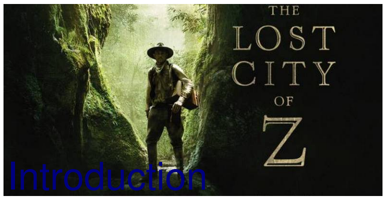
The Lost Telescope of Z

Electronic Computational Homotopy Theory Seminar

March 9, 2017



Doug Ravenel 1.1
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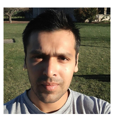
1 Introduction

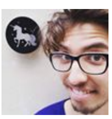
This talk began in discussions last summer with

Agnes Beaudry

Mark Behrens



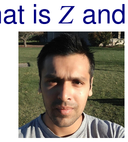
Prasit Bhattacharya 1.2



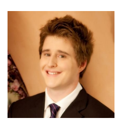
Dominic Culver

Zhouli Xu

What is Z and what is its telescope?



Z is a finite CW spectrum constructed recently by Prasit Bhattacharya and Philip Egger.



It has 32 cells in dimensions ranging from 0 to 16. Mahowald would say it is “half of $A(2)$.”

It admits a self map $\Sigma^6 Z \to Z$ realizing multiplication by v_2. Its telescope is the colimit obtained by iterating this map.

The homotopy of its $K(2)$-localization is very nice.

It could be an interesting test case for the Telescope Conjecture, which says that its telescope and $K(2)$-localization are the same.

Z might have a motivic analog. This could lead to additional structure in its Adams spectral sequence. 1.3

What is the Telescope Conjecture?

I first made the Telescope Conjecture in the late ’70s and published it in 1984

LOCALIZATION WITH RESPECT TO CERTAIN PERIODIC HOMOLOGY THEORIES

By Douglas C. Ravenel*

It has a version for each prime p and each integer $n \geq 0$.

The $n = 1$ case is due to Mahowald for $p = 2$ and to Miller for odd primes.

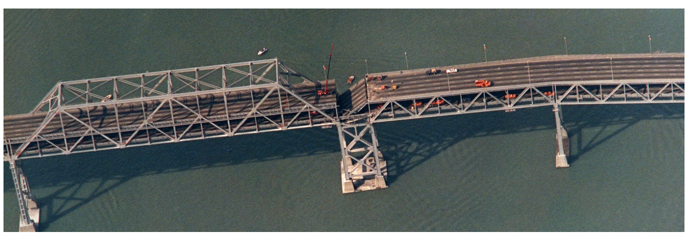
1.4

What is the Telescope Conjecture? (continued)

In 1989 there was a homotopy theory program at MSRI.

MSRI Mathematical Sciences Research Institute

Something happened there that led me to think I could disprove the conjecture for $n \geq 2$.



Earthquake of October 17, 1989

1.5

What is the Telescope Conjecture? (continued)

A few years later the proof fell through.

In 1999 I wrote a paper about it with Mark Mahowald and Paul Shick.

THE TRIPLE LOOP SPACE APPROACH TO THE TELESCOPE CONJECTURE

MARK MAHOWALD, DOUGLAS RAVENEL AND PAUL SHICK

DISCLAIMER: Having bet on both sides of this question, my credibility now stands at ZERO.

1.6

2 The triple loop space approach

The triple loop space approach

Recall that the mod 2 dual Steenrod algebra is

$$A_* = \mathbf{Z}/2[\xi_1, \xi_2, \dots] \qquad \text{with } |\xi_n| = 2^n - 1.$$

Mahowald had a spectrum Y with $H_*Y = \mathbf{Z}/2[\xi_1]/(\xi_1^4)$ or "half" of $A(1)_* = \mathbf{Z}/2[\xi_1, \xi_2]/(\xi_1^4, \xi_2^2)$. It has a self map

$$\Sigma^2 Y \xrightarrow{v_1} Y \longrightarrow C_{v_1} = \text{cofiber}$$

with

$$H_*C_{v_1} = A(1)_* = \mathbf{Z}/2[\xi_1, \xi_2]/(\xi_1^4, \xi_2^2).$$

The Bhattacharya-Egger spectrum Z has

$$H_*Z = \mathbf{Z}/2[\xi_1, \xi_2]/(\xi_1^8, \xi_2^4).$$

and a self map

$$\Sigma^6 Z \xrightarrow{v_2} Z \longrightarrow C_{v_2} = \text{cofiber}$$

with

$$H_*C_{v_2} = \mathbf{Z}/2[\xi_1, \xi_2, \xi_3]/(\xi_1^8, \xi_2^4, \xi_3^2) = A(2)_*.$$

1.7

The triple loop space approach (continued)

The Bhattacharya-Egger spectrum Z has

$$H_*Z = \mathbf{Z}/2[\xi_1, \xi_2]/(\xi_1^8, \xi_2^4).$$

and a self map

$$\Sigma^6 Z \xrightarrow{v_2} Z \longrightarrow C_{v_2} = \text{cofiber}.$$

with

$$H_*C_{v_2} = \mathbf{Z}/2[\xi_1, \xi_2, \xi_3]/(\xi_1^8, \xi_2^4, \xi_3^2) = A(2)_*.$$

In MRS we have spectra $y(n)$ for all $n > 0$ with

$$H_*y(n) = \mathbf{Z}/2[\xi_1, \xi_2, \dots, \xi_n].$$

Unlike Y and Z, it is an associative ring spectrum. 1.8

The triple loop space approach (continued)

In MRS we have associative ring spectra $y(n)$ for all $n > 0$ with

$$H_*y(n) = \mathbf{Z}/2[\xi_1, \xi_2, \dots, \xi_n].$$

It has a self-map

$$\Sigma^{2(2^n-1)} y(n) \xrightarrow{v_n} y(n)$$

inducing an isomorphism in $K(n)_*(-)$, the nth Morava K-theory.

The Telescope Conjecture says that $v_n^{-1}y(n)$, the colimit or telescope obtained by iterating the self map, and $L_{K(n)}y(n)$, the Bousfield localization with respect $K(n)$, are the same.

We have ways to study the homotopy groups of both of them. 1.9

3 The construction of $y(n)$

The construction of $y(n)$

Consider the diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & BO \\ & {}_{i}\searrow \quad \nearrow_{g} & \\ & \Omega^2 S^3 & \end{array}$$

where

- f represents the nontrivial element of $\pi_1 BO = \mathbf{Z}/2$,
- i is the adjoint of the identity map on $\Sigma^2 S^1 = S^3$ and
- g is the extension of f given by the infinite loop space structure on BO.

We know that

$$H_* \Omega^2 S^3 = \mathbf{Z}/2[u_1, u_2, \ldots] \qquad \text{with } |u_n| = 2^n - 1 = |\xi_n|.$$

1.10

The construction of $y(n)$ (continued)

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & BO \\ & {}_{i}\searrow \quad \nearrow_{g} & \\ & \Omega^2 S^3 & \end{array}$$

$$H_* \Omega^2 S^3 = \mathbf{Z}/2[u_1, u_2, \ldots] \qquad \text{with } |u_n| = 2^n - 1 = |\xi_n|.$$

Let $y(\infty)$ denote the Thom spectrum induced by g. Long ago Mahowald showed that it is the mod 2 Eilenberg-Mac Lane spectrum $H\mathbf{Z}/2$.

We will construct subspaces W_n of $\Omega^2 S^3$ with

$$H_* W_n = \mathbf{Z}/2[u_1, u_2, \ldots, u_n],$$

and $y(n)$ will be the corresponding Thom spectrum. 1.11

The construction of $y(n)$ (continued)

In the early 50s Ioan James defined the reduced product $J_k X$ (for any space X) as a certain quotient of $X^{\times k}$ and showed that $J_\infty X$ is equivalent to $\Omega\Sigma X$.

He showed there is a 2-local fiber sequence

$$\Omega^2 S^{2^{n+1}+1} \to J_{2^n-1} S^2 \to \Omega S^3 \to \Omega S^{2^{n+1}+1}.$$

Note that ΩS^3 is equivalent to a CW complex with a single cell in each even dimension. $J_{2^n-1} S^2$ is its $(2^{n+1}-1)$-skeleton.

Our space W_n is $\Omega J_{2^n-1} S^2$, so it maps to $\Omega^2 S^3$ as desired. The MRS spectrum $y(n)$ is the Thomification of

$$\Omega J_{2^n-1} S^2 \longrightarrow \Omega^2 S^3 \xrightarrow{g} BO.$$

1.12

The construction of $y(n)$ (continued)

The MRS spectrum $y(n)$ is the Thomification of

$$\Omega J_{2^n-1}S^2 \longrightarrow \Omega^2 S^3 \xrightarrow{g} BO.$$

From James' 2-local fiber sequence

$$\Omega^3 S^{2^{n+1}+1} \to \Omega J_{2^n-1}S^2 \to \Omega^2 S^3$$

we get maps of spectra

$$\Sigma^\infty S^{|v_n|} \to \Sigma^\infty \Omega^3 S^{2^{n+1}+1} \to y(n) \to H\mathbf{Z}/2.$$

where the map $S^{|v_n|} \to \Omega^3 S^{2^{n+1}+1}$ is the inclusion of the bottom cell. Since $y(n)$ is the Thom spectrum for a loop map, it is an associative ring spectrum. The composite map above leads to the desired v_n-self map of $y(n)$. 1.13

4 The Adams-Novikov spectral sequence for $L_{K(n)}y(n)$

The Adams-Novikov spectral sequence for $L_{K(n)}y(n)$

Let $Y(n)$ denote the telescope associated with $y(n)$. Then we have

$$\begin{aligned}
BP_* &= \mathbf{Z}_{(2)}[v_1, v_2, \ldots] \quad \text{where } |v_i| = 2^{i+1}-2 \\
BP_*(BP) &= BP_*[t_1, t_2, \ldots] \quad \text{where } |t_i| = 2^{i+1}-2 \\
BP_*(y(n)) &= (BP_*/I_n)[t_1, t_2, \ldots t_n] \\
&\qquad \text{where } I_n = (2, v_1, ..v_{n-1}) \\
BP_*(Y(n)) &= BP_*(L_{K(n)}y(n)) = v_n^{-1}BP_*(y(n))
\end{aligned}$$

The Adams-Novikov E_2-term for $L_{K(n)}y(n)$ is

$$E_2 = \mathbf{Z}/2[v_n^{\pm 1}, v_{n+1}, \ldots v_{2n}] \otimes E(h_{n+i,j} \colon 1 \le i \le n, 0 \le j < n)$$

where $h_{n+i,j} = [t_{n+i}^{2^j}]$. The second factor is an exterior algebra on n^2 generators. This E_2-term is finitely generated as a module over the ring

$$R(n) = \mathbf{Z}/2[v_n^{\pm 1}, v_{n+1}, \ldots v_{2n}].$$

1.14

5 The Adams spectral sequences for $y(n)$ and $Y(n)$

The Adams spectral sequences for $y(n)$ and $Y(n)$

Since

$$H_*y(n) = \mathbf{Z}/2[\xi_1, \xi_2, \ldots, \xi_n],$$

a standard change-or-rings argument shows that

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/2, H_*y(n)) \cong \mathrm{Ext}_{A[n]_*}(\mathbf{Z}/2, \mathbf{Z}/2)$$

where

$$A[n]_* = \mathbf{Z}/2[\xi_{n+1}, \xi_{n+2}, \ldots].$$

This leads to an Adams E_1-term of the form

$$E_1 = P(v_n, v_{n+1}, \ldots) \otimes P(h_{n+i,j} \colon i > 0, j \ge 0)$$

where, for such i and j,

$$\begin{aligned} v_{n+i-1} &= [\xi_{n+i}] \in E_1^{1,2^{n+i}-1}, \\ h_{n+i,j} &= [\xi_{n+i}^{2^{j+1}}] \in E_1^{1,2^j(2^{n+i}-1)} \\ \text{and} \qquad d(v_{2n+i}^{2^j}) &= \sum_{0\le k<i} v_{n+k}^{2^j} h_{n+i+j-k,n+k} = v_n^{2^j} h_{n+i+j,n} + \dots \end{aligned}$$

1.15

Localizing the Adams spectral sequence for $y(n)$

The Adams spectral sequence for a spectrum X is based on an Adams resolution, which is a diagram of the form

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \dots$$

with certain properties. When $X = y(2)$, the self map $\Sigma^6 X_i \to X_i$ lifts to X_{i+1}, and we get a diagram

$$\begin{array}{cccccccccccc} & & X_0 & \leftarrow & X_1 & \leftarrow & X_2 & \leftarrow & X_3 & \leftarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & \Sigma^{-6}X_0 \leftarrow & \Sigma^{-6}X_1 & \leftarrow & \Sigma^{-6}X_2 & \leftarrow & \Sigma^{-6}X_3 & \leftarrow & \Sigma^{-6}X_4 & \leftarrow & \dots \\ & \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \Sigma^{-12}X_0 \leftarrow & \Sigma^{-12}X_1 \leftarrow & \Sigma^{-12}X_2 & \leftarrow & \Sigma^{-12}X_3 & \leftarrow & \Sigma^{-12}X_4 & \leftarrow & \Sigma^{-12}X_5 & \leftarrow & \dots \\ \downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots & & \end{array}$$

This leads to a localized Adams spectral sequence converging to the homotopy of

$$Y(n) = v_n^{-1} y(n).$$

1.16

Localizing the Adams spectral sequence for $y(n)$ (continued)

This localization converts

$$E_1 = P(v_n, v_{n+1}, \dots) \otimes P(h_{n+i,j} \colon i > 0, j \ge 0)$$

converging to $\pi_* y(n)$ to

$$E_2 = P(v_n^{\pm 1}, v_{n+1}, \dots, v_{2n}) \otimes P(h_{n+i,j} \colon i > 0, 0 \le j < n)$$

converging to $\pi_* Y(n)$. For $n = 2$ this reads

$$E_2 = P(v_2^{\pm 1}, v_3, v_4) \otimes P(h_{2+i,0}, h_{2+i,1} \colon i > 0).$$

It is likely that for $i > 0$ there are Adams differentials

$$\begin{aligned} d_2 h_{4+i,0} &= v_2 h_{2+i,1}^2 \\ d_4 h_{3+i,1} &= v_2 h_{2+i,0}^4. \end{aligned}$$

1.17

Localizing the Adams spectral sequence for $y(n)$ (continued)

In the localized Adams spectral sequence for $Y(2)$ we have

$$E_2 = P(v_2^{\pm 1}, v_3, v_4) \otimes P(h_{2+i,0}, h_{2+i,1} \colon i > 0).$$

with likely differentials

$$d_2 h_{4+i,0} = v_2 h_{2+i,1}^2 \qquad \text{and} \qquad d_4 h_{3+i,1} = v_2 h_{2+i,0}^4.$$

This would leave

$$E_5 = E_\infty = P(v_2^{\pm 1}, v_3, v_4) \otimes E(h_{3,0}, h_{3,1}, h_{4,0}) \otimes E(b_{3,0}, b_{4,0}, b_{5,0}, \dots)$$

where $b_{i,0} = h_{i,0}^2$. This is infinitely generated over the ring

$$R(2) = P(v_2^{\pm 1}, v_3, v_4)$$

while $\pi_* L_{K(2)} y(2)$ is finitely generated over it. 1.18

6 Disproving the Telescope Conjecture for $n \geq 2$?

Disproving the Telescope Conjecture for $n \geq 2$?

We have just seen that, if all goes according to plan, the Adams-Novikov spectral sequence shows that

$$\pi_* L_{K(2)} y(2) = P(v_2^{\pm 1}, v_3, v_4) \otimes E(h_{3,0}, h_{3,1}, h_{4,0}, h_{4,1})$$

while the localized Adams spectral sequence shows that

$$\pi_* Y(2) = P(v_2^{\pm 1}, v_3, v_4) \otimes E(h_{3,0}, h_{3,1}, h_{4,0}) \otimes E(b_{3,0}, b_{4,0}, b_{5,0}, \dots).$$

There is a similar story for $n > 2$ and for odd primes. The Telescope Conjecture says these two graded groups are the same, so this appears to disprove it.

What could go wrong? We do not have complete control over differentials in the localized Adams spectral sequence. The ones we "know" about could be preempted by others that we don't know about. Mahowald, Shick and I were unable to rule out this possibility. 1.19

7 Going equivariant

Going equivariant

If this approach is to succeed, we need some more structure in the localized Adams spectral sequence for $Y(n)$. Here I will outline a way to get $y(n)$ and $Y(n)$ into a C_2-equivariant setting.

Recall that the construction of $y(n)$ involved the diagram

$$\begin{array}{ccccc} S^1 & \xrightarrow{i} & \Omega^2 S^3 & \xrightarrow{g} & BO \\ & & \uparrow & & \\ & & \Omega J_{2^n-1} S^2 & & \end{array}$$

We can add another space and get

$$\begin{array}{ccccc} S^1 & \xrightarrow{i} & \Omega^2 S^3 & \xrightarrow{g} & BO \\ & & \uparrow & & \uparrow \\ & & \Omega J_{2^n-1} S^2 & \rightarrow & \Omega(SU(k+1)/SO(k+1)) \end{array} \qquad \text{for } k \gg 0.$$

1.20

Going equivariant (continued)

$$\begin{array}{ccccc} S^1 & \xrightarrow{i} & \Omega^2 S^3 & \xrightarrow{g} & BO \\ & & \uparrow & & \uparrow a_k \\ & & \Omega J_{2^n-1} S^2 & \xrightarrow{g_n} & \Omega(SU(k+1)/SO(k+1)). \end{array}$$

The map a_k is related to Bott's proof of his Periodicity Theorem. In mod 2 homology we have

$$H_* BO = \mathbf{Z}/2[b_1, b_2, \dots] \quad \text{where } |b_i| = i,$$
$$H_* \Omega(SU(k+1)/SO(k+1)) = \mathbf{Z}/2[b_1, \dots b_k]$$

and the loop map g_n exists for $k \geq 2^n - 1$. Thomifying the square on the right gives

$$\begin{array}{ccc} H\mathbf{Z}/2 & \longrightarrow & MO \\ \uparrow & & \uparrow \\ y(n) & \longrightarrow & w(k), \end{array}$$

where $w(k)$ is the Thom spectrum induced by the map a_k. 1.21

Going equivariant (continued)

One can show that

$$\begin{array}{ccccc} S^1 & \xrightarrow{i} & \Omega^2 S^3 & \xrightarrow{g} & BO \\ & & \uparrow & & \uparrow a_k \\ & & \Omega J_{2^n-1} S^2 & \xrightarrow{g_n} & \Omega(SU(k+1)/SO(k+1)). \end{array}$$

is the fixed point set of the following diagram of C_2-spaces:

$$\begin{array}{ccccc} S^\rho & \xrightarrow{i} & \Omega^{1+\rho} S^{1+2\rho} & \xrightarrow{g} & BU_{\mathbf{R}} \\ & & \uparrow & & \uparrow a_k \\ & & \Omega^\rho J_{2^n-1} S^{2\rho} & \xrightarrow{g_n} & \Omega^\sigma SU(k+1)_{\mathbf{R}} \end{array}$$

where

- $BU_{\mathbf{R}}$ and $SU_{\mathbf{R}}$ denote the spaces BU and SU equipped with a C_2-action related to complex conjugation,
- σ denotes the sign representation of C_2 and
- $\rho = 1+\sigma$ denotes its regular representation. 1.22

Going equivariant (continued)

Here is our C_2-diagram again.

$$\begin{array}{cccccc} S^\rho & \xrightarrow{i} & \Omega^{1+\rho} S^{1+2\rho} & \xrightarrow{g} & BU_{\mathbf{R}} & \qquad MU_{\mathbf{R}} \\ & & \uparrow & & \uparrow i_k & \qquad \uparrow \\ & & \Omega^\rho J_{2^n-1} S^{2\rho} & \xrightarrow{g_n} & \Omega^\sigma SU(k+1)_{\mathbf{R}} & \qquad X(k)_{\mathbf{R}} \end{array}$$

with Thom spectra indicated on the right. Taking 2-local fibers of the vertical maps in the square gives

$$\begin{array}{ccc} \Omega^{1+\rho} S^{1+2\rho} & \xrightarrow{g} & BU_{\mathbf{R}} \\ \uparrow & & \uparrow a_k \\ \Omega^\rho J_{2^n-1} S^{2\rho} & \xrightarrow{g_n} & \Omega^\sigma SU(k+1)_{\mathbf{R}} \\ \uparrow & & \uparrow \\ \Omega^{2+\rho} S^{1+2^{n+1}\rho} & \longrightarrow & \Omega^\rho (SU/SU(k+1))_{\mathbf{R}} \end{array}$$

The two fibers have the same connectivity when $k = 2^{n+1}-2$. 1.23

Going equivariant (continued)

$$\begin{array}{ccc} \Omega^{1+\rho} S^{1+2\rho} & \xrightarrow{g} & BU_{\mathbf{R}} \\ \uparrow & & \uparrow a_{|v_n|} \\ \Omega^\rho J_{2^n-1} S^{2\rho} & \xrightarrow{g_n} & \Omega^\sigma SU(1+|v_n|)_{\mathbf{R}} \\ \uparrow & & \uparrow \\ \Omega^{2+\rho} S^{1+2^{n+1}\rho} & \longrightarrow & \Omega^\rho (SU/SU(1+|v_n|))_{\mathbf{R}} \end{array}$$

It follows that we have a map $y(n) \to w(|v_n|)$ inducing a monomorphism in mod 2 homology, and therefore maps

$$S^{|v_n|} \to \Omega^3 S^{2^{n+1}+1} \to y(n) \to w(|v_n|),$$

where $w(k)$ is the geometric fixed point set of the Thom spectrum $X(k)_{\mathbf{R}}$. The above composite leads to a telescope $W(|v_n|)$ which is the geometric fixed point spectrum of the telescope for a map

$$\Sigma^{(1+|v_n|)\rho-1} X(|v_n|)_{\mathbf{R}} \to X(|v_n|)_{\mathbf{R}}.$$

The underlying spectrum of this telescope is contractible because the underlying map is known to be nilpotent. 1.24