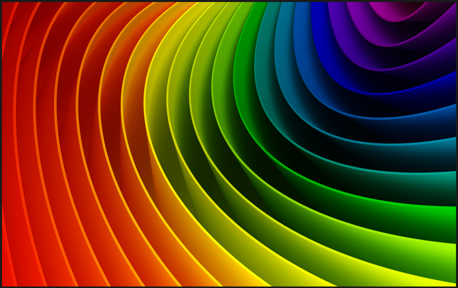

The Chromatic Conjectures in Homotopy Theory

Doug Ravenel
University of Rochester

INTERDISCIPLINARY COLLOQUIUM SERIES
July 21, 2024

Some topology

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Two maps $f_0, f_1 : X \to Y$ are homotopic*, $f_0 \simeq f_1$, if there is a continuous map*

$$h : X \times [0,1] \to Y \qquad \text{with } h(x,t) = f_t(x) \text{ for } t = 0, 1.$$

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Homotopy is an equivalence relation among such maps, and we get a set $[X, Y]$ of homotopy classes of maps from X to Y.

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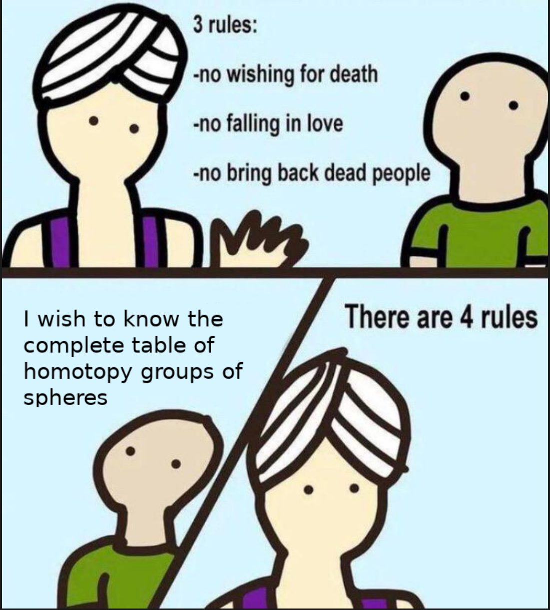

Joke by Yuri Sulyma

Three developments circa 1970

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You can read about it in my first book, Complex cobordism and stable homotopy groups of spheres.

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We know that $K(n)_*V(n-1) \neq 0$ and w_n is a $K(n)$-equivalence. These lead to the construction of the v_n-periodic families aka Greek letter elements

$$\begin{array}{ll} \alpha_t \in \pi_{t|v_1|-1}\mathbb{S} & \text{for } p \geq 3 \\ \beta_t \in \pi_{t|v_2|-2p}\mathbb{S} & \text{for } p \geq 5 \\ \gamma_t \in \pi_{t|v_3|-2p^2-2p+1}\mathbb{S} & \text{for } p \geq 7 \end{array}$$

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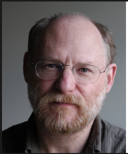
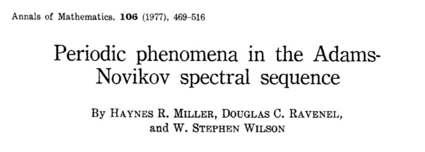
Annals of Mathematics, **106** (1977), 469–516

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By Haynes R. Miller, Douglas C. Ravenel, and W. Stephen Wilson


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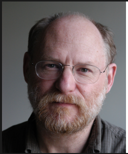

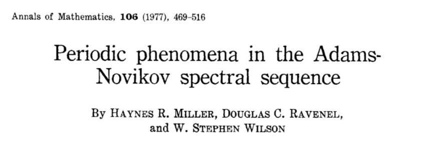

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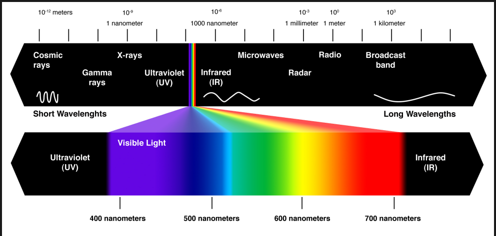

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What would happen if we replace $I_n = (p, \ldots v_{n-1})$ by a smaller invariant regular ideal with n generators, and look for a self map inducing multiplication by some power of v_n?

The chromatic resolution

Recall that $BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, where $|v_n| = 2(p^n - 1)$,

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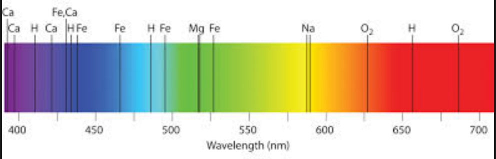

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The chromatic resolution (continued)

$$\begin{tikzcd}
0 \arrow[r] & N^n \arrow[r, "v_n^{-1}"] \arrow[d, equal] & M^n \arrow[r] \arrow[d, equal] & N^{n+1} \arrow[r] \arrow[d, equal] & 0 \\
 & BP_*/(p^\infty, \ldots v_{n-1}^\infty) & v_n^{-1}BP_*/(p^\infty, \ldots v_{n-1}^\infty) & BP_*/(p^\infty, \ldots v_n^\infty) &
\end{tikzcd}$$

The chromatic resolution

$$0 \to BP_* \to M^0 \to M^1 \to M^2 \to M^3 \to \cdots$$

is obtained by splicing together these short exact sequence for all $n \geq 0$.

This construction is purely algebraic. It takes place in the category of $BP_*(BP)$-comodules.

IS THERE A SIMILAR CONSTRUCTION IN THE STABLE HOMOTOPY CATEGORY?

The chromatic resolution (continued)

$$0 \longrightarrow N^n \xrightarrow{v_n^{-1}} M^n \longrightarrow N^{n+1} \longrightarrow 0$$

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IS THERE A SIMILAR CONSTRUCTION, AND THE BEAUTIFUL ALGEBRA THAT GOES ALONG WITH IT, IN THE STABLE HOMOTOPY CATEGORY?

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OR IS IT JUST AN ARTIFACT OF COMPLEX COBORDISM THEORY?

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This question occupied me for several years.

The chromatic resolution (continued)

This paper appeared in 1984.

Doug Ravenel

The chromatic resolution (continued)

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LOCALIZATION WITH RESPECT TO CERTAIN PERIODIC HOMOLOGY THEORIES

By Douglas C. Ravenel*

This paper represents an attempt, only partially successful, to get at what appear to be some deep and hitherto unexamined properties of the stable homotopy category. This work was motivated by the discovery of the pervasive manifestation of various types of periodicity in the E_2-term of the Adams-Novikov spectral sequence converging to the stable homotopy groups of spheres. In section 3 of [34] and section 8 of [41], we introduced the chromatic spectral sequence, which converges to the above E_2-term. Unlike most spectral sequences, its input is in some sense more interesting than its output, as the former displays many appealing patterns which are somewhat hidden in the latter (see section 8 of [41] for a more detailed discussion). It is not so much a computational aid (although it has been used [34] for computing the Novikov 2-line) as a conceptual tool for understanding certain qualitative aspects of the Novikov E_2-term.

Since the Novikov E_2-term is a reasonably good approximation to sta-

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This was easy enough for $n = 0$. We knew then how to invert a prime p homotopically. The resulting N^1 is the Moore spectrum for the group $\mathbb{Q}/\mathbb{Z}_{(p)}$. But how would we invert v_1 to do the next step?

Bousfield localization (continued)

As luck would have it,

Bousfield localization (continued)

As luck would have it, Bousfield localization had just been invented!

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THE LOCALIZATION OF SPECTRA WITH RESPECT TO HOMOLOGY

A. K. Bousfield†

(*Received 3 January* 1979)

Bousfield localization (continued)

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Theorem (Bousfield localization of spectra 1979)

For a given E there is a coaugmented functor L_E such that for any spectrum X, $L_E X$ is E-local and the map $X \to L_E X$ is an E_*-equivalence.

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It turns out that when E and X are both connective, then $L_E X$ can be described in arithmetic terms. It is either obtained from X by inverting some set of primes, or it is the p-adic completion for a single prime p.

Things can be much more interesting when either E or X (or both) fail to be connective.

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It is closely related to the fancier Morava spectrum E_n, but the latter had not been invented yet. It turns out that both lead to the same localization functor.

Bousfield equivalence

Recall that a spectrum Z is E-local if, whenever $f : A \to B$ is an E_*-equivalence, that is a map inducing an isomorphism $E_*A \to E_*B$, then the induced map

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$$\langle E \rangle \vee \langle F \rangle := \langle E \vee F \rangle \text{ and } \langle E \rangle \wedge \langle F \rangle := \langle E \wedge F \rangle.$$

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Bousfield equivalence (continued)

Theorem (Formal properties of Bousfield classes)

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Bousfield equivalence (continued)

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Then $\langle X\rangle = \langle \widehat{X}\rangle \vee \langle C_v\rangle$ and $\langle \widehat{X}\rangle \wedge \langle C_v\rangle = \langle *\rangle$.

Bousfield equivalence (continued)

Theorem (Some Bousfield equivalence classes)

Some Bousfield equivalences.

Bousfield equivalence (continued)

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1

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It turns out that the functor L_n satisfies a stronger condition,

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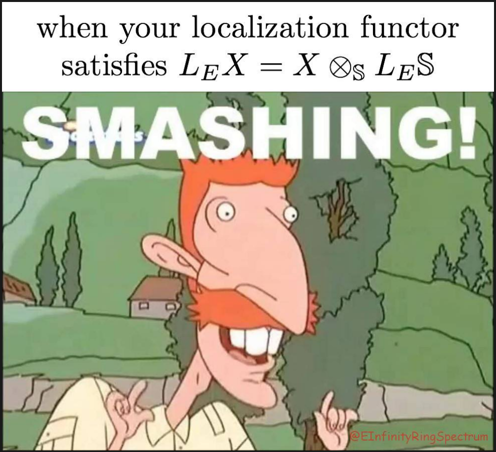

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The Bousfield class of a p-local finite spectrum X is determined by its chromatic type, i.e., the smallest n for which $K(n)_*X \neq 0$. In particular if H_*X is not all torsion, then $\langle X\rangle = \langle \mathbb{S}_{(p)}\rangle$.

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Note that it is independent of the choice of v. Since v is a $K(n)$-equivalence and therefore an $E(n)$-equivalence, we have maps

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This is trivially true for $n = 0$, and for $n = 1$ it was proved around 1980 by Mahowald for $p = 2$ and by Miller for p odd.

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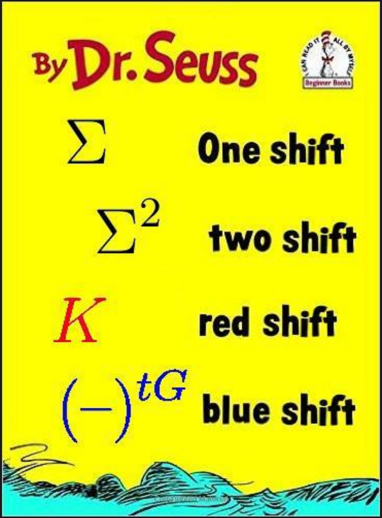
Jeremy, Tomer, myself, Ishan and Robert at Oxford University, June 9, 2023.
Photo by Matteo Barucco.


By Dr. Seuss
Beginner Books
Σ One shift
Σ^2 two shift
K red shift
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THANK YOU!