array}{ccc} \mathrm{Free}_{E_2}(\mathbb{S}) & \xrightarrow{\bar{0}} & \mathbb{S} \\ \bar{2} \downarrow & & \downarrow \\ \mathbb{S} & \longrightarrow & H\mathbb{F}_2 \end{array}$$

which gives

$$\pi_0 \mathrm{Map}_{E_2}(H\mathbb{F}_2, \Gamma(M\mathbb{F}_2[\rho^{-1}])) \cong \pi_1 \Gamma(M\mathbb{F}_2[\rho^{-1}]) = \mathbb{F}_2$$

There is thus a unique such E_2 -map not homotopic to the unit map η . We conclude by observing that the Tate-valued Frobenius

$$\varphi_2 : H\mathbb{F}_2 \rightarrow H\mathbb{F}_2^{tC_2}$$

of Nikolaus–Scholze is an E_∞ map that factors thru the connective cover

$$H\mathbb{F}_2[y] \simeq \Gamma(M\mathbb{F}_2[\rho^{-1}]) \simeq \Phi^{C_2} H\underline{\mathbb{F}}_2$$

and admits the above description by [31, IV.1.5]. \square

3.2. The Mayer–Vietoris sequence for $i_* H\mathbb{F}_2 \otimes BPGL$

Theorem 3.10 may be used to describe explicitly the gluing map in the arithmetic square for $i_* H\mathbb{F}_2 \otimes BPGL$. To determine its effect on homotopy, we need the following lemma.

Lemma 3.11. *The map induced on homotopy by*

$$H\mathbb{F}_2 \otimes H\mathbb{F}_2 \xrightarrow{1 \otimes Sq^i} H\mathbb{F}_2 \otimes \Sigma^i H\mathbb{F}_2$$

is the cap product

$$- \cap Sq^i : \mathcal{A}_* \xrightarrow{\Delta} \mathcal{A}_* \otimes \mathcal{A}_* \xrightarrow{1 \otimes \langle -, Sq^i \rangle} \mathcal{A}_{*-i}$$

Proof. The map induced on homotopy by $1 \otimes Sq^i$ is an \mathcal{A}_* -comodule map, and \mathcal{A}_{*-i} is a cofree \mathcal{A}_* -comodule. It suffices to show the composite

$$\mathcal{A}_* \xrightarrow{(1 \otimes Sq^i)_*} \mathcal{A}_{*-i} \xrightarrow{\epsilon} \mathbb{F}_2[i]$$

coincides with the composite

$$\mathcal{A}_* \xrightarrow{\Delta} \mathcal{A}_* \otimes \mathcal{A}_* \xrightarrow{1 \otimes \langle -, Sq^i \rangle} \mathcal{A}_{*-i} \xrightarrow{\epsilon} \mathbb{F}_2[i]$$

where $\epsilon : \mathcal{A}_* \rightarrow \mathbb{F}_2$ is the coidentity map in the Hopf algebra \mathcal{A}_* . Under the isomorphism $\mathcal{A} \cong \text{Hom}_{\mathbb{F}_2}(\mathcal{A}_*, \mathbb{F}_2)$, the latter map corresponds to Sq^i . The same is true of the former map as the pairing $\mathcal{A}_* \otimes_{\mathbb{F}_2} \mathcal{A} \rightarrow \mathbb{F}_2$ is induced by the map in $\mathcal{S}p$

$$(H\mathbb{F}_2 \otimes H\mathbb{F}_2) \otimes_{H\mathbb{F}_2} F(H\mathbb{F}_2, H\mathbb{F}_2) \rightarrow H\mathbb{F}_2 \otimes H\mathbb{F}_2 \xrightarrow{\mu} H\mathbb{F}_2 \quad \square$$

Proposition 3.12. *The map*

$$\pi_{*,*}^{\mathbb{R}}(i_* H\mathbb{F}_2 \otimes BPGL)[\rho^{-1}] \rightarrow \pi_{*,*}^{\mathbb{R}}(M\mathbb{F}_2 \otimes BPGL)_{\rho}^{\wedge}[\tau^{-1}, \rho^{-1}]$$

is given by the map

$$\varphi : \mathcal{A}_*[\rho^{\pm}] \rightarrow \mathbb{F}_2[\tau^{\pm}, \rho, \bar{t}_i]_{\rho}^{\wedge}[\rho^{-1}]$$

of $\mathbb{F}_2[\rho^{\pm}]$ -algebras sending

$$\varphi(\xi_i) = \rho^{2^i-1} \bar{t}_i + \left(\frac{\tau^{2^i-1}}{\rho} \right) \bar{t}_{i-1}$$

Proof. The diagram

$$\begin{array}{ccc} (i_* H\mathbb{F}_2 \otimes BPGL)[\rho^{-1}] & \longrightarrow & (M\mathbb{F}_2 \otimes M\mathbb{F}_2)[\rho^{-1}] \\ \downarrow & & \downarrow \\ (M\mathbb{F}_2 \otimes BPGL)_{\rho}^{\wedge}[\tau^{-1}, \rho^{-1}] & \longrightarrow & (M\mathbb{F}_2 \otimes M\mathbb{F}_2)_{\rho}^{\wedge}[\tau^{-1}, \rho^{-1}] \end{array}$$

is given on bigraded homotopy by

$$\begin{array}{ccc} \mathcal{A}_*[\rho^{\pm}] & \longrightarrow & \mathcal{A}_*[\tau, \eta_R(\tau), \rho^{\pm}] \\ \downarrow \varphi & & \downarrow \\ \mathbb{F}_2[\rho, \tau^{\pm}, \bar{t}_i]_{\rho}^{\wedge}[\rho^{-1}] & \longrightarrow & \mathcal{A}_*[\tau^{\pm}, \eta_R(\tau)^{\pm}, \rho]_{\rho}^{\wedge}[\rho^{-1}] \end{array}$$

Note that the top map is *not* a map of \mathcal{A}_* -algebras; this is the content of Proposition 3.8. The bottom and righthand maps are injective, so it suffices to determine the top map. Combining Theorem 3.10 and Lemma 3.11, this is the map of $\mathbb{F}_2[\rho^{\pm}]$ -algebras sending

$$\xi_i \mapsto \sum_{j \geq 0} (\xi_i \cap Sq^j) \cdot \left(\frac{\eta_R(\tau)}{\rho} \right)^j$$

Using the coproduct formula on \mathcal{A}_* and the fact that Sq^j is dual to ξ_1^j , we have

$$\xi_i \mapsto \xi_i + \xi_{i-1}^2 \frac{\eta_R(\tau)}{\rho}$$

We conclude by applying the relations of Proposition 3.7 \square

We move now to weight zero and set $\chi_i := \tau^{2^i-1} \bar{t}_i$ and $z := \rho/\tau = y^{-1}$. We may summarize the results of Section 3 thus far as follows.

Proposition 3.13. *The arithmetic square of $i_* H\mathbb{F}_2 \otimes BPGL$ is given by*

$$\begin{array}{ccc} H_*\Gamma(BPGL) & \longrightarrow & \mathcal{A}_* \\ \downarrow & & \downarrow \varphi \\ \mathbb{F}_2[z, \chi_i | i \geq 1]_z^\wedge & \hookrightarrow & \mathbb{F}_2[z, \chi_i | i \geq 1]_z^\wedge[z^{-1}] \end{array} \quad \begin{array}{c} \xi_i \\ \downarrow \\ \chi_i z^{2^i-1} + \chi_{i-1} z^{-1} \end{array}$$

with $|z| = -1$ and $|\chi_i| = 2(2^i - 1)$.

To compute the associated Mayer–Vietoris sequence, we establish the following.

Proposition 3.14. *The arithmetic square of Proposition 3.13 satisfies the following properties:*

- (1) *The map φ is injective.*
- (2) *The map $\mathbb{F}_2[z, \chi_i | i \geq 1]_z^\wedge \oplus \mathcal{A}_* \rightarrow \mathbb{F}_2[z, \chi_i | i \geq 1]_z^\wedge[z^{-1}]$ is injective in nonzero degrees.*
- (3) *The maps*

$$H_*\Gamma(BPGL) \rightarrow H_*\Gamma(BPGL[\rho^{-1}]) = \mathcal{A}_*$$

and

$$H_*\Gamma(BPGL) \rightarrow \pi_{*,0}^{\mathbb{R}}(i_* H\mathbb{F}_2 \otimes BPGL)_\rho^\wedge[\tau^{-1}]$$

are both zero in nonzero degrees

- (4) *In positive degrees j , the following sequence is short exact*

$$0 \rightarrow (\mathbb{F}_2[z, \chi_i | i \geq 1]_z^\wedge)_{j+1} \oplus \mathcal{A}_{j+1} \rightarrow (\mathbb{F}_2[z, \chi_i | i \geq 1]_z^\wedge[z^{-1}])_{j+1} \xrightarrow{\partial} H_j\Gamma(BPGL) \rightarrow 0$$

Proof. For (1), we have a diagram

$$\begin{array}{ccc} \pi_{*,*}^{\mathbb{R}}(i_* H\mathbb{F}_2 \otimes BPGL)[\rho^{-1}] & \longrightarrow & \pi_{*,*}^{\mathbb{R}}(i_* H\mathbb{F}_2 \otimes M\mathbb{F}_2)[\rho^{-1}] \\ \downarrow \varphi & & \downarrow \\ \pi_{*,*}^{\mathbb{R}}(M\mathbb{F}_2 \otimes BPGL)_\rho^\wedge[\tau^{-1}, \rho^{-1}] & \longrightarrow & \pi_{*,*}^{\mathbb{R}}(M\mathbb{F}_2 \otimes M\mathbb{F}_2)_\rho^\wedge[\tau^{-1}, \rho^{-1}] \end{array}$$

and the righthand and bottom maps are injective. We have seen the top map is induced by the map

$$i_* H\mathbb{F}_2 \otimes i_* H\mathbb{F}_2 \xrightarrow{1 \otimes 1 \otimes i_*(\varphi_2)} i_* H\mathbb{F}_2 \otimes i_* H\mathbb{F}_2[y]$$

φ_2 differs from the unit map $H\mathbb{F}_2 \xrightarrow{\eta} H\mathbb{F}_2[y]$ by an automorphism of $H\mathbb{F}_2[y]$ and hence is the inclusion of a retract.

For (2), since φ is injective, the map

$$\mathbb{F}_2[z, \chi_i | i \geq 1]_z^\wedge \oplus \mathcal{A}_* \rightarrow \mathbb{F}_2[z, \chi_i | i \geq 1]_z^\wedge [z^{-1}]$$

fails to be injective in degree $j \neq 0$ if and only if there exists a class $0 \neq x \in \mathcal{A}_j$ such that

$$\varphi(x) \in \text{image} \left(\mathbb{F}_2[z, \chi_i | i \geq 1]_z^\wedge \hookrightarrow \mathbb{F}_2[z, \chi_i | i \geq 1]_z^\wedge [z^{-1}] \right)$$

Note, however, that since $\varphi(\xi_i) = z^{-1}(\chi_i z^{2^i} + \chi_{i-1})$, a monomial $\xi_{i_1}^{j_1} \cdots \xi_{i_n}^{j_n}$ is sent to

$$\begin{aligned} \varphi(\xi_{i_1}^{j_1} \cdots \xi_{i_n}^{j_n}) &= z^{-j_1}(\chi_{i_1} z^{2^{i_1}} + \chi_{i_1-1})^{j_1} \cdots z^{-j_n}(\chi_{i_n} z^{2^{i_n}} + \chi_{i_n-1})^{j_n} \\ &= z^{-(j_1 + \cdots + j_n)} \chi_{i_1-1}^{j_1} \cdots \chi_{i_n-1}^{j_n} + \cdots \end{aligned}$$

where the omitted terms are of the form $z^m r(\chi_i)$ with $m > -(j_1 + \cdots + j_n)$, where $r(\chi_i)$ is some polynomial in the χ_i 's. In particular, for a polynomial

$$p(\xi_i) = \sum_{k=1}^N \xi_{i_{k,1}}^{j_{k,1}} \cdots \xi_{i_{k,n_k}}^{j_{k,n_k}}$$

in \mathcal{A}_* , let

$$\begin{aligned} M &:= \max\{j_{k,1} + \cdots + j_{k,n_k} : 1 \leq k \leq N\} \\ q(\xi_i) &:= \sum_{k : j_{k,1} + \cdots + j_{k,n_k} = M} \xi_{i_{k,1}}^{j_{k,1}} \cdots \xi_{i_{k,n_k}}^{j_{k,n_k}} \end{aligned}$$

so that $p(\xi_i) = q(\xi_i) + r(\xi_i)$ where

$$r(\xi_i) := \sum_{k : j_{k,1} + \cdots + j_{k,n_k} < M} \xi_{i_{k,1}}^{j_{k,1}} \cdots \xi_{i_{k,n_k}}^{j_{k,n_k}}$$

Then

$$\varphi(p(\xi_i)) = z^{-M} q(\chi_{i-1}) + \cdots$$

where the omitted terms are of the form $z^m s(\chi_i)$ with $m > -M$, where $s(\chi_i)$ is some polynomial in the χ_i 's. Note that $q(\chi_{i-1}) \neq 0$, so this sum cannot be in the image of the bottom map in the arithmetic square, as $M > 0$. (3) and (4) are immediate from the Mayer–Vietoris sequence. \square

This is already enough to describe $H_*\Gamma(BPGL)$ as an \mathcal{A}_* -comodule. For the following discussion, cf. [23]. Let $K(\rho)$ be defined by the cofiber sequence

$$K(\rho) \rightarrow S^{0,0} \rightarrow S^{0,0}[\rho^{-1}]$$

We have a long exact sequence of left \mathcal{A}_* -comodules

$$\cdots \xrightarrow{\partial} H_*\Gamma(BPGL \otimes K(\rho)) \rightarrow H_*\Gamma(BPGL) \rightarrow \mathcal{A}_* \xrightarrow{\partial} H_{*-1}\Gamma(BPGL \otimes K(\rho)) \rightarrow \cdots$$

Proposition 3.14 gives a splitting of left \mathcal{A}_* -comodules

$$H_*\Gamma(BPGL) \cong \mathbb{F}_2\{1\} \oplus \text{coker}(\partial)$$

To describe $\text{coker}(\partial)$, since our arithmetic square is a pullback in weight zero by Proposition 2.17 and Corollary 2.20, we have a diagram

$$\begin{array}{ccc} \mathcal{A}_* & \xrightarrow{\partial} & H_{*-1}\Gamma(BPGL \otimes K(\rho)) \\ \downarrow \varphi & & \downarrow = \\ \mathbb{F}_2[z, \chi_i]_z^\wedge[z^{-1}] & \longrightarrow & H_{*-1}\Gamma(BPGL \otimes K(\rho)) \end{array}$$

Since $\mathbb{F}_2[z, \chi_i]_z^\wedge \rightarrow \mathbb{F}_2[z, \chi_i]_z^\wedge[z^{-1}]$ is an injection, we find that

$$\begin{aligned} \Sigma H_*\Gamma(BPGL \otimes K(\rho)) &= \text{coker}(\mathbb{F}_2[z, \chi_i]_z^\wedge \rightarrow \mathbb{F}_2[z, \chi_i]_z^\wedge[z^{-1}]) \\ &= \mathbb{F}_2[z^{-1}, \chi_i]\{z^{-1}\} \end{aligned}$$

Together with our explicit description of φ , we conclude:

Theorem 3.15. $H_*\Gamma(BPGL)$ splits as a left \mathcal{A}_* -comodule

$$H_*\Gamma(BPGL) = \mathbb{F}_2\{1\} \oplus \text{coker}(\partial)$$

where ∂ may be described as the map

$$\begin{aligned} \partial : \mathcal{A}_* &\rightarrow \Sigma^{-1}\mathbb{F}_2[z^{-1}, \chi_i]\{z^{-1}\} \\ \xi_i &\mapsto \chi_i z^{2^i-1} + \zeta_{i-1} z^{-1} \end{aligned}$$

which is to be understood as multiplicative on the generators ξ_i , while in the result, all nonnegative powers of y are set to 0. The map ∂ is injective in positive degrees.

Remark 3.16. It is possible to describe the \mathcal{A}_* -comodule structure on $\text{coker}(\partial)$ with explicit formulas. We omit these formulas, however, as we believe they are too complicated

to be useful. By contrast, as we will see, the HSSS of $BPGL$ describes $H_*\Gamma(BPGL)$ as a subquotient of an explicit and straightforward \mathcal{A}_* -comodule algebra.

We remark that it is also possible to describe $H_*\Gamma(BPGL\langle m \rangle)$ as an \mathcal{A}_* -comodule somewhat explicitly in terms of the boundary map in the isotropy separation sequence, as in Theorem 3.15. Due to the complicated nature of these formulas, however, we again prefer to use the HSSS to describe the global structure of homology.

3.3. The image of the edge homomorphism for $BPGL\langle m \rangle$

We will see in Section 5 that in the HSSS for $BPGL\langle m \rangle$, the map $E_2 \rightarrow E_2[\rho^{-1}]$ to the ρ -localized spectral sequence is an injection on E_2 , and this allows us to determine the image of the edge homomorphism from that of the localized spectral sequence, using our results on the arithmetic square.

The edge homomorphism is natural, giving a commutative diagram

$$\begin{array}{ccc} \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes BPGL\langle m \rangle) & \longrightarrow & \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes BPGL\langle m \rangle)[\rho^{-1}] \\ \downarrow & & \downarrow \\ \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z}) & \longrightarrow & \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})[\rho^{-1}] \end{array}$$

In the lefthand side of this square, we have an isomorphism in weight zero

$$\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2 \cong H_*H\mathbb{Z} \cong \pi_{*,0}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})$$

coming from the canonical map $i_*H\mathbb{Z} \rightarrow M\mathbb{Z}$. On the righthand side, using the equivalence

$$i_*H\mathbb{F}_2[y^{2^{m+1}}][\rho^{-1}] \simeq BPGL\langle m \rangle[\rho^{-1}]$$

of Lemma 3.6, the righthand map becomes the canonical inclusion

$$\mathcal{A}_*[y^{2^{m+1}}, \rho^{\pm}] \hookrightarrow \mathcal{A}_*[y^2, \rho^{\pm}]$$

Describing the bottom map in terms of these identifications therefore places us in the context of Theorem 3.10, which determines two identifications

$$H_*\Gamma(M\mathbb{F}_2[\rho^{-1}]) \cong \mathcal{A}_*[y]$$

To be explicit, the canonical map $i_*H\mathbb{F}_2[\rho^{-1}] \rightarrow M\mathbb{F}_2[\rho^{-1}]$ gives an equivalence $H\mathbb{F}_2[y] \simeq \Gamma(M\mathbb{F}_2[\rho^{-1}])$, which determines an isomorphism of the above form denoted \cong_{can} . On the other hand, the map $i_*H\mathbb{F}_2[\rho^{-1}] \simeq BPGL[\rho^{-1}] \rightarrow M\mathbb{F}_2[\rho^{-1}]$ determines an isomorphism of the same form, which we denote \cong_{BP} . Theorem 3.10 and the proof of Proposition 3.12 tell us how to translate between these identifications.

Proposition 3.17. *The composition*

$$\mathcal{A}_*[y] \cong_{\text{can}} H_*\Gamma(M\mathbb{F}_2[\rho^{-1}]) \cong_{BP} \mathcal{A}_*[y]$$

sends

$$\begin{aligned} y &\mapsto y \\ \zeta_i &\mapsto \zeta_i + \zeta_{i-1}y^{2^{i-1}} \end{aligned}$$

Proof. Theorem 3.10 and Lemma 3.11 describe the inverse of this composite as a sum of cap products. Using the congruence

$$\zeta_m \equiv \xi_1^{2^m-1} \pmod{(\xi_2, \xi_3, \dots)}$$

one has that the inverse of this composite sends

$$\zeta_m \mapsto \sum_{i=0}^m \zeta_i y^{2^m-2^i}$$

which implies the claimed description. \square

Corollary 3.18. *The image of the edge homomorphism in the HSSS of $BPGL\langle m \rangle$ contains $\mathcal{A}_*\square_{\mathcal{A}(m)_*}\mathbb{F}_2 \subset H_*H\mathbb{Z}$.*

Proof. We claim that the image of each of the generators

$$\zeta_1^{2^{m+1}}, \zeta_2^{2^m}, \dots, \zeta_{m+1}^2, \zeta_{m+2}, \dots,$$

of $\mathcal{A}_*\square_{\mathcal{A}(m)_*}\mathbb{F}_2$ along the map $E_2 \rightarrow E_2[\rho^{-1}]$ is in the image of the edge homomorphism for the ρ -localized spectral sequence. Indeed, Proposition 3.17 implies that the localization map sends

$$\begin{aligned} \zeta_i^{2^{m+2-i}} &\mapsto \zeta_i^{2^{m+2-i}} + \zeta_{i-1}^{2^{m+2-i}} y^{2^{m+1}} & \text{for } i \leq m+2 \\ \zeta_i &\mapsto \zeta_i + \zeta_{i-1}y^{2^{i-1}} & \text{for } i > m+2 \end{aligned}$$

so that each of these generators lands in the subalgebra $\mathcal{A}_*[y^{2^{m+1}}] \subset E_2[\rho^{-1}]$.

We show in Corollary 4.5 that $E_2 \rightarrow E_2[\rho^{-1}]$ is an injection, and it therefore suffices to show that each of the classes

$$\begin{aligned} \zeta_i^{2^{m+2-i}} + \zeta_{i-1}^{2^{m+2-i}} y^{2^{m+1}} & \text{for } i \leq m+2 \\ \zeta_i + \zeta_{i-1}y^{2^{i-1}} & \text{for } i > m+2 \end{aligned}$$

admits a lift along the map

$$\pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes BPGL\langle m \rangle) \rightarrow \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes BPGL\langle m \rangle)[\rho^{-1}]$$

The Mayer–Vietoris sequence for the arithmetic square of $i_*H\mathbb{F}_2 \otimes BPGL\langle m \rangle$ implies that it suffices to show that the image of each of these classes along the map

$$\varphi_m : \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes BPGL\langle m \rangle)[\rho^{-1}] \rightarrow \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes BPGL\langle m \rangle)_{\rho}^{\wedge}[\tau^{-1}, \rho^{-1}]$$

is in the image of the map

$$\pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes BPGL\langle m \rangle)_{\rho}^{\wedge}[\tau^{-1}] \rightarrow \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes BPGL\langle m \rangle)_{\rho}^{\wedge}[\tau^{-1}, \rho^{-1}]$$

The map φ_m is described by Proposition 3.12 along with the fact that it sends $y^{2^{m+1}}$ to $(\eta_R(\tau)/\rho)^{2^{m+1}}$.

Embedding into the case $m = -1$, the map φ sends these classes back along the inverse of the composition in Proposition 3.17, i.e. to the image of each of the generators above along the map

$$i_* : \mathcal{A}_* \rightarrow \pi_{*,*}^{\mathbb{R}}(M\mathbb{F}_2 \otimes M\mathbb{F}_2)_{\rho}^{\wedge}[\tau^{-1}, \rho^{-1}]$$

The image of the map

$$\pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes BPGL\langle m \rangle)_{\rho}^{\wedge}[\tau^{-1}] \rightarrow \pi_{*,*}^{\mathbb{R}}(M\mathbb{F}_2 \otimes M\mathbb{F}_2)_{\rho}^{\wedge}[\tau^{-1}, \rho^{-1}]$$

is a sub- \mathcal{A}_* -comodule, so it suffices to show that its image contains

$$i_*(\zeta_{m+2}), i_*(\zeta_{m+3}), \dots$$

Conjugating the relations [27, Theorem 2.12], one has

$$i_*(\zeta_i) = \frac{1}{\tau} (c(\tau_{i-1})\eta_R(\tau)^{2^{i-1}} + \rho^{2^i}c(\tau_i))$$

The result now follows from Proposition 3.2. \square

Arguing similarly, we find the following classes in the image of the edge homomorphism in weight -1 .

Corollary 3.19. *For all $m \geq 0$, the class $\rho\xi_m$ admits a lift along the map*

$$\pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes BPGL) \rightarrow \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes BPGL)[\rho^{-1}]$$

Therefore any lift of $\rho\xi_m \in E_2[\rho^{-1}]$ along $E_2 \rightarrow E_2[\rho^{-1}]$ is in the image of the edge homomorphism.

Proof. By Proposition 3.12, we have

$$\varphi(\rho\xi_m) = \rho^{2^m}\bar{t}_m + \tau^{2^{m-1}}\bar{t}_{m-1} \quad \square$$

4. The HSSS for $BPGL\langle m \rangle$

4.1. The slice E_2 page

By Proposition 2.22, the HSSS of $BPGL\langle m \rangle$ has E_2 page

$$\pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z}[\bar{v}_1, \dots, \bar{v}_m]) = \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})[\bar{v}_1, \dots, \bar{v}_m]$$

We therefore proceed to calculate the bigraded homotopy ring

$$\pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})$$

This has some unexpected multiplicative features; we first determine the additive structure.

Proposition 4.1. *The map $M\mathbb{Z} \rightarrow i_*H\mathbb{F}_2 \otimes M\mathbb{Z}$ determines an isomorphism of left \mathcal{A}_* -comodules*

$$\pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z}) \cong (\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2)[\tau^2] \oplus \bigoplus_{i \geq 0, j > 0} \mathcal{A}_*\{\tau^{2i}\rho^j\}$$

Proof. Voevodsky's computation [41] of $\pi_{*,*}^{\mathbb{R}}M\mathbb{Z}$ (see also [15]) implies that the $H\mathbb{Z}$ -module

$$\Gamma\left(\bigoplus_{b \in \mathbb{Z}} \Sigma^{0,b}M\mathbb{Z}\right)$$

splits as

$$\bigoplus_{i \geq 0} H\mathbb{Z}\{\tau^{2i}\} \oplus \bigoplus_{i \geq 0, j > 0} H\mathbb{F}_2\{\tau^{2i}\rho^j\}$$

and the result follows upon taking homology. \square

There are subtleties in computing the products in this ring, and these arise from the following class.

Definition 4.2. Let x_1 denote the class $[\rho\zeta_1]$ in bidegree $(0, -1)$. We use brackets to emphasize that x_1 is indecomposable in

$$\pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})$$

In particular, there is no class ζ_1 , as $\zeta_1 \notin \mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2$, and x_1 is not divisible by ρ .

Most of the ring structure in $\pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})$ is determined by the ring maps

$$\pi_*(H\mathbb{F}_2 \otimes H\mathbb{Z}) \rightarrow \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})$$

and

$$\pi_{*,*}^{\mathbb{R}}(M\mathbb{Z}) \rightarrow \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})$$

However, the class x_1 is not in the subalgebra generated by the images of these maps. To determine products involving x_1 , we instead work ρ -locally.

Proposition 4.3. *There is an isomorphism of bigraded rings*

$$\mathcal{A}_*[y_2, \rho^{\pm}] \rightarrow \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})[\rho^{-1}]$$

where $|y_2| = 2$, and the localization map

$$\phi : \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z}) \rightarrow \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})[\rho^{-1}]$$

sends

$$\tau^2 \mapsto \rho^2 y_2 \quad x_1 \mapsto \zeta_1 \rho \quad \zeta_1^2 \mapsto \zeta_1^2 + y_2 \quad \zeta_i \mapsto \zeta_i + \zeta_{i-1} y_2^{2^{i-2}}$$

where ζ_1^2, ζ_i are regarded as the elements in the

$$(\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2)[\tau^2]$$

summand described in the previous proposition.

Proof. We define the map

$$\mathcal{A}_*[y_2, \rho^{\pm}] \rightarrow \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})[\rho^{-1}]$$

via the map

$$\mathcal{A}_*[\rho^{\pm}] \cong \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes BPGL)[\rho^{-1}] \rightarrow \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})[\rho^{-1}]$$

by sending y_2 to the image of τ^2/ρ^2 along the right unit map

$$\pi_{*,*}^{\mathbb{R}}(M\mathbb{Z})[\rho^{-1}] \rightarrow \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})[\rho^{-1}]$$

This is an isomorphism as $M\mathbb{Z}[\rho^{-1}] \simeq i_*H\mathbb{F}_2[y_2][\rho^{-1}]$.

For the claims about the map ϕ , note that Proposition 4.1 implies that

$$\pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})$$

is ρ -torsion free, so ϕ is an injection. The map

$$\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2 \cong \pi_{*,0}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z}) \rightarrow \pi_{*,0}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})[\rho^{-1}] \cong \mathcal{A}_*[y_2]$$

is described by Proposition 3.17 since the left and right presentations come from the canonical and $BPGL$ $H\mathbb{F}_2$ -algebra structures, respectively. Now $x_1 \mapsto \rho\xi_1$ since $\rho\xi_1$ is the only nonzero class in this bidegree. \square

Corollary 4.4. *In the ring $\pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})$, we have the relation*

$$x_1^2 = \zeta_1^2 \rho^2 + \tau^2$$

Proof. The map ϕ of the previous proposition is an injective ring map, and one has

$$\phi(\zeta_1^2 \rho^2 + \tau^2) = \zeta_1^2 \rho^2 + y_2 \rho^2 + y_2 \rho^2 = \phi(x_1^2) \quad \square$$

Corollary 4.5. *The map*

$$(\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2)[\rho, x_1] \rightarrow \pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes M\mathbb{Z})$$

is an isomorphism of left \mathcal{A}_ -comodule algebras, where ρ is primitive, and*

$$\psi(x_1) = 1 \otimes x_1 + \xi_1 \otimes \rho$$

Proof. The source and target of the map are bigraded \mathbb{F}_2 -vector spaces of the same (finite) graded dimension. It therefore suffices to show the map is surjective, and this follows from Corollary 4.4, which implies that τ^2 is in the image. \square

4.2. Differentials on subalgebras of the Steenrod algebra

Corollary 4.5 gives a description of our E_2 -page

$$E_2^{*,*,*}(BPGL\langle m \rangle; i_*H\mathbb{F}_2) \cong (\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2)[\rho, x_1, \bar{v}_1, \dots, \bar{v}_m]$$

The classes $\rho, x_1, \bar{v}_1, \dots, \bar{v}_m$ are all permanent cycles, and the subalgebra

$$\mathcal{A}_* \square_{\mathcal{A}(m)_*} \mathbb{F}_2 \subset \mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2$$

consists of permanent cycles by Corollary 3.18. In this section, we describe a family of differentials in this spectral sequence on $\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2$ that interpolate between $\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2$ and $\mathcal{A}_* \square_{\mathcal{A}(m)_*} \mathbb{F}_2$.

For all i , $\mathcal{A}_* \square_{\mathcal{A}(i-1)_*} \mathbb{F}_2$ is a free $\mathcal{A}_* \square_{\mathcal{A}(i)_*} \mathbb{F}_2$ -module given by

$$\mathcal{A}_* \square_{\mathcal{A}(i-1)_*} \mathbb{F}_2 \cong (\mathcal{A}_* \square_{\mathcal{A}(i)_*} \mathbb{F}_2) \otimes E(\zeta_1^{2^i}, \zeta_2^{2^{i-1}}, \dots, \zeta_{i+1})$$

We show that, for $i \leq m$ each of these generators $\zeta_j^{2^{i+1-j}}$ supports a $d_{2^{i+1}-1}$ (Theorem 4.16). Each of the squares of these generators is then a cycle and we are left with $\mathcal{A}_* \square_{\mathcal{A}(i)_*} \mathbb{F}_2$ on $E_{2^{i+1}}$ along with various Massey products (see Definition 4.7) due to the fact that the values of $d_{2^{i+1}-1}$ on these generators are not algebraically independent. We deduce from these differentials that the image of the edge homomorphism is *precisely* $\mathcal{A}_* \square_{\mathcal{A}(m)_*} \mathbb{F}_2$ (Theorem 4.18).

To determine these differentials, we use an inductive argument on m and Corollary 3.18 to deduce that the differential $d_{2^{m+1}-1}$ must take classes in $\mathcal{A}_* \square_{\mathcal{A}(m-1)_*} \mathbb{F}_2$ to a ρ -torsion free part of $E_{2^{m+1}-1}$. The differential is then determined by the structure of the ρ -localized spectral sequence, which we determine completely (Proposition 4.6).

In this section, we use the following notation for our HSS's.

$$\begin{aligned} E_r^{*,*,*} \langle m \rangle &:= E_r^{*,*,*}(BPGL \langle m \rangle; i_* H\mathbb{F}_2) \\ E_r^{*,*,*} \langle m \rangle [\rho^{-1}] &:= E_r^{*,*,*}(BPGL \langle m \rangle; i_* H\mathbb{F}_2[\rho^{-1}]) \end{aligned}$$

4.2.1. The localized spectral sequence

We begin by determining the structure of the localized spectral sequence

$$E_r[\rho^{-1}] = E_r^{*,*,*}(BPGL; i_* H\mathbb{F}_2[\rho^{-1}])$$

The spectral sequence $E_r \langle m \rangle [\rho^{-1}]$ for m finite can be read off from this by setting $f_i = 0$ for $i > m$.

Proposition 4.6. *In the spectral sequence $E_r[\rho^{-1}]$, we have*

- (1) $E_2[\rho^{-1}] \cong \mathcal{A}_*[y_2, f_1, f_2, \dots][\rho^{\pm}]$, where
 - $f_m = \rho^{2^m-1} \bar{v}_m$ so that $|f_m| = (2^m - 1, 0, 2^m - 1)$
 - \mathcal{A}_* is the image of the edge homomorphism

$$\mathcal{A}_* \cong \pi_{*,*}^{\mathbb{R}}(i_* H\mathbb{F}_2 \otimes BPGL)[\rho^{-1}] \rightarrow \pi_{*,*}^{\mathbb{R}}(i_* H\mathbb{F}_2 \otimes M\mathbb{Z})[\rho^{-1}] = \mathcal{A}_*[y_2]$$

and y_2 is the image of the class τ^2/ρ^2 along the map

$$\pi_{*,*}^{\mathbb{R}}(M\mathbb{Z})[\rho^{-1}] \rightarrow \pi_{*,*}^{\mathbb{R}}(i_* H\mathbb{F}_2 \otimes M\mathbb{Z})[\rho^{-1}]$$

- If $q \in \mathcal{A}_i$, the tridegree of q is $(i, 0, -i)$, and $|y_2| = (2, 0, -2)$

(2) The differential $d_k[\rho^{-1}] = 0$ unless $k = 2^{m+1} - 1$ for $m \geq 1$, and

$$E_{2^{m+1}-1}[\rho^{-1}] = \mathcal{A}_*[y_2^{2^{m-1}}, f_m, f_{m+1}, \dots][\rho^{\pm}]$$

(3) The differential $d_{2^{m+1}-1}[\rho^{-1}]$ is determined by

$$d_{2^{m+1}-1}[\rho^{-1}](y_2^{2^{m-1}}) = f_m$$

Proof. The description of $E_2[\rho^{-1}]$ follows from Proposition 4.3. The image of the edge homomorphism consists of permanent cycles, so the only algebra generator that is not a permanent cycle is y_2 , and since the spectral sequence must converge to

$$\pi_{*,*}^{\mathbb{R}}(i_*H\mathbb{F}_2 \otimes BPGL)[\rho^{-1}] \cong \mathcal{A}_*[\rho^{\pm}]$$

each f_i must be hit by a differential. For degree reasons, the claimed pattern of differentials is the only possibility. \square

Proposition 4.3 also gives us a description of the map $E_2 \rightarrow E_2[\rho^{-1}]$, and there are important permanent cycles in $E_2^{*,*,*}(BPGL; i_*H\mathbb{F}_2)$ that lift the classes $\rho\xi_m$ along this map.

Definition 4.7. Define $x_0 = \rho$, x_1 as in Definition 4.2, and inductively

$$x_m = \sum_{i=0}^{m-1} x_i \zeta_{m-i}^{2^i}$$

for $m > 1$. The tridegree of x_m is $(2^m - 2, -1, 2 - 2^m)$ in $E_r^{*,*,*}(BPGL; i_*H\mathbb{F}_2)$.

Proposition 4.8. The map $E_2 \rightarrow E_2[\rho^{-1}]$ sends

$$\zeta_1^2 \mapsto \zeta_1^2 + y_2 \quad \zeta_i \mapsto \zeta_i + \zeta_{i-1}y_2^{2^{i-2}} \quad x_m \mapsto \rho\xi_m$$

Proof. The first two follow immediately from Proposition 4.3, and the claim

$$x_m \mapsto \rho\xi_m$$

follows by induction on m . \square

The fact that $E_2 \rightarrow E_2[\rho^{-1}]$ is an injection also gives us the coactions on these classes.

Corollary 4.9. *On E_2 , the classes x_m satisfy the following coaction formula*

$$\begin{aligned}\psi(x_0) &= 1 \otimes x_0 \\ \psi(x_m) &= \sum_{i=0}^m \xi_i^{2^{m-i}} \otimes x_{m-i}\end{aligned}$$

4.2.2. Differentials on $(\mathcal{A}(m)//\mathcal{A}(m-1))^*$

The cofiber sequences

$$\Sigma^{2(2^m-1), 2^m-1} BPGL\langle m \rangle \xrightarrow{\bar{v}_m} BPGL\langle m \rangle \rightarrow BPGL\langle m-1 \rangle$$

allow us to set up an inductive argument to determine a family of differentials in $E_r^{*,*,*}\langle m \rangle$. We first state a standard lemma on morphisms of spectral sequences.

Lemma 4.10. *Suppose $E_r^{s,t}(1) \rightarrow E_r^{s,t}(2)$ is a morphism of spectral sequences with the property that the map is an isomorphism for $t < N$ when $r = 2$. Then*

- (1) *The map $E_r^{s,t}(1) \rightarrow E_r^{s,t}(2)$ is an isomorphism whenever $t + r < N + 2$.*
- (2) *The map $E_r^{s,t}(1) \rightarrow E_r^{s,t}(2)$ is an injection whenever $t < N$.*

In our case, we have the following.

Proposition 4.11. *For $t < 2^{m+1} - 2$, the map*

$$P^t BPGL\langle m \rangle \rightarrow P^t BPGL\langle m-1 \rangle$$

is an equivalence.

Proof. We have a cofiber sequence of $BPGL$ -modules

$$\Sigma^{2(2^m-1), 2^m-1} BPGL\langle m \rangle \xrightarrow{\bar{v}_m} BPGL\langle m \rangle \rightarrow BPGL\langle m-1 \rangle$$

The source of \bar{v}_m -multiplication is slice $2(2^m - 1)$ -connective, and the result follows. \square

We begin by showing that each of the classes $\zeta_j^{2^{i+1}-j}$ survives to $E_{2^{i+1}-1}$.

Proposition 4.12. *For $m \geq 1$, in $E_r^{*,*,*}\langle m \rangle$,*

$$d_r(\zeta_j^{2^{i+1}-j}) = 0$$

for all $1 \leq i \leq m$, $1 \leq j \leq i+1$, and $r < 2^{i+1} - 1$.

Proof. We proceed by induction on m . When $m = 1$, this follows from the fact that $d_2 = 0$; in fact $d_{2k} = 0$ for all k and m as the odd slices of $BPGL\langle m \rangle$ vanish. By Proposition 4.11 and Lemma 4.10, the map of spectral sequences

$$E_r^{s,0,t}\langle m \rangle \rightarrow E_r^{s,0,t}\langle m-1 \rangle$$

is an injection whenever $t < 2^{m+1} - 2$. The classes $\zeta_j^{2^{i+1-j}}$ have $t = 0$, and thus the $i < m$ case of the proposition follows by the inductive hypothesis. The $i = m$ case then follows in the same way from the fact that the classes $\zeta_j^{2^{m+1-j}}$ are permanent cycles in $E_r^{*,*,*}\langle m-1 \rangle$ by Corollary 3.18. \square

We know, therefore, that the first possible nonzero differential on $\zeta_j^{2^{i+1-j}}$ is a $d_{2^{i+1}-1}$. We show that this differential is nonzero and can be read off from the corresponding differential in the localized spectral sequence $E_r^{*,*,*}\langle m \rangle[\rho^{-1}]$. For this, we need to know that the differential lands in an ρ -torsion free part of the spectral sequence, for which we need the following lemma.

Lemma 4.13. *In $E_k^{*,*,*}\langle m \rangle$, for $k < 2^{m+1} - 1$, there are no differentials of the form*

$$d_k(x) = \bar{v}_m p(\rho, x_1) q(\zeta_i) + r(\bar{v}_i, \rho, x_1) s(\zeta_i)$$

where

$$\begin{aligned} q(\zeta_i), s(\zeta_i) &\in \mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2 \\ p(\rho, x_1) &\in \mathbb{F}_2[\rho, x_1] \\ r(\bar{v}_i, \rho, x_1) &\in \mathbb{F}_2[\rho, x_1, \bar{v}_1, \dots, \bar{v}_{m-1}] \end{aligned}$$

such that pq is nonzero.

Proof. Suppose to the contrary that, for some $k < 2^{m+1} - 1$, there is some nonzero differential $d_k(x) = y$ for y of the above form, and let k be minimal with respect to this property. It follows that y is ρ -torsion free; since y is ρ -torsion free on E_2 , if we had $\rho^N y = 0 \in E_k$, we must have a nonzero differential $d_{k'}(x') = \rho^N y$ for some $k' < k$, contradicting minimality of k .

Since $y \in E_k$ is ρ -torsion free, its image in the localized spectral sequence $E_k^{*,*,*}\langle m \rangle[\rho^{-1}]$ is nonzero. By naturality, this determines a nonzero differential d_k in the localized spectral sequence. This contradicts the description of the differentials given in Proposition 4.6. \square

Remark 4.14. In particular, letting $rs = 0$, it follows that the subspace

$$B_m := \bar{v}_m \cdot \text{Sym}^{2^m-1}(\mathbb{F}_2\{\rho, x_1\}) \cdot (\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2)$$

of E_2 receives no differentials d_r for $r < 2^{m+1} - 1$. In particular, the subspace of B_m of d_1, \dots, d_r cycles for $r < 2^{m+1} - 1$ is a ρ -torsion free subspace of $E_{2^{m+1}-1}$, since B_m is ρ -torsion free on E_2 .

Proposition 4.15. *In $E_{2^{m+1}-1}\langle m \rangle$, the composition*

$$\mathcal{A}_* \square_{\mathcal{A}(m-1)_*} \mathbb{F}_2 \hookrightarrow E_{2^{m+1}-1} \xrightarrow{d_{2^{m+1}-1}} E_{2^{m+1}-1}$$

lands in the subspace of

$$\bar{v}_m \cdot \text{Sym}^{2^m-1}(\mathbb{F}_2\{\rho, x_1\}) \cdot (\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2)$$

that survives to $E_{2^{m+1}-1}$.

Proof. The class $z := d_{2^{m+1}-1}(\zeta_i^{2^{m+1}-i})$ may be written as a sum of monomials of the form

$$y := \rho^a x_1^b m(\bar{v}_i) q(\zeta_i)$$

written as an element in a subquotient of E_2 , where $m(\bar{v}_i)$ is a monomial in $\mathbb{F}_2[\bar{v}_1, \dots, \bar{v}_m]$ and $q(\zeta_i)$ is a monomial in $\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2$. Note that the class $\zeta_i^{2^{m+1}-i}$ has $t = 0$, hence y has $t = 2^{m+1} - 2$. It follows that $\bar{v}_m \cdot \bar{v}_j$ does not divide $m(\bar{v}_i)$ for any $j \leq m$.

We therefore have that z can be written as

$$\bar{v}_m p(\rho, x_1) q(\zeta_i) + r(\rho, x_1, \bar{v}_i) s(\zeta_i)$$

as in the lemma. The class $\bar{v}_m p(\rho, x_1) q(\zeta_i)$ lies in

$$\bar{v}_m \cdot \text{Sym}^{2^m-1}(\mathbb{F}_2\{\rho, x_1\}) \cdot (\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2)$$

since z is in weight zero. Note that pq is nonzero by naturality and the fact that

$$d_{2^{m+1}-1}[\rho^{-1}](y_2^{2^{m-1}}) = f_m$$

by Proposition 4.6. We claim that $r(\bar{v}_i, \rho, x_1) s(\zeta_i) = 0 \in E_{2^{m+1}-1}$, completing the proof.

If $r(\bar{v}_i, \rho, x_1) s(\zeta_i) \neq 0 \in E_{2^{m+1}-1}$, since

$$\zeta_i^{2^{m+1}-i} \in E_r^{*,*,*} \langle m-1 \rangle$$

is a permanent cycle, we must have that $r(\bar{v}_i, \rho, x_1) s(\zeta_i) = 0$ on $E_{2^{m+1}-1}^{*,*,*} \langle m-1 \rangle$. Hence there must be a differential in the latter spectral sequence of the form

$$d_k(x) = r(\bar{v}_i, \rho, x_1) s(\zeta_i)$$

for $k < 2^{m+1} - 1$. The class x lies in

$$E_k^{2^{m+1-i}-1-k, 0, 2^{m+1}-1-k} \langle m-1 \rangle$$

and by Lemma 4.10 (1), the map

$$E_k^{2^{m+1-i}-1-k, 0, 2^{m+1}-1-k} \langle m \rangle \rightarrow E_k^{2^{m+1-i}-1-k, 0, 2^{m+1}-1-k} \langle m-1 \rangle$$

is an isomorphism. Therefore, the differential $d_k(x)$ lifts to a differential in $E_k^{*,*,*} \langle m \rangle$ of the form

$$d_k(\tilde{x}) = \bar{v}_m p'(\rho, x_1) q'(\zeta_i) + r'(\rho, x_1, \bar{v}_i) s'(\zeta_i)$$

But we must have $p'q' = 0$ by Lemma 4.13, and $E_k^{*,*,*} \langle m \rangle \rightarrow E_k^{*,*,*} \langle m-1 \rangle$ sends

$$r'(\rho, x_1, \bar{v}_i) s'(\zeta_i) \mapsto r(\rho, x_1, \bar{v}_i) s(\zeta_i)$$

An argument similar to that of Lemma 4.13 shows that $r's' = rs \in E_k \langle m \rangle$. \square

Theorem 4.16. In $E_{2^{i+1}-1}^{*,*,*}(BPGL \langle m \rangle; i_* HF_2)$, we have the differentials

$$d_{2^{i+1}-1}(\zeta_j^{2^{i+1-j}}) = \bar{v}_i \rho^{2^i-1} \left(p_{j-1} \left(\frac{x_1}{\rho}, \dots, \frac{x_{j-1}}{\rho} \right) \right)^{2^{i+1-j}}$$

for all $1 \leq i \leq m$ and $1 \leq j \leq i+1$, where p_j is the polynomial

$$\zeta_j = p_j(\xi_1, \dots, \xi_j)$$

given by the inversion formulas in the Hopf algebra \mathcal{A}_* .

Proof. We know from Lemma 4.13 and Proposition 4.15 that these differentials land in an ρ -torsion free part of the spectral sequence, so we use naturality and Proposition 4.6. In particular, the map $E_2 \rightarrow E_2[\rho^{-1}]$ sends

$$\zeta_j^{2^{i+1-j}} \mapsto \zeta_j^{2^{i+1-j}} + \zeta_{j-1}^{2^{i+1-j}} y_2^{2^{i-1}}$$

and

$$d_{2^{i+1}-1}[\rho^{-1}](\zeta_j^{2^{i+1-j}} + \zeta_{j-1}^{2^{i+1-j}} y_2^{2^{i-1}}) = \zeta_{j-1}^{2^{i+1-j}} f_i \quad \square$$

Remark 4.17. There is another way to view these differentials. The composition

$$\mathcal{A}_* \square_{\mathcal{A}(i-1)_*} \mathbb{F}_2 \hookrightarrow E_{2^{i+1}-1} \xrightarrow{d_{2^{i+1}-1}} \Sigma E_{2^{i+1}-1} \rightarrow \Sigma E_{2^{i+1}-1}[\rho^{-1}]$$

lands in $\Sigma \mathcal{A}_* \cdot f_i \cong \mathcal{A}_{*-2^i}$, and the last map in the composition is an injection on

$$d_{2^{i+1}-1}(\mathcal{A}_* \square_{\mathcal{A}(i-1)_*} \mathbb{F}_2)$$

by Lemma 4.13 and Proposition 4.15. The formulas given in the theorem tell us this map coincides with the \mathcal{A}_* -comodule map

$$\mathcal{A}_* \square_{\mathcal{A}(i-1)_*} \mathbb{F}_2 \xrightarrow{-\cap Sq^{2^i}} \mathcal{A}_{*-2^i}$$

Indeed, we have

$$\begin{aligned} \Delta(\zeta_j^{2^{i+1}-j}) &= \sum_{l+k=j} \zeta_l^{2^{i+1}-j} \otimes \zeta_k^{2^{i+1}-j+l} \\ &\equiv \sum_{l+k=j} \zeta_l^{2^{i+1}-j} \otimes (\xi_1^{2^k}-1)^{2^{i+1}-j+l} \in \mathcal{A}_* \otimes \mathcal{A}_*/(\xi_2, \xi_3, \dots) \\ &\equiv \sum_{l+k=i} \zeta_l^{2^{i+1}-j} \otimes \xi_1^{2^{i+1}-2^{i+1}-j+l} \in \mathcal{A}_* \otimes \mathcal{A}_*/(\xi_2, \xi_3, \dots) \end{aligned}$$

where we have used the congruence

$$\zeta_k \equiv \xi_1^{2^k-1} \pmod{(\xi_2, \xi_3, \dots)}$$

The righthand tensor factor is $\xi_1^{2^i}$ if and only if $l = j - 1$, so pairing on the right with Sq^{2^i} gives $\zeta_{j-1}^{2^{i+1}-j}$.

Theorem 4.18. *The image of the edge homomorphism in weight zero in*

$$E_{\infty}^{*,*,*}(BPGL\langle m \rangle; i_* H\mathbb{F}_2)$$

is precisely $\mathcal{A}_ \square_{\mathcal{A}(m)_*} \mathbb{F}_2$.*

Proof. We proceed by induction, where the case $m = 0$ is clear. By induction, Proposition 4.12, and Lemma 4.10 (1), we see that a class $x \in \mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2$ is a $d_2, \dots, d_{2^{m+1}-2}$ -cycle if and only if $x \in \mathcal{A}_* \square_{\mathcal{A}(m-1)_*} \mathbb{F}_2$. As in Remark 4.17, the restriction of $d_{2^{m+1}-1}$ to $\mathcal{A}_* \square_{\mathcal{A}(m-1)_*} \mathbb{F}_2$ may be identified with the map

$$\mathcal{A}_* \square_{\mathcal{A}(m-1)_*} \mathbb{F}_2 \xrightarrow{-\cap Sq^{2^m}} \mathcal{A}_{*-2^m}$$

The kernel of this map is precisely $\mathcal{A}_* \square_{\mathcal{A}(m)_*} \mathbb{F}_2$ because $\mathcal{A}_* \square_{\mathcal{A}(m)_*} \mathbb{F}_2$ is dual to the quotient $\mathcal{A}/\mathcal{A}(m)$, and $\mathcal{A}(m)$ is the subalgebra of the Steenrod algebra generated by the classes

$$\{Sq^{2^i} : 0 \leq i \leq m\} \quad \square$$

5. The homology of $\Gamma(BPGL\langle m \rangle)$ for $m \leq 3$

In Section 4, we determined the E_2 page of the HSSS for $BPGL\langle m \rangle$, the image of the edge homomorphism, and a family of differentials. Modulo comodule algebra extension problems, this reduces the computation of $H_*\Gamma(BPGL\langle m \rangle)$ to two problems:

- (1) Show that the differentials of Theorem 4.16 are *all* the differentials in the HSSS for $BPGL\langle m \rangle$. In particular, show that this spectral sequence collapses on E_{2m+1} .
- (2) Compute homology with respect to these differentials.

We know of no way to solve problem (1) except by way of our knowledge of the image of the edge homomorphism from Section 3.3 and sparseness arguments afforded by solving problem (2) for small values of m . Problem (2) is purely algebraic; however, the computation quickly becomes very complicated as m grows, and it requires the use of algebraic Bockstein spectral sequences.

Nevertheless we solve both problems for $m \leq 3$, completely determining the E_∞ page of the HSSS as a comodule algebra. Moreover, we solve all comodule algebra extension problems for $m \leq 2$. We can solve the extension problems for $m = 3$ case by case in an ad hoc manner, but the number of such extensions is very large, so we do not attempt to include such a computation here.

When $m = 1$, kgl is a form of $BPGL\langle 1 \rangle$ and $\Gamma(kgl) \simeq ko$, so this gives a quick computation of H_*ko that does not require knowledge of π_*ko or the Wood cofiber sequence. When $m = 2$, we have an equivalence

$$\Gamma(BPGL\langle 2 \rangle) \simeq BP_{\mathbb{R}}\langle 2 \rangle^{C_2}$$

by Proposition 2.18, and it is a theorem of Hill and Meier that $tmf_1(3)$ is a form of $BP_{\mathbb{R}}\langle 2 \rangle$ [21]. Since $tmf_1(3)^{C_2} \simeq tmf_0(3)$, we give a complete computation of the comodule algebra

$$H_*tmf_0(3)$$

which is new (Theorem 5.11). We use this to deduce a Wood-type splitting of tmf -modules

$$tmf \wedge X \simeq tmf_0(3)$$

where X is a certain 10-cell complex that was predicted by Davis and Mahowald [10].

When $m = 3$, it is not known if $\Gamma(BPGL\langle 3 \rangle)$ even admits a ring structure. However, we may use the ring structure of $BPGL$ and the fact that the HSSS of $BPGL\langle 3 \rangle$ is a module over that of $BPGL$.

5.1. The homology of $\Gamma(BPGL\langle 1 \rangle)$

Combining Corollary 4.5, Theorem 4.16, and Corollary 3.18, we have the following.

Theorem 5.1. *There is a spectral sequence of \mathcal{A}_* -comodule algebras with E_2 -term given by*

$$E_2^{*,*,*} = \mathbb{F}_2[\zeta_1^2, \zeta_2, \dots][\rho, x_1, \bar{v}_1]$$

that converges to $H_*\Gamma(BPGL\langle 1 \rangle)$. We have d_3 -differentials

$$d_3(\zeta_1^2) = \rho \bar{v}_1 \quad d_3(\zeta_2) = x_1 \bar{v}_1$$

and the subalgebra generated by

$$\rho, x_1, x_2, \bar{v}_1, \zeta_1^4, \zeta_2^2, \zeta_3, \dots$$

consists of permanent cycles.

We display the E_3 page of this spectral sequence in Fig. 2. The black dots represent classes in the dual Steenrod algebra, and brown dots represent classes divisible by \bar{v}_1 . Brown structure lines represent multiplication by $\rho \bar{v}_1$, which detects $\eta \in \pi_1 S^0$ (see [27]), and we have drawn the vanishing lines of Proposition 2.6 in green. It is straightforward to compute homology with respect to these differentials.

Corollary 5.2. *In the HSSS of $BPGL\langle 1 \rangle$, we have*

$$E_\infty = E_4 = \frac{\mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \dots][\rho, x_1, x_2, \bar{v}_1]}{(\rho \bar{v}_1, x_1 \bar{v}_1, x_2 \bar{v}_1, x_2^2 + \rho^2 \zeta_2^2 + x_1^2 \zeta_1^4)}$$

In weight zero, the relations imply that there is no contribution on E_∞ from the generators in nonzero weights, and we recover the classical computation of the homology of ko (displayed in Fig. 3). Each nonzero stem has exactly one nonzero filtration, so there are no nontrivial comodule algebra extensions.

Corollary 5.3. *There is an isomorphism of \mathcal{A}_* -comodule algebras*

$$H_*(ko) \cong \mathcal{A}_* \square_{\mathcal{A}(1)_*} \mathbb{F}_2 = \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \dots]$$

5.2. The homology of $\Gamma(BPGL\langle 2 \rangle)$

We move to height 2, and we display the E_7 page of the HSSS in Fig. 8. As before, black dots represent classes in the dual Steenrod algebra, and red dots represent classes divisible by \bar{v}_1 . Orange structure lines represent multiplication by $\rho^3 \bar{v}_2$, which detects $\nu \in \pi_3 S^0$, and we have drawn the vanishing lines of Proposition 2.6 in green.

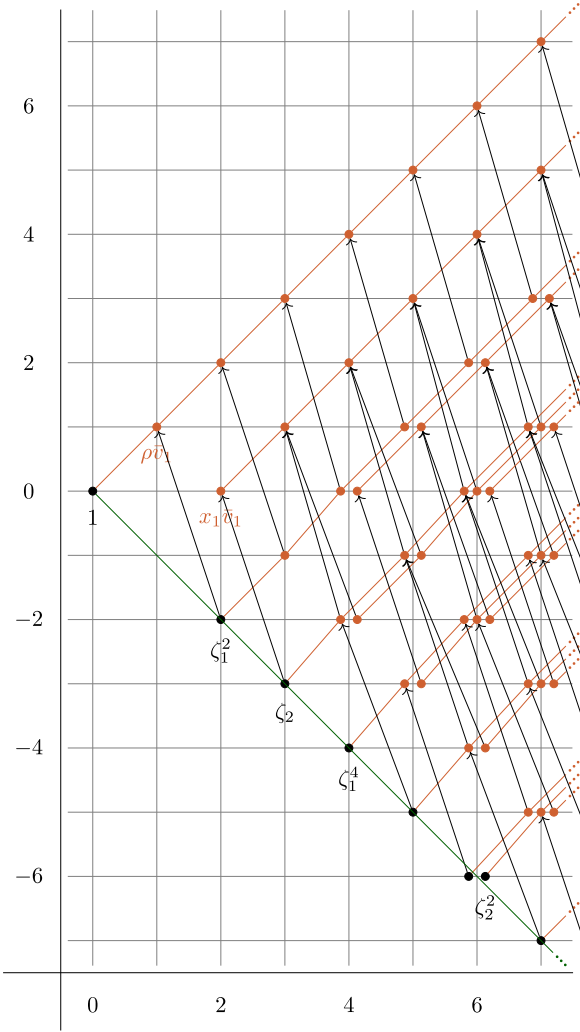


Fig. 2. The E_3 page of the HSSS for $BPGL\langle 1 \rangle$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Theorem 5.4. *There is a spectral sequence of \mathcal{A}_* -comodule algebras with E_2 -term given by*

$$E_2^{*,*,*} = \mathbb{F}_2[\zeta_1^2, \zeta_2, \dots][\rho, x_1, \bar{v}_1, \bar{v}_2]$$

that converges to $H_\Gamma(BPGL\langle 2 \rangle)$. We have differentials*

$$\begin{aligned} d_3(\zeta_1^2) &= \rho\bar{v}_1 & d_3(\zeta_2) &= x_1\bar{v}_1 & d_7(\zeta_1^4) &= \rho^3\bar{v}_2 \\ d_7(\zeta_2^2) &= \rho x_1^2\bar{v}_2 & d_7(\zeta_3) &= (x_1^3 + \rho^2 x_2)\bar{v}_2 \end{aligned}$$

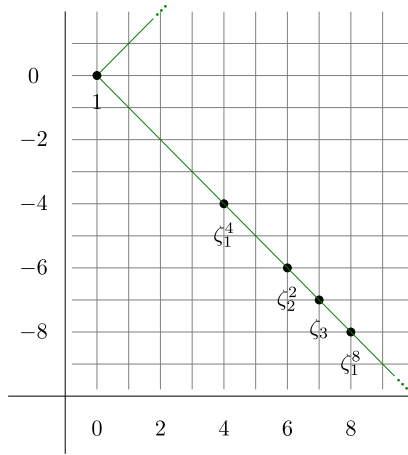


Fig. 3. The E_∞ page of the HSSS for $BPGL\langle 1 \rangle$.

The subalgebra of E_2 generated by the classes

$$\rho, x_1, x_2, x_3, \bar{v}_1, \bar{v}_2, \zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \dots$$

consists of permanent cycles.

The formulas for d_3 imply that, as a dga, one has an isomorphism

$$E_3^{*,*,*}(BPGL\langle 2 \rangle; i_* H\mathbb{F}_2) \cong E_3^{*,*,*}(BPGL\langle 1 \rangle; i_* H\mathbb{F}_2)[\bar{v}_2]$$

and we deduce the following.

Corollary 5.5. *The $E_4 = E_7$ page is given by*

$$\frac{\mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \dots][\rho, x_1, x_2, \bar{v}_1, \bar{v}_2]}{(\rho \bar{v}_1, x_1 \bar{v}_1, x_2 \bar{v}_1, x_2^2 + \rho^2 \zeta_2^2 + x_1^2 \zeta_1^4)}$$

Computing homology with respect to d_7 is much trickier as - unlike E_3 - E_7 is not ρ -torsion free, and hence there are a number of Massey products one must take into account. For example, we have the d_7 -cycle

$$\bar{v}_1 \zeta_1^4 \in \langle \bar{v}_1, \rho^3, \bar{v}_2 \rangle$$

We are interested primarily in the computation in weight zero, however, and none of these classes can contribute to stems of weight zero. Indeed, we observe the following.

Proposition 5.6. *The map of dga's*

$$E_7 \rightarrow E_7/(\bar{v}_1) = \frac{\mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \dots][\rho, x_1, x_2, \bar{v}_2]}{(x_2^2 + \rho^2 \zeta_2^2 + x_1^2 \zeta_1^4)}$$

is an isomorphism in weights ≤ 0 .

Proof. The relations $\rho\bar{v}_1 = x_1\bar{v}_1 = x_2\bar{v}_1 = 0$ imply the map is an isomorphism in weights ≤ 0 . It is a map of dga's because the ideal $(\bar{v}_1) \subset E_7$ is contained in $\ker(d_7)$. \square

We therefore instead compute the homology of the simpler dga $(E_7/(\bar{v}_1), d_7)$. We claim that $\ker(d_7)$ is the subalgebra of $E_7/(\bar{v}_1)$ generated by the classes

$$\rho, x_1, x_2, x_3, \bar{v}_2, \zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \dots$$

It follows that we have the presentation

$$\ker(d_7) = \frac{\mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \dots][\rho, x_1, x_2, x_3, \bar{v}_2]}{(x_2^4 + \rho^4\zeta_2^4 + x_1^4\zeta_1^8, x_3^2 + \rho^2\zeta_3^2 + x_1^2\zeta_2^4 + x_2^2\zeta_1^8)}$$

Writing our differentials in terms of this description of the kernel, we have

$$\begin{aligned} d_7(\zeta_1^4) &= \rho^3\bar{v}_2 & d_7(\zeta_2^2) &= \rho x_1^2\bar{v}_2 \\ d_7(\zeta_3) &= (x_1^3 + \rho^2x_2)\bar{v}_2 & d_7(\zeta_1^4\zeta_2^2) &= \rho x_2^2\bar{v}_2 \\ d_7(\zeta_1^4\zeta_3) &= (x_1x_2^2 + \rho^2x_3)\bar{v}_2 & d_7(\zeta_2^2\zeta_3) &= (x_1^2x_3 + x_2^3)\bar{v}_2 \\ d_7(\zeta_1^4\zeta_2^2\zeta_3) &= (x_1^2x_2\zeta_1^8 + x_2^2x_3 + \rho^2x_1\zeta_2^4)\bar{v}_2 \end{aligned}$$

Letting I_2 be the ideal in $\ker(d_7)$ generated by this list of relations, it follows that

$$H_*(E_7/(\bar{v}_1)) = \frac{\mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \dots][\rho, x_1, x_2, x_3, \bar{v}_2]}{I_2 + (x_2^4 + \rho^4\zeta_2^4 + x_1^4\zeta_1^8, x_3^2 + \rho^2\zeta_3^2 + x_1^2\zeta_2^4 + x_2^2\zeta_1^8)}$$

It remains to verify that the claimed list of elements indeed generates $\ker(d_7)$. We achieve this by running the ρ -Bockstein spectral sequence, which converges since $E_7/(\bar{v}_1)$ is (graded) ρ -complete. We begin by computing the associated graded of the ρ -adic filtration.

Proposition 5.7. *The E_0 -page of the ρ -Bockstein spectral sequence is given by*

$$\frac{\mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \dots][\rho, x_1, x_2, \bar{v}_2]}{(x_2^2 + x_1^2\zeta_1^4)}$$

This spectral sequence converges to the homology of $E_7/(\bar{v}_1)$. We have differentials

$$\delta_0(\zeta_3) = x_1^3\bar{v}_2 \quad \delta_1(\zeta_2^2) = \rho x_1^2\bar{v}_2 \quad \delta_3(\zeta_1^4) = \rho^3\bar{v}_2$$

The annihilator ideal of $\delta_0(\zeta_3)$ in E_0 is trivial, so δ_0 does not create any Massey product cycles. We conclude the following.

Proposition 5.8. *The E_1 -page of the ρ -Bockstein spectral sequence is given by*

$$\frac{\mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3^2, \zeta_4, \dots][\rho, x_1, x_2, \bar{v}_2]}{(x_2^2 + x_1^2 \zeta_1^4, x_1^3 \bar{v}_2)}$$

To compute the E_2 -page, we note that the class x_3 is a cycle in $E_7/(\bar{v}_1)$ and therefore it projects to a permanent cycle in the ρ -Bockstein spectral sequence. We also denote its projection

$$x_1 \zeta_2^2 + x_2 \zeta_1^4$$

by x_3 .

Proposition 5.9. *The E_3 -page of the ρ -Bockstein spectral sequence is given by*

$$\frac{\mathbb{F}_2[\zeta_1^4, \zeta_2^4, \zeta_3^2, \zeta_4, \dots][\rho, x_1, x_2, x_3, \bar{v}_2]}{(x_3^2 + x_2^2 \zeta_1^8 + x_1^2 \zeta_2^4, x_2^2 + x_1^2 \zeta_1^4, x_1^3 \bar{v}_2, \rho x_1^2 \bar{v}_2, (x_1^2 x_3 + x_2^3) \bar{v}_2)}$$

Proof. The kernel of δ_1 is generated as an algebra by the classes

$$\rho, x_1, x_2, \bar{v}_2, \zeta_1^4, \zeta_2^4, \zeta_3^2, \zeta_4, \dots$$

and $\zeta_2^2 \cdot \text{ann}_{E_1}(\rho x_1^2 \bar{v}_2)$, and we claim $\text{ann}_{E_1}(\rho x_1^2 \bar{v}_2) = (x_1)$. To see this, let

$$R := \frac{\mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3^2, \zeta_4, \dots][\rho, x_1, x_2, \bar{v}_2]}{(x_2^2 + x_1^2 \zeta_1^4)}$$

There is a surjection $\pi : R \rightarrow R/(x_1^3 \bar{v}_2) = E_1$, so if

$$\pi(r) \cdot \rho x_1^2 \bar{v}_2 = 0$$

then $r \rho x_1^2 \bar{v}_2 \in (x_1^3 \bar{v}_2) \subset (x_1^3)$ in R . Using that R is of the form $T[\rho, \bar{v}_2]$ for T an x_1 -torsion free ring, it follows that $r \in (x_1)$.

Therefore, the map

$$S := \frac{\mathbb{F}_2[\zeta_1^4, \zeta_2^4, \zeta_3^2, \zeta_4, \dots][\rho, x_1, x_2, x_3, \bar{v}_2]}{(x_3^2 + x_2^2 \zeta_1^8 + x_1^2 \zeta_2^4, x_2^2 + x_1^2 \zeta_1^4, x_1^3 \bar{v}_2, (x_1^2 x_3 + x_2^3) \bar{v}_2)} \rightarrow \ker(\delta_1)$$

is surjective. To see that it is injective, note that the map

$$S \rightarrow S[y]/(y^2 + \zeta_2^4, x_1 y + x_3 + x_2 \zeta_1^4)$$

is injective, and the latter is easily identified with E_1 by setting $y = \zeta_2^2$. Now, $\text{im}(\delta_1)$ is the ideal in S generated by $\rho x_1^2 \bar{v}_2$, and the result follows. \square

Proposition 5.10. *The E_4 page of the ρ -Bockstein spectral sequence is given by the quotient of*

$$\mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \dots][\rho, x_1, x_2, x_3, \bar{v}_2]$$

by the ideal generated by the elements

$$\begin{array}{lll} x_3^2 + x_2^2 \zeta_2^4 + x_1^2 \zeta_1^8 & x_2^4 + x_1^4 \zeta_1^8 & x_1^3 \bar{v}_2 \\ \rho x_1^2 \bar{v}_2 & \rho^3 \bar{v}_2 & (x_1^2 x_3 + x_2^3) \bar{v}_2 \\ x_2^2 x_3 \bar{v}_2 + \zeta_1^8 x_1^2 x_2 \bar{v}_2 & x_2^2 \rho \bar{v}_2 & x_1 x_2^2 \bar{v}_2 \end{array}$$

Proof. The kernel of δ_3 is generated by the classes

$$\rho, x_1, x_2, x_3, \bar{v}_2, \zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \dots$$

and $\zeta_1^4 \cdot \text{ann}_{E_3}(\rho^3 \bar{v}_2)$, and we claim $\text{ann}_{E_3}(\rho^3 \bar{v}_2) = (x_1^2)$. To see this, set

$$R := \frac{\mathbb{F}_2[\zeta_1^4, \zeta_2^4, \zeta_3^2, \zeta_4, \dots][\rho, x_1, x_2, x_3, \bar{v}_2]}{(x_3^2 + x_1^2 \zeta_2^4 + x_2^2 \zeta_1^8, x_2^2 + x_1^2 \zeta_1^4)}$$

We have a surjection

$$\pi : R \rightarrow R/(\rho x_1^2 \bar{v}_2, x_1^3 \bar{v}_2, (x_1^2 x_3 + x_2^3) \bar{v}_2) = E_3$$

So if $\pi(r) \cdot \rho^3 \bar{v}_2 = 0$, then

$$r \rho^3 \bar{v}_2 \in (\rho x_1^2 \bar{v}_2, x_1^3 \bar{v}_2, (x_1^2 x_3 + x_2^3) \bar{v}_2) \subset (x_1^2)$$

in R . Using that R is of the form $T[\rho, \bar{v}_2]$ for T an x_1 -torsion free ring, we see that $r \in (x_1^2)$.

In particular, we do not pick up any new Massey product cycles as $x_1^2 \zeta_1^4 = x_2^2$, and therefore the map

$$S := \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \dots][\rho, x_1, x_2, x_3, \bar{v}_2]/J \rightarrow \ker(\partial_7)$$

is surjective, where J is the ideal generated by the list in the statement of the proposition, with the element $\rho^3 \bar{v}_2$ removed. To see that this map is injective, note that the map

$$S \rightarrow S[y]/(y^2 + \zeta_1^8, x_1^2 y + x_2^2)$$

is injective, and the latter is easily identified with E_3 by setting $y = \zeta_1^4$. \square

The remaining algebra generators are all permanent cycles, and so the ρ -Bockstein spectral sequence collapses on E_4 . This verifies the discussion following Proposition 5.6.

5.2.1. The homology of $\mathrm{tmf}_0(3)$

Moving to weight zero, we have the subalgebra $\mathcal{A}_* \square_{\mathcal{A}(2)_*} \mathbb{F}_2$ along with the following ten elements

$$\begin{array}{lll} 1 \in H_0 & \rho^2 x_1 \bar{v}_2 \in H_4 & \rho^2 x_2 \bar{v}_2 \in H_6 \\ \rho x_1 x_2 \bar{v}_2 \in H_7 & x_1^2 x_2 \bar{v}_2 \in H_8 & \rho^2 x_3 \bar{v}_2 \in H_{10} \\ \rho x_1 x_3 \bar{v}_2 \in H_{11} & x_1^2 x_3 \bar{v}_2 \in H_{12} & \rho x_2 x_3 \bar{v}_2 \in H_{13} \\ x_1 x_2 x_3 \bar{v}_2 \in H_{14} & & \end{array}$$

These form an $\mathcal{A}(2)_*$ -comodule algebra M_2 of dimension 10 with coactions determined by Corollary 4.9 and the fact that ρ and \bar{v}_2 are primitive. We display M_2 in Fig. 1, where we have omitted the unit class as it generates a trivial comodule summand.

Theorem 5.11. *The homology of $\mathrm{tmf}_0(3) \simeq \Gamma(BPGL\langle 2 \rangle)$ is isomorphic as an \mathcal{A}_* -comodule algebra to*

$$\mathcal{A}_* \square_{\mathcal{A}(2)_*} M_2$$

where \overline{M}_2 is as in Fig. 1, and the multiplication in \overline{M}_2 is square zero.

Proof. The description follows for the E_∞ page of the HSSS for $BPGL\langle 2 \rangle$ by the discussion following Proposition 5.6. This page is displayed in Fig. 9.

It suffices now to observe that there are no nontrivial comodule algebra extensions in the HSSS. Indeed, each class in M_2 is in the highest filtration in its respective stem so the projection $H_* \mathrm{tmf}_0(3) \rightarrow M_2$ is a map of $\mathcal{A}(2)_*$ -comodule algebras, which is adjoint to an isomorphism of \mathcal{A}_* -comodule algebras

$$H_* \mathrm{tmf}_0(3) \rightarrow \mathcal{A}_* \square_{\mathcal{A}(2)_*} M_2 \quad \square$$

Prior to the results of Hill–Lawson [20], it was not known how to construct a connective model $\mathrm{tmf}_0(3)$ of the periodic spectrum $\mathrm{TMF}_0(3)$ via a derived algebraic geometry approach. Nonetheless, computational aspects of such a spectrum were studied in detail, and Davis–Mahowald proposed several definitions of such a spectrum. In particular, they constructed a certain 10-cell complex X and considered the spectrum $\mathrm{tmf} \wedge X$ as a good connective model. They give an explicit construction of X , but we show that it exists by Toda obstruction theory.

Proposition 5.12. *Let N be the \mathcal{A} -module with one generator in each dimension 3, 5, 6, 7, 9, 10, 11, 12, 13, where the following Steenrod squares are nonzero on the generator g of dimension 3*

$$\mathrm{Sq}^2, \mathrm{Sq}^3, \mathrm{Sq}^4, \mathrm{Sq}^4 \mathrm{Sq}^2, \mathrm{Sq}^5 \mathrm{Sq}^2, \mathrm{Sq}^6 \mathrm{Sq}^2 = \mathrm{Sq}^8, \mathrm{Sq}^6 \mathrm{Sq}^3, \mathrm{Sq}^7 \mathrm{Sq}^3$$

and $\mathrm{Sq}^6(g) = 0$. Then there exists a unique 2-complete bounded below spectrum Y with $H^*Y \cong N$. Moreover, there is a map $Y \rightarrow S^0$ extending 2ν .

Proof. Toda obstruction theory (see [8, Theorem 3.2] or [39]) implies that it suffices to show that the -2 stem in

$$\mathrm{Ext}_{\mathcal{A}}^{*,*}(N, N)$$

is trivial. This is easily checked using Bruner's software, and we display the chart in Fig. 4. The -1 -stem contains no classes in filtration higher than 1, so uniqueness follows. The existence of the map $Y \rightarrow S^0$ may also be checked directly from the Adams spectral sequence, and this is the argument used in [10, Theorem 2.1(b)]. \square

It is easy to check that the \mathcal{A} -module N has the property that its restriction to $\mathcal{A}(2)$ is dual to the $\mathcal{A}(2)_*$ -comodule $\Sigma^{-1}\overline{M}_2$. We conclude the following.

Corollary 5.13. *Let X be the cofiber of the map $Y \rightarrow S^0$ constructed in Proposition 5.12. There is a 2-local equivalence of tmf -modules*

$$\mathrm{tmf} \wedge X \rightarrow \mathrm{tmf}_0(3)$$

Proof. The homotopy groups of tmf and $\mathrm{tmf}_0(3)$ are degreewise finitely generated, so it suffices to produce a 2-complete equivalence, which follows from the existence of any map $X \rightarrow \mathrm{tmf}_0(3)$ that induces the map of \mathcal{A}_* -comodules $H_*X \rightarrow \mathcal{A}_* \square_{\mathcal{A}(2)_*} M_2$ adjoint to the identity map of the $\mathcal{A}(2)_*$ -comodule M_2 .

For this, it suffices to show that in the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}_*}(H_*X, H_*\mathrm{tmf}_0(3)) \cong \mathrm{Ext}_{\mathcal{A}(2)_*}(M_2, M_2) \implies [X, \mathrm{tmf}_0(3)]_2^\wedge$$

the identity map ι is a permanent cycle. By Proposition 5.12, the 4-skeleton of X is the 2-cell complex $C(2\nu)$. Since $2\nu = 0 \in \pi_3\mathrm{tmf}_0(3)$ (see [21] or [27]), the unit map $S^0 \rightarrow \mathrm{tmf}_0(3)$ extends over $C(2\nu)$, which implies that the map of Adams spectral sequences

$$\mathrm{Ext}_{\mathcal{A}_*}(H_*X, H_*\mathrm{tmf}_0(3)) \rightarrow \mathrm{Ext}_{\mathcal{A}_*}(H_*C(2\nu), H_*\mathrm{tmf}_0(3))$$

sends ι to a permanent cycle.

We use Bruner's ext software to produce the former in Fig. 5 and the latter in Fig. 6a. The map is an isomorphism in bidegrees $(0, 0)$, $(-1, 2)$ and $(-1, 3)$, sending ι to the generator of the lefthand h_0 -tower in Fig. 6a. It follows by naturality that ι does not support a $d_{<4}$ in Fig. 5. We claim the generator of the righthand h_0 -tower in Fig. 6a supports a nontrivial d_2 . Given this, the nontrivial d_2 lifts to Fig. 5, and it follows from h_0 -linearity that the -1 stem is then trivial on the E_3 -page.

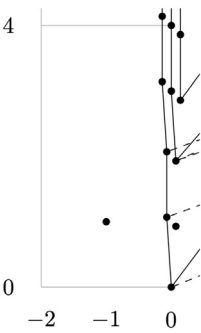


Fig. 4. $\text{Ext}_{\mathcal{A}}(N, N)$.

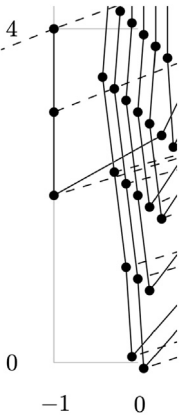
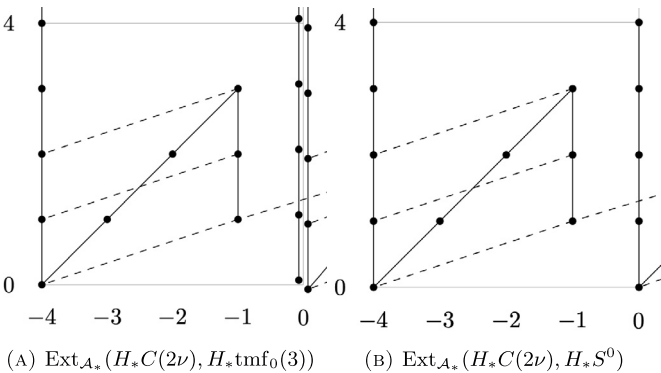


Fig. 5. $\text{Ext}_{\mathcal{A}_*}(H_*X, H_*\text{tmf}_0(3))$.

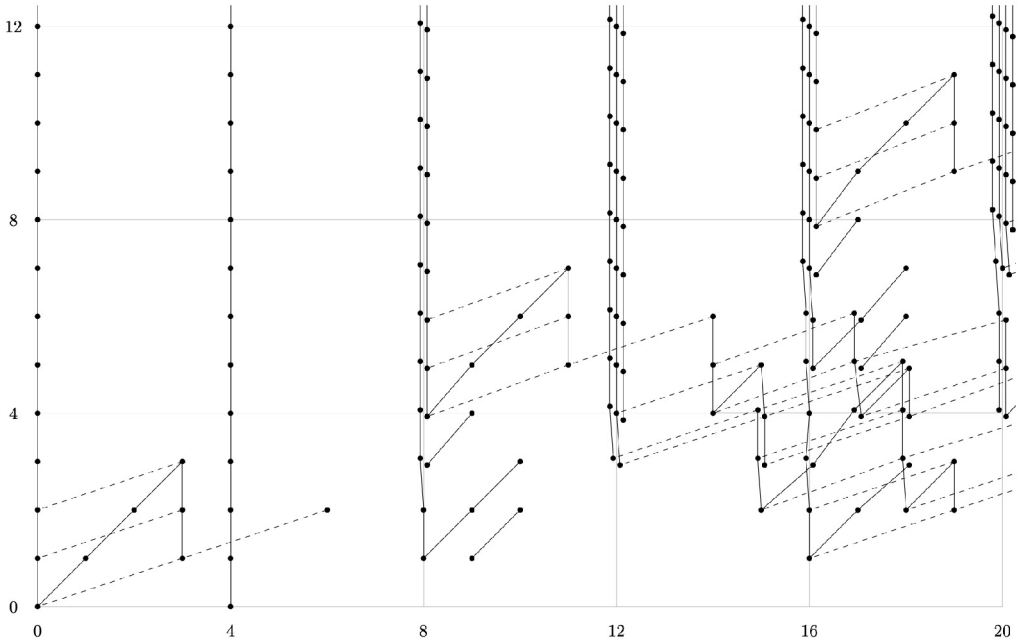


(A) $\text{Ext}_{\mathcal{A}_*}(H_*C(2\nu), H_*\text{tmf}_0(3))$ (B) $\text{Ext}_{\mathcal{A}_*}(H_*C(2\nu), H_*S^0)$

Fig. 6.

It remains to show the claimed d_2 in Fig. 6a. This follows by naturality via the map

$$\text{Ext}_{\mathcal{A}_*}(H_*C(2\nu), \mathbb{F}_2) \rightarrow \text{Ext}_{\mathcal{A}_*}(H_*C(2\nu), H_*\text{tmf}_0(3))$$

Fig. 7. $\mathrm{Ext}_{A_*}(\mathbb{F}_2, H_*\mathrm{tmf}_0(3))$.

and we display the former E_2 page in Fig. 6b. Since

$$D(C(2\nu)) \not\cong S^{-4} \vee S^0$$

the class in bidegree $(0,0)$ must support a differential. It follows easily from the long exact sequence in homotopy that $\pi_{-1}D(C(2\nu)) = \pi_3C(2\nu) = \mathbb{Z}/2$, which implies the claimed d_2 . \square

Corollary 5.14. *The Adams spectral sequence for $\mathrm{tmf}_0(3)$ does not collapse on E_2 . In particular, we have $d_2(x) = h_0h_2$, where x is the nontrivial class in bidegree $(4,0)$.*

Proof. The claimed d_2 follows by naturality via the map $C(2\nu) \rightarrow \mathrm{tmf}_0(3)$, where the Adams spectral sequence for $C(2\nu)$ supports the nontrivial d_2 discussed in the proof of Corollary 5.13. We display the Adams E_2 -page for $\mathrm{tmf}_0(3)$ in Fig. 7. \square

Remark 5.15. Corollary 5.14 stands in contrast to the motivic Adams spectral sequence, which collapses for $BPGL\langle m \rangle$ for all m (similarly for the C_2 -equivariant Adams spectral sequence for $BP_{\mathbb{R}}\langle m \rangle$).

5.3. The homology of $\Gamma(BPGL\langle 3 \rangle)$

We follow arguments similar to the $m = 2$ case to compute the HSSS for $BPGL\langle 3 \rangle$. The E_{15} page is displayed in Fig. 10, with conventions as before, where \bar{v}_3 classes are

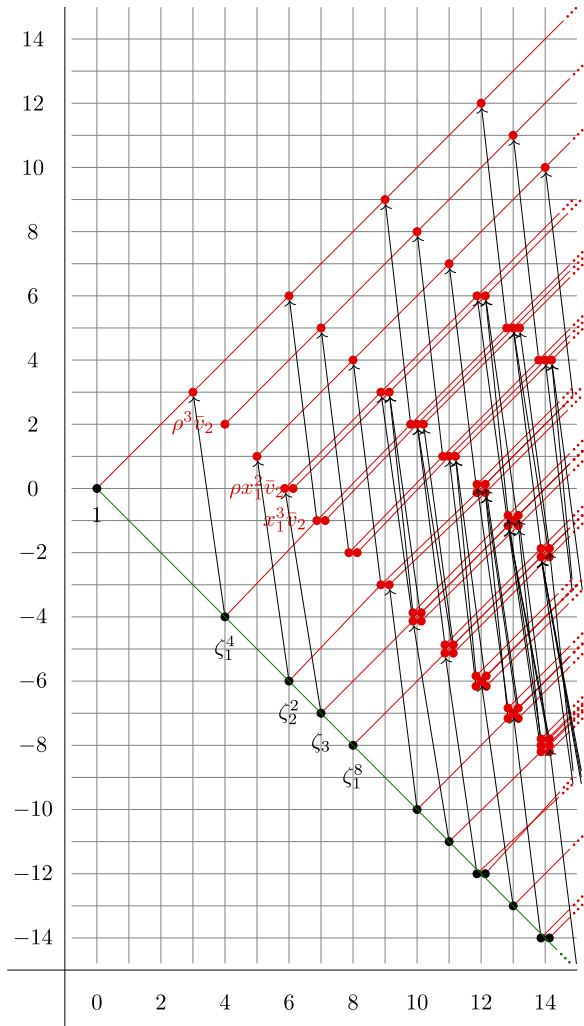


Fig. 8. The E_7 page of the HSSS for $BPGL\langle 2 \rangle$.

represented by blue dots, and blue structure lines indicate multiplication by $\rho^7 \bar{v}_3$, which detects $\sigma \in \pi_7 S^0$.

Theorem 5.16. *There is a spectral sequence of \mathcal{A}_* -comodule algebras with E_2 -term given by*

$$E_2^{*,*,*} = \mathbb{F}_2[\zeta_1^2, \zeta_2, \dots][\rho, x_1, \bar{v}_1, \bar{v}_2, \bar{v}_3]$$

We have differentials

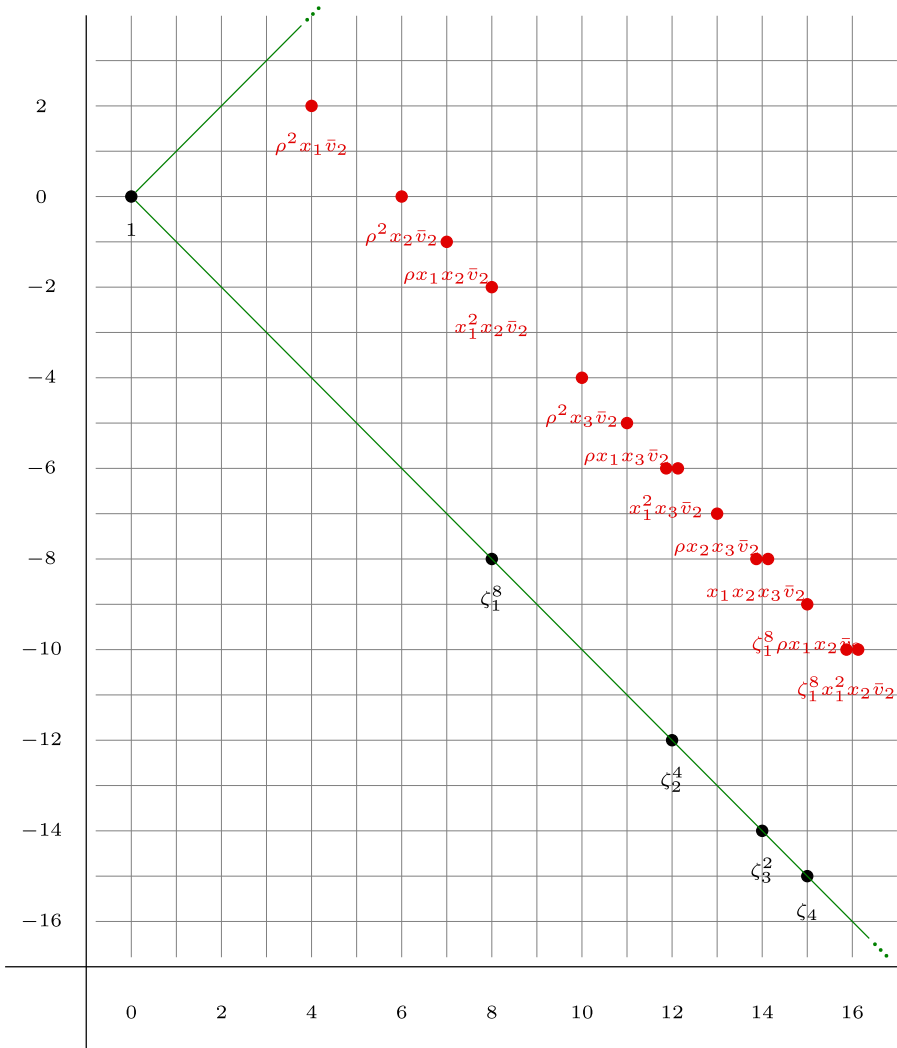


Fig. 9. The E_∞ page of the HSSS for $BPGL\langle 2 \rangle$.

$$\begin{aligned}
 d_3(\zeta_1^2) &= \rho \bar{v}_1 & d_3(\zeta_2) &= x_1 \bar{v}_1 & d_7(\zeta_1^4) &= \rho^3 \bar{v}_2 \\
 d_7(\zeta_2^2) &= \rho x_1^2 \bar{v}_2 & d_7(\zeta_3) &= (x_1^3 + \rho^2 x_2) \bar{v}_2 & d_{15}(\zeta_1^8) &= \rho^7 \bar{v}_3 \\
 d_{15}(\zeta_2^4) &= \rho^3 x_1^4 \bar{v}_3 & d_{15}(\zeta_3^2) &= (\rho x_1^6 + \rho^5 x_2^2) \bar{v}_3 & d_{15}(\zeta_4) &= (x_1^7 + \rho^2 x_1^4 x_2 + \rho^4 x_1 x_2^2 + \rho^6 x_3) \bar{v}_3
 \end{aligned}$$

The subalgebra of E_2 generated by the classes

$$\rho, x_1, x_2, x_3, x_4, \bar{v}_1, \bar{v}_2, \bar{v}_3, \zeta_1^{16}, \zeta_2^8, \zeta_3^4, \zeta_4^2, \zeta_5, \dots$$

consists of permanent cycles.

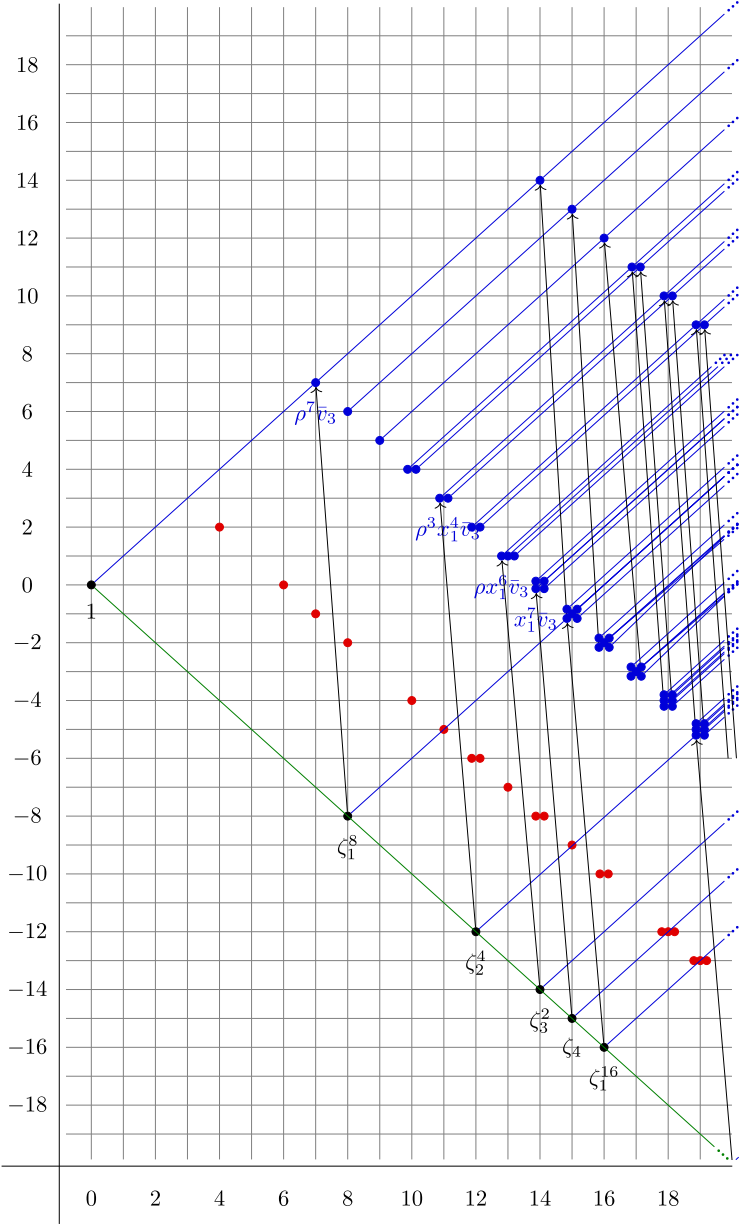


Fig. 10. The E_{15} page of the HSSS for $BPGL\langle 3 \rangle$.

The formulas for $d_{\leq 7}$ imply that, as a dga, one has an isomorphism

$$E_{15}^{*,*,*}(BPGL\langle 3 \rangle; i_* H\mathbb{F}_2) \cong E_8^{*,*,*}(BPGL\langle 2 \rangle; i_* H\mathbb{F}_2)[\bar{v}_3]$$

The ideal $(\bar{v}_1, \bar{v}_2) \subset E_{15}$ is contained in $\ker(d_{15})$, and we have maps of dga's

$$E_{15} \rightarrow E_{15}/(\bar{v}_1, \bar{v}_2^2) \rightarrow E_{15}/(\bar{v}_1, \bar{v}_2)$$

There is nothing in weight 0 contributed by \bar{v}_1 because of the d_3 's, and there is nothing in weight 0 contributed by \bar{v}_2^2 because of the d_7 's. In fact, the first map is an isomorphism in non-positive weights, and the second map is surjective with square zero kernel consisting of d_{15} -cycles (for degree reasons). Using this, it is not difficult to recover the E_{16} page from $H_*(E_{15}/(\bar{v}_1, \bar{v}_2))$, and we have an isomorphism

$$E_{15}/(\bar{v}_1, \bar{v}_2) \cong \frac{\mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \dots][\rho, x_1, x_2, x_3, \bar{v}_3]}{(x_2^4 + \zeta_1^8 x_1^4 + \zeta_2^4 \rho^4, x_3^2 + \zeta_1^8 x_2^2 + \zeta_2^4 x_1^2 + \zeta_3^2 \rho^2)}$$

with d_{15} determined by the four explicit differentials given above.

We claim that $\ker(d_{15})$ is the subalgebra of $E_{15}/(\bar{v}_1, \bar{v}_2)$ generated by the classes

$$\zeta_1^{16}, \zeta_2^8, \zeta_3^4, \zeta_4^2, \zeta_5, \dots, \rho, x_1, x_2, x_3, x_4, \bar{v}_3$$

It follows that we have the presentation

$$\ker(d_{15}) = \frac{\mathbb{F}_2[\zeta_1^{16}, \zeta_2^8, \zeta_3^4, \zeta_4^2, \zeta_5, \dots][\rho, x_1, x_2, x_3, x_4, \bar{v}_3]}{(x_2^8 + \rho^8 \zeta_2^8 + x_1^8 \zeta_1^{16}, x_3^4 + \rho^4 \zeta_3^4 + x_1^4 \zeta_1^8 + x_2^4 \zeta_1^{16}, x_4^2 + \rho^2 \zeta_4^2 + x_1^2 \zeta_1^4 + x_2^2 \zeta_2^8 + x_3^2 \zeta_1^{16})}$$

Writing our differentials in terms of this description of the kernel, we have

$$\begin{aligned} d_{15}(\zeta_1^8) &= \rho^7 \bar{v}_3 \\ d_{15}(\zeta_2^4) &= \rho^3 x_1^4 \bar{v}_3 \\ d_{15}(\zeta_3^2) &= (\rho x_1^6 + \rho^5 x_2^2) \bar{v}_3 \\ d_{15}(\zeta_4) &= (x_1^7 + \rho^2 x_1^4 x_2 + \rho^4 x_1 x_2^2 + \rho^6 x_3) \bar{v}_3 \\ d_{15}(\zeta_1^8 \zeta_2^4) &= \rho^3 x_2^4 \bar{v}_3 \\ d_{15}(\zeta_1^8 \zeta_3^2) &= (\rho^5 x_3^2 + \rho x_1^2 x_2^4) \bar{v}_3 \\ d_{15}(\zeta_1^8 \zeta_4) &= (x_1^3 x_2^4 + \rho^2 x_2^5 + \rho^4 x_1 x_3^2 + \rho^6 x_4) \bar{v}_3 \\ d_{15}(\zeta_2^4 \zeta_3^2) &= (\rho x_1^4 x_3^2 + \rho x_2^6) \bar{v}_3 \\ d_{15}(\zeta_2^4 \zeta_4) &= (x_1^5 x_3^2 + x_1 x_2^6 + \rho^2 x_1^4 x_4 + \rho^2 x_2^4 x_3) \bar{v}_3 \\ d_{15}(\zeta_3^2 \zeta_4) &= (x_2^7 + x_1^6 x_4 + x_1^4 x_2 x_3^2 + x_1^2 x_2^4 x_3 + \rho^4 x_2^2 x_4 + \rho^4 x_3^3) \bar{v}_3 \\ d_{15}(\zeta_1^8 \zeta_2^4 \zeta_3^2) &= (\rho x_2^4 x_3^2 + \rho x_1^4 x_2^2 \zeta_1^{16} + \rho^5 x_1^2 \zeta_2^8) \bar{v}_3 \\ d_{15}(\zeta_1^8 \zeta_2^4 \zeta_4) &= (x_1 x_2^4 x_3^2 + x_1^5 x_2^2 \zeta_1^{16} + \rho^2 x_2^4 x_4 + \rho^2 x_1^4 x_3 \zeta_1^{16} + \rho^4 x_1^3 \zeta_2^8 + \rho^6 x_2 \zeta_2^8) \bar{v}_3 \\ d_{15}(\zeta_1^8 \zeta_3^2 \zeta_4) &= (x_1^2 x_2^4 x_4 + x_2^5 x_3^2 + x_1^4 x_2^3 \zeta_1^{16} + \rho^4 x_3^3 x_4 + \rho^4 x_2^2 x_3 \zeta_1^{16} + \rho^4 x_1^2 x_2 \zeta_2^8 \\ &\quad + \rho^6 x_1 \zeta_3^4) \bar{v}_3 \\ d_{15}(\zeta_2^4 \zeta_3^2 \zeta_4) &= (x_1^4 x_3^2 x_4 + x_2^6 x_4 + x_2^4 x_3^3 + x_1^4 x_2^2 x_3 \zeta_1^{16} + (x_1^6 x_2 + \rho^4 x_1^2 x_3 + \rho^4 x_2^3) \zeta_2^8) \bar{v}_3 \end{aligned}$$

$$\begin{aligned}
& + \rho^2 x_1^5 \zeta_3^4 \bar{v}_3 \\
d_{15}(\zeta_1^8 \zeta_2^4 \zeta_3^2 \zeta_4) &= (x_2^4 x_3^2 x_4 + x_1^2 x_2^5 \zeta_2^8 + x_1^4 x_2^2 x_4 \zeta_1^{16} + x_1^4 x_3^3 \zeta_1^{16} + x_2^6 x_3 \zeta_1^{16} + \rho^2 x_1 x_2^4 \zeta_3^4 \\
& + \rho^4 x_1^2 x_4 \zeta_2^8 + \rho^4 x_2 x_3^2 \zeta_2^8) \bar{v}_3
\end{aligned}$$

Letting I_3 be the ideal in $\ker(d_{15})$ generated by this list of relations, it follows that $H_*(E_{15}/(\bar{v}_1, \bar{v}_2)) = \ker(d_{15})/I_3$. It remains to verify that the claimed list of elements indeed generates $\ker(d_{15})$. We achieve this by running the ρ -Bockstein spectral sequence as before, and our arguments follow closely those of the $m = 2$ case.

Proposition 5.17. *The E_0 -page of the ρ -Bockstein spectral sequence is given by*

$$\frac{\mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \dots][\rho, x_1, x_2, x_3, \bar{v}_3]}{(x_2^4 + \zeta_1^8 x_1^4, x_3^2 + \zeta_1^8 x_2^2 + \zeta_2^4 x_1^2)}$$

We have differentials

$$\delta_0(\zeta_4) = x_1^7 \bar{v}_3 \quad \delta_1(\zeta_3^2) = \rho x_1^6 \bar{v}_3 \quad \delta_3(\zeta_2^4) = \rho^3 x_1^4 \bar{v}_3 \quad \delta_7(\zeta_1^8) = \rho^7 \bar{v}_3$$

The annihilator ideal of $\delta_0(\zeta_4)$ in E_0 is trivial, so δ_0 does not create any Massey product cycles. We conclude the following.

Proposition 5.18. *The E_1 -page of the ρ -Bockstein spectral sequence is given by*

$$\frac{\mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4^2, \dots][\rho, x_1, x_2, x_3, \bar{v}_3]}{(x_2^4 + \zeta_1^8 x_1^4, x_3^2 + \zeta_1^8 x_2^2 + \zeta_2^4 x_1^2, x_1^7 \bar{v}_3)}$$

The annihilator ideal of $\delta_1(\zeta_3^2)$ in E_1 is (x_1) , which gives our Massey product x_4 .

Proposition 5.19. *The E_3 -page of the ρ -Bockstein spectral sequence is given by the quotient of the polynomial ring*

$$\mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^4, \zeta_4^2, \zeta_5, \dots][\rho, x_1, x_2, x_3, x_4, \bar{v}_3]$$

by the ideal generated by the elements

$$\begin{aligned}
x_2^4 + \zeta_1^8 x_1^4, & & x_3^2 + \zeta_1^8 x_2^2 + \zeta_2^4 x_1^2, \\
x_4^2 + x_1^2 \zeta_3^4 + x_2^2 \zeta_2^8 + x_3^2 \zeta_1^{16}, & & x_1^7 \bar{v}_3, \\
(x_2^7 + x_1^6 x_4 + x_1^4 x_2 x_3^2 + x_1^2 x_2^4 x_3) \bar{v}_3, & & \rho x_1^6 \bar{v}_3.
\end{aligned}$$

Proof. The description of $\text{ann}_{E_1}(\rho x_1^6 \bar{v}_3)$ implies that the map

$$S := \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^4, \zeta_4^2, \zeta_5, \dots][\rho, x_1, x_2, x_3, x_4, \bar{v}_3]/J \rightarrow \ker(\delta_1)$$

is surjective, where J is the ideal generated by the list of elements in the statement of the proposition, with $\rho x_1^6 \bar{v}_3$ removed. To see that it is injective, note that the map

$$S \rightarrow S[y]/(y^2 + \zeta_3^4, x_1 y + x_2 \zeta_2^4 + x_3 \zeta_1^8 + x_4)$$

is injective, and the latter is easily identified with E_1 by setting $y = \zeta_3^2$. \square

Proposition 5.20. *The E_7 -page of the ρ -Bockstein spectral sequence is given by the quotient of the polynomial ring*

$$\mathbb{F}_2[\zeta_1^8, \zeta_2^8, \zeta_3^4, \zeta_4^2, \zeta_5, \dots][\rho, x_1, x_2, x_3, x_4, \bar{v}_3]$$

by the ideal generated by the elements

$$\begin{array}{ll} x_2^4 + \zeta_1^8 x_1^4, & x_3^4 + \zeta_1^{16} x_2^4 + \zeta_2^8 x_1^4, \\ x_4^2 + x_1^2 \zeta_3^4 + x_2^2 \zeta_2^8 + x_3^2 \zeta_1^{16}, & x_1^7 \bar{v}_3, \\ (x_2^7 + x_1^6 x_4 + x_1^4 x_2 x_3^2 + x_1^2 x_2^4 x_3) \bar{v}_3, & \rho x_1^6 \bar{v}_3, \\ \rho^3 x_1^4 \bar{v}_3, & (x_1^5 x_3^2 + x_1 x_2^6) \bar{v}_3, \\ (\rho x_1^4 x_3^2 + \rho x_2^6) \bar{v}_3, & (x_1^4 x_3^2 x_4 + x_2^6 x_4 + x_2^4 x_3^3 + x_1^4 x_2^2 x_3 \zeta_1^{16} + x_1^6 x_2 \zeta_2^8) \bar{v}_3. \end{array}$$

Proof. We claim that $\text{ann}_{E_3}(\rho^3 x_1^4 \bar{v}_3) = (x_1^2)$. To see this, let

$$R = \frac{\mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^4, \zeta_4^2, \dots][\rho, x_1, x_2, x_3, x_4, \bar{v}_3]}{(x_2^4 + \zeta_1^8 x_1^4, x_3^2 + \zeta_1^8 x_2^2 + \zeta_2^4 x_1^2, x_4^2 + x_1^2 \zeta_3^4 + x_2^2 \zeta_2^8 + x_3^2 \zeta_1^{16})}$$

There is a surjection

$$\pi : R \rightarrow R/(x_1^7 \bar{v}_3, \rho x_1^6 \bar{v}_3, (x_2^7 + x_1^6 x_4 + x_1^4 x_2 x_3^2 + x_1^2 x_2^4 x_3) \bar{v}_3) \cong E_3$$

So if $\pi(r) \cdot \rho^3 x_1^4 \bar{v}_3 = 0$, we have

$$r \rho^3 x_1^4 \bar{v}_3 \in (x_1^7 \bar{v}_3, \rho x_1^6 \bar{v}_3, (x_2^7 + x_1^6 x_4 + x_1^4 x_2 x_3^2 + x_1^2 x_2^4 x_3) \bar{v}_3) \subset (x_1^6)$$

Using that $R = T[\rho, \bar{v}_3]$ for T an x_1 -torsion free ring, it follows that $r x_1^4 \in (x_1^6)$ and therefore $r \in (x_1^2)$. Since $x_2^2 \zeta_3^4 = x_3^2 + \zeta_1^8 x_2^2$, this creates no new cycles.

It follows that the map

$$S := \mathbb{F}_2[\zeta_1^8, \zeta_2^8, \zeta_3^4, \zeta_4^2, \zeta_5, \dots][\rho, x_1, x_2, x_3, x_4, \bar{v}_3]/J \rightarrow \ker(\delta_3)$$

is surjective, where J is the ideal generated by the list of elements in the statement of the proposition, with $\rho^3 x_1^4 \bar{v}_3$ removed. To see that it is injective, note that the map

$$S \rightarrow S[y]/(y^2 + \zeta_2^8, x_1^2 y + x_2^2 \zeta_1^8 + x_3^2)$$

is injective, and the latter is easily identified with E_3 by setting $y = \zeta_2^4$. \square

Proposition 5.21. *The E_8 -page of the ρ -Bockstein spectral sequence is given by the quotient of the polynomial ring*

$$\mathbb{F}_2[\zeta_1^{16}, \zeta_2^8, \zeta_3^4, \zeta_4^2, \zeta_5, \dots][\rho, x_1, x_2, x_3, x_4, \bar{v}_3]$$

by the ideal generated by the elements

$$\begin{aligned} & x_2^8 + \zeta_1^{16} x_1^8, & x_3^4 + \zeta_1^{16} x_2^4 + \zeta_2^8 x_1^4, \\ & x_4^2 + x_1^2 \zeta_3^4 + x_2^2 \zeta_2^8 + x_3^2 \zeta_1^{16}, & x_1^7 \bar{v}_3, \\ & (x_2^7 + x_1^6 x_4 + x_1^4 x_2 x_3^2 + x_1^2 x_2^4 x_3) \bar{v}_3, & \rho x_1^6 \bar{v}_3, \\ & \rho^3 x_1^4 \bar{v}_3, & (x_1^5 x_3^2 + x_1 x_2^6) \bar{v}_3, \\ & (\rho x_1^4 x_3^2 + \rho x_2^6) \bar{v}_3, & (x_1^4 x_3^2 x_4 + x_2^6 x_4 + x_2^4 x_3^3 + x_1^4 x_2 x_3 \zeta_1^{16} + x_1^6 x_2 \zeta_2^8) \bar{v}_3, \\ & \rho^7 \bar{v}_3, & \rho^3 x_2^4 \bar{v}_3, \\ & (\rho^5 x_3^2 + \rho x_1^2 x_2^4) \bar{v}_3, & (x_1^3 x_2^4 + \rho^2 x_2^5 + \rho^4 x_1 x_3^2 + \rho^6 x_4) \bar{v}_3, \\ & (\rho x_2^4 x_3^2 + \rho x_1^4 x_2^2 \zeta_1^{16} + \rho^5 x_1^2 \zeta_2^8) \bar{v}_3. \\ & (x_1 x_2^4 x_3^2 + x_1^5 x_2^2 \zeta_1^{16} + \rho^2 x_2^4 x_4 + \rho^2 x_1^4 x_3 \zeta_1^{16} + \rho^4 x_1^3 \zeta_2^8 + \rho^6 x_2 \zeta_2^8) \bar{v}_3 \\ & (x_1^2 x_2^4 x_4 + x_2^5 x_3^2 + x_1^4 x_2^3 \zeta_1^{16} + \rho^4 x_3^3 x_4 + \rho^4 x_2^2 x_3 \zeta_1^{16} + \rho^4 x_1^2 x_2 \zeta_2^8 + \rho^6 x_1 \zeta_3^4) \bar{v}_3 \\ & (x_2^4 x_3^2 x_4 + x_1^2 x_2^5 \zeta_2^8 + x_1^4 x_2^2 x_4 \zeta_1^{16} + x_1^4 x_3^3 \zeta_1^{16} + x_2^6 x_3 \zeta_1^{16} + \rho^2 x_1 x_2^4 \zeta_3^4 + \rho^4 x_1^2 x_4 \zeta_2^8 + \rho^4 x_2 x_3^2 \zeta_2^8) \bar{v}_3 \end{aligned}$$

Proof. We claim that $\text{ann}_{E_7}(\rho^7 \bar{v}_3) = (x_1^4)$. To see this, let

$$R = \frac{\mathbb{F}_2[\zeta_1^8, \zeta_2^8, \zeta_3^4, \zeta_4^2, \dots][\rho, x_1, x_2, x_3, x_4, \bar{v}_3]}{(x_2^4 + \zeta_1^8 x_1^4, x_3^4 + \zeta_1^{16} x_2^4 + \zeta_2^8 x_1^4, x_4^2 + x_1^2 \zeta_3^4 + x_2^2 \zeta_2^8 + x_3^2 \zeta_1^{16})}$$

There is a surjection $\pi : R \rightarrow R/I \cong E_7$ where I is the ideal generated by the elements

$$\begin{aligned} & x_1^7 \bar{v}_3, & (x_2^7 + x_1^6 x_4 + x_1^4 x_2 x_3^2 + x_1^2 x_2^4 x_3) \bar{v}_3, \\ & \rho x_1^6 \bar{v}_3, & \rho^3 x_1^4 \bar{v}_3, \\ & (x_1^5 x_3^2 + x_1 x_2^6) \bar{v}_3, & (\rho x_1^4 x_3^2 + \rho x_2^6) \bar{v}_3, \\ & (x_1^4 x_3^2 x_4 + x_2^6 x_4 + x_2^4 x_3^3 + x_1^4 x_2 x_3 \zeta_1^{16} + x_1^6 x_2 \zeta_2^8) \bar{v}_3. \end{aligned}$$

So if $\pi(r) \cdot \rho^7 \bar{v}_3 = 0 \in E_7$, then $r \rho^7 \bar{v}_3 \in I$. It is not hard to check that $I \subset (x_1^4)$, so since R is of the form $T[\rho, \bar{v}_3]$ for T an x_1 -torsion free ring we have $r \in (x_1^4)$. Arguing as before, we arrive at the claimed presentation. \square

The remaining algebra generators are all permanent cycles, and so the ρ -Bockstein spectral sequence collapses on E_8 . This verifies the discussion preceding Proposition 5.17, so that $H_*(E_{15}/(\bar{v}_1, \bar{v}_2))$ has the claimed description as an \mathcal{A}_* -comodule algebra. In weight zero, we have

$$H_*(E_{15}/(\bar{v}_1, \bar{v}_2)) \cong \mathcal{A}_* \square_{\mathcal{A}(3)_*} M_3$$

and machine computation gives the following basis of the $\mathcal{A}(3)_*$ -comodule algebra M_3 , which has dimension 165.

$1 \in H_0$	$\rho^6 x_1 \bar{v}_3 \in H_8$	$\rho^5 x_1^2 \bar{v}_3 \in H_9$
$\rho^6 x_2 \bar{v}_3 \in H_{10}$	$\rho^4 x_1^3 \bar{v}_3 \in H_{10}$	$\rho^5 x_1 x_2 \bar{v}_3 \in H_{11}$
$\rho^4 x_1^2 x_2 \bar{v}_3 \in H_{12}$	$\rho^2 x_1^5 \bar{v}_3 \in H_{12}$	$\rho^5 x_2^2 \bar{v}_3 \in H_{13}$
$\rho^3 x_1^3 x_2 \bar{v}_3 \in H_{13}$	$\rho^6 x_3 \bar{v}_3 \in H_{14}$	$\rho^4 x_1 x_2^2 \bar{v}_3 \in H_{14}$
$\rho^2 x_1^4 x_2 \bar{v}_3 \in H_{14}$	$\rho^5 x_1 x_3 \bar{v}_3 \in H_{15}$	$\rho^3 x_1^2 x_2^2 \bar{v}_3 \in H_{15}$
$\rho x_1^5 x_2 \bar{v}_3 \in H_{15}$	$\rho^4 x_1^2 x_3 \bar{v}_3 \in H_{16}$	$\rho^4 x_2^3 \bar{v}_3 \in H_{16}$
$\rho^2 x_1^3 x_2^2 \bar{v}_3 \in H_{16}$	$x_1^6 x_2 \bar{v}_3 \in H_{16}$	$\rho^5 x_2 x_3 \bar{v}_3 \in H_{17}$
$\rho^3 x_1^3 x_3 \bar{v}_3 \in H_{17}$	$\rho^3 x_1 x_2^3 \bar{v}_3 \in H_{17}$	$\rho x_1^4 x_2^2 \bar{v}_3 \in H_{17}$
$\rho^4 x_1 x_2 x_3 \bar{v}_3 \in H_{18}$	$\rho^2 x_1^4 x_3 \bar{v}_3 \in H_{18}$	$\rho^2 x_1^2 x_2^3 \bar{v}_3 \in H_{18}$
$x_1^5 x_2^2 \bar{v}_3 \in H_{18}$	$\rho^3 x_1^2 x_2 x_3 \bar{v}_3 \in H_{19}$	$\rho x_1^5 x_3 \bar{v}_3 \in H_{19}$
$\rho x_1^3 x_2^3 \bar{v}_3 \in H_{19}$	$\rho^4 x_2^2 x_3 \bar{v}_3 \in H_{20}$	$\rho^2 x_1^3 x_2 x_3 \bar{v}_3 \in H_{20}$
$\rho^2 x_1 x_2^4 \bar{v}_3 \in H_{20}$	$x_1^6 x_3 \bar{v}_3 \in H_{20}$	$x_1^4 x_2^3 \bar{v}_3 \in H_{20}$
$\rho^5 x_2^3 \bar{v}_3 \in H_{21}$	$\rho^3 x_1 x_2^2 x_3 \bar{v}_3 \in H_{21}$	$\rho x_1^4 x_2 x_3 \bar{v}_3 \in H_{21}$
$\rho^6 x_4 \bar{v}_3 \in H_{22}$	$\rho^4 x_1 x_2^2 \bar{v}_3 \in H_{22}$	$\rho^2 x_1^2 x_2^2 x_3 \bar{v}_3 \in H_{22}$
$\rho^2 x_2^5 \bar{v}_3 \in H_{22}$	$x_1^5 x_2 x_3 \bar{v}_3 \in H_{22}$	$\rho^5 x_1 x_4 \bar{v}_3 \in H_{23}$
$\rho^3 x_1^2 x_3^2 \bar{v}_3 \in H_{23}$	$\rho^3 x_2^3 x_3 \bar{v}_3 \in H_{23}$	$\rho x_1^3 x_2^2 x_3 \bar{v}_3 \in H_{23}$
$\rho x_1 x_2^5 \bar{v}_3 \in H_{23}$	$\rho^4 x_1^2 x_4 \bar{v}_3 \in H_{24}$	$\rho^4 x_2 x_3^2 \bar{v}_3 \in H_{24}$
$\rho^2 x_1^3 x_3^2 \bar{v}_3 \in H_{24}$	$\rho^2 x_1 x_2^3 x_3 \bar{v}_3 \in H_{24}$	$x_1^4 x_2^2 x_3 \bar{v}_3 \in H_{24}$
$x_1^2 x_2^5 \bar{v}_3 \in H_{24}$	$\rho^5 x_2 x_4 \bar{v}_3 \in H_{25}$	$\rho^3 x_3^3 x_4 \bar{v}_3 \in H_{25}$
$\rho^3 x_1 x_2 x_3^2 \bar{v}_3 \in H_{25}$	$\rho x_1^4 x_3^2 \bar{v}_3 \in H_{25}$	$\rho x_1^2 x_2^3 x_3 \bar{v}_3 \in H_{25}$
$\rho^4 x_1 x_2 x_4 \bar{v}_3 \in H_{26}$	$\rho^2 x_1^4 x_4 \bar{v}_3 \in H_{26}$	$\rho^2 x_1^2 x_2 x_3^2 \bar{v}_3 \in H_{26}$
$\rho^2 x_2^4 x_3 \bar{v}_3 \in H_{26}$	$x_1^5 x_3^2 \bar{v}_3 \in H_{26}$	$x_1^3 x_2^3 x_3 \bar{v}_3 \in H_{26}$
$\rho^3 x_1^2 x_2 x_4 \bar{v}_3 \in H_{27}$	$\rho^3 x_2^2 x_3^2 \bar{v}_3 \in H_{27}$	$\rho x_1^5 x_4 \bar{v}_3 \in H_{27}$
$\rho x_1^3 x_2 x_3^2 \bar{v}_3 \in H_{27}$	$\rho x_1 x_2^4 x_3 \bar{v}_3 \in H_{27}$	$\rho^4 x_3^3 \bar{v}_3 \in H_{28}$
$\rho^2 x_1^3 x_2 x_4 \bar{v}_3 \in H_{28}$	$\rho^2 x_1 x_2^2 x_3^2 \bar{v}_3 \in H_{28}$	$x_1^6 x_4 \bar{v}_3 \in H_{28}$
$x_1^4 x_2 x_3^2 \bar{v}_3 \in H_{28}$	$x_1^2 x_2^4 x_3 \bar{v}_3 \in H_{28}$	$\rho^4 x_2^2 x_4 \bar{v}_3 \in H_{28}$
$\rho^3 x_1 x_2^2 x_4 \bar{v}_3 \in H_{29}$	$\rho^5 x_3 x_4 \bar{v}_3 \in H_{29}$	$\rho^3 x_1 x_3^3 \bar{v}_3 \in H_{29}$
$\rho x_1^4 x_2 x_4 \bar{v}_3 \in H_{29}$	$\rho x_1^2 x_2^2 x_3^2 \bar{v}_3 \in H_{29}$	$\rho x_2^5 x_3 \bar{v}_3 \in H_{29}$
$\rho^2 x_1^2 x_2^2 x_4 \bar{v}_3 \in H_{30}$	$\rho^4 x_1 x_3 x_4 \bar{v}_3 \in H_{30}$	$\rho^2 x_1^2 x_3^3 \bar{v}_3 \in H_{30}$

$$\begin{array}{lll}
\rho^2 x_2^3 x_3^2 \bar{v}_3 \in H_{30} & x_1^5 x_2 x_4 \bar{v}_3 \in H_{30} & x_1^3 x_2^2 x_3^2 \bar{v}_3 \in H_{30} \\
x_1 x_2^5 x_3 \bar{v}_3 \in H_{30} & \rho^3 x_1^2 x_3 x_4 \bar{v}_3 \in H_{31} & \rho^3 x_2 x_3^3 \bar{v}_3 \in H_{31} \\
\rho x_1^3 x_2^2 x_4 \bar{v}_3 \in H_{31} & \rho x_1^3 x_3^3 \bar{v}_3 \in H_{31} & \rho x_1 x_2^3 x_3^2 \bar{v}_3 \in H_{31} \\
\rho^3 x_2^3 x_4 \bar{v}_3 \in H_{31} & \rho^4 x_2 x_3 x_4 \bar{v}_3 \in H_{32} & \rho^2 x_1^3 x_3 x_4 \bar{v}_3 \in H_{32} \\
\rho^2 x_1 x_2^3 x_4 \bar{v}_3 \in H_{32} & \rho^2 x_1 x_2 x_3^3 \bar{v}_3 \in H_{32} & x_1^4 x_2^2 x_4 \bar{v}_3 \in H_{32} \\
x_1^4 x_3^3 \bar{v}_3 \in H_{32} & x_1^2 x_2^3 x_3^2 \bar{v}_3 \in H_{32} & x_2^6 x_3 \bar{v}_3 \in H_{32} \\
\rho^3 x_1 x_2 x_3 x_4 \bar{v}_3 \in H_{33} & \rho x_1^4 x_3 x_4 \bar{v}_3 \in H_{33} & \rho x_1^2 x_3^2 x_4 \bar{v}_3 \in H_{33} \\
\rho x_1^2 x_2 x_3^3 \bar{v}_3 \in H_{33} & \rho^2 x_1^2 x_2 x_3 x_4 \bar{v}_3 \in H_{34} & \rho^2 x_2^2 x_3^3 \bar{v}_3 \in H_{34} \\
x_1^5 x_3 x_4 \bar{v}_3 \in H_{34} & x_1^3 x_2^3 x_4 \bar{v}_3 \in H_{34} & x_1^3 x_2 x_3^3 \bar{v}_3 \in H_{34} \\
\rho^2 x_2^4 x_4 \bar{v}_3 \in H_{34} & \rho x_1^3 x_2 x_3 x_4 \bar{v}_3 \in H_{35} & \rho x_1 x_2^4 x_4 \bar{v}_3 \in H_{35} \\
\rho x_1 x_2^2 x_3^3 \bar{v}_3 \in H_{35} & \rho^3 x_2^2 x_3 x_4 \bar{v}_3 \in H_{35} & \rho^4 x_2^2 x_4 \bar{v}_3 \in H_{36} \\
x_1^4 x_2 x_3 x_4 \bar{v}_3 \in H_{36} & x_1^2 x_2^4 x_4 \bar{v}_3 \in H_{36} & x_1^2 x_2^2 x_3^3 \bar{v}_3 \in H_{36} \\
\rho^2 x_1 x_2^2 x_3 x_4 \bar{v}_3 \in H_{36} & \rho^3 x_1 x_2^2 x_4 \bar{v}_3 \in H_{37} & \rho x_1^2 x_2^2 x_3 x_4 \bar{v}_3 \in H_{37} \\
\rho x_2^5 x_4 \bar{v}_3 \in H_{37} & \rho x_2^3 x_3^3 \bar{v}_3 \in H_{37} & \rho^2 x_1^2 x_2^2 x_3 \bar{v}_3 \in H_{38} \\
x_1^3 x_2^2 x_3 x_4 \bar{v}_3 \in H_{38} & x_1 x_2^5 x_4 \bar{v}_3 \in H_{38} & x_1 x_2^2 x_3^3 \bar{v}_3 \in H_{38} \\
\rho^2 x_2^3 x_3 x_4 \bar{v}_3 \in H_{38} & \rho^3 x_2 x_3^2 x_4 \bar{v}_3 \in H_{39} & \rho x_1^3 x_2^2 x_4 \bar{v}_3 \in H_{39} \\
\rho x_1 x_2^3 x_3 x_4 \bar{v}_3 \in H_{39} & \rho^2 x_1 x_2 x_3^2 x_4 \bar{v}_3 \in H_{40} & x_1^4 x_2^2 x_4 \bar{v}_3 \in H_{40} \\
x_1^2 x_2^3 x_3 x_4 \bar{v}_3 \in H_{40} & x_2^6 x_4 \bar{v}_3 \in H_{40} & \rho^6 x_1^3 x_2^2 x_3^3 \bar{v}_3^2 \in H_{40} \\
\rho x_1^2 x_2 x_3^2 x_4 \bar{v}_3 \in H_{41} & \rho x_2^4 x_3 x_4 \bar{v}_3 \in H_{41} & x_1^3 x_2 x_3^2 x_4 \bar{v}_3 \in H_{42} \\
x_1 x_2^4 x_3 x_4 \bar{v}_3 \in H_{42} & \rho^2 x_2^2 x_3^2 x_4 \bar{v}_3 \in H_{42} & \rho^3 x_3^3 x_4 \bar{v}_3 \in H_{43} \\
\rho x_1 x_2^2 x_3^2 x_4 \bar{v}_3 \in H_{43} & \rho^2 x_1 x_3^3 x_4 \bar{v}_3 \in H_{44} & x_1^2 x_2^2 x_3^2 x_4 \bar{v}_3 \in H_{44} \\
x_2^5 x_3 x_4 \bar{v}_3 \in H_{44} & \rho^6 x_1^3 x_2^2 x_3^3 \bar{v}_3^2 \in H_{44} & \rho x_1^2 x_3^3 x_4 \bar{v}_3 \in H_{45} \\
\rho x_2^3 x_3^2 x_4 \bar{v}_3 \in H_{45} & \rho^2 x_2 x_3^3 x_4 \bar{v}_3 \in H_{46} & x_1^3 x_3^3 x_4 \bar{v}_3 \in H_{46} \\
x_1 x_2^3 x_3^2 x_4 \bar{v}_3 \in H_{46} & \rho^6 x_1^2 x_2^3 x_3^3 \bar{v}_3^2 \in H_{46} & \rho x_1 x_2 x_3^3 x_4 \bar{v}_3 \in H_{47} \\
\rho^5 x_1^3 x_2^3 x_3^3 \bar{v}_3^2 \in H_{47} & x_1^2 x_2 x_3^3 x_4 \bar{v}_3 \in H_{48} & \rho^6 x_1^3 x_2^3 x_3 x_4 \bar{v}_3^2 \in H_{48} \\
\rho x_2^2 x_3^3 x_4 \bar{v}_3 \in H_{49} & x_1 x_2^2 x_3^3 x_4 \bar{v}_3 \in H_{50} & x_2^3 x_3^3 x_4 \bar{v}_3 \in H_{52}
\end{array}$$

Note that multiplication in M_3 is not square zero, as there are five generators divisible by \bar{v}_3^2 . For example, there is the nontrivial product $H_8 \otimes H_{32} \rightarrow H_{40}$

$$\rho^6 x_1 \bar{v}_3 \cdot x_1^2 x_2^3 x_3^2 \bar{v}_3 = \rho^6 x_1^3 x_2^3 x_3^2 \bar{v}_3^2$$

However, all products in M_3 are determined by the relations in the discussion preceding Proposition 5.17. The coactions follow from Corollary 4.9 and the fact that ρ and \bar{v}_3 are primitive.

For the following, we refer to Fig. 11. Here we use rectangles in a given bidegree if it has rank ≥ 3 .

Theorem 5.22. *The HSSS for $BPGL\langle 3 \rangle$ collapses on E_{16} , and there is an isomorphism of \mathcal{A}_* -comodule algebras (in weight zero)*

$$E_\infty \cong (\mathcal{A}_* \square_{\mathcal{A}(3)_*} M_3) \oplus (\mathcal{A}_* \square_{\mathcal{A}(2)_*} \overline{M}_2)$$

where the latter is a square zero extension with $\overline{M}_3 \cdot \overline{M}_2 = 0$, where \overline{M}_i denotes the augmentation ideal in M_i .

Proof. We run the \bar{v}_2 -Bockstein spectral sequence to calculate E_{16} , using that we have an isomorphism in weight zero

$$E_{16} \cong H_*(E_{15}/(\bar{v}_1, \bar{v}_2^2))$$

The description of the d_{15} 's implies that, on E_{15}

$$\text{image}(d_{15}) \cap (\bar{v}_2) = 0$$

and so this spectral sequence collapses. All of the relations in $H_*(E_{15}/(\bar{v}_2))$ hold in E_{16} again by the differentials given in Theorem 5.16, and the claimed description follows for E_{16} .

It remains to show that the HSSS collapses on E_{16} , and the only classes on E_{16} that are not in the subalgebra of permanent cycles described in Theorem 5.16 lie in the subspace

$$V \cdot E(\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4)$$

where V is the subspace of E_{16} generated by classes divisible by \bar{v}_2 , which lie on the line of red dots (the line $y = 6 - x$) in Fig. 11. It suffices therefore to show that the subspace

$$\bar{v}_2 \cdot E(\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4)$$

of E_2 consists of permanent cycles. Since this is a spectral sequence of \mathcal{A}_* -comodules, the coproduct formulas imply that it suffices to show the class

$$\bar{v}_2 \zeta_1^8 \zeta_2^4 \zeta_3^2 \zeta_4$$

is a permanent cycle. It follows for degree reasons that the only possible differential on this class is a d_{23} with target a sum of monomials of the form

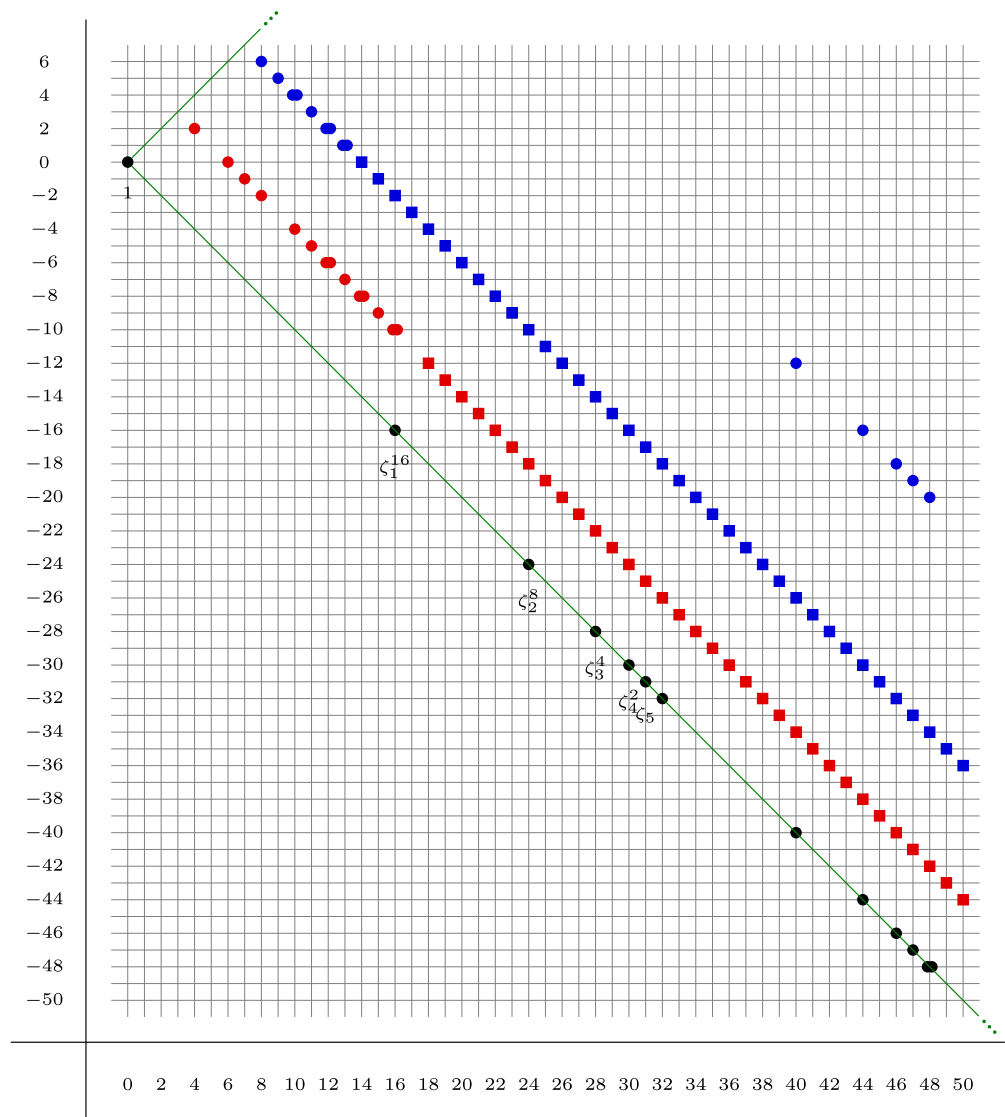


Fig. 11. The E_∞ page of the HSSS for $BPGL\langle 3 \rangle$.

$$p(\zeta_i)\rho^ax_1^b\bar{v}_3^2$$

written as an element of E_2 . It is straightforward to compute the \bar{v}_3^2 part of E_{16} in this weight following closely the computation above, from which a machine computation shows there is no class in the required degree on E_{16} . \square

Remark 5.23. Modulo comodule algebra extension problems, this describes

$$H_*\Gamma(BPGL\langle 3 \rangle)$$

Unlike the $m = 2$ computation, there is room here for nontrivial comodule algebra extensions. We can resolve these case by case using ad hoc restriction and transfer arguments, but we do not include this, as the size of M_3 makes this prohibitively lengthy.

For the same reasons, we do not include a schematic for the comodule M_3 , although all of its structure may be determined from the coactions on the x_m 's described in Corollary 4.9.

6. The HSSS in equivariant homotopy

In the equivariant stable homotopy category $\mathcal{S}p^G$ for a finite group G , Hill–Hopkins–Ravenel [19] constructed a filtration analogous to Voevodsky's slice filtration. Later, Ullman [40] constructed a filtration with better multiplicative properties called the *regular* slice filtration. To fix notions, we work with Ullman's filtration, though for the spectra we consider, the filtrations coincide.

Most of our results on the arithmetic square and the HSSS for $BPGL\langle m \rangle$ apply without change in the equivariant setting for $BP_{\mathbb{R}}\langle m \rangle$ by applying Betti realization and using Corollary 2.20. As in Section 2, we can smash the slice tower of E with any G -spectrum K , obtaining a spectral sequence. The proofs in Section 2 go through in this setting essentially verbatim, so we omit details. Whereas the slice spectral sequence gives a very powerful tool to compute the homotopy groups of the fixed points E^G of a G -spectrum, our spectral sequence computes the *homology* of E^G .

Computations in \mathbb{R} -motivic homotopy and C_2 -equivariant homotopy often differ due to the presence of the negative cone in the equivariant homology of a point. We fully compute the $RO(C_2)$ -graded HSSS for $k_{\mathbb{R}}$ and we find an exotic differential (Corollary 6.9) of the form

$$d_5(2u_{2\sigma}^{-1} \cdot \zeta_1^2 \zeta_2) = \bar{v}_1^2$$

This differential, in particular, originates in the negative cone and lands in the positive cone. We deduce this differential from the following fact.

Proposition 6.1. *The map*

$$\pi_* BP_{\mathbb{R}} \rightarrow \pi_*(i_* H\mathbb{F}_2 \otimes BP_{\mathbb{R}})$$

sends $\bar{v}_m^{2^m} \mapsto 0$.

Proof. By Theorem 4.16, we have a differential

$$d_{2^{m+1}-1}(\zeta_1^{2^m}) = a_{\sigma}^{2^m-1} \bar{v}_m$$

on $E_{2^{m+1}-1}(BP_{\mathbb{R}}; i_* H\mathbb{F}_2)$. There is nothing in higher filtration in this stem, so we have the relation

$$a_{\sigma}^{2^m-1}\bar{v}_m = 0 \in \pi_{\star}(i_{\star}H\mathbb{F}_2 \otimes BP_{\mathbb{R}})$$

By induction, we may assume that $a_{\sigma}^{2^m-i}\bar{v}_m^i = 0$ for $1 \leq i < 2^m$, which implies that the class $a_{\sigma}^{2^m-(i+1)}\bar{v}_m^i$ is in the image of the transfer. Since the restriction of \bar{v}_m is zero, the Frobenius relation implies that $a_{\sigma}^{2^m-(i+1)}\bar{v}_m^{i+1} = 0$. \square

Remark 6.2. Note that the map

$$\pi_{\star}BP_{\mathbb{R}} \rightarrow \pi_{\star}(H\underline{\mathbb{F}}_2 \otimes BP_{\mathbb{R}})$$

sends $\bar{v}_m \mapsto 0$, since the latter is a constant Mackey functor in degrees $\ast\rho$, and $v_m = 0$ in $H_{\ast}(BP; \mathbb{F}_2)$. However, it is not hard to see from the HSSS that

$$\bar{v}_m \neq 0 \in \pi_{\star}(i_{\star}H\mathbb{F}_2 \otimes BP_{\mathbb{R}})$$

It seems likely that 2^m is the minimal nilpotence degree for all m .

The computation of the E_2 -page of the homological slice spectral sequence in the equivariant setting follows in the same way, where now we include the negative cone. We recall Stong's computation of $\pi_{\star}H\underline{\mathbb{Z}}$.

Proposition 6.3. *The $RO(C_2)$ -graded homotopy $\pi_{\star}H\underline{\mathbb{Z}}$ is the square zero extension of the ring $\mathbb{Z}[a_{\sigma}, u_{2\sigma}]/(2a_{\sigma})$ by the module*

$$NC := \mathbb{Z}\{e_{2n\sigma} | n > 0\} \oplus \mathbb{F}_2\{e_{(2n+1)\sigma}/a_{\sigma}^j | n > 0, j \geq 0\}$$

where $e_{k\sigma} = \text{tr}(u_{\sigma}^{-k})$. The module structure on NC is determined by the relations

$$a_{\sigma}e_{k\sigma} = u_{2\sigma}e_{2\sigma} = u_{2\sigma}e_{3\sigma} = 0$$

and

$$u_{2\sigma}e_{k\sigma} = e_{(k-2)\sigma}$$

for $k \geq 4$.

Proposition 6.4. *There is an isomorphism of $RO(C_2)$ -graded rings*

$$\pi_{\star}(i_{\star}H\mathbb{F}_2 \otimes H\underline{\mathbb{Z}}) \cong (\mathcal{A}_{\star} \square_{\mathcal{A}(0)_{\star}} \mathbb{F}_2)[a_{\sigma}, x_1] \oplus NC$$

where the latter ring is a square zero extension of the ring $(\mathcal{A}_{\star} \square_{\mathcal{A}(0)_{\star}} \mathbb{F}_2)[a_{\sigma}, x_1]$ by the module

$$NC := (\mathcal{A}_{\star} \square_{\mathcal{A}(0)_{\star}} \mathbb{F}_2)\{e_{(2i+1)\sigma}/a_{\sigma}^j, x_1 \cdot e_{(2i+1)\sigma}/a_{\sigma}^j : i \geq 1, j \geq 0\}$$

defined by the relation

$$x_1 \cdot (x_1 \cdot e_{(2i+1)\sigma} / a_\sigma^j) = \zeta_1^2 \cdot e_{(2i+1)\sigma} / a_\sigma^{j-2} + e_{(2i-1)\sigma} / a_\sigma^j$$

and the usual relations in $\pi_* H\mathbb{Z}$.

Proof. This follows from Proposition 6.3 and the isomorphisms

$$\begin{aligned} \mathcal{A}_* \{e_{(2i+1)\sigma} / a_\sigma^j\} &\cong (\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2) \{e_{(2i+1)\sigma} / a_\sigma^j\} \oplus \zeta_1 \cdot (\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2) \{e_{(2i+1)\sigma} / a_\sigma^j\} \\ &\cong (\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2) \{e_{(2i+1)\sigma} / a_\sigma^j\} \oplus x_1 \cdot (\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2) \{e_{(2i+1)\sigma} / a_\sigma^{j+1}\} \end{aligned}$$

and

$$(\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2) \{e_{2i\sigma}\} \cong (\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2) \{x_1 \cdot e_{(2i+1)\sigma}\}$$

For the latter, we are using that

$$x_1 \cdot e_{(2i+1)\sigma} = e_{2i\sigma}$$

as follows from the Frobenius relation and that x_1 restricts to u_σ . The relations (and the fact that NC is square zero) then follow from the fact that

$$\pi_* H\mathbb{Z} \rightarrow \pi_*(i_* H\mathbb{F}_2 \otimes H\mathbb{Z})$$

is a ring map, along with Corollary 4.4. \square

Corollary 6.5. *We have an isomorphism*

$$E_2^{*,*}(BP_{\mathbb{R}}\langle m \rangle; i_* H\mathbb{F}_2) \cong \left((\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2)[a_\sigma, x_1] \oplus NC \right) [\bar{v}_1, \dots, \bar{v}_m]$$

An element $p \in (\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2)_i$ has bidegree $(i, -i)$, and we have the following bidegrees

$$\begin{aligned} |\bar{v}_i| &= ((2^i - 1)\rho, 0) \\ |a_\sigma| &= (-\sigma, 1) \\ |x_1| &= (1 - \sigma, 0) \\ |e_{(2i+1)\sigma} / a_\sigma^j| &= -(2i + 1) + (2i + j + 1)\sigma, -j \end{aligned}$$

Proposition 6.6. *In the spectral sequence*

$$E_r^{*,*}(k_{\mathbb{R}}; i_* H\mathbb{F}_2)$$

the subalgebra

$$\left((\mathcal{A}_* \square_{\mathcal{A}(1)_*} \mathbb{F}_2)[a_\sigma, x_1] \oplus NC \right) [\bar{v}_1, \dots, \bar{v}_m]$$

of the E_2 -page consists of permanent cycles.

Proof. This follows from Corollary 3.19, Theorem 4.18, and the fact that the negative cone in $\pi_* H\mathbb{Z}$ consists of permanent cycles in the slice spectral sequence for $BP_{\mathbb{R}}$. \square

By Theorem 4.16, the first differentials we see are

$$\begin{aligned} d_3(\zeta_1^2) &= a_\sigma \bar{v}_1 \\ d_3(\zeta_2) &= x_1 \bar{v}_1 \end{aligned}$$

and d_3 vanishes on the rest of the algebra generators by Proposition 6.6. This gives the following.

Proposition 6.7. $E_4^{*,*}(k_{\mathbb{R}}; i_* H\mathbb{F}_2)$ is the subalgebra of

$$\frac{\left((\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2)[a_\sigma, x_1] \oplus NC \right) [\bar{v}_1]}{(a_\sigma \bar{v}_1, x_1 \bar{v}_1)}$$

generated by

$$\begin{aligned} & \frac{\left((\mathcal{A}_* \square_{\mathcal{A}(1)_*} \mathbb{F}_2)[a_\sigma, x_1] \oplus NC \right) [\bar{v}_1]}{(a_\sigma \bar{v}_1, x_1 \bar{v}_1)} \\ & x_2 := x_1 \zeta_1^2 + a_\sigma \zeta_2 \in \langle a_\sigma, \bar{v}_1, x_1 \rangle \\ & y_n := e_{n\sigma} \zeta_1^2 = \text{tr}(u_\sigma^{-n} \zeta_1^2) \\ & z_m := x_1 \cdot \frac{e_{-3\sigma}}{a_\sigma^m} \zeta_2 \\ & w := e_{2\sigma} \zeta_1^2 \zeta_2 = \text{tr}(u_\sigma^{-2} \zeta_1^2 \zeta_2) \end{aligned}$$

for $n \geq 2$ and $m \geq 0$

The subalgebra

$$\frac{\left((\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2)[a_\sigma, x_1] \oplus NC \right) [\bar{v}_1]}{(a_\sigma \bar{v}_1, x_1 \bar{v}_1)}$$

consists of permanent cycles, x_2 is a permanent cycle by Corollary 3.19, the y_n 's are permanent cycles since they are transfers of permanent cycles, and we show that the z_m 's are permanent cycles for degree reasons.

Proposition 6.8. *The classes z_m are permanent cycles for all $m \geq 0$.*

Proof. For this, note that

$$|z_m| = (1 + (m + 2)\sigma, -3 - m)$$

So that

$$|d_r(z_m)| = ((m + 2)\sigma, -3 - m + r)$$

Using our description of E_4 , and the relations

$$\bar{v}_1 x_2 = \bar{v}_1 y_n = \bar{v}_1 z_m = \bar{v}_1 w = 0$$

the target $d_r(z_m)$ can be written as a sum of monomials of the form

$$m(\zeta_i) \cdot x_1^\epsilon \cdot \frac{e_{(2i+1)\sigma}}{a_\sigma^j}$$

written as elements in a subquotient of E_2 , for $\epsilon = 0, 1$ and $m(\zeta_i)$ a monomial in $\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2$. If $m \in (\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2)_k$, this monomial has bidegree

$$(k, -k) + (-(2i + 1 - \epsilon) + (2i + 1 - \epsilon + j)\sigma, -j)$$

It follows that $k = 2i + 1 - \epsilon$ so that the filtration is $-(2i + 1 - \epsilon + j)$. On the other hand, looking at the σ degree, we have

$$2i + 1 - \epsilon + j = m + 2$$

This is a contradiction, as $r \geq 4$ tells us the filtration must be at least $1 - m$. \square

Corollary 6.9. *The differential d_5 is determined by*

$$d_5(w) = \bar{v}_1^2$$

Proof. By Proposition 6.8, the only algebra generator of E_4 that is not a permanent cycle is w . By Proposition 6.1, there must be a differential killing \bar{v}_1^2 , and for degree reasons, the stated differential is the only possibility. \square

Corollary 6.10. *The spectral sequence $E_r^{*,*}(k_{\mathbb{R}}; i_* H\mathbb{F}_2)$ collapses on E_6 , which is the subalgebra of*

$$\frac{\left((\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2)[a_\sigma, x_1] \oplus NC \right) [\bar{v}_1]}{(a_\sigma \bar{v}_1, x_1 \bar{v}_1, \bar{v}_1^2)}$$

generated by

$$\frac{\left((\mathcal{A}_* \square_{\mathcal{A}(1)_*} \mathbb{F}_2)[a_\sigma, x_1] \oplus NC \right) [\bar{v}_1]}{(a_\sigma \bar{v}_1, x_1 \bar{v}_1, \bar{v}_1^2)}$$

and the classes x_2, y_n, z_m for $n \geq 2$ and $m \geq 0$.

Proof. There are no new cycles because

$$w^2 = a_\sigma w = u_\sigma w = wz_m = wy_n = wx_2 = 0 \in E_6$$

since these relations hold on E_2 . The generators are all permanent cycles, so $E_6 = E_\infty$. \square

Remark 6.11. In this section, we relied heavily on ad hoc degree arguments to control the algebra generators of the $RO(C_2)$ -graded HSSS for $BP_{\mathbb{R}}\langle 1 \rangle$. For larger values of m , it becomes harder to determine when classes coming from the negative cone in the HSSS for $BP_{\mathbb{R}}\langle m \rangle$ are permanent cycles, so it seems that one needs additional input in these cases. However, by Proposition 6.1 there must be some pattern of differentials leaving the negative cone that kill powers of \bar{v}_m .

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