These are the vogages of the starship *Cofibrant* . . .

with

- its transfinite warp drive
- its small object photon torpedoes
- its adjunction replicator
- its fibrant replacement transporter beam
- and ???

Mike Hill
Mike Hopkins
Doug Ravenel

Mike Hill
Mike Hopkins
Doug Ravenel

Model categories and spectra

University of Chicago

May 15, 2017

Mike Hill
UCLA
Mike Hopkins
Harvard University
Doug Ravenel
University of Rochester

Introduction

The purpose of this talk is to describe a theorem about a cofibrantly generated Quillen model structure on certain categories of spectra.

Mike Hill
Mike Hopkins
Doug Ravenel

Introduction

The purpose of this talk is to describe a theorem about a cofibrantly generated Quillen model structure on certain categories of spectra. It came up in the process of writing a book about equivariant stable homotopy theory.

Mike Hill
Mike Hopkins
Doug Ravenel

Introduction

The purpose of this talk is to describe a theorem about a cofibrantly generated Quillen model structure on certain categories of spectra. It came up in the process of writing a book about equivariant stable homotopy theory.

A spectrum X was originally defined to be a sequence of pointed spaces or simplicial sets $\{X_0, X_1, X_2, \ldots\}$

Mike Hill
Mike Hopkins
Doug Ravenel

Introduction

The purpose of this talk is to describe a theorem about a cofibrantly generated Quillen model structure on certain categories of spectra. It came up in the process of writing a book about equivariant stable homotopy theory.

A spectrum X was originally defined to be a sequence of pointed spaces or simplicial sets $\{X_0, X_1, X_2, \ldots\}$ with structure maps $\epsilon_n^X : \Sigma X_n \to X_{n+1}$.

Mike Hill
Mike Hopkins
Doug Ravenel

Introduction

The purpose of this talk is to describe a theorem about a cofibrantly generated Quillen model structure on certain categories of spectra. It came up in the process of writing a book about equivariant stable homotopy theory.

A spectrum X was originally defined to be a sequence of pointed spaces or simplicial sets $\{X_0, X_1, X_2, \ldots\}$ with structure maps $\epsilon_n^X : \Sigma X_n \to X_{n+1}$. A map of spectra $f : X \to Y$ is a collection of pointed maps $f_n : X_n \to Y_n$ compatible with the structure maps.

Introduction

The purpose of this talk is to describe a theorem about a cofibrantly generated Quillen model structure on certain categories of spectra. It came up in the process of writing a book about equivariant stable homotopy theory.

A spectrum X was originally defined to be a sequence of pointed spaces or simplicial sets $\{X_0, X_1, X_2, \ldots\}$ with structure maps $\epsilon_n^X : \Sigma X_n \to X_{n+1}$. A map of spectra $f : X \to Y$ is a collection of pointed maps $f_n : X_n \to Y_n$ compatible with the structure maps.

There are two different notions of weak equivalence in the category of spectra $\mathcal{S}p$:

Mike Hill
Mike Hopkins
Doug Ravenel

Introduction

The purpose of this talk is to describe a theorem about a cofibrantly generated Quillen model structure on certain categories of spectra. It came up in the process of writing a book about equivariant stable homotopy theory.

A spectrum X was originally defined to be a sequence of pointed spaces or simplicial sets $\{X_0, X_1, X_2, \ldots\}$ with structure maps $\epsilon_n^X : \Sigma X_n \to X_{n+1}$. A map of spectra $f : X \to Y$ is a collection of pointed maps $f_n : X_n \to Y_n$ compatible with the structure maps.

There are two different notions of weak equivalence in the category of spectra $\mathcal{S}p$:

- $f : X \to Y$ is a strict equivalence if each map f_n is a weak equivalence.

Introduction

The purpose of this talk is to describe a theorem about a cofibrantly generated Quillen model structure on certain categories of spectra. It came up in the process of writing a book about equivariant stable homotopy theory.

A spectrum X was originally defined to be a sequence of pointed spaces or simplicial sets $\{X_0, X_1, X_2, \ldots\}$ with structure maps $\epsilon_n^X : \Sigma X_n \to X_{n+1}$. A map of spectra $f : X \to Y$ is a collection of pointed maps $f_n : X_n \to Y_n$ compatible with the structure maps.

There are two different notions of weak equivalence in the category of spectra $\mathcal{S}p$:

- $f : X \to Y$ is a strict equivalence if each map f_n is a weak equivalence.
- $f : X \to Y$ is a stable equivalence if . . .

Mike Hill
Mike Hopkins
Doug Ravenel

Introduction (continued)

There are two different notions of weak equivalence in the category of spectra $\mathcal{S}p$:

- $f : X \to Y$ is a strict equivalence if each map f_n is a weak equivalence.
- $f : X \to Y$ is a stable equivalence if . . .

Mike Hill
Mike Hopkins
Doug Ravenel

Introduction (continued)

There are two different notions of weak equivalence in the category of spectra $\mathcal{S}p$:

- $f : X \to Y$ is a strict equivalence if each map f_n is a weak equivalence.
- $f : X \to Y$ is a stable equivalence if ...

There are at least two different ways to finish the definition of stable equivalence:

Mike Hill
Mike Hopkins
Doug Ravenel

Introduction (continued)

There are two different notions of weak equivalence in the category of spectra $\mathcal{S}p$:

- $f : X \to Y$ is a strict equivalence if each map f_n is a weak equivalence.
- $f : X \to Y$ is a stable equivalence if ...

There are at least two different ways to finish the definition of stable equivalence:

(i) Define stable homotopy groups of spectra

Mike Hill
Mike Hopkins
Doug Ravenel

Introduction (continued)

There are two different notions of weak equivalence in the category of spectra $\mathcal{S}p$:

- $f : X \to Y$ is a strict equivalence if each map f_n is a weak equivalence.
- $f : X \to Y$ is a stable equivalence if ...

There are at least two different ways to finish the definition of stable equivalence:

(i) Define stable homotopy groups of spectra and require $\pi_* f$ to be an isomorphism.

Mike Hill
Mike Hopkins
Doug Ravenel

Introduction (continued)

There are two different notions of weak equivalence in the category of spectra $\mathcal{S}p$:

- $f : X \to Y$ is a strict equivalence if each map f_n is a weak equivalence.
- $f : X \to Y$ is a stable equivalence if ...

There are at least two different ways to finish the definition of stable equivalence:

(i) Define stable homotopy groups of spectra and require $\pi_* f$ to be an isomorphism.

(ii) Define a functor $\Lambda : \mathcal{S}p \to \mathcal{S}p$

Mike Hill
Mike Hopkins
Doug Ravenel

Introduction (continued)

There are two different notions of weak equivalence in the category of spectra $\mathcal{S}p$:

- $f : X \to Y$ is a strict equivalence if each map f_n is a weak equivalence.
- $f : X \to Y$ is a stable equivalence if ...

There are at least two different ways to finish the definition of stable equivalence:

(i) Define stable homotopy groups of spectra and require $\pi_* f$ to be an isomorphism.

(ii) Define a functor $\Lambda : \mathcal{S}p \to \mathcal{S}p$ where $(\Lambda X)_n$ is the colimit of

$$X_n \to \Omega X_{n+1} \to \Omega^2 X_{n+2} \to \dots$$

Introduction (continued)

There are two different notions of weak equivalence in the category of spectra $\mathcal{S}p$:

- $f : X \to Y$ is a strict equivalence if each map f_n is a weak equivalence.
- $f : X \to Y$ is a stable equivalence if ...

There are at least two different ways to finish the definition of stable equivalence:

(i) Define stable homotopy groups of spectra and require $\pi_* f$ to be an isomorphism.

(ii) Define a functor $\Lambda : \mathcal{S}p \to \mathcal{S}p$ where $(\Lambda X)_n$ is the colimit of

$$X_n \to \Omega X_{n+1} \to \Omega^2 X_{n+2} \to \dots$$

and then require Λf to be a strict equivalence.

Mike Hill
Mike Hopkins
Doug Ravenel

Introduction (continued)

There are two different notions of weak equivalence in the category of spectra $\mathcal{S}p$:

- $f : X \to Y$ is a strict equivalence if each map f_n is a weak equivalence.
- $f : X \to Y$ is a stable equivalence if ...

There are at least two different ways to finish the definition of stable equivalence:

(i) Define stable homotopy groups of spectra and require $\pi_* f$ to be an isomorphism.

(ii) Define a functor $\Lambda : \mathcal{S}p \to \mathcal{S}p$ where $(\Lambda X)_n$ is the colimit of

$$X_n \to \Omega X_{n+1} \to \Omega^2 X_{n+2} \to \dots$$

and then require Λf to be a strict equivalence.

Classically these two definitions are equivalent,

Introduction (continued)

There are two different notions of weak equivalence in the category of spectra $\mathcal{S}p$:

- $f : X \to Y$ is a strict equivalence if each map f_n is a weak equivalence.
- $f : X \to Y$ is a stable equivalence if ...

There are at least two different ways to finish the definition of stable equivalence:

(i) Define stable homotopy groups of spectra and require $\pi_* f$ to be an isomorphism.

(ii) Define a functor $\Lambda : \mathcal{S}p \to \mathcal{S}p$ where $(\Lambda X)_n$ is the colimit of

$$X_n \to \Omega X_{n+1} \to \Omega^2 X_{n+2} \to \dots$$

and then require Λf to be a strict equivalence.

Classically these two definitions are equivalent, but in certain variants of the definition of spectra themselves,

Mike Hill
Mike Hopkins
Doug Ravenel

Introduction (continued)

There are two different notions of weak equivalence in the category of spectra $\mathcal{S}p$:

- $f : X \to Y$ is a strict equivalence if each map f_n is a weak equivalence.
- $f : X \to Y$ is a stable equivalence if ...

There are at least two different ways to finish the definition of stable equivalence:

(i) Define stable homotopy groups of spectra and require $\pi_* f$ to be an isomorphism.

(ii) Define a functor $\Lambda : \mathcal{S}p \to \mathcal{S}p$ where $(\Lambda X)_n$ is the colimit of

$$X_n \to \Omega X_{n+1} \to \Omega^2 X_{n+2} \to \dots$$

and then require Λf to be a strict equivalence.

Classically these two definitions are equivalent, but in certain variants of the definition of spectra themselves, they are different.

Introduction (continued)

There are two different notions of weak equivalence in the category of spectra $\mathcal{S}p$:

- $f : X \to Y$ is a strict equivalence if each map f_n is a weak equivalence.
- $f : X \to Y$ is a stable equivalence if ...

There are at least two different ways to finish the definition of stable equivalence:

(i) Define stable homotopy groups of spectra and require $\pi_* f$ to be an isomorphism.

(ii) Define a functor $\Lambda : \mathcal{S}p \to \mathcal{S}p$ where $(\Lambda X)_n$ is the colimit of

$$X_n \to \Omega X_{n+1} \to \Omega^2 X_{n+2} \to \dots$$

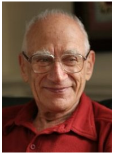
and then require Λf to be a strict equivalence.

Classically these two definitions are equivalent, but in certain variants of the definition of spectra themselves, they are different. They differ in the category $\mathcal{S}p^{\Sigma}$ of symmetric spectra of Hovey-Shipley-Smith.

Mike Hill
Mike Hopkins
Doug Ravenel

Iintroduction (continued)

Dan Quillen
1940-2011



Dan Kan
1928-2013

Pete
Bousfield

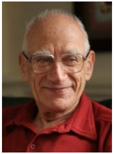
Max Kelly
1930-2007

In order to understand this better we need to discuss

Mike Hill
Mike Hopkins
Doug Ravenel

Iintroduction (continued)


Dan Quillen
1940-2011


Dan Kan
1928-2013


Pete
Bousfield


Max Kelly
1930-2007

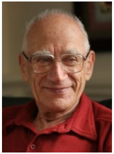
In order to understand this better we need to discuss

- Quillen model categories

Mike Hill
Mike Hopkins
Doug Ravenel

Iintroduction (continued)

Dan Quillen
1940-2011



Dan Kan
1928-2013

Pete
Bousfield

Max Kelly
1930-2007

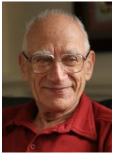
In order to understand this better we need to discuss

- Quillen model categories
- Fibrant and cofibrant replacement

Mike Hill
Mike Hopkins
Doug Ravenel

Iintroduction (continued)

Dan Quillen
1940-2011



Dan Kan
1928-2013

Pete
Bousfield

Max Kelly
1930-2007

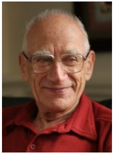
In order to understand this better we need to discuss

- Quillen model categories
- Fibrant and cofibrant replacement
- Cofibrant generation

Mike Hill
Mike Hopkins
Doug Ravenel

Iintroduction (continued)

Dan Quillen
1940-2011



Dan Kan
1928-2013

Pete
Bousfield

Max Kelly
1930-2007

In order to understand this better we need to discuss

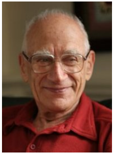
- Quillen model categories
- Fibrant and cofibrant replacement
- Cofibrant generation
- Bousfield localization

Mike Hill
Mike Hopkins
Doug Ravenel

Mike Hill
Mike Hopkins
Doug Ravenel

Iintroduction (continued)

Dan Quillen
1940-2011



Dan Kan
1928-2013

Pete
Bousfield

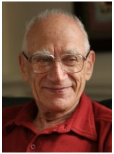
Max Kelly
1930-2007

In order to understand this better we need to discuss

- Quillen model categories
- Fibrant and cofibrant replacement
- Cofibrant generation
- Bousfield localization
- Enriched category theory

Iintroduction (continued)

Dan Quillen
1940-2011



Dan Kan
1928-2013

Pete
Bousfield

Max Kelly
1930-2007

In order to understand this better we need to discuss

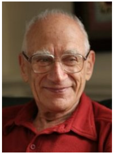
- Quillen model categories
- Fibrant and cofibrant replacement
- Cofibrant generation
- Bousfield localization
- Enriched category theory

We will see that the passage from strict equivalence to stable equivalence is a form of Bousfield localization.

Mike Hill
Mike Hopkins
Doug Ravenel

Iintroduction (continued)

Dan Quillen
1940-2011



Dan Kan
1928-2013

Pete
Bousfield

Max Kelly
1930-2007

In order to understand this better we need to discuss

- Quillen model categories
- Fibrant and cofibrant replacement
- Cofibrant generation
- Bousfield localization
- Enriched category theory

We will see that the passage from strict equivalence to stable equivalence is a form of Bousfield localization. We will give an explicit description of the cofibrant generating sets for the stable category.

Quillen model categories

Definition

A Quillen model category $\mathcal{M}$ *is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations,*

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories

Definition

A Quillen model category $\mathcal{M}$ *is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations, each of which includes all isomorphisms,*

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories

Definition

A Quillen model category $\mathcal{M}$ *is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations, each of which includes all isomorphisms, satisfying the following five axioms:*

Quillen model categories

Definition

A Quillen model category $\mathcal{M}$ *is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations, each of which includes all isomorphisms, satisfying the following five axioms:*

MC1 **Bicompleteness axiom.** $\mathcal{M}$ *has all small limits and colimits.*

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories

Definition

A Quillen model category $\mathcal{M}$ *is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations, each of which includes all isomorphisms, satisfying the following five axioms:*

MC1 **Bicompleteness axiom.** $\mathcal{M}$ *has all small limits and colimits. These include products,*

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories

Definition

A Quillen model category *$\mathcal{M}$ is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations, each of which includes all isomorphisms, satisfying the following five axioms:*

MC1 **Bicompleteness axiom.** *$\mathcal{M}$ has all small limits and colimits. These include products, coproducts,*

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories

Definition

A Quillen model category $\mathcal{M}$ *is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations, each of which includes all isomorphisms, satisfying the following five axioms:*

MC1 **Bicompleteness axiom.** $\mathcal{M}$ *has all small limits and colimits. These include products, coproducts, pullbacks*

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories

Definition

A Quillen model category $\mathcal{M}$ *is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations, each of which includes all isomorphisms, satisfying the following five axioms:*

MC1 **Bicompleteness axiom.** $\mathcal{M}$ *has all small limits and colimits. These include products, coproducts, pullbacks and pushouts.*

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories

Definition

A Quillen model category $\mathcal{M}$ *is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations, each of which includes all isomorphisms, satisfying the following five axioms:*

MC1 **Bicompleteness axiom.** $\mathcal{M}$ *has all small limits and colimits. These include products, coproducts, pullbacks and pushouts. This implies that* $\mathcal{M}$ *has initial and terminal objects.*

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories

Definition

A Quillen model category $\mathcal{M}$ *is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations, each of which includes all isomorphisms, satisfying the following five axioms:*

MC1 Bicompleteness axiom. $\mathcal{M}$ *has all small limits and colimits. These include products, coproducts, pullbacks and pushouts. This implies that* $\mathcal{M}$ *has initial and terminal objects.*

MC2 2-out-of-3 axiom. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be morphisms in* $\mathcal{M}$*.*

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories

Definition

A Quillen model category $\mathcal{M}$ *is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations, each of which includes all isomorphisms, satisfying the following five axioms:*

MC1 **Bicompleteness axiom.** $\mathcal{M}$ *has all small limits and colimits. These include products, coproducts, pullbacks and pushouts. This implies that* $\mathcal{M}$ *has initial and terminal objects.*

MC2 **2-out-of-3 axiom.** *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be morphisms in* $\mathcal{M}$*. Then if any two of* f*,* g *and* gf *are weak equivalences, so is the third.*

Quillen model categories

Definition

A Quillen model category $\mathcal{M}$ *is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations, each of which includes all isomorphisms, satisfying the following five axioms:*

MC1 **Bicompleteness axiom.** *$\mathcal{M}$ has all small limits and colimits. These include products, coproducts, pullbacks and pushouts. This implies that $\mathcal{M}$ has initial and terminal objects.*

MC2 **2-out-of-3 axiom.** *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in $\mathcal{M}$. Then if any two of f, g and gf are weak equivalences, so is the third.*

MC3 **Retract axiom.** *A retract of a weak equivalence, fibration or cofibration*

Quillen model categories

Definition

A Quillen model category $\mathcal{M}$ *is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations, each of which includes all isomorphisms, satisfying the following five axioms:*

MC1 **Bicompleteness axiom.** $\mathcal{M}$ *has all small limits and colimits. These include products, coproducts, pullbacks and pushouts. This implies that* $\mathcal{M}$ *has initial and terminal objects.*

MC2 **2-out-of-3 axiom.** *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be morphisms in* $\mathcal{M}$*. Then if any two of* f*,* g *and* gf *are weak equivalences, so is the third.*

MC3 **Retract axiom.** *A retract of a weak equivalence, fibration or cofibration is again a weak equivalence, fibration or cofibration.*

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories

Definition

A Quillen model category $\mathcal{M}$ *is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations, each of which includes all isomorphisms, satisfying the following five axioms:*

MC1 **Bicompleteness axiom.** $\mathcal{M}$ *has all small limits and colimits. These include products, coproducts, pullbacks and pushouts. This implies that* $\mathcal{M}$ *has initial and terminal objects.*

MC2 **2-out-of-3 axiom.** *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be morphisms in* $\mathcal{M}$. *Then if any two of* f, g *and* gf *are weak equivalences, so is the third.*

MC3 **Retract axiom.** *A retract of a weak equivalence, fibration or cofibration is again a weak equivalence, fibration or cofibration.*

We say that a fibration or cofibration is trivial (or acyclic)

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories

Definition

A Quillen model category *$\mathcal{M}$ is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations, each of which includes all isomorphisms, satisfying the following five axioms:*

MC1 **Bicompleteness axiom.** *$\mathcal{M}$ has all small limits and colimits. These include products, coproducts, pullbacks and pushouts. This implies that $\mathcal{M}$ has initial and terminal objects.*

MC2 **2-out-of-3 axiom.** *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in $\mathcal{M}$. Then if any two of f, g and gf are weak equivalences, so is the third.*

MC3 **Retract axiom.** *A retract of a weak equivalence, fibration or cofibration is again a weak equivalence, fibration or cofibration.*

We say that a fibration or cofibration is trivial (or acyclic) if it is also a weak equivalence.

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories (continued)

Definition

MC4 **Lifting axiom.** *Given a commutative diagram*

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories (continued)

Definition

MC4 **Lifting axiom.** *Given a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y, \end{array}$$

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories (continued)

Definition

MC4 **Lifting axiom.** *Given a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle \text{cofibration}}\; i \big\downarrow & \overset{h}{\nearrow} & \big\downarrow p \;{\scriptstyle \text{trivial fibration}} \\ B & \xrightarrow[g]{} & Y, \end{array}$$

a morphism h exists for i and p as indicated.

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories (continued)

Definition

MC4 **Lifting axiom.** *Given a commutative diagram*

$$\begin{array}{ccccc} & A & \xrightarrow{\ f\ } & X & \\ \textit{trivial cofibration} & {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} & \textit{fibration} \\ & B & \xrightarrow[\ g\ ]{} & Y, & \end{array}$$

a morphism h exists for i and p as indicated.

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories (continued)

Definition

MC4 **Lifting axiom.** *Given a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y, \end{array}$$

cofibration — trivial fibration

trivial cofibration — fibration

a morphism h exists for i and p as indicated.

Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories (continued)

Definition

MC4 **Lifting axiom.** *Given a commutative diagram*

$$\begin{array}{ccccc} & A & \xrightarrow{f} & X & \\ \color{blue}{\text{cofibration}} & i \downarrow & \overset{h}{\nearrow} & \downarrow p & \color{blue}{\text{trivial fibration}} \\ \color{red}{\text{trivial cofibration}} & B & \xrightarrow[g]{} & Y, & \color{red}{\text{fibration}} \end{array}$$

a morphism h exists for i and p as indicated.

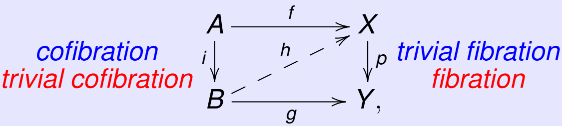
MC5 **Factorization axiom.** *Any morphism $f : X \to Y$ can be functorially factored in two ways as*

$$X \xrightarrow{f} Y$$

Quillen model categories (continued)

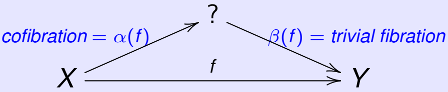
Definition

MC4 Lifting axiom. *Given a commutative diagram*



a morphism h exists for i and p as indicated.

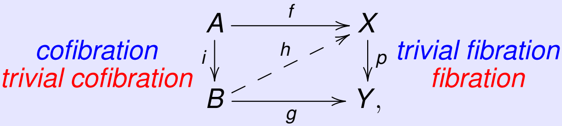
MC5 Factorization axiom. *Any morphism $f : X \to Y$ can be functorially factored in two ways as*



Quillen model categories (continued)

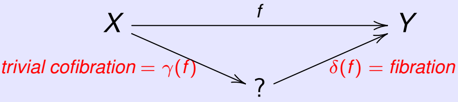
Definition

MC4 Lifting axiom. *Given a commutative diagram*



a morphism h exists for i and p as indicated.

MC5 Factorization axiom. *Any morphism $f : X \to Y$ can be functorially factored in two ways as*

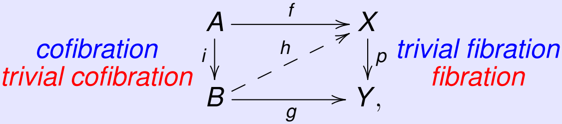


Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories (continued)

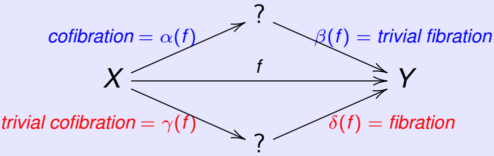
Definition

MC4 Lifting axiom. *Given a commutative diagram*



a morphism h exists for i and p as indicated.

MC5 Factorization axiom. *Any morphism $f : X \to Y$ can be functorially factored in two ways as*

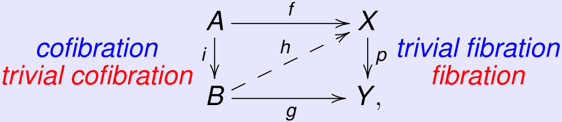


Mike Hill
Mike Hopkins
Doug Ravenel

Quillen model categories (continued)

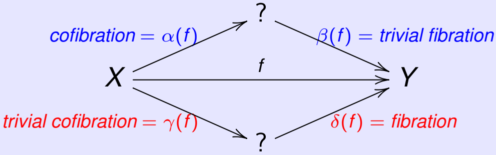
Definition

MC4 Lifting axiom. *Given a commutative diagram*



a morphism h exists for i and p as indicated.

MC5 Factorization axiom. *Any morphism* $f : X \to Y$ *can be functorially factored in two ways as*



This is the hardest axiom to verify in practice.

Mike Hill
Mike Hopkins
Doug Ravenel

Some examples

A toy example. The category $\mathcal{S}et$ of sets

Mike Hill
Mike Hopkins
Doug Ravenel

Some examples

A toy example. The category $\mathcal{S}et$ of sets with bijections as weak equivalences and

Mike Hill
Mike Hopkins
Doug Ravenel

Some examples

A toy example. The category $\mathcal{S}et$ of sets with bijections as weak equivalences and all morphisms as fibrations and cofibrations

Mike Hill
Mike Hopkins
Doug Ravenel

Some examples

A toy example. The category $\mathcal{S}et$ of sets with bijections as weak equivalences and all morphisms as fibrations and cofibrations satisfies Quillen's axioms.

Mike Hill
Mike Hopkins
Doug Ravenel

Some examples

A toy example. The category $\mathcal{S}et$ of sets with bijections as weak equivalences and all morphisms as fibrations and cofibrations satisfies Quillen's axioms.

A classical example. Let $\mathcal{T}op$ denote the category of (compactly generated weak Hausdorff) topological spaces.

Mike Hill
Mike Hopkins
Doug Ravenel

Some examples

A toy example. The category $\mathcal{S}et$ of sets with bijections as weak equivalences and all morphisms as fibrations and cofibrations satisfies Quillen's axioms.

A classical example. Let $\mathcal{T}op$ denote the category of (compactly generated weak Hausdorff) topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups.

Mike Hill
Mike Hopkins
Doug Ravenel

Some examples

A toy example. The category $\mathcal{S}et$ of sets with bijections as weak equivalences and all morphisms as fibrations and cofibrations satisfies Quillen's axioms.

A classical example. Let $\mathcal{T}op$ denote the category of (compactly generated weak Hausdorff) topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups. Fibrations are Serre fibrations,

Mike Hill
Mike Hopkins
Doug Ravenel

Some examples

A toy example. The category $\mathcal{S}et$ of sets with bijections as weak equivalences and all morphisms as fibrations and cofibrations satisfies Quillen's axioms.

A classical example. Let $\mathcal{T}op$ denote the category of (compactly generated weak Hausdorff) topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups. Fibrations are Serre fibrations, that is is maps $p : X \to Y$ with the right lifting property

$$\begin{array}{ccc} I^n & \xrightarrow{f} & X \\ {\scriptstyle j_n}\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle p} \\ I^{n+1} & \xrightarrow[g]{} & Y, \end{array} \qquad \text{for each } n \geq 0.$$

Some examples

A toy example. The category $\mathcal{S}et$ of sets with bijections as weak equivalences and all morphisms as fibrations and cofibrations satisfies Quillen's axioms.

A classical example. Let $\mathcal{T}op$ denote the category of (compactly generated weak Hausdorff) topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups. Fibrations are Serre fibrations, that is is maps $p : X \to Y$ with the right lifting property

$$\begin{array}{ccc} I^n & \xrightarrow{\quad f \quad} & X \\ {\scriptstyle j_n}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ I^{n+1} & \xrightarrow[\quad g \quad]{} & Y, \end{array} \qquad \text{for each } n \geq 0.$$

Cofibrations are maps (such as $i_n : S^{n-1} \to D^n$ for $n \geq 0$) having the left lifting property with respect to all trivial Serre fibrations.

Mike Hill
Mike Hopkins
Doug Ravenel

Some definitions

We will denote the initial and terminal objects of $\mathcal{M}$ by $\emptyset$ and $*$. When they are the same, we say that $\mathcal{M}$ is pointed.

Mike Hill
Mike Hopkins
Doug Ravenel

Some definitions

We will denote the initial and terminal objects of $\mathcal{M}$ by $\emptyset$ and $*$. When they are the same, we say that $\mathcal{M}$ is pointed.

Definition

An object X is cofibrant if the unique map $\emptyset \to X$ is a cofibration. It X is fibrant if the unique map $X \to *$ is a fibration.

Mike Hill
Mike Hopkins
Doug Ravenel

Some definitions

We will denote the initial and terminal objects of $\mathcal{M}$ by $\emptyset$ and $*$. When they are the same, we say that $\mathcal{M}$ is pointed.

Definition

An object X is cofibrant if the unique map $\emptyset \to X$ is a cofibration. It X is fibrant if the unique map $X \to *$ is a fibration.

All objects in $\mathcal{T}$ and $\mathcal{T}op$ are fibrant.

Mike Hill
Mike Hopkins
Doug Ravenel

Some definitions

We will denote the initial and terminal objects of $\mathcal{M}$ by $\emptyset$ and $*$. When they are the same, we say that $\mathcal{M}$ is pointed.

Definition

An object X is cofibrant if the unique map $\emptyset \to X$ is a cofibration. It X is fibrant if the unique map $X \to *$ is a fibration.

All objects in $\mathcal{T}$ and $\mathcal{T}op$ are fibrant. The cofibrant objects are the CW-complexes.

Mike Hill
Mike Hopkins
Doug Ravenel

Some definitions

We will denote the initial and terminal objects of $\mathcal{M}$ by $\emptyset$ and $*$. When they are the same, we say that $\mathcal{M}$ is pointed.

Definition

An object X is cofibrant if the unique map $\emptyset \to X$ is a cofibration. It X is fibrant if the unique map $X \to *$ is a fibration.

All objects in $\mathcal{T}$ and $\mathcal{T}op$ are fibrant. The cofibrant objects are the CW-complexes.

By **MC5**, for any object X, the unique maps $\emptyset \to X$ and $X \to *$ have factorizations

$$\emptyset \to QX \to X \qquad \text{and} \qquad X \to RX \to *$$

Mike Hill
Mike Hopkins
Doug Ravenel

Some definitions

We will denote the initial and terminal objects of $\mathcal{M}$ by $\emptyset$ and $*$. When they are the same, we say that $\mathcal{M}$ is pointed.

Definition

An object X is cofibrant if the unique map $\emptyset \to X$ is a cofibration. It X is fibrant if the unique map $X \to *$ is a fibration.

All objects in $\mathcal{T}$ and $\mathcal{T}op$ are fibrant. The cofibrant objects are the CW-complexes.

By **MC5**, for any object X, the unique maps $\emptyset \to X$ and $X \to *$ have factorizations

$$\emptyset \to QX \to X \qquad \text{and} \qquad X \to RX \to *$$

where QX is a cofibrant object weakly equivalent to X,

Mike Hill
Mike Hopkins
Doug Ravenel

Some definitions

We will denote the initial and terminal objects of $\mathcal{M}$ by $\emptyset$ and $*$. When they are the same, we say that $\mathcal{M}$ is pointed.

Definition

An object X is cofibrant if the unique map $\emptyset \to X$ is a cofibration. It X is fibrant if the unique map $X \to *$ is a fibration.

All objects in $\mathcal{T}$ and $\mathcal{T}op$ are fibrant. The cofibrant objects are the CW-complexes.

By **MC5**, for any object X, the unique maps $\emptyset \to X$ and $X \to *$ have factorizations

$$\emptyset \to QX \to X \qquad \text{and} \qquad X \to RX \to *$$

where QX is a cofibrant object weakly equivalent to X, and RX is a fibrant object weakly equivalent to X.

Mike Hill
Mike Hopkins
Doug Ravenel

Some definitions (continued)

By **MC5**, for any object X, the unique maps $\emptyset \to X$ and $X \to *$ have factorizations

$$\emptyset \to QX \to X \qquad \text{and} \qquad X \to RX \to *$$

where QX is a cofibrant object weakly equivalent to X, and RX is a fibrant object weakly equivalent to X.

Mike Hill
Mike Hopkins
Doug Ravenel

Some definitions (continued)

By **MC5**, for any object X, the unique maps $\emptyset \to X$ and $X \to *$ have factorizations

$$\emptyset \to QX \to X \qquad \text{and} \qquad X \to RX \to *$$

where QX is a cofibrant object weakly equivalent to X, and RX is a fibrant object weakly equivalent to X.

These maps to and from X are called cofibrant and fibrant approximations.

Mike Hill
Mike Hopkins
Doug Ravenel

Some definitions (continued)

By **MC5**, for any object X, the unique maps $\emptyset \to X$ and $X \to *$ have factorizations

$$\emptyset \to QX \to X \qquad \text{and} \qquad X \to RX \to *$$

where QX is a cofibrant object weakly equivalent to X, and RX is a fibrant object weakly equivalent to X.

These maps to and from X are called cofibrant and fibrant approximations. The objects QX and RX are called cofibrant and fibrant replacements of X.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generation

Example

In $\mathcal{T}op$, let

$$\mathcal{I} = \{i_n : S^{n-1} \to D^n, n \geq 0\} \text{ and } \mathcal{J} = \{j_n : I^n \to I^{n+1}, n \geq 0\}.$$

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generation

Example

In $\mathcal{T}op$, let

$$\mathcal{I} = \{i_n : S^{n-1} \to D^n, n \geq 0\} \text{ and } \mathcal{J} = \{j_n : I^n \to I^{n+1}, n \geq 0\}.$$

It is known that every (trivial) cofibration in $\mathcal{T}op$ can be derived from the ones in ($\mathcal{J}$) $\mathcal{I}$ by iterating certain elementary constructions.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generation

Example

In $\mathcal{T}op$, let

$$\mathcal{I} = \{i_n : S^{n-1} \to D^n, n \geq 0\} \text{ and } \mathcal{J} = \{j_n : I^n \to I^{n+1}, n \geq 0\}.$$

It is known that every (trivial) cofibration in $\mathcal{T}op$ can be derived from the ones in ($\mathcal{J}$) $\mathcal{I}$ by iterating certain elementary constructions. A map is a (trivial) fibration iff it has the right lifting property with respect to each map in ($\mathcal{I}$) $\mathcal{J}$.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generation

Example

In $\mathcal{T}op$, let

$$\mathcal{I} = \{ i_n : S^{n-1} \to D^n, n \geq 0 \} \text{ and } \mathcal{J} = \{ j_n : I^n \to I^{n+1}, n \geq 0 \}.$$

It is known that every (trivial) cofibration in $\mathcal{T}op$ can be derived from the ones in ($\mathcal{J}$) $\mathcal{I}$ by iterating certain elementary constructions. A map is a (trivial) fibration iff it has the right lifting property with respect to each map in ($\mathcal{I}$) $\mathcal{J}$.

Definition

A cofibrantly generated model category $\mathcal{M}$

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generation

Example

In $\mathcal{T}op$, let

$$\mathcal{I} = \left\{ i_n : S^{n-1} \to D^n, n \geq 0 \right\} \text{ and } \mathcal{J} = \left\{ j_n : I^n \to I^{n+1}, n \geq 0 \right\}.$$

It is known that every (trivial) cofibration in $\mathcal{T}op$ can be derived from the ones in ($\mathcal{J}$) $\mathcal{I}$ by iterating certain elementary constructions. A map is a (trivial) fibration iff it has the right lifting property with respect to each map in ($\mathcal{I}$) $\mathcal{J}$.

Definition

A cofibrantly generated model category $\mathcal{M}$ is one with morphism sets $\mathcal{I}$ and $\mathcal{J}$ having properties as above.

Cofibrant generation

Example

In $\mathcal{T}op$, let

$$\mathcal{I} = \left\{ i_n : S^{n-1} \to D^n, n \geq 0 \right\} \text{ and } \mathcal{J} = \left\{ j_n : I^n \to I^{n+1}, n \geq 0 \right\}.$$

It is known that every (trivial) cofibration in $\mathcal{T}op$ can be derived from the ones in ($\mathcal{J}$) $\mathcal{I}$ by iterating certain elementary constructions. A map is a (trivial) fibration iff it has the right lifting property with respect to each map in ($\mathcal{I}$) $\mathcal{J}$.

Definition

A ***cofibrantly generated model category*** *$\mathcal{M}$ is one with morphism sets $\mathcal{I}$ and $\mathcal{J}$ having properties as above. $\mathcal{I}$ ($\mathcal{J}$) is a* ***generating set of (trivial) cofibrations****.*

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generation

Example

In $\mathcal{T}op$, let

$$\mathcal{I} = \{i_n : S^{n-1} \to D^n, n \geq 0\} \text{ and } \mathcal{J} = \{j_n : I^n \to I^{n+1}, n \geq 0\}.$$

It is known that every (trivial) cofibration in $\mathcal{T}op$ can be derived from the ones in ($\mathcal{J}$) $\mathcal{I}$ by iterating certain elementary constructions. A map is a (trivial) fibration iff it has the right lifting property with respect to each map in ($\mathcal{I}$) $\mathcal{J}$.

Definition

A cofibrantly generated model category $\mathcal{M}$ is one with morphism sets $\mathcal{I}$ and $\mathcal{J}$ having properties as above. $\mathcal{I}$ ($\mathcal{J}$) is a generating set of (trivial) cofibrations.

In practice, defining weak equivalences and specifying generating sets $\mathcal{I}$ and $\mathcal{J}$ is the most convenient way to describe a model category.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generation (continued)

Definition

A *cofibrantly generated model category* $\mathcal{M}$ *is one with morphism sets* $\mathcal{I}$ *and* $\mathcal{J}$ *having similar properties to the ones in* $\mathcal{T}op$. $\mathcal{I}$ ($\mathcal{J}$) *is a* *generating set of (trivial) cofibrations*.

In practice, specifying the generating sets $\mathcal{I}$ and $\mathcal{J}$, and defining weak equivalences is the most convenient way to describe a model category.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generation (continued)

Definition

A cofibrantly generated model category $\mathcal{M}$ is one with morphism sets $\mathcal{I}$ and $\mathcal{J}$ having similar properties to the ones in $\mathcal{T}op$. $\mathcal{I}$ ($\mathcal{J}$) is a generating set of (trivial) cofibrations.

In practice, specifying the generating sets $\mathcal{I}$ and $\mathcal{J}$, and defining weak equivalences is the most convenient way to describe a model category.

The Kan Recognition Theorem gives four necessary and sufficient conditions for morphism sets $\mathcal{I}$ and $\mathcal{J}$ to be generating sets as above,

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generation (continued)

Definition

A *cofibrantly generated model category* $\mathcal{M}$ *is one with morphism sets* $\mathcal{I}$ *and* $\mathcal{J}$ *having similar properties to the ones in* $\mathcal{T}op$. $\mathcal{I}$ ($\mathcal{J}$) *is a* *generating set of (trivial) cofibrations*.

In practice, specifying the generating sets $\mathcal{I}$ and $\mathcal{J}$, and defining weak equivalences is the most convenient way to describe a model category.

The Kan Recognition Theorem gives four necessary and sufficient conditions for morphism sets $\mathcal{I}$ and $\mathcal{J}$ to be generating sets as above, assuming that weak equivalences have already been defined.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generation (continued)

Definition

A *cofibrantly generated model category* $\mathcal{M}$ *is one with morphism sets* $\mathcal{I}$ *and* $\mathcal{J}$ *having similar properties to the ones in* $\mathcal{T}op$*.* $\mathcal{I}$ *(*$\mathcal{J}$*) is a* *generating set of (trivial) cofibrations.*

In practice, specifying the generating sets $\mathcal{I}$ and $\mathcal{J}$, and defining weak equivalences is the most convenient way to describe a model category.

The Kan Recognition Theorem gives four necessary and sufficient conditions for morphism sets $\mathcal{I}$ and $\mathcal{J}$ to be generating sets as above, assuming that weak equivalences have already been defined. They are too technical for this talk.

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Suppose we have a model category $\mathcal{M}$, and we wish to change the model structure

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Suppose we have a model category $\mathcal{M}$, and we wish to change the model structure (without altering the underlying category)

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Suppose we have a model category $\mathcal{M}$, and we wish to change the model structure (without altering the underlying category) as follows.

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Suppose we have a model category $\mathcal{M}$, and we wish to change the model structure (without altering the underlying category) as follows.

- Enlarge the class of weak equivalences in some way.

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Suppose we have a model category $\mathcal{M}$, and we wish to change the model structure (without altering the underlying category) as follows.

- Enlarge the class of weak equivalences in some way.
- Keep the same class of cofibrations as before.

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Suppose we have a model category $\mathcal{M}$, and we wish to change the model structure (without altering the underlying category) as follows.

- Enlarge the class of weak equivalences in some way.
- Keep the same class of cofibrations as before.
- Define fibrations in terms of right lifting properties with respect to the newly defined trivial cofibrations.

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Suppose we have a model category $\mathcal{M}$, and we wish to change the model structure (without altering the underlying category) as follows.

- Enlarge the class of weak equivalences in some way.
- Keep the same class of cofibrations as before.
- Define fibrations in terms of right lifting properties with respect to the newly defined trivial cofibrations. The class of trivial fibrations remains unaltered.

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Suppose we have a model category $\mathcal{M}$, and we wish to change the model structure (without altering the underlying category) as follows.

- Enlarge the class of weak equivalences in some way.
- Keep the same class of cofibrations as before.
- Define fibrations in terms of right lifting properties with respect to the newly defined trivial cofibrations. The class of trivial fibrations remains unaltered.

Since there are more weak equivalences,

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Suppose we have a model category $\mathcal{M}$, and we wish to change the model structure (without altering the underlying category) as follows.

- Enlarge the class of weak equivalences in some way.
- Keep the same class of cofibrations as before.
- Define fibrations in terms of right lifting properties with respect to the newly defined trivial cofibrations. The class of trivial fibrations remains unaltered.

Since there are more weak equivalences, there are more trivial cofibrations.

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Suppose we have a model category $\mathcal{M}$, and we wish to change the model structure (without altering the underlying category) as follows.

- Enlarge the class of weak equivalences in some way.
- Keep the same class of cofibrations as before.
- Define fibrations in terms of right lifting properties with respect to the newly defined trivial cofibrations. The class of trivial fibrations remains unaltered.

Since there are more weak equivalences, there are more trivial cofibrations. Hence there are fewer fibrations

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Suppose we have a model category $\mathcal{M}$, and we wish to change the model structure (without altering the underlying category) as follows.

- Enlarge the class of weak equivalences in some way.
- Keep the same class of cofibrations as before.
- Define fibrations in terms of right lifting properties with respect to the newly defined trivial cofibrations. The class of trivial fibrations remains unaltered.

Since there are more weak equivalences, there are more trivial cofibrations. Hence there are fewer fibrations and fewer fibrant objects.

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Suppose we have a model category $\mathcal{M}$, and we wish to change the model structure (without altering the underlying category) as follows.

- Enlarge the class of weak equivalences in some way.
- Keep the same class of cofibrations as before.
- Define fibrations in terms of right lifting properties with respect to the newly defined trivial cofibrations. The class of trivial fibrations remains unaltered.

Since there are more weak equivalences, there are more trivial cofibrations. Hence there are fewer fibrations and fewer fibrant objects. This could make the fibrant replacement functor much more interesting.

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Suppose we have a model category $\mathcal{M}$, and we wish to change the model structure (without altering the underlying category) as follows.

- Enlarge the class of weak equivalences in some way.
- Keep the same class of cofibrations as before.
- Define fibrations in terms of right lifting properties with respect to the newly defined trivial cofibrations. The class of trivial fibrations remains unaltered.

Since there are more weak equivalences, there are more trivial cofibrations. Hence there are fewer fibrations and fewer fibrant objects. This could make the fibrant replacement functor much more interesting.

The hardest part of this is showing that the new classes of weak equivalences and fibrations,

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Suppose we have a model category $\mathcal{M}$, and we wish to change the model structure (without altering the underlying category) as follows.

- Enlarge the class of weak equivalences in some way.
- Keep the same class of cofibrations as before.
- Define fibrations in terms of right lifting properties with respect to the newly defined trivial cofibrations. The class of trivial fibrations remains unaltered.

Since there are more weak equivalences, there are more trivial cofibrations. Hence there are fewer fibrations and fewer fibrant objects. This could make the fibrant replacement functor much more interesting.

The hardest part of this is showing that the new classes of weak equivalences and fibrations, along with the original class of cofibrations,

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Suppose we have a model category $\mathcal{M}$, and we wish to change the model structure (without altering the underlying category) as follows.

- Enlarge the class of weak equivalences in some way.
- Keep the same class of cofibrations as before.
- Define fibrations in terms of right lifting properties with respect to the newly defined trivial cofibrations. The class of trivial fibrations remains unaltered.

Since there are more weak equivalences, there are more trivial cofibrations. Hence there are fewer fibrations and fewer fibrant objects. This could make the fibrant replacement functor much more interesting.

The hardest part of this is showing that the new classes of weak equivalences and fibrations, along with the original class of cofibrations, satisfy the Factorization Axiom **MC5**.

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea.

Suppose we have a model category $\mathcal{M}$, and we wish to change the model structure (without altering the underlying category) as follows.

- Enlarge the class of weak equivalences in some way.
- Keep the same class of cofibrations as before.
- Define fibrations in terms of right lifting properties with respect to the newly defined trivial cofibrations. The class of trivial fibrations remains unaltered.

Since there are more weak equivalences, there are more trivial cofibrations. Hence there are fewer fibrations and fewer fibrant objects. This could make the fibrant replacement functor much more interesting.

The hardest part of this is showing that the new classes of weak equivalences and fibrations, along with the original class of cofibrations, satisfy the Factorization Axiom **MC5**. The proof involves some delicate set theory.

Mike Hill
Mike Hopkins
Doug Ravenel

Three examples of Bousfield localization

Three examples of Bousfield localization

Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

Mike Hill
Mike Hopkins
Doug Ravenel

Three examples of Bousfield localization

Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

1. Choose an integer $n > 0$.

Mike Hill
Mike Hopkins
Doug Ravenel

Three examples of Bousfield localization

Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

1. Choose an integer $n > 0$. Define a map f to be a weak equivalence if $\pi_k f$ is an isomorphism for $k \leq n$.

Mike Hill
Mike Hopkins
Doug Ravenel

Three examples of Bousfield localization

Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

1. Choose an integer $n > 0$. Define a map f to be a weak equivalence if $\pi_k f$ is an isomorphism for $k \leq n$. Then the fibrant objects are the spaces with no homotopy above dimension n.

Mike Hill
Mike Hopkins
Doug Ravenel

Three examples of Bousfield localization

Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

1. Choose an integer $n > 0$. Define a map f to be a weak equivalence if $\pi_k f$ is an isomorphism for $k \leq n$. Then the fibrant objects are the spaces with no homotopy above dimension n. The fibrant replacement functor is the nth Postnikov section.

Mike Hill
Mike Hopkins
Doug Ravenel

Three examples of Bousfield localization

Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

1. Choose an integer $n > 0$. Define a map f to be a weak equivalence if $\pi_k f$ is an isomorphism for $k \leq n$. Then the fibrant objects are the spaces with no homotopy above dimension n. The fibrant replacement functor is the nth Postnikov section. It was originally constructed by attaching cells to kill all homotopy above dimension n.

Mike Hill
Mike Hopkins
Doug Ravenel

Three examples of Bousfield localization

Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

1. Choose an integer $n > 0$. Define a map f to be a weak equivalence if $\pi_k f$ is an isomorphism for $k \leq n$. Then the fibrant objects are the spaces with no homotopy above dimension n. The fibrant replacement functor is the nth Postnikov section. It was originally constructed by attaching cells to kill all homotopy above dimension n.
2. Choose a prime p.

Mike Hill
Mike Hopkins
Doug Ravenel

Three examples of Bousfield localization

Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

1. Choose an integer $n > 0$. Define a map f to be a weak equivalence if $\pi_k f$ is an isomorphism for $k \leq n$. Then the fibrant objects are the spaces with no homotopy above dimension n. The fibrant replacement functor is the nth Postnikov section. It was originally constructed by attaching cells to kill all homotopy above dimension n.
2. Choose a prime p. Define a map to be a weak equivalence if it induces an isomorphism in mod p homology.

Mike Hill
Mike Hopkins
Doug Ravenel

Three examples of Bousfield localization

Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

1. Choose an integer $n > 0$. Define a map f to be a weak equivalence if $\pi_k f$ is an isomorphism for $k \leq n$. Then the fibrant objects are the spaces with no homotopy above dimension n. The fibrant replacement functor is the nth Postnikov section. It was originally constructed by attaching cells to kill all homotopy above dimension n.
2. Choose a prime p. Define a map to be a weak equivalence if it induces an isomorphism in mod p homology. On simply connected spaces, the fibrant replacement functor is p-adic completion.

Mike Hill
Mike Hopkins
Doug Ravenel

Three examples of Bousfield localization

Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

1. Choose an integer $n > 0$. Define a map f to be a weak equivalence if $\pi_k f$ is an isomorphism for $k \leq n$. Then the fibrant objects are the spaces with no homotopy above dimension n. The fibrant replacement functor is the nth Postnikov section. It was originally constructed by attaching cells to kill all homotopy above dimension n.
2. Choose a prime p. Define a map to be a weak equivalence if it induces an isomorphism in mod p homology. On simply connected spaces, the fibrant replacement functor is p-adic completion.
3. Choose a generalized homology theory h_*.

Mike Hill
Mike Hopkins
Doug Ravenel

Three examples of Bousfield localization

Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

1. Choose an integer $n > 0$. Define a map f to be a weak equivalence if $\pi_k f$ is an isomorphism for $k \leq n$. Then the fibrant objects are the spaces with no homotopy above dimension n. The fibrant replacement functor is the nth Postnikov section. It was originally constructed by attaching cells to kill all homotopy above dimension n.
2. Choose a prime p. Define a map to be a weak equivalence if it induces an isomorphism in mod p homology. On simply connected spaces, the fibrant replacement functor is p-adic completion.
3. Choose a generalized homology theory h_*. Define a map f to be a weak equivalence if $h_* f$ is an isomorphism.

Mike Hill
Mike Hopkins
Doug Ravenel

Three examples of Bousfield localization

Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

1. Choose an integer $n > 0$. Define a map f to be a weak equivalence if $\pi_k f$ is an isomorphism for $k \leq n$. Then the fibrant objects are the spaces with no homotopy above dimension n. The fibrant replacement functor is the nth Postnikov section. It was originally constructed by attaching cells to kill all homotopy above dimension n.
2. Choose a prime p. Define a map to be a weak equivalence if it induces an isomorphism in mod p homology. On simply connected spaces, the fibrant replacement functor is p-adic completion.
3. Choose a generalized homology theory h_*. Define a map f to be a weak equivalence if $h_* f$ is an isomorphism. The resulting fibrant replacement functor is Bousfield localization with respect to h_*.

Mike Hill
Mike Hopkins
Doug Ravenel

Three examples of Bousfield localization

Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

1. Choose an integer $n > 0$. Define a map f to be a weak equivalence if $\pi_k f$ is an isomorphism for $k \leq n$. Then the fibrant objects are the spaces with no homotopy above dimension n. The fibrant replacement functor is the nth Postnikov section. It was originally constructed by attaching cells to kill all homotopy above dimension n.
2. Choose a prime p. Define a map to be a weak equivalence if it induces an isomorphism in mod p homology. On simply connected spaces, the fibrant replacement functor is p-adic completion.
3. Choose a generalized homology theory h_*. Define a map f to be a weak equivalence if $h_* f$ is an isomorphism. The resulting fibrant replacement functor is Bousfield localization with respect to h_*. One can do the same with the category of spectra,

Mike Hill
Mike Hopkins
Doug Ravenel

Three examples of Bousfield localization

Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

1. Choose an integer $n > 0$. Define a map f to be a weak equivalence if $\pi_k f$ is an isomorphism for $k \leq n$. Then the fibrant objects are the spaces with no homotopy above dimension n. The fibrant replacement functor is the nth Postnikov section. It was originally constructed by attaching cells to kill all homotopy above dimension n.
2. Choose a prime p. Define a map to be a weak equivalence if it induces an isomorphism in mod p homology. On simply connected spaces, the fibrant replacement functor is p-adic completion.
3. Choose a generalized homology theory h_*. Define a map f to be a weak equivalence if $h_* f$ is an isomorphism. The resulting fibrant replacement functor is Bousfield localization with respect to h_*. One can do the same with the category of spectra, once we have the right model structure on it.

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization in a cofibrantly generated model category

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization in a cofibrantly generated model category

Suppose $\mathcal{M}$ is a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$.

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization in a cofibrantly generated model category

Suppose $\mathcal{M}$ is a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$. Let $\mathcal{M}'$ denote its Bousfield localization of $\mathcal{M}$ with respect to some expanded class of weak equivalences.

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization in a cofibrantly generated model category

Suppose $\mathcal{M}$ is a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$. Let $\mathcal{M}'$ denote its Bousfield localization of $\mathcal{M}$ with respect to some expanded class of weak equivalences. What are its generating sets $\mathcal{I}'$? and $\mathcal{J}'$?

Bousfield localization in a cofibrantly generated model category

Suppose $\mathcal{M}$ is a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$. Let $\mathcal{M}'$ denote its Bousfield localization of $\mathcal{M}$ with respect to some expanded class of weak equivalences. What are its generating sets $\mathcal{I}'$? and $\mathcal{J}'$?

Since $\mathcal{M}'$ has the same class of cofibrations as $\mathcal{M}$,

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization in a cofibrantly generated model category

Suppose $\mathcal{M}$ is a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$. Let $\mathcal{M}'$ denote its Bousfield localization of $\mathcal{M}$ with respect to some expanded class of weak equivalences. What are its generating sets $\mathcal{I}'$? and $\mathcal{J}'$?

Since $\mathcal{M}'$ has the same class of cofibrations as $\mathcal{M}$, we can set $\mathcal{I}' = \mathcal{I}$.

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization in a cofibrantly generated model category

Suppose $\mathcal{M}$ is a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$. Let $\mathcal{M}'$ denote its Bousfield localization of $\mathcal{M}$ with respect to some expanded class of weak equivalences. What are its generating sets $\mathcal{I}'$? and $\mathcal{J}'$?

Since $\mathcal{M}'$ has the same class of cofibrations as $\mathcal{M}$, we can set $\mathcal{I}' = \mathcal{I}$.

Since $\mathcal{M}'$ has the more trivial cofibrations than $\mathcal{M}$,

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization in a cofibrantly generated model category

Suppose $\mathcal{M}$ is a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$. Let $\mathcal{M}'$ denote its Bousfield localization of $\mathcal{M}$ with respect to some expanded class of weak equivalences. What are its generating sets $\mathcal{I}'$? and $\mathcal{J}'$?

Since $\mathcal{M}'$ has the same class of cofibrations as $\mathcal{M}$, we can set $\mathcal{I}' = \mathcal{I}$.

Since $\mathcal{M}'$ has the more trivial cofibrations than $\mathcal{M}$, we need to make $\mathcal{J}'$ bigger tthan $\mathcal{J}$.

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization in a cofibrantly generated model category

Suppose $\mathcal{M}$ is a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$. Let $\mathcal{M}'$ denote its Bousfield localization of $\mathcal{M}$ with respect to some expanded class of weak equivalences. What are its generating sets $\mathcal{I}'$? and $\mathcal{J}'$?

Since $\mathcal{M}'$ has the same class of cofibrations as $\mathcal{M}$, we can set $\mathcal{I}' = \mathcal{I}$.

Since $\mathcal{M}'$ has the more trivial cofibrations than $\mathcal{M}$, we need to make $\mathcal{J}'$ bigger tthan $\mathcal{J}$. There is a theorem saying when such a $\mathcal{J}'$ exists,

Bousfield localization in a cofibrantly generated model category

Suppose $\mathcal{M}$ is a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$. Let $\mathcal{M}'$ denote its Bousfield localization of $\mathcal{M}$ with respect to some expanded class of weak equivalences. What are its generating sets $\mathcal{I}'$? and $\mathcal{J}'$?

Since $\mathcal{M}'$ has the same class of cofibrations as $\mathcal{M}$, we can set $\mathcal{I}' = \mathcal{I}$.

Since $\mathcal{M}'$ has the more trivial cofibrations than $\mathcal{M}$, we need to make $\mathcal{J}'$ bigger tthan $\mathcal{J}$. There is a theorem saying when such a $\mathcal{J}'$ exists, but there is no known general description of it.

Mike Hill
Mike Hopkins
Doug Ravenel

Bousfield localization in a cofibrantly generated model category

Suppose $\mathcal{M}$ is a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$. Let $\mathcal{M}'$ denote its Bousfield localization of $\mathcal{M}$ with respect to some expanded class of weak equivalences. What are its generating sets $\mathcal{I}'$? and $\mathcal{J}'$?

Since $\mathcal{M}'$ has the same class of cofibrations as $\mathcal{M}$, we can set $\mathcal{I}' = \mathcal{I}$.

Since $\mathcal{M}'$ has the more trivial cofibrations than $\mathcal{M}$, we need to make $\mathcal{J}'$ bigger tthan $\mathcal{J}$. There is a theorem saying when such a $\mathcal{J}'$ exists, but there is no known general description of it.

We will give such a description in a certain case related to stable homotopy theory.

Enriched category theory

Mike Hill
Mike Hopkins
Doug Ravenel

A symmetric monoidal structure on a category $\mathcal{V}_0$ is a functor

$$\mathcal{V}_0 \times \mathcal{V}_0 \xrightarrow{\otimes} \mathcal{V}_0$$

Enriched category theory

A symmetric monoidal structure on a category $\mathcal{V}_0$ is a functor

$$\mathcal{V}_0 \times \mathcal{V}_0 \xrightarrow{\otimes} \mathcal{V}_0$$

sending a pair of objects (X, Y) to a third object $X \otimes Y$. It is required to have suitable properties including

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory

A symmetric monoidal structure on a category $\mathcal{V}_0$ is a functor

$$\mathcal{V}_0 \times \mathcal{V}_0 \xrightarrow{\otimes} \mathcal{V}_0$$

sending a pair of objects (X, Y) to a third object $X \otimes Y$. It is required to have suitable properties including

- a natural isomorphism $t : X \otimes Y \to Y \otimes X$ and

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory

A symmetric monoidal structure on a category $\mathcal{V}_0$ is a functor

$$\mathcal{V}_0 \times \mathcal{V}_0 \xrightarrow{\otimes} \mathcal{V}_0$$

sending a pair of objects (X, Y) to a third object $X \otimes Y$. It is required to have suitable properties including

- a natural isomorphism $t : X \otimes Y \to Y \otimes X$ and
- a unit object $\mathbf{1}$ such that $\mathbf{1} \otimes X$ is naturally isomorphic to X.

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory

A symmetric monoidal structure on a category $\mathcal{V}_0$ is a functor

$$\mathcal{V}_0 \times \mathcal{V}_0 \xrightarrow{\otimes} \mathcal{V}_0$$

sending a pair of objects (X, Y) to a third object $X \otimes Y$. It is required to have suitable properties including

- a natural isomorphism $t : X \otimes Y \to Y \otimes X$ and
- a unit object $\mathbf{1}$ such that $\mathbf{1} \otimes X$ is naturally isomorphic to X.

We denote this by $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$.

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory

A symmetric monoidal structure on a category $\mathcal{V}_0$ is a functor

$$\mathcal{V}_0 \times \mathcal{V}_0 \xrightarrow{\otimes} \mathcal{V}_0$$

sending a pair of objects (X, Y) to a third object $X \otimes Y$. It is required to have suitable properties including

- a natural isomorphism $t : X \otimes Y \to Y \otimes X$ and
- a unit object $\mathbf{1}$ such that $\mathbf{1} \otimes X$ is naturally isomorphic to X.

We denote this by $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$.

Familiar examples include $(\mathcal{S}et, \times, *)$,

Enriched category theory

A symmetric monoidal structure on a category $\mathcal{V}_0$ is a functor

$$\mathcal{V}_0 \times \mathcal{V}_0 \xrightarrow{\otimes} \mathcal{V}_0$$

sending a pair of objects (X, Y) to a third object $X \otimes Y$. It is required to have suitable properties including

- a natural isomorphism $t : X \otimes Y \to Y \otimes X$ and
- a unit object $\mathbf{1}$ such that $\mathbf{1} \otimes X$ is naturally isomorphic to X.

We denote this by $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$.

Familiar examples include $(\mathcal{S}et, \times, *)$, $(\mathcal{T}op, \times, *)$,

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory

A symmetric monoidal structure on a category $\mathcal{V}_0$ is a functor

$$\mathcal{V}_0 \times \mathcal{V}_0 \xrightarrow{\otimes} \mathcal{V}_0$$

sending a pair of objects (X, Y) to a third object $X \otimes Y$. It is required to have suitable properties including

- a natural isomorphism $t : X \otimes Y \to Y \otimes X$ and
- a unit object $\mathbf{1}$ such that $\mathbf{1} \otimes X$ is naturally isomorphic to X.

We denote this by $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$.

Familiar examples include $(\mathcal{S}et, \times, *)$, $(\mathcal{T}op, \times, *)$, $(\mathcal{T}, \wedge, S^0)$,

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory

A symmetric monoidal structure on a category $\mathcal{V}_0$ is a functor

$$\mathcal{V}_0 \times \mathcal{V}_0 \xrightarrow{\otimes} \mathcal{V}_0$$

sending a pair of objects (X, Y) to a third object $X \otimes Y$. It is required to have suitable properties including

- a natural isomorphism $t : X \otimes Y \to Y \otimes X$ and
- a unit object $\mathbf{1}$ such that $\mathbf{1} \otimes X$ is naturally isomorphic to X.

We denote this by $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$.

Familiar examples include $(\mathcal{S}et, \times, *)$, $(\mathcal{T}op, \times, *)$, $(\mathcal{T}, \wedge, S^0)$, where $\mathcal{T}$ is the category of pointed topological spaces,

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory

A symmetric monoidal structure on a category $\mathcal{V}_0$ is a functor

$$\mathcal{V}_0 \times \mathcal{V}_0 \xrightarrow{\otimes} \mathcal{V}_0$$

sending a pair of objects (X, Y) to a third object $X \otimes Y$. It is required to have suitable properties including

- a natural isomorphism $t : X \otimes Y \to Y \otimes X$ and
- a unit object $\mathbf{1}$ such that $\mathbf{1} \otimes X$ is naturally isomorphic to X.

We denote this by $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$.

Familiar examples include $(\mathcal{S}et, \times, *)$, $(\mathcal{T}op, \times, *)$, $(\mathcal{T}, \wedge, S^0)$, where $\mathcal{T}$ is the category of pointed topological spaces, and $(\mathcal{S}et_\Delta, \times, *)$,

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory

A symmetric monoidal structure on a category $\mathcal{V}_0$ is a functor

$$\mathcal{V}_0 \times \mathcal{V}_0 \xrightarrow{\otimes} \mathcal{V}_0$$

sending a pair of objects (X, Y) to a third object $X \otimes Y$. It is required to have suitable properties including

- a natural isomorphism $t : X \otimes Y \to Y \otimes X$ and
- a unit object $\mathbf{1}$ such that $\mathbf{1} \otimes X$ is naturally isomorphic to X.

We denote this by $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$.

Familiar examples include $(\mathcal{S}et, \times, *)$, $(\mathcal{T}op, \times, *)$, $(\mathcal{T}, \wedge, S^0)$, where $\mathcal{T}$ is the category of pointed topological spaces, and $(\mathcal{S}et_\Delta, \times, *)$, where $\mathcal{S}et_\Delta$ is the category of simplicial sets.

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a symmetric monoidal category as above.

Definition

A $\mathcal{V}$*-category*

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a symmetric monoidal category as above.

Definition

A $\mathcal{V}$-category (or a category enriched over $\mathcal{V}$)

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a symmetric monoidal category as above.

Definition

A $\mathcal{V}$-category (or a category enriched over $\mathcal{V}$) consists of

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a symmetric monoidal category as above.

Definition

A $\mathcal{V}$-category (or a category enriched over $\mathcal{V}$) consists of

- *a collection of objects,*

Enriched category theory (continued)

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a symmetric monoidal category as above.

Definition

A $\mathcal{V}$-category (or a category enriched over $\mathcal{V}$) consists of

- *a collection of objects,*
- *for each pair of objects (X, Y) a morphism object $\mathcal{C}(X, Y)$ in $\mathcal{V}_0$*

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a symmetric monoidal category as above.

Definition

A $\mathcal{V}$-category (or a category enriched over $\mathcal{V}$) consists of

- *a collection of objects,*
- *for each pair of objects (X, Y) a morphism object $\mathcal{C}(X, Y)$ in $\mathcal{V}_0$ (instead of a set of morphisms $X \to Y$),*

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a symmetric monoidal category as above.

Definition

A $\mathcal{V}$-category (or a category enriched over $\mathcal{V}$) consists of

- *a collection of objects,*
- *for each pair of objects (X, Y) a morphism object $\mathcal{C}(X, Y)$ in $\mathcal{V}_0$ (instead of a set of morphisms $X \to Y$),*
- *for each triple of objects (X, Y, Z) a composition morphism in $\mathcal{V}_0$*

$$c_{X,Y,Z} : \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$$

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a symmetric monoidal category as above.

Definition

A $\mathcal{V}$-category (or a category enriched over $\mathcal{V}$) consists of

- *a collection of objects,*
- *for each pair of objects (X, Y) a morphism object $\mathcal{C}(X, Y)$ in $\mathcal{V}_0$ (instead of a set of morphisms $X \to Y$),*
- *for each triple of objects (X, Y, Z) a composition morphism in $\mathcal{V}_0$*

$$c_{X,Y,Z} : \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$$

(replacing the usual composition)

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a symmetric monoidal category as above.

Definition

A $\mathcal{V}$-category (or a category enriched over $\mathcal{V}$) consists of

- *a collection of objects,*
- *for each pair of objects (X, Y) a morphism object $\mathcal{C}(X, Y)$ in $\mathcal{V}_0$ (instead of a set of morphisms $X \to Y$),*
- *for each triple of objects (X, Y, Z) a composition morphism in $\mathcal{V}_0$*

$$c_{X,Y,Z} : \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$$

  (replacing the usual composition) and
- *for each object X, an identity morphism in $\mathcal{V}_0$ $\mathbf{1} \to \mathcal{C}(X, X)$,*

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a symmetric monoidal category as above.

Definition

A $\mathcal{V}$-category (or a category enriched over $\mathcal{V}$) consists of

- *a collection of objects,*
- *for each pair of objects (X, Y) a morphism object $\mathcal{C}(X, Y)$ in $\mathcal{V}_0$ (instead of a set of morphisms $X \to Y$),*
- *for each triple of objects (X, Y, Z) a composition morphism in $\mathcal{V}_0$*

$$c_{X,Y,Z} : \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$$

  (replacing the usual composition) and
- *for each object X, an identity morphism in $\mathcal{V}_0$ $\mathbf{1} \to \mathcal{C}(X, X)$, replacing the usual identity morphism $X \to X$.*

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a symmetric monoidal category as above.

Definition

A $\mathcal{V}$-category (or a category enriched over $\mathcal{V}$) consists of

- *a collection of objects,*
- *for each pair of objects (X, Y) a morphism object $\mathcal{C}(X, Y)$ in $\mathcal{V}_0$ (instead of a set of morphisms $X \to Y$),*
- *for each triple of objects (X, Y, Z) a composition morphism in $\mathcal{V}_0$*

$$c_{X,Y,Z} : \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$$

  (replacing the usual composition) and
- *for each object X, an identity morphism in $\mathcal{V}_0$ $\mathbf{1} \to \mathcal{C}(X, X)$, replacing the usual identity morphism $X \to X$.*

There is an underlying ordinary category $\mathcal{C}_0$ with the same objects as $\mathcal{C}$

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a symmetric monoidal category as above.

Definition

A $\mathcal{V}$-category (or a category enriched over $\mathcal{V}$) consists of

- *a collection of objects,*
- *for each pair of objects (X, Y) a morphism object $\mathcal{C}(X, Y)$ in $\mathcal{V}_0$ (instead of a set of morphisms $X \to Y$),*
- *for each triple of objects (X, Y, Z) a composition morphism in $\mathcal{V}_0$*

$$c_{X,Y,Z} : \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$$

  (replacing the usual composition) and
- *for each object X, an identity morphism in $\mathcal{V}_0$ $\mathbf{1} \to \mathcal{C}(X, X)$, replacing the usual identity morphism $X \to X$.*

There is an underlying ordinary category $\mathcal{C}_0$ with the same objects as $\mathcal{C}$ and morphism sets

$$\mathcal{C}_0(X, Y) = \mathcal{V}_0(\mathbf{1}, \mathcal{C}(X, Y)).$$

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

One can define enriched functors ($\mathcal{V}$-functors) between $\mathcal{V}$-categories

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

One can define enriched functors ($\mathcal{V}$-functors) between $\mathcal{V}$-categories and enriched natural transformations ($\mathcal{V}$-natural transformations) between them.

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

One can define enriched functors ($\mathcal{V}$-functors) between $\mathcal{V}$-categories and enriched natural transformations ($\mathcal{V}$-natural transformations) between them.

In this language, an ordinary category is enriched over $\mathcal{S}et$.

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

One can define enriched functors ($\mathcal{V}$-functors) between $\mathcal{V}$-categories and enriched natural transformations ($\mathcal{V}$-natural transformations) between them.

In this language, an ordinary category is enriched over $\mathcal{S}et$.

A topological category is one that is enriched over $\mathcal{T}op$.

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

One can define enriched functors ($\mathcal{V}$-functors) between $\mathcal{V}$-categories and enriched natural transformations ($\mathcal{V}$-natural transformations) between them.

In this language, an ordinary category is enriched over $\mathcal{S}et$.

A topological category is one that is enriched over $\mathcal{T}op$.

A simplicial category is one that is enriched over $\mathcal{S}et_\Delta$, the category of simplicial sets.

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

One can define enriched functors ($\mathcal{V}$-functors) between $\mathcal{V}$-categories and enriched natural transformations ($\mathcal{V}$-natural transformations) between them.

In this language, an ordinary category is enriched over $\mathcal{S}et$.

A topological category is one that is enriched over $\mathcal{T}op$.

A simplicial category is one that is enriched over $\mathcal{S}et_\Delta$, the category of simplicial sets.

A symmetric monoidal category $\mathcal{V}_0$ is closed if it enriched over itself.

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

One can define enriched functors ($\mathcal{V}$-functors) between $\mathcal{V}$-categories and enriched natural transformations ($\mathcal{V}$-natural transformations) between them.

In this language, an ordinary category is enriched over $\mathcal{S}et$.

A topological category is one that is enriched over $\mathcal{T}op$.

A simplicial category is one that is enriched over $\mathcal{S}et_\Delta$, the category of simplicial sets.

A symmetric monoidal category $\mathcal{V}_0$ is closed if it enriched over itself. This means that for each pair of objects (X, Y)

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

One can define enriched functors ($\mathcal{V}$-functors) between $\mathcal{V}$-categories and enriched natural transformations ($\mathcal{V}$-natural transformations) between them.

In this language, an ordinary category is enriched over $\mathcal{S}et$.

A topological category is one that is enriched over $\mathcal{T}op$.

A simplicial category is one that is enriched over $\mathcal{S}et_\Delta$, the category of simplicial sets.

A symmetric monoidal category $\mathcal{V}_0$ is closed if it enriched over itself. This means that for each pair of objects (X, Y) there is an internal Hom object $\mathcal{V}(X, Y)$

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory (continued)

One can define enriched functors ($\mathcal{V}$-functors) between $\mathcal{V}$-categories and enriched natural transformations ($\mathcal{V}$-natural transformations) between them.

In this language, an ordinary category is enriched over $\mathcal{S}et$.

A topological category is one that is enriched over $\mathcal{T}op$.

A simplicial category is one that is enriched over $\mathcal{S}et_\Delta$, the category of simplicial sets.

A symmetric monoidal category $\mathcal{V}_0$ is closed if it enriched over itself. This means that for each pair of objects (X, Y) there is an internal Hom object $\mathcal{V}(X, Y)$ with natural isomorphisms

$$\mathcal{V}_0(X \otimes Y, Z) \cong \mathcal{V}_0(X, \mathcal{V}(Y, Z)).$$

Enriched category theory (continued)

One can define enriched functors ($\mathcal{V}$-functors) between $\mathcal{V}$-categories and enriched natural transformations ($\mathcal{V}$-natural transformations) between them.

In this language, an ordinary category is enriched over $\mathcal{S}et$.

A topological category is one that is enriched over $\mathcal{T}op$.

A simplicial category is one that is enriched over $\mathcal{S}et_\Delta$, the category of simplicial sets.

A symmetric monoidal category $\mathcal{V}_0$ is closed if it enriched over itself. This means that for each pair of objects (X, Y) there is an internal Hom object $\mathcal{V}(X, Y)$ with natural isomorphisms

$$\mathcal{V}_0(X \otimes Y, Z) \cong \mathcal{V}_0(X, \mathcal{V}(Y, Z)).$$

The symmetric monoidal categories $\mathcal{S}et$, $\mathcal{T}op$, $\mathcal{T}$ and $\mathcal{S}et_\Delta$ are each closed.

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors

Recall that a spectrum X was originally defined to be a sequence of pointed spaces $\{X_n\}$

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors

Recall that a spectrum X was originally defined to be a sequence of pointed spaces $\{X_n\}$ with structure maps $\Sigma X_n \to X_{n+1}$.

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors

Recall that a spectrum X was originally defined to be a sequence of pointed spaces $\{X_n\}$ with structure maps $\Sigma X_n \to X_{n+1}$. We will redefine it to be an enriched $\mathcal{T}$-valued functor on a small $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$.

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors

Recall that a spectrum X was originally defined to be a sequence of pointed spaces $\{X_n\}$ with structure maps $\Sigma X_n \to X_{n+1}$. We will redefine it to be an enriched $\mathcal{T}$-valued functor on a small $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$. This will make the structure maps built in to the functor.

Spectra as enriched functors

Recall that a spectrum X was originally defined to be a sequence of pointed spaces $\{X_n\}$ with structure maps $\Sigma X_n \to X_{n+1}$. We will redefine it to be an enriched $\mathcal{T}$-valued functor on a small $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$. This will make the structure maps built in to the functor. Maps between spectra will be enriched natural transformations.

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors

Recall that a spectrum X was originally defined to be a sequence of pointed spaces $\{X_n\}$ with structure maps $\Sigma X_n \to X_{n+1}$. We will redefine it to be an enriched $\mathcal{T}$-valued functor on a small $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$. This will make the structure maps built in to the functor. Maps between spectra will be enriched natural transformations.

Definition

The indexing category $\mathscr{J}^{\mathbf{N}}$ has natural numbers $n \geq 0$ as objects with

$$\mathscr{J}^{\mathbf{N}}(m,n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{otherwise.} \end{cases}$$

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors

Recall that a spectrum X was originally defined to be a sequence of pointed spaces $\{X_n\}$ with structure maps $\Sigma X_n \to X_{n+1}$. We will redefine it to be an enriched $\mathcal{T}$-valued functor on a small $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$. This will make the structure maps built in to the functor. Maps between spectra will be enriched natural transformations.

Definition

The indexing category $\mathscr{J}^{\mathbf{N}}$ has natural numbers $n \geq 0$ as objects with

$$\mathscr{J}^{\mathbf{N}}(m, n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{otherwise.} \end{cases}$$

For $m \leq m \leq p$, the composition morphism

$$j_{m,n,p} : S^{p-n} \wedge S^{n-m} \to S^{p-m}$$

is the standard homeomorphism.

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

We can define a spectrum X to be an enriched functor $X : \mathscr{J}^{\mathbf{N}} \to \mathcal{T}$.

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

We can define a spectrum X to be an enriched functor $X : \mathscr{J}^{\mathbf{N}} \to \mathcal{T}$. We denote its value at n by X_n.

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

We can define a spectrum X to be an enriched functor $X : \mathscr{J}^{\mathbf{N}} \to \mathcal{T}$. We denote its value at n by X_n. Functoriality means that for each $m, n \geq 0$

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

We can define a spectrum X to be an enriched functor $X : \mathscr{J}^{\mathbf{N}} \to \mathcal{T}$. We denote its value at n by X_n. Functoriality means that for each $m, n \geq 0$ there is a continuous structure map

$$\epsilon^X_{m,n} : \mathscr{J}^{\mathbf{N}}(m,n) \wedge X_m \to X_n.$$

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

We can define a spectrum X to be an enriched functor $X : \mathscr{J}^{\mathbf{N}} \to \mathcal{T}$. We denote its value at n by X_n. Functoriality means that for each $m, n \geq 0$ there is a continuous structure map

$$\epsilon^X_{m,n} : \mathscr{J}^{\mathbf{N}}(m,n) \wedge X_m \to X_n.$$

Since

$$\mathscr{J}^{\mathbf{N}}(m,n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{otherwise,} \end{cases}$$

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

We can define a spectrum X to be an enriched functor $X : \mathscr{J}^{\mathbf{N}} \to \mathcal{T}$. We denote its value at n by X_n. Functoriality means that for each $m, n \geq 0$ there is a continuous structure map

$$\epsilon^X_{m,n} : \mathscr{J}^{\mathbf{N}}(m,n) \wedge X_m \to X_n.$$

Since

$$\mathscr{J}^{\mathbf{N}}(m,n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{otherwise,} \end{cases}$$

for $m \leq n$ we get the expected map $\Sigma^{n-m} X_m \to X_n$.

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

We can define a spectrum X to be an enriched functor $X : \mathscr{J}^{\mathbf{N}} \to \mathcal{T}$. We denote its value at n by X_n. Functoriality means that for each $m, n \geq 0$ there is a continuous structure map

$$\epsilon^X_{m,n} : \mathscr{J}^{\mathbf{N}}(m,n) \wedge X_m \to X_n.$$

Since

$$\mathscr{J}^{\mathbf{N}}(m,n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{otherwise,} \end{cases}$$

for $m \leq n$ we get the expected map $\Sigma^{n-m} X_m \to X_n$.

Definition

For $m \geq 0$,

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

We can define a spectrum X to be an enriched functor $X : \mathscr{J}^{\mathbf{N}} \to \mathcal{T}$. We denote its value at n by X_n. Functoriality means that for each $m, n \geq 0$ there is a continuous structure map

$$\epsilon^X_{m,n} : \mathscr{J}^{\mathbf{N}}(m,n) \wedge X_m \to X_n.$$

Since

$$\mathscr{J}^{\mathbf{N}}(m,n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{otherwise,} \end{cases}$$

for $m \leq n$ we get the expected map $\Sigma^{n-m} X_m \to X_n$.

Definition

For $m \geq 0$, the **Yoneda spectrum** $\text{よ}^m = S^{-m}$ *is given by*

$$(S^{-m})_n = \mathscr{J}^{\mathbf{N}}(m,n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{otherwise.} \end{cases}$$

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

We can define a spectrum X to be an enriched functor $X : \mathscr{J}^{\mathbf{N}} \to \mathcal{T}$. We denote its value at n by X_n. Functoriality means that for each $m, n \geq 0$ there is a continuous structure map

$$\epsilon^X_{m,n} : \mathscr{J}^{\mathbf{N}}(m,n) \wedge X_m \to X_n.$$

Since

$$\mathscr{J}^{\mathbf{N}}(m,n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{otherwise,} \end{cases}$$

for $m \leq n$ we get the expected map $\Sigma^{n-m} X_m \to X_n$.

Definition

For $m \geq 0$, the Yoneda spectrum $\text{よ}^m = S^{-m}$ is given by

$$(S^{-m})_n = \mathscr{J}^{\mathbf{N}}(m,n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{otherwise.} \end{cases}$$

In particular, S^{-0} is the sphere spectrum,

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

We can define a spectrum X to be an enriched functor $X : \mathscr{J}^{\mathbf{N}} \to \mathcal{T}$. We denote its value at n by X_n. Functoriality means that for each $m, n \geq 0$ there is a continuous structure map

$$\epsilon^X_{m,n} : \mathscr{J}^{\mathbf{N}}(m,n) \wedge X_m \to X_n.$$

Since

$$\mathscr{J}^{\mathbf{N}}(m,n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{otherwise,} \end{cases}$$

for $m \leq n$ we get the expected map $\Sigma^{n-m} X_m \to X_n$.

Definition

For $m \geq 0$, the Yoneda spectrum $\text{よ}^m = S^{-m}$ is given by

$$(S^{-m})_n = \mathscr{J}^{\mathbf{N}}(m,n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{otherwise.} \end{cases}$$

In particular, S^{-0} is the sphere spectrum, and S^{-m} is its formal mth desuspension.

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

Warning The catgeory $\mathscr{J}^{\mathbf{N}}$ is monoidal (under addition) but not symmetric monoidal.

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

Warning The catgeory $\mathscr{J}^{\mathbf{N}}$ is monoidal (under addition) but not symmetric monoidal. It admits an embedding functor into $\mathcal{T}$,

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

Warning The catgeory $\mathscr{J}^{\mathbf{N}}$ is monoidal (under addition) but not symmetric monoidal. It admits an embedding functor into $\mathcal{T}$, namely the Yoneda functor よ^0 given by

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

Warning The catgeory $\mathscr{J}^{\mathbf{N}}$ is monoidal (under addition) but not symmetric monoidal. It admits an embedding functor into $\mathcal{T}$, namely the Yoneda functor よ^0 given by

$$n \mapsto \mathscr{J}^{\mathbf{N}}(0, n) = S^n$$

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

Warning The catgeory $\mathscr{J}^{\mathbf{N}}$ is monoidal (under addition) but not symmetric monoidal. It admits an embedding functor into $\mathcal{T}$, namely the Yoneda functor よ^0 given by

$$n \mapsto \mathscr{J}^{\mathbf{N}}(0, n) = S^n$$

$\mathcal{T}$ is symmetric monoidal,

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

Warning The catgeory $\mathscr{J}^{\mathbf{N}}$ is monoidal (under addition) but not symmetric monoidal. It admits an embedding functor into $\mathcal{T}$, namely the Yoneda functor よ^0 given by

$$n \mapsto \mathscr{J}^{\mathbf{N}}(0, n) = S^n$$

$\mathcal{T}$ is symmetric monoidal, and there is a twist isomorphism

$$t : S^m \wedge S^n \to S^n \wedge S^m.$$

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

Warning The catgeory $\mathscr{J}^{\mathbf{N}}$ is monoidal (under addition) but not symmetric monoidal. It admits an embedding functor into $\mathcal{T}$, namely the Yoneda functor よ^0 given by

$$n \mapsto \mathscr{J}^{\mathbf{N}}(0, n) = S^n$$

$\mathcal{T}$ is symmetric monoidal, and there is a twist isomorphism

$$t : S^m \wedge S^n \to S^n \wedge S^m.$$

However this morphism is not in the image of the functor よ^0.

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

Warning The catgeory $\mathscr{J}^{\mathbf{N}}$ is monoidal (under addition) but not symmetric monoidal. It admits an embedding functor into $\mathcal{T}$, namely the Yoneda functor $よ^0$ given by

$$n \mapsto \mathscr{J}^{\mathbf{N}}(0, n) = S^n$$

$\mathcal{T}$ is symmetric monoidal, and there is a twist isomorphism

$$t : S^m \wedge S^n \to S^n \wedge S^m.$$

However this morphism is not in the image of the functor $よ^0$. There is no twist isomorphism in $\mathscr{J}^{\mathbf{N}}$,

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

Warning The catgeory $\mathscr{J}^{\mathbf{N}}$ is monoidal (under addition) but not symmetric monoidal. It admits an embedding functor into $\mathcal{T}$, namely the Yoneda functor よ^0 given by

$$n \mapsto \mathscr{J}^{\mathbf{N}}(0, n) = S^n$$

$\mathcal{T}$ is symmetric monoidal, and there is a twist isomorphism

$$t : S^m \wedge S^n \to S^n \wedge S^m.$$

However this morphism is not in the image of the functor よ^0. There is no twist isomorphism in $\mathscr{J}^{\mathbf{N}}$, so its monoidal structure is not symmetric.

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

Warning The catgeory $\mathscr{J}^{\mathbf{N}}$ is monoidal (under addition) but not symmetric monoidal. It admits an embedding functor into $\mathcal{T}$, namely the Yoneda functor よ^0 given by

$$n \mapsto \mathscr{J}^{\mathbf{N}}(0, n) = S^n$$

$\mathcal{T}$ is symmetric monoidal, and there is a twist isomorphism

$$t : S^m \wedge S^n \to S^n \wedge S^m.$$

However this morphism is not in the image of the functor よ^0. There is no twist isomorphism in $\mathscr{J}^{\mathbf{N}}$, so its monoidal structure is not symmetric.

This is the reason that the category of spectra $\mathcal{S}p$ defined in this way

Spectra as enriched functors (continued)

Warning The catgeory $\mathscr{J}^{\mathbf{N}}$ is monoidal (under addition) but not symmetric monoidal. It admits an embedding functor into $\mathcal{T}$, namely the Yoneda functor $よ^{0}$ given by

$$n \mapsto \mathscr{J}^{\mathbf{N}}(0, n) = S^n$$

$\mathcal{T}$ is symmetric monoidal, and there is a twist isomorphism

$$t : S^m \wedge S^n \to S^n \wedge S^m.$$

However this morphism is not in the image of the functor $よ^{0}$. There is no twist isomorphism in $\mathscr{J}^{\mathbf{N}}$, so its monoidal structure is not symmetric.

This is the reason that the category of spectra $\mathcal{S}p$ defined in this way does not have a convenient smash product.

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

Warning The catgeory $\mathscr{J}^{\mathbf{N}}$ is monoidal (under addition) but not symmetric monoidal. It admits an embedding functor into $\mathcal{T}$, namely the Yoneda functor よ^0 given by

$$n \mapsto \mathscr{J}^{\mathbf{N}}(0, n) = S^n$$

$\mathcal{T}$ is symmetric monoidal, and there is a twist isomorphism

$$t : S^m \wedge S^n \to S^n \wedge S^m.$$

However this morphism is not in the image of the functor よ^0. There is no twist isomorphism in $\mathscr{J}^{\mathbf{N}}$, so its monoidal structure is not symmetric.

This is the reason that the category of spectra $\mathcal{S}p$ defined in this way does not have a convenient smash product. This was a headache in the subject for decades!

Spectra as enriched functors (continued)

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

However we can define the smash product of a spectrum X and a pointed space K by

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

However we can define the smash product of a spectrum X and a pointed space K by

$$(X \wedge K)_n = X_n \wedge K.$$

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

However we can define the smash product of a spectrum X and a pointed space K by

$$(X \wedge K)_n = X_n \wedge K.$$

The categorical term for this is that $\mathcal{S}p$ is tensored over $\mathcal{T}$.

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

However we can define the smash product of a spectrum X and a pointed space K by

$$(X \wedge K)_n = X_n \wedge K.$$

The categorical term for this is that $\mathcal{S}p$ is tensored over $\mathcal{T}$.

The category of spectra is also cotensored over $\mathcal{T}$,

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

However we can define the smash product of a spectrum X and a pointed space K by

$$(X \wedge K)_n = X_n \wedge K.$$

The categorical term for this is that $\mathcal{S}p$ is tensored over $\mathcal{T}$.

The category of spectra is also cotensored over $\mathcal{T}$, meaning we can define a spectrum X^K by

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

However we can define the smash product of a spectrum X and a pointed space K by

$$(X \wedge K)_n = X_n \wedge K.$$

The categorical term for this is that $\mathcal{S}p$ is tensored over $\mathcal{T}$.

The category of spectra is also cotensored over $\mathcal{T}$, meaning we can define a spectrum X^K by

$$(X^K)_n = X_n^K.$$

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

However we can define the smash product of a spectrum X and a pointed space K by

$$(X \wedge K)_n = X_n \wedge K.$$

The categorical term for this is that $\mathcal{S}p$ is tensored over $\mathcal{T}$.

The category of spectra is also cotensored over $\mathcal{T}$, meaning we can define a spectrum X^K by

$$(X^K)_n = X_n^K.$$

More generally when a $\mathcal{V}$-category is both tensored and cotensored over $\mathcal{V}$,

Mike Hill
Mike Hopkins
Doug Ravenel

Spectra as enriched functors (continued)

However we can define the smash product of a spectrum X and a pointed space K by

$$(X \wedge K)_n = X_n \wedge K.$$

The categorical term for this is that $\mathcal{S}p$ is tensored over $\mathcal{T}$.

The category of spectra is also cotensored over $\mathcal{T}$, meaning we can define a spectrum X^K by

$$(X^K)_n = X_n^K.$$

More generally when a $\mathcal{V}$-category is both tensored and cotensored over $\mathcal{V}$, we say it is bitensored over $\mathcal{V}$.

Mike Hill
Mike Hopkins
Doug Ravenel

The projective model structure on the category of spectra

Mike Hill
Mike Hopkins
Doug Ravenel

The projective model structure on the category of spectra

We can define the category of spectra to be $[\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$, the category of $\mathcal{T}$-valued $\mathcal{T}$-functors on the $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$.

Mike Hill
Mike Hopkins
Doug Ravenel

The projective model structure on the category of spectra

We can define the category of spectra to be $[\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$, the category of $\mathcal{T}$-valued $\mathcal{T}$-functors on the $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$. We define the projective model structure on it as follows.

Mike Hill
Mike Hopkins
Doug Ravenel

The projective model structure on the category of spectra

We can define the category of spectra to be $[\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$, the category of $\mathcal{T}$-valued $\mathcal{T}$-functors on the $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$. We define the projective model structure on it as follows.

- A map $f : X \to Y$ is a weak equivalence or fibration if $f_n : X_n \to Y_n$ is one for each $n \geq 0$.

Mike Hill
Mike Hopkins
Doug Ravenel

Mike Hill
Mike Hopkins
Doug Ravenel

The projective model structure on the category of spectra

We can define the category of spectra to be $[\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$, the category of $\mathcal{T}$-valued $\mathcal{T}$-functors on the $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$. We define the projective model structure on it as follows.

- A map $f : X \to Y$ is a weak equivalence or fibration if $f_n : X_n \to Y_n$ is one for each $n \geq 0$. In other words, weak equivalences and fibrations are strict weak equivalences and fibrations.

The projective model structure on the category of spectra

We can define the category of spectra to be $[\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$, the category of $\mathcal{T}$-valued $\mathcal{T}$-functors on the $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$. We define the projective model structure on it as follows.

- A map $f : X \to Y$ is a weak equivalence or fibration if $f_n : X_n \to Y_n$ is one for each $n \geq 0$. In other words, weak equivalences and fibrations are strict weak equivalences and fibrations.
- Cofibrations are defined in terms of left lifting properties.

Mike Hill
Mike Hopkins
Doug Ravenel

The projective model structure on the category of spectra

We can define the category of spectra to be $[\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$, the category of $\mathcal{T}$-valued $\mathcal{T}$-functors on the $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$. We define the projective model structure on it as follows.

- A map $f : X \to Y$ is a weak equivalence or fibration if $f_n : X_n \to Y_n$ is one for each $n \geq 0$. In other words, weak equivalences and fibrations are strict weak equivalences and fibrations.
- Cofibrations are defined in terms of left lifting properties.

This model structure is known to be cofibrantly generated with the following generating sets.

$$\mathcal{I}^{proj} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{I}_+$$

$$\mathcal{J}^{proj} = \left\{ S^{-m} \wedge (j_{n+} : I_+^n \to I_+^{n+1}) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{J}_+$$

The projective model structure on the category of spectra

We can define the category of spectra to be $[\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$, the category of $\mathcal{T}$-valued $\mathcal{T}$-functors on the $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$. We define the projective model structure on it as follows.

- A map $f : X \to Y$ is a weak equivalence or fibration if $f_n : X_n \to Y_n$ is one for each $n \geq 0$. In other words, weak equivalences and fibrations are strict weak equivalences and fibrations.
- Cofibrations are defined in terms of left lifting properties.

This model structure is known to be cofibrantly generated with the following generating sets.

$$\mathcal{I}^{proj} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{I}_+$$

$$\mathcal{J}^{proj} = \left\{ S^{-m} \wedge (j_{n+} : I_+^n \to I_+^{n+1}) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{J}_+$$

where $f_+ : X_+ \to Y_+$ denotes $f : X \to Y$ with disjoint base points added to X and Y.

Mike Hill
Mike Hopkins
Doug Ravenel

The projective model structure on the category of spectra

We can define the category of spectra to be $[\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$, the category of $\mathcal{T}$-valued $\mathcal{T}$-functors on the $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$. We define the projective model structure on it as follows.

- A map $f : X \to Y$ is a weak equivalence or fibration if $f_n : X_n \to Y_n$ is one for each $n \geq 0$. In other words, weak equivalences and fibrations are strict weak equivalences and fibrations.
- Cofibrations are defined in terms of left lifting properties.

This model structure is known to be cofibrantly generated with the following generating sets.

$$\mathcal{I}^{proj} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{I}_+$$
$$\mathcal{J}^{proj} = \left\{ S^{-m} \wedge (j_{n+} : I_+^n \to I_+^{n+1}) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{J}_+$$

where $f_+ : X_+ \to Y_+$ denotes $f : X \to Y$ with disjoint base points added to X and Y. $\mathcal{I}_+$ and $\mathcal{J}_+$ are generating sets for $\mathcal{T}$.

Mike Hill
Mike Hopkins
Doug Ravenel

The projective model structure on the category of spectra

We can define the category of spectra to be $[\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$, the category of $\mathcal{T}$-valued $\mathcal{T}$-functors on the $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$. We define the projective model structure on it as follows.

- A map $f : X \to Y$ is a weak equivalence or fibration if $f_n : X_n \to Y_n$ is one for each $n \geq 0$. In other words, weak equivalences and fibrations are strict weak equivalences and fibrations.
- Cofibrations are defined in terms of left lifting properties.

This model structure is known to be cofibrantly generated with the following generating sets.

$$\mathcal{I}^{proj} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{I}_+$$

$$\mathcal{J}^{proj} = \left\{ S^{-m} \wedge (j_{n+} : I_+^n \to I_+^{n+1}) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{J}_+$$

where $f_+ : X_+ \to Y_+$ denotes $f : X \to Y$ with disjoint base points added to X and Y. $\mathcal{I}_+$ and $\mathcal{J}_+$ are generating sets for $\mathcal{T}$. They are the pointed analogs of $\mathcal{I}$ and $\mathcal{J}$,

The projective model structure on the category of spectra

We can define the category of spectra to be $[\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$, the category of $\mathcal{T}$-valued $\mathcal{T}$-functors on the $\mathcal{T}$-category $\mathscr{J}^{\mathbf{N}}$. We define the projective model structure on it as follows.

- A map $f : X \to Y$ is a weak equivalence or fibration if $f_n : X_n \to Y_n$ is one for each $n \geq 0$. In other words, weak equivalences and fibrations are strict weak equivalences and fibrations.
- Cofibrations are defined in terms of left lifting properties.

This model structure is known to be cofibrantly generated with the following generating sets.

$$\mathcal{I}^{proj} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{I}_+$$

$$\mathcal{J}^{proj} = \left\{ S^{-m} \wedge (j_{n+} : I_+^n \to I_+^{n+1}) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{J}_+$$

where $f_+ : X_+ \to Y_+$ denotes $f : X \to Y$ with disjoint base points added to X and Y. $\mathcal{I}_+$ and $\mathcal{J}_+$ are generating sets for $\mathcal{T}$. They are the pointed analogs of $\mathcal{I}$ and $\mathcal{J}$, the generating sets for $\mathcal{T}op$.

Mike Hill
Mike Hopkins
Doug Ravenel

A generalization

The above can be generalized as follows.

Mike Hill
Mike Hopkins
Doug Ravenel

A generalization

The above can be generalized as follows.

- Replace $\mathcal{T}$ by a pointed cofibrantly generated model category $\mathcal{M}$ with a closed symmetic monoidal structure

Mike Hill
Mike Hopkins
Doug Ravenel

A generalization

The above can be generalized as follows.

- Replace $\mathcal{T}$ by a pointed cofibrantly generated model category $\mathcal{M}$ with a closed symmetic monoidal structure (sometimes called a cofibrantly generated Quillen ring)

Mike Hill
Mike Hopkins
Doug Ravenel

A generalization

The above can be generalized as follows.

- Replace $\mathcal{T}$ by a pointed cofibrantly generated model category $\mathcal{M}$ with a closed symmetic monoidal structure (sometimes called a cofibrantly generated Quillen ring) and generating sets $\mathcal{I}$ an $\mathcal{J}$.

A generalization

The above can be generalized as follows.

- Replace $\mathcal{T}$ by a pointed cofibrantly generated model category $\mathcal{M}$ with a closed symmetic monoidal structure (sometimes called a cofibrantly generated Quillen ring) and generating sets $\mathcal{I}$ an $\mathcal{J}$. For example, $\mathcal{M}$ could be $\mathcal{T}^G$, the category of pointed G-spaces with the Bredon model structure.

A generalization

The above can be generalized as follows.

- Replace $\mathcal{T}$ by a pointed cofibrantly generated model category $\mathcal{M}$ with a closed symmetic monoidal structure (sometimes called a cofibrantly generated Quillen ring) and generating sets $\mathcal{I}$ an $\mathcal{J}$. For example, $\mathcal{M}$ could be $\mathcal{T}^G$, the category of pointed G-spaces with the Bredon model structure.
- Replace the suspension functor $\Sigma = S^1 \wedge -$

Mike Hill
Mike Hopkins
Doug Ravenel

A generalization

The above can be generalized as follows.

- Replace $\mathcal{T}$ by a pointed cofibrantly generated model category $\mathcal{M}$ with a closed symmetic monoidal structure (sometimes called a cofibrantly generated Quillen ring) and generating sets $\mathcal{I}$ an $\mathcal{J}$. For example, $\mathcal{M}$ could be $\mathcal{T}^G$, the category of pointed G-spaces with the Bredon model structure.
- Replace the suspension functor $\Sigma = S^1 \wedge -$ by the functor $K \wedge -$ for a fixed cofibrant object K,

Mike Hill
Mike Hopkins
Doug Ravenel

A generalization

The above can be generalized as follows.

- Replace $\mathcal{T}$ by a pointed cofibrantly generated model category $\mathcal{M}$ with a closed symmetic monoidal structure (sometimes called a cofibrantly generated Quillen ring) and generating sets $\mathcal{I}$ an $\mathcal{J}$. For example, $\mathcal{M}$ could be $\mathcal{T}^G$, the category of pointed G-spaces with the Bredon model structure.
- Replace the suspension functor $\Sigma = S^1 \wedge -$ by the functor $K \wedge -$ for a fixed cofibrant object K, such as S^{ρ_G},

Mike Hill
Mike Hopkins
Doug Ravenel

A generalization

The above can be generalized as follows.

- Replace $\mathcal{T}$ by a pointed cofibrantly generated model category $\mathcal{M}$ with a closed symmetic monoidal structure (sometimes called a cofibrantly generated Quillen ring) and generating sets $\mathcal{I}$ an $\mathcal{J}$. For example, $\mathcal{M}$ could be $\mathcal{T}^G$, the category of pointed G-spaces with the Bredon model structure.
- Replace the suspension functor $\Sigma = S^1 \wedge -$ by the functor $K \wedge -$ for a fixed cofibrant object K, such as S^{ρ_G}, the sphere associated with the regular representation of the finite group G.

Mike Hill
Mike Hopkins
Doug Ravenel

A generalization

The above can be generalized as follows.

- Replace $\mathcal{T}$ by a pointed cofibrantly generated model category $\mathcal{M}$ with a closed symmetic monoidal structure (sometimes called a cofibrantly generated Quillen ring) and generating sets $\mathcal{I}$ an $\mathcal{J}$. For example, $\mathcal{M}$ could be $\mathcal{T}^G$, the category of pointed G-spaces with the Bredon model structure.
- Replace the suspension functor $\Sigma = S^1 \wedge -$ by the functor $K \wedge -$ for a fixed cofibrant object K, such as S^{ρ_G}, the sphere associated with the regular representation of the finite group G.
- Replace $\mathscr{J}^{\mathbf{N}}$ by the $\mathcal{M}$-category $\mathscr{J}_K^{\mathbf{N}}$ with morphism objects

Mike Hill
Mike Hopkins
Doug Ravenel

A generalization

The above can be generalized as follows.

- Replace $\mathcal{T}$ by a pointed cofibrantly generated model category $\mathcal{M}$ with a closed symmetic monoidal structure (sometimes called a cofibrantly generated Quillen ring) and generating sets $\mathcal{I}$ an $\mathcal{J}$. For example, $\mathcal{M}$ could be $\mathcal{T}^G$, the category of pointed G-spaces with the Bredon model structure.
- Replace the suspension functor $\Sigma = S^1 \wedge -$ by the functor $K \wedge -$ for a fixed cofibrant object K, such as S^{ρ_G}, the sphere associated with the regular representation of the finite group G.
- Replace $\mathscr{J}^{\mathbf{N}}$ by the $\mathcal{M}$-category $\mathscr{J}^{\mathbf{N}}_K$ with morphism objects

$$\mathscr{J}^{\mathbf{N}}_K(m, n) = \begin{cases} K^{\wedge(n-m)} & \text{for } n \geq m \\ * & \text{otherwise.} \end{cases}$$

Mike Hill
Mike Hopkins
Doug Ravenel

A generalization

The above can be generalized as follows.

- Replace $\mathcal{T}$ by a pointed cofibrantly generated model category $\mathcal{M}$ with a closed symmetic monoidal structure (sometimes called a cofibrantly generated Quillen ring) and generating sets $\mathcal{I}$ an $\mathcal{J}$. For example, $\mathcal{M}$ could be $\mathcal{T}^G$, the category of pointed G-spaces with the Bredon model structure.
- Replace the suspension functor $\Sigma = S^1 \wedge -$ by the functor $K \wedge -$ for a fixed cofibrant object K, such as S^{ρ_G}, the sphere associated with the regular representation of the finite group G.
- Replace $\mathscr{J}^{\mathbf{N}}$ by the $\mathcal{M}$-category $\mathscr{J}_K^{\mathbf{N}}$ with morphism objects

$$\mathscr{J}_K^{\mathbf{N}}(m,n) = \begin{cases} K^{\wedge(n-m)} & \text{for } n \geq m \\ * & \text{otherwise.} \end{cases}$$

- Replace the Yoneda spectrum S^{-m} by the functor $K^{-m} : \mathscr{J}_K^{\mathbf{N}} \to \mathcal{M}$ given by

Mike Hill
Mike Hopkins
Doug Ravenel

A generalization

The above can be generalized as follows.

- Replace $\mathcal{T}$ by a pointed cofibrantly generated model category $\mathcal{M}$ with a closed symmetic monoidal structure (sometimes called a cofibrantly generated Quillen ring) and generating sets $\mathcal{I}$ an $\mathcal{J}$. For example, $\mathcal{M}$ could be $\mathcal{T}^G$, the category of pointed G-spaces with the Bredon model structure.
- Replace the suspension functor $\Sigma = S^1 \wedge -$ by the functor $K \wedge -$ for a fixed cofibrant object K, such as S^{ρ_G}, the sphere associated with the regular representation of the finite group G.
- Replace $\mathscr{I}^{\mathbf{N}}$ by the $\mathcal{M}$-category $\mathscr{I}_K^{\mathbf{N}}$ with morphism objects

$$\mathscr{I}_K^{\mathbf{N}}(m,n) = \begin{cases} K^{\wedge(n-m)} & \text{for } n \geq m \\ * & \text{otherwise.} \end{cases}$$

- Replace the Yoneda spectrum S^{-m} by the functor $K^{-m} : \mathscr{I}_K^{\mathbf{N}} \to \mathcal{M}$ given by

$$(K^{-m})_n = \mathscr{I}_K^{\mathbf{N}}(m,n).$$

Mike Hill
Mike Hopkins
Doug Ravenel

A generalization (continued)

Mike Hill
Mike Hopkins
Doug Ravenel

A generalization (continued)

Then we can define the projective model structure on the enriched functor category $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$ as follows.

Mike Hill
Mike Hopkins
Doug Ravenel

A generalization (continued)

Then we can define the projective model structure on the enriched functor category $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$ as follows.

- A map $f : X \to Y$ is a weak equivalence or fibration if $f_n : X_n \to Y_n$ is one for each $n \geq 0$.

Mike Hill
Mike Hopkins
Doug Ravenel

A generalization (continued)

Then we can define the projective model structure on the enriched functor category $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$ as follows.

- A map $f : X \to Y$ is a weak equivalence or fibration if $f_n : X_n \to Y_n$ is one for each $n \geq 0$.
- Cofibrations are defined in terms of left lifting properties.

Mike Hill
Mike Hopkins
Doug Ravenel

A generalization (continued)

Then we can define the projective model structure on the enriched functor category $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$ as follows.

- A map $f : X \to Y$ is a weak equivalence or fibration if $f_n : X_n \to Y_n$ is one for each $n \geq 0$.
- Cofibrations are defined in terms of left lifting properties.

This model structure is known to be cofibrantly generated with generating sets

$$\mathcal{I}^{proj} = \{K^{-m} \colon m \geq 0\} \wedge \mathcal{I}$$

and

$$\mathcal{J}^{proj} = \{K^{-m} \colon m \geq 0\} \wedge \mathcal{J}.$$

Mike Hill
Mike Hopkins
Doug Ravenel

MOre about Bousfield localization

Mike Hill
Mike Hopkins
Doug Ravenel

MOre about Bousfield localization

In order to discuss Bousfield localization more precisely, it helps to start with a model category that is enriched over a Quillen ring $\mathcal{M}$

Mike Hill
Mike Hopkins
Doug Ravenel

MOre about Bousfield localization

In order to discuss Bousfield localization more precisely, it helps to start with a model category that is enriched over a Quillen ring $\mathcal{M}$ (possibly but not necessarily the catgeory we want to localize),

Mike Hill
Mike Hopkins
Doug Ravenel

MOre about Bousfield localization

In order to discuss Bousfield localization more precisely, it helps to start with a model category that is enriched over a Quillen ring $\mathcal{M}$ (possibly but not necessarily the catgeory we want to localize), so we can speak of weak equivalences of morphisms objects.

Mike Hill
Mike Hopkins
Doug Ravenel

MOre about Bousfield localization

In order to discuss Bousfield localization more precisely, it helps to start with a model category that is enriched over a Quillen ring $\mathcal{M}$ (possibly but not necessarily the catgeory we want to localize), so we can speak of weak equivalences of morphisms objects. Recall that a Quillen ring $\mathcal{M}$ is model category with a closed symmetic monoidal structure.

Mike Hill
Mike Hopkins
Doug Ravenel

MOre about Bousfield localization

In order to discuss Bousfield localization more precisely, it helps to start with a model category that is enriched over a Quillen ring $\mathcal{M}$ (possibly but not necessarily the catgeory we want to localize), so we can speak of weak equivalences of morphisms objects. Recall that a Quillen ring $\mathcal{M}$ is model category with a closed symmetic monoidal structure. A Quillen $\mathcal{M}$-module

Mike Hill
Mike Hopkins
Doug Ravenel

MOre about Bousfield localization

In order to discuss Bousfield localization more precisely, it helps to start with a model category that is enriched over a Quillen ring $\mathcal{M}$ (possibly but not necessarily the catgeory we want to localize), so we can speak of weak equivalences of morphisms objects. Recall that a Quillen ring $\mathcal{M}$ is model category with a closed symmetic monoidal structure. A Quillen $\mathcal{M}$-module is a model category $\mathcal{N}$ that is enriched and bitensored over $\mathcal{M}$.

Mike Hill
Mike Hopkins
Doug Ravenel

More about Bousfield localization (continued)

Mike Hill
Mike Hopkins
Doug Ravenel

More about Bousfield localization (continued)

Definition

Let $\mathcal{N}$ be a module over Quillen ring $\mathcal{M}$ as above,

Mike Hill
Mike Hopkins
Doug Ravenel

More about Bousfield localization (continued)

Definition

Let $\mathcal{N}$ be a module over Quillen ring $\mathcal{M}$ as above, and let S be a set of morphisms in $\mathcal{N}$.

Mike Hill
Mike Hopkins
Doug Ravenel

More about Bousfield localization (continued)

Definition

Let $\mathcal{N}$ be a module over Quillen ring $\mathcal{M}$ as above, and let S be a set of morphisms in $\mathcal{N}$.

An object Z is S-local if for each $f : A \to B$ in S, the map

$$f^* : \mathcal{N}(B, Z) \to \mathcal{N}(A, Z)$$

is a weak equivalence in $\mathcal{M}$.

Mike Hill
Mike Hopkins
Doug Ravenel

More about Bousfield localization (continued)

Definition

Let $\mathcal{N}$ be a module over Quillen ring $\mathcal{M}$ as above, and let S be a set of morphisms in $\mathcal{N}$.

An object Z is S-local if for each $f : A \to B$ in S, the map

$$f^* : \mathcal{N}(B, Z) \to \mathcal{N}(A, Z)$$

is a weak equivalence in $\mathcal{M}$.

A morphism $g : X \to Y$ in $\mathcal{N}$ is an S-equivalence if for each S-local object Z the map

Mike Hill
Mike Hopkins
Doug Ravenel

More about Bousfield localization (continued)

Definition

Let $\mathcal{N}$ be a module over Quillen ring $\mathcal{M}$ as above, and let S be a set of morphisms in $\mathcal{N}$.

An object Z is S-local if for each $f : A \to B$ in S, the map

$$f^* : \mathcal{N}(B, Z) \to \mathcal{N}(A, Z)$$

is a weak equivalence in $\mathcal{M}$.

A morphism $g : X \to Y$ in $\mathcal{N}$ is an S-equivalence if for each S-local object Z the map

$$g^* : \mathcal{N}(Y, Z) \to \mathcal{N}(X, Z)$$

is a weak equivalence in $\mathcal{M}$.

Mike Hill
Mike Hopkins
Doug Ravenel

More about Bousfield localization (continued)

Mike Hill
Mike Hopkins
Doug Ravenel

More about Bousfield localization (continued)

It is easy to verify that every weak equivalence is an S-equivalence,

Mike Hill
Mike Hopkins
Doug Ravenel

More about Bousfield localization (continued)

It is easy to verify that every weak equivalence is an S-equivalence, that a retract of an S-equivalence is an S-equivalence,

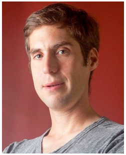
Mike Hill
Mike Hopkins
Doug Ravenel

More about Bousfield localization (continued)

It is easy to verify that every weak equivalence is an S-equivalence, that a retract of an S-equivalence is an S-equivalence, and that S-equivalences have the 2-of-3 property.

Clark Barwick

Phil Hirschhorn



Jacob Lurie

Jeff Smith

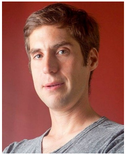
Mike Hill
Mike Hopkins
Doug Ravenel

More about Bousfield localization (continued)

It is easy to verify that every weak equivalence is an S-equivalence, that a retract of an S-equivalence is an S-equivalence, and that S-equivalences have the 2-of-3 property.

Clark Barwick

Phil Hirschhorn



Jacob Lurie

Jeff Smith

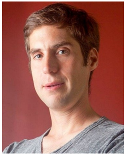
The four shown above have shown that under various mild hypotheses on $\mathcal{N}$,

More about Bousfield localization (continued)

It is easy to verify that every weak equivalence is an S-equivalence, that a retract of an S-equivalence is an S-equivalence, and that S-equivalences have the 2-of-3 property.

Clark Barwick

Phil Hirschhorn



Jacob Lurie

Jeff Smith

The four shown above have shown that under various mild hypotheses on $\mathcal{N}$, the class of S-equivalences leads to a new model structure on $\mathcal{N}$

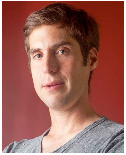
Mike Hill
Mike Hopkins
Doug Ravenel

More about Bousfield localization (continued)

It is easy to verify that every weak equivalence is an S-equivalence, that a retract of an S-equivalence is an S-equivalence, and that S-equivalences have the 2-of-3 property.


Clark Barwick


Phil Hirschhorn


Jacob Lurie


Jeff Smith

The four shown above have shown that under various mild hypotheses on $\mathcal{N}$, the class of S-equivalences leads to a new model structure on $\mathcal{N}$ for any morphism set S.

Mike Hill
Mike Hopkins
Doug Ravenel

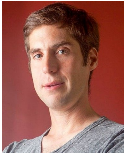
Mike Hill
Mike Hopkins
Doug Ravenel

More about Bousfield localization (continued)

It is easy to verify that every weak equivalence is an S-equivalence, that a retract of an S-equivalence is an S-equivalence, and that S-equivalences have the 2-of-3 property.

Clark Barwick

Phil Hirschhorn



Jacob Lurie

Jeff Smith

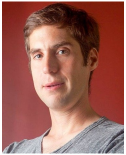
The four shown above have shown that under various mild hypotheses on $\mathcal{N}$, the class of S-equivalences leads to a new model structure on $\mathcal{N}$ for any morphism set S. We denote this new model category by $L_S\mathcal{N}$.

More about Bousfield localization (continued)

It is easy to verify that every weak equivalence is an S-equivalence, that a retract of an S-equivalence is an S-equivalence, and that S-equivalences have the 2-of-3 property.


Clark Barwick


Phil Hirschhorn


Jacob Lurie


Jeff Smith

The four shown above have shown that under various mild hypotheses on $\mathcal{N}$, the class of S-equivalences leads to a new model structure on $\mathcal{N}$ for any morphism set S. We denote this new model category by $L_S\mathcal{N}$. We also denote its fibrant replacement functor by L_S.

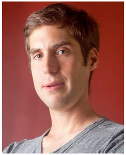
Mike Hill
Mike Hopkins
Doug Ravenel

More about Bousfield localization (continued)

It is easy to verify that every weak equivalence is an S-equivalence, that a retract of an S-equivalence is an S-equivalence, and that S-equivalences have the 2-of-3 property.

Clark Barwick

Phil Hirschhorn



Jacob Lurie

Jeff Smith

The four shown above have shown that under various mild hypotheses on $\mathcal{N}$, the class of S-equivalences leads to a new model structure on $\mathcal{N}$ for any morphism set S. We denote this new model category by $L_S\mathcal{N}$. We also denote its fibrant replacement functor by L_S. The fibrant objects of $L_S\mathcal{N}$ are the S-local objects of $\mathcal{N}$.

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure

We will define a set S of morphisms in $\mathcal{S}p = [\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$ (and more generally in $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$)

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure

We will define a set S of morphisms in $\mathcal{S}p = [\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$ (and more generally in $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$)such that S-equivalences are stable equivalences.

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure

We will define a set S of morphisms in $\mathcal{S}p = [\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$ (and more generally in $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$)such that S-equivalences are stable equivalences.

For each $m \geq 0$, let the mth stabilizing map

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure

We will define a set S of morphisms in $\mathcal{S}p = [\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$ (and more generally in $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$)such that S-equivalences are stable equivalences.

For each $m \geq 0$, let the mth stabilizing map

$$s_m : S^{-1-m} \wedge S^1 \to S^{-m}$$

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure

We will define a set S of morphisms in $\mathcal{S}p = [\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$ (and more generally in $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$)such that S-equivalences are stable equivalences.

For each $m \geq 0$, let the mth stabilizing map

$$s_m : S^{-1-m} \wedge S^1 \to S^{-m}$$

be the one whose nth component is

$$\begin{cases} * \to * & \text{for } n < m \\ * \to S^0 & \text{for } n = m \\ S^{n-m-1} \wedge S^1 \to S^{n-m} & \text{otherwise} \end{cases}$$

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure

We will define a set S of morphisms in $\mathcal{S}p = [\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$ (and more generally in $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$)such that S-equivalences are stable equivalences.

For each $m \geq 0$, let the *m*th stabilizing map

$$s_m : S^{-1-m} \wedge S^1 \to S^{-m}$$

be the one whose nth component is

$$\begin{cases} * \to * & \text{for } n < m \\ * \to S^0 & \text{for } n = m \\ S^{n-m-1} \wedge S^1 \to S^{n-m} & \text{otherwise} \end{cases}$$

Since this is a homeomorphism,

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure

We will define a set S of morphisms in $\mathcal{S}p = [\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$ (and more generally in $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$)such that S-equivalences are stable equivalences.

For each $m \geq 0$, let the *m*th stabilizing map

$$s_m : S^{-1-m} \wedge S^1 \to S^{-m}$$

be the one whose nth component is

$$\begin{cases} * \to * & \text{for } n < m \\ * \to S^0 & \text{for } n = m \\ S^{n-m-1} \wedge S^1 \to S^{n-m} & \text{otherwise} \end{cases}$$

Since this is a homeomorphism, and hence a weak equivalence, for large n,

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure

We will define a set S of morphisms in $\mathcal{S}p = [\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$ (and more generally in $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$)such that S-equivalences are stable equivalences.

For each $m \geq 0$, let the mth stabilizing map

$$s_m : S^{-1-m} \wedge S^1 \to S^{-m}$$

be the one whose nth component is

$$\begin{cases} * \to * & \text{for } n < m \\ * \to S^0 & \text{for } n = m \\ S^{n-m-1} \wedge S^1 \to S^{n-m} & \text{otherwise} \end{cases}$$

Since this is a homeomorphism, and hence a weak equivalence, for large n, s_m is a stable equivalence.

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure

We will define a set S of morphisms in $\mathcal{S}p = [\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$ (and more generally in $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$)such that S-equivalences are stable equivalences.

For each $m \geq 0$, let the mth stabilizing map

$$s_m : S^{-1-m} \wedge S^1 \to S^{-m}$$

be the one whose nth component is

$$\begin{cases} * \to * & \text{for } n < m \\ * \to S^0 & \text{for } n = m \\ S^{n-m-1} \wedge S^1 \to S^{n-m} & \text{otherwise} \end{cases}$$

Since this is a homeomorphism, and hence a weak equivalence, for large n, s_m is a stable equivalence.

The morphism set we want is

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure

We will define a set S of morphisms in $\mathcal{S}p = [\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$ (and more generally in $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$)such that S-equivalences are stable equivalences.

For each $m \geq 0$, let the mth stabilizing map

$$s_m : S^{-1-m} \wedge S^1 \to S^{-m}$$

be the one whose nth component is

$$\begin{cases} * \to * & \text{for } n < m \\ * \to S^0 & \text{for } n = m \\ S^{n-m-1} \wedge S^1 \to S^{n-m} & \text{otherwise} \end{cases}$$

Since this is a homeomorphism, and hence a weak equivalence, for large n, s_m is a stable equivalence.

The morphism set we want is

$$S = \{s_m : m \geq 0\}.$$

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

The morphism set we want is

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

The morphism set we want is

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\}.$$

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

The morphism set we want is

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\}.$$

What are the S-local objects?

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

The morphism set we want is

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\}.$$

What are the S-local objects? Now for the fun part!

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

The morphism set we want is

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\}.$$

What are the S-local objects? Now for the fun part! The Yoneda lemma implies that for any space K and spectrum Z,

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

The morphism set we want is

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\}.$$

What are the S-local objects? Now for the fun part! The Yoneda lemma implies that for any space K and spectrum Z,

$$\mathcal{S}p(S^{-n} \wedge K, Z) \cong (Z_n)^K.$$

Stabilizing maps and the stable model structure (continued)

The morphism set we want is

$$S = \{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \}.$$

What are the S-local objects? Now for the fun part! The Yoneda lemma implies that for any space K and spectrum Z,

$$\mathcal{S}p(S^{-n} \wedge K, Z) \cong (Z_n)^K.$$

This means that s_m^* is the map

$$\eta_m^Z : Z_m \to \Omega Z_{m+1},$$

the adjoint of the structure map $\epsilon_m^Z : \Sigma Z_m \to Z_{m+1}$.

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

The morphism set we want is

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\}.$$

What are the S-local objects? Now for the fun part! The Yoneda lemma implies that for any space K and spectrum Z,

$$\mathcal{S}p(S^{-n} \wedge K, Z) \cong (Z_n)^K.$$

This means that s_m^* is the map

$$\eta_m^Z : Z_m \to \Omega Z_{m+1},$$

the adjoint of the structure map $\epsilon_m^Z : \Sigma Z_m \to Z_{m+1}$.

The spectrum Z is S-local iff the map η_m^Z is a weak equivalence for each $m \geq 0$,

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

The morphism set we want is

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\}.$$

What are the S-local objects? Now for the fun part! The Yoneda lemma implies that for any space K and spectrum Z,

$$\mathcal{S}p(S^{-n} \wedge K, Z) \cong (Z_n)^K.$$

This means that s_m^* is the map

$$\eta_m^Z : Z_m \to \Omega Z_{m+1},$$

the adjoint of the structure map $\epsilon_m^Z : \Sigma Z_m \to Z_{m+1}$.

The spectrum Z is S-local iff the map η_m^Z is a weak equivalence for each $m \geq 0$, i.e., Z is an Ω-spectrum as classically defined.

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

The morphism set we want is

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\}.$$

What are the S-local objects? Now for the fun part! The Yoneda lemma implies that for any space K and spectrum Z,

$$\mathcal{S}p(S^{-n} \wedge K, Z) \cong (Z_n)^K.$$

This means that s_m^* is the map

$$\eta_m^Z : Z_m \to \Omega Z_{m+1},$$

the adjoint of the structure map $\epsilon_m^Z : \Sigma Z_m \to Z_{m+1}$.

The spectrum Z is S-local iff the map η_m^Z is a weak equivalence for each $m \geq 0$, i.e., Z is an Ω-spectrum as classically defined. The observation that the fibrant objects are the Ω-spectra

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

The morphism set we want is

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\}.$$

What are the S-local objects? Now for the fun part! The Yoneda lemma implies that for any space K and spectrum Z,

$$\mathcal{S}p(S^{-n} \wedge K, Z) \cong (Z_n)^K.$$

This means that s_m^* is the map

$$\eta_m^Z : Z_m \to \Omega Z_{m+1},$$

the adjoint of the structure map $\epsilon_m^Z : \Sigma Z_m \to Z_{m+1}$.

The spectrum Z is S-local iff the map η_m^Z is a weak equivalence for each $m \geq 0$, i.e., Z is an Ω-spectrum as classically defined. The observation that the fibrant objects are the Ω-spectra is originally due to Bousfield-Friedlander, 1978.

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

For

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\},$$

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

For

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\},$$

a spectrum Z is S-local iff it is an Ω-spectrum.

Stabilizing maps and the stable model structure (continued)

For

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\},$$

a spectrum Z is S-local iff it is an Ω-spectrum.

What are the S-equivalences?

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

For

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\},$$

a spectrum Z is S-local iff it is an Ω-spectrum.

What are the S-equivalences? A map $g : X \to Y$ is an S-equivalence if

$$g^* : \mathcal{S}p(Y, Z) \to \mathcal{S}p(X, Z)$$

is a weak equivalence for every Ω-psectrum Z,

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

For

$$S = \{s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0\},$$

a spectrum Z is S-local iff it is an Ω-spectrum.

What are the S-equivalences? A map $g : X \to Y$ is an S-equivalence if

$$g^* : \mathcal{S}p(Y, Z) \to \mathcal{S}p(X, Z)$$

is a weak equivalence for every Ω-psectrum Z, if g induces an isomorphism in every generalized cohomology theory.

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

For

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\},$$

a spectrum Z is S-local iff it is an Ω-spectrum.

What are the S-equivalences? A map $g : X \to Y$ is an S-equivalence if

$$g^* : \mathcal{S}p(Y, Z) \to \mathcal{S}p(X, Z)$$

is a weak equivalence for every Ω-psectrum Z, if g induces an isomorphism in every generalized cohomology theory. This coincides with a classical definition of stable equivalence.

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

For

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\},$$

a spectrum Z is S-local iff it is an Ω-spectrum.

What are the S-equivalences? A map $g : X \to Y$ is an S-equivalence if

$$g^* : \mathcal{S}p(Y, Z) \to \mathcal{S}p(X, Z)$$

is a weak equivalence for every Ω-psectrum Z, if g induces an isomorphism in every generalized cohomology theory. This coincides with a classical definition of stable equivalence.

This means that the Bousfield localization $L_S \mathcal{S}p$ is the category of classically define spectra in which weak equivalences are stable equivalences.

Mike Hill
Mike Hopkins
Doug Ravenel

Stabilizing maps and the stable model structure (continued)

For

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\},$$

a spectrum Z is S-local iff it is an Ω-spectrum.

What are the S-equivalences? A map $g : X \to Y$ is an S-equivalence if

$$g^* : \mathcal{S}p(Y, Z) \to \mathcal{S}p(X, Z)$$

is a weak equivalence for every Ω-psectrum Z, if g induces an isomorphism in every generalized cohomology theory. This coincides with a classical definition of stable equivalence.

This means that the Bousfield localization $L_S\mathcal{S}p$ is the category of classically define spectra in which weak equivalences are stable equivalences. Its homotopy cetegory is the one described long ago by Boardman and Vogt.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category

Recall that the projective (or strict) model structure on $\mathcal{S}p$ has cofibrant generating sets

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category

Recall that the projective (or strict) model structure on $\mathcal{S}p$ has cofibrant generating sets

$$\mathcal{I}^{proj} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{I}_+$$

$$\mathcal{J}^{proj} = \left\{ S^{-m} \wedge (j_{n+} : I_+^n \to I_+^{n+1}) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{J}_+$$

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category

Recall that the projective (or strict) model structure on $\mathcal{S}p$ has cofibrant generating sets

$$\mathcal{I}^{proj} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{I}_+$$

$$\mathcal{J}^{proj} = \left\{ S^{-m} \wedge (j_{n+} : I_+^n \to I_+^{n+1}) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{J}_+$$

We can define $\mathcal{I}^{stable}$ to be $\mathcal{I}^{proj}$,

Mike Hill
Mike Hopkins
Doug Ravenel

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category

Recall that the projective (or strict) model structure on $\mathcal{S}p$ has cofibrant generating sets

$$\mathcal{I}^{proj} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{I}_+$$

$$\mathcal{J}^{proj} = \left\{ S^{-m} \wedge (j_{n+} : I_+^n \to I_+^{n+1}) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{J}_+$$

We can define $\mathcal{I}^{stable}$ to be $\mathcal{I}^{proj}$, but we must enlarge $\mathcal{J}^{proj}$ in some way to get $\mathcal{J}^{stable}$.

Cofibrant generating sets for the stable category

Recall that the projective (or strict) model structure on $\mathcal{S}p$ has cofibrant generating sets

$$\mathcal{I}^{proj} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{I}_+$$

$$\mathcal{J}^{proj} = \left\{ S^{-m} \wedge (j_{n+} : I_+^n \to I_+^{n+1}) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{J}_+$$

We can define $\mathcal{I}^{stable}$ to be $\mathcal{I}^{proj}$, but we must enlarge $\mathcal{J}^{proj}$ in some way to get $\mathcal{J}^{stable}$. To describe this we need the following.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Definition

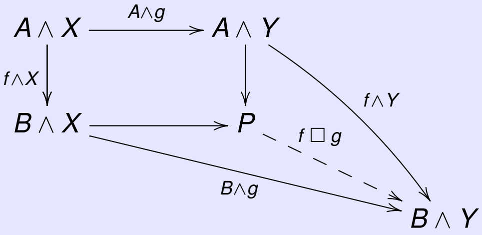
Let $\mathcal{M}$ be a Quillen ring with a morphism $g : X \to Y$,

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Definition

Let $\mathcal{M}$ be a Quillen ring with a morphism $g : X \to Y$, and $\mathcal{N}$ a Quillen $\mathcal{M}$-module with a morphism $f : A \to B$. Consider the diagram

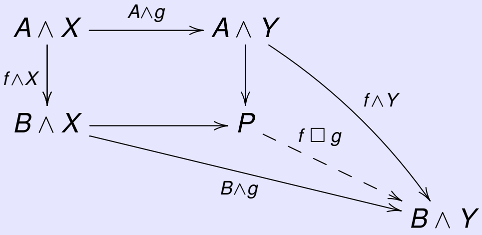


Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Definition

Let $\mathcal{M}$ be a Quillen ring with a morphism $g : X \to Y$, and $\mathcal{N}$ a Quillen $\mathcal{M}$-module with a morphism $f : A \to B$. Consider the diagram

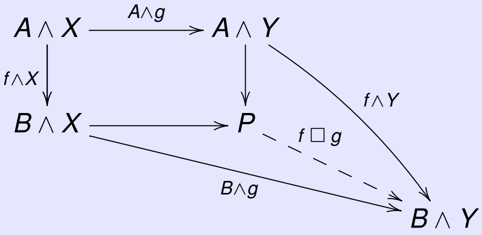


where P is the pushout of the two maps from $A \wedge X$.

Cofibrant generating sets for the stable category (continued)

Definition

Let $\mathcal{M}$ be a Quillen ring with a morphism $g : X \to Y$, and $\mathcal{N}$ a Quillen $\mathcal{M}$-module with a morphism $f : A \to B$. Consider the diagram



where P is the pushout of the two maps from $A \wedge X$. Then the pushout corner map (or pushout smash product) $f \,\Box\, g$ is the unique map $P \to B \wedge Y$ that makes the diagram commute.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

An easy example of a pushout corner map.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

An easy example of a pushout corner map. Let
$\mathcal{M} = \mathcal{N} = \mathcal{T}op$,

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

An easy example of a pushout corner map. Let $\mathcal{M} = \mathcal{N} = \mathcal{T}op$, let M and N be manifolds with boundary,

Mike Hill
Mike Hopkins
Doug Ravenel

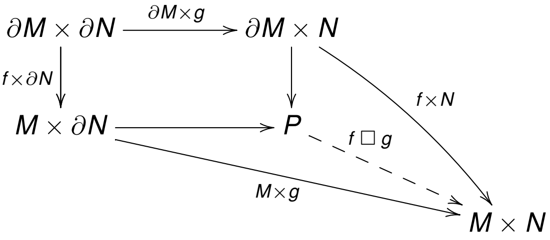
Cofibrant generating sets for the stable category (continued)

An easy example of a pushout corner map. Let $\mathcal{M} = \mathcal{N} = \mathcal{T}op$, let M and N be manifolds with boundary, and consider the morphisms $f : \partial M \to M$ and $g : \partial N \to N$, the inclusions of the boundaries.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

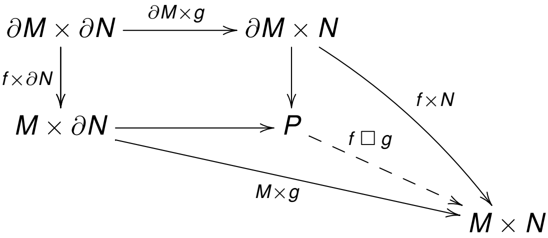
An easy example of a pushout corner map. Let $\mathcal{M} = \mathcal{N} = \mathcal{T}op$, let M and N be manifolds with boundary, and consider the morphisms $f : \partial M \to M$ and $g : \partial N \to N$, the inclusions of the boundaries. Then the diagram is



Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

An easy example of a pushout corner map. Let $\mathcal{M} = \mathcal{N} = \mathcal{T}op$, let M and N be manifolds with boundary, and consider the morphisms $f : \partial M \to M$ and $g : \partial N \to N$, the inclusions of the boundaries. Then the diagram is

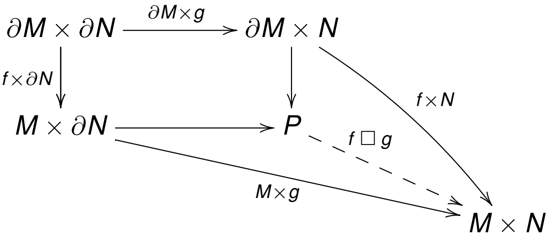


In this case the pushout is

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

An easy example of a pushout corner map. Let $\mathcal{M} = \mathcal{N} = \mathcal{T}op$, let M and N be manifolds with boundary, and consider the morphisms $f : \partial M \to M$ and $g : \partial N \to N$, the inclusions of the boundaries. Then the diagram is



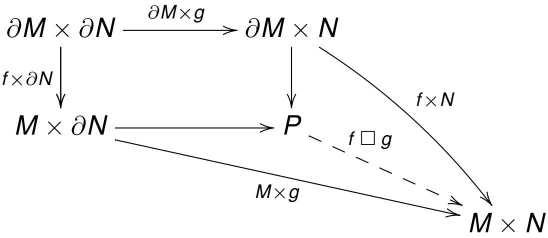
In this case the pushout is

$$P = (\partial M \times N) \cup_{\partial M \times \partial N} (M \times \partial N) = \partial(M \times N),$$

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

An easy example of a pushout corner map. Let $\mathcal{M} = \mathcal{N} = \mathcal{T}op$, let M and N be manifolds with boundary, and consider the morphisms $f : \partial M \to M$ and $g : \partial N \to N$, the inclusions of the boundaries. Then the diagram is



In this case the pushout is

$$P = (\partial M \times N) \cup_{\partial M \times \partial N} (M \times \partial N) = \partial(M \times N),$$

and $f \,\square\, g$ is the inclusion $\partial(M \times N) \to M \times N$.

Cofibrant generating sets for the stable category (continued)

Now we can describe the cofibrant generating sets for $L_s\mathcal{S}p$.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Now we can describe the cofibrant generating sets for $L_s\mathcal{S}p$. Recall again that

$$\mathcal{I}^{proj} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{I}_+$$

$$\mathcal{J}^{proj} = \left\{ S^{-m} \wedge (j_{n+} : I_+^n \to I_+^{n+1}) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{J}_+$$

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Now we can describe the cofibrant generating sets for $L_s\mathcal{S}p$. Recall again that

$$\mathcal{I}^{proj} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{I}_+$$
$$\mathcal{J}^{proj} = \left\{ S^{-m} \wedge (j_{n+} : I_+^n \to I_+^{n+1}) \colon m, n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathcal{J}_+$$

Theorem

The following are cofibrant generating sets for $L_S\mathcal{S}p$.

$$\begin{aligned} \mathcal{I}^{stable} &= \mathcal{I}^{proj} \\ \mathcal{J}^{stable} &= \mathcal{J}^{proj} \cup \{ s_m \square i_{n+} \colon m, n \geq 0 \} \\ &= \mathcal{J}^{proj} \cup (S \square \mathcal{I}_+). \end{aligned}$$

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Now we can describe the cofibrant generating sets for $L_s\mathcal{S}p$. Recall again that

$$\mathcal{I}^{proj} = \left\{S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0\right\} = \left\{S^{-m}\right\} \wedge \mathcal{I}_+$$
$$\mathcal{J}^{proj} = \left\{S^{-m} \wedge (j_{n+} : I_+^n \to I_+^{n+1}) \colon m, n \geq 0\right\} = \left\{S^{-m}\right\} \wedge \mathcal{J}_+$$

Theorem

The following are cofibrant generating sets for $L_S\mathcal{S}p$.

$$\begin{aligned} \mathcal{I}^{stable} &= \mathcal{I}^{proj} \\ \mathcal{J}^{stable} &= \mathcal{J}^{proj} \cup \{s_m \square i_{n+} \colon m, n \geq 0\} \\ &= \mathcal{J}^{proj} \cup (S \square \mathcal{I}_+). \end{aligned}$$

The proof consists of showing that these two sets satisfy the four (unnamed) technical conditions of the Kan Recognition Theorem.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Now we can describe the cofibrant generating sets for $L_s\mathcal{S}p$. Recall again that

$$\mathcal{I}^{proj} = \left\{S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0\right\} = \left\{S^{-m}\right\} \wedge \mathcal{I}_+$$
$$\mathcal{J}^{proj} = \left\{S^{-m} \wedge (j_{n+} : I_+^n \to I_+^{n+1}) \colon m, n \geq 0\right\} = \left\{S^{-m}\right\} \wedge \mathcal{J}_+$$

Theorem

The following are cofibrant generating sets for $L_S\mathcal{S}p$.

$$\begin{aligned}\mathcal{I}^{stable} &= \mathcal{I}^{proj}\\ \mathcal{J}^{stable} &= \mathcal{J}^{proj} \cup \{s_m \square i_{n+} \colon m, n \geq 0\}\\ &= \mathcal{J}^{proj} \cup (S \square \mathcal{I}_+).\end{aligned}$$

The proof consists of showing that these two sets satisfy the four (unnamed) technical conditions of the Kan Recognition Theorem. Most of it is routine.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Theorem

The following are cofibrant generating sets for $L_S\mathcal{S}p$.

$$\begin{aligned}\mathcal{I}^{stable} &= \mathcal{I}^{proj}\\ \mathcal{J}^{stable} &= \mathcal{J}^{proj} \cup \{s_m \square i_{n+} :\ m, n \geq 0\}\\ &= \mathcal{J}^{proj} \cup (S \square \mathcal{I}_+).\end{aligned}$$

The proof consists of showing that these two sets satisfy the four (unnamed) technical conditions of the Kan Recognition Theorem.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Theorem

The following are cofibrant generating sets for $L_S\mathcal{S}p$.

$$\begin{aligned}\mathcal{I}^{stable} &= \mathcal{I}^{proj}\\ \mathcal{J}^{stable} &= \mathcal{J}^{proj} \cup \{s_m \square i_{n+} : m, n \geq 0\}\\ &= \mathcal{J}^{proj} \cup (S \square \mathcal{I}_+).\end{aligned}$$

The proof consists of showing that these two sets satisfy the four (unnamed) technical conditions of the Kan Recognition Theorem. Most of it is routine.

Cofibrant generating sets for the stable category (continued)

Theorem

The following are cofibrant generating sets for $L_S\mathcal{S}p$.

$$\begin{aligned} \mathcal{I}^{stable} &= \mathcal{I}^{proj} \\ \mathcal{J}^{stable} &= \mathcal{J}^{proj} \cup \{s_m \square i_{n+} : m, n \geq 0\} \\ &= \mathcal{J}^{proj} \cup (S \square \mathcal{I}_+). \end{aligned}$$

The proof consists of showing that these two sets satisfy the four (unnamed) technical conditions of the Kan Recognition Theorem. Most of it is routine.

The most difficult point is to show that a stable equivalence with the right lifting property with respect to $\mathcal{J}^{stable}$ also has it with respect to $\mathcal{I}^{stable}$,

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Theorem

The following are cofibrant generating sets for $L_S\mathcal{S}p$.

$$\begin{aligned}\mathcal{I}^{stable} &= \mathcal{I}^{proj}\\ \mathcal{J}^{stable} &= \mathcal{J}^{proj} \cup \{s_m \square i_{n+} : m, n \geq 0\}\\ &= \mathcal{J}^{proj} \cup (S \square \mathcal{I}_+).\end{aligned}$$

The proof consists of showing that these two sets satisfy the four (unnamed) technical conditions of the Kan Recognition Theorem. Most of it is routine.

The most difficult point is to show that a stable equivalence with the right lifting property with respect to $\mathcal{J}^{stable}$ also has it with respect to $\mathcal{I}^{stable}$, which means it is a trivial fibration.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Again, the key point is to show that a stable equivalence $p : X \to Y$ with the right lifting property with respect to

$$\mathcal{J}^{stable} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\}$$
$$\cup \{ s_m \square i_{n+} \colon m, n \geq 0 \}$$

Cofibrant generating sets for the stable category (continued)

Again, the key point is to show that a stable equivalence $p: X \to Y$ with the right lifting property with respect to

$$\mathcal{J}^{stable} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\}$$
$$\cup \left\{ s_m \,\square\, i_{n+} \colon m, n \geq 0 \right\}$$

also has it with respect to

$$\mathcal{I}^{proj} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\}.$$

Cofibrant generating sets for the stable category (continued)

Again, the key point is to show that a stable equivalence $p: X \to Y$ with the right lifting property with respect to

$$\mathcal{J}^{stable} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\} \cup \{ s_m \,\square\, i_{n+} \colon m, n \geq 0 \}$$

also has it with respect to

$$\mathcal{I}^{proj} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\}.$$

Hence we are looking at a strict fibration that has the right lifting property with respect to each pushout corner map $s_m \,\square\, i_{n+}$.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Again, the key point is to show that a stable equivalence $p : X \to Y$ with the right lifting property with respect to

$$\mathcal{J}^{stable} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\}$$
$$\cup \left\{ s_m \,\square\, i_{n+} \colon m, n \geq 0 \right\}$$

also has it with respect to

$$\mathcal{I}^{proj} = \left\{ S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m, n \geq 0 \right\}.$$

Hence we are looking at a strict fibration that has the right lifting property with respect to each pushout corner map $s_m \,\square\, i_{n+}$.

The latter condition is equivalent to the diagram

$$\begin{array}{ccc} X_m & \xrightarrow{\quad p_m \quad} & Y_m \\ {\scriptstyle \eta_m^X}\downarrow & & \downarrow{\scriptstyle \eta_m^Y} \\ \Omega X_{m+1} & \xrightarrow{\ \Omega p_{m+1}\ } & \Omega Y_{m+1} \end{array}$$

being homotopy Cartesian.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Recall the functor $\Lambda : \mathcal{S}p \to \mathcal{S}p$ for which $(\Lambda X)_m$ is the colimit of

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Recall the functor $\Lambda : \mathcal{S}p \to \mathcal{S}p$ for which $(\Lambda X)_m$ is the colimit of

$$X_m \xrightarrow{\eta_m^X} \Omega X_{m+1} \xrightarrow{\Omega\eta_{m+1}^X} \Omega^2 X_{m+2} \xrightarrow{\Omega^2\eta_{m+2}^X} \dots$$

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Recall the functor $\Lambda : \mathcal{S}p \to \mathcal{S}p$ for which $(\Lambda X)_m$ is the colimit of

$$X_m \xrightarrow{\eta_m^X} \Omega X_{m+1} \xrightarrow{\Omega\eta_{m+1}^X} \Omega^2 X_{m+2} \xrightarrow{\Omega^2\eta_{m+2}^X} \dots$$

We know that the corner map condition on our strict fibration $p : X \to Y$ implies that the diagram

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Recall the functor $\Lambda : \mathcal{S}p \to \mathcal{S}p$ for which $(\Lambda X)_m$ is the colimit of

$$X_m \xrightarrow{\eta_m^X} \Omega X_{m+1} \xrightarrow{\Omega\eta_{m+1}^X} \Omega^2 X_{m+2} \xrightarrow{\Omega^2\eta_{m+2}^X} \cdots$$

We know that the corner map condition on our strict fibration $p : X \to Y$ implies that the diagram

$$\begin{array}{ccc} X_m & \xrightarrow{p_m} & Y_m \\ \downarrow & & \downarrow \\ (\Lambda X)_m & \xrightarrow{\Lambda p_m} & (\Lambda Y)_m \end{array}$$

is homotopy Cartesian.

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Recall the functor $\Lambda : \mathcal{S}p \to \mathcal{S}p$ for which $(\Lambda X)_m$ is the colimit of

$$X_m \xrightarrow{\eta_m^X} \Omega X_{m+1} \xrightarrow{\Omega\eta_{m+1}^X} \Omega^2 X_{m+2} \xrightarrow{\Omega^2\eta_{m+2}^X} \cdots$$

We know that the corner map condition on our strict fibration $p : X \to Y$ implies that the diagram

$$\begin{array}{ccc} X_m & \xrightarrow{p_m} & Y_m \\ \downarrow & & \downarrow \\ (\Lambda X)_m & \xrightarrow{\Lambda p_m} & (\Lambda Y)_m \end{array}$$

is homotopy Cartesian. It is known that Λ converts stable equivalences to strict ones,

Mike Hill
Mike Hopkins
Doug Ravenel

Cofibrant generating sets for the stable category (continued)

Recall the functor $\Lambda : \mathcal{S}p \to \mathcal{S}p$ for which $(\Lambda X)_m$ is the colimit of

$$X_m \xrightarrow{\eta_m^X} \Omega X_{m+1} \xrightarrow{\Omega\eta_{m+1}^X} \Omega^2 X_{m+2} \xrightarrow{\Omega^2\eta_{m+2}^X} \dots$$

We know that the corner map condition on our strict fibration $p : X \to Y$ implies that the diagram

$$\begin{array}{ccc} X_m & \xrightarrow{p_m} & Y_m \\ \downarrow & & \downarrow \\ (\Lambda X)_m & \xrightarrow{\Lambda p_m} & (\Lambda Y)_m \end{array}$$

is homotopy Cartesian. It is known that Λ converts stable equivalences to strict ones, so p_m is a weak equivalence,

Cofibrant generating sets for the stable category (continued)

Recall the functor $\Lambda : \mathcal{S}p \to \mathcal{S}p$ for which $(\Lambda X)_m$ is the colimit of

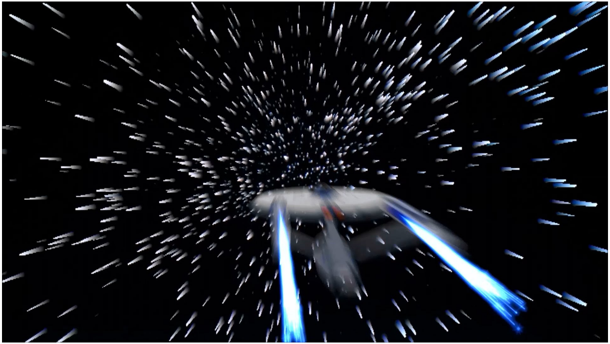
$$X_m \xrightarrow{\eta_m^X} \Omega X_{m+1} \xrightarrow{\Omega \eta_{m+1}^X} \Omega^2 X_{m+2} \xrightarrow{\Omega^2 \eta_{m+2}^X} \cdots$$

We know that the corner map condition on our strict fibration $p : X \to Y$ implies that the diagram

$$\begin{array}{ccc} X_m & \xrightarrow{p_m} & Y_m \\ \downarrow & & \downarrow \\ (\Lambda X)_m & \xrightarrow{\Lambda p_m} & (\Lambda Y)_m \end{array}$$

is homotopy Cartesian. It is known that Λ converts stable equivalences to strict ones, so p_m is a weak equivalence, which makes p a trivial fibration as desired.

Mike Hill
Mike Hopkins
Doug Ravenel



Thank you!

Mike Hill
Mike Hopkins
Doug Ravenel