Hot Topics: Life after the Telescope Conjecture

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Why the Telescope Conjecture?

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Three developments in the early 70s

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3. Associativity: $F(F(x, y), z) = F(x, F(y, z))$. This implies complicated relations among the $a_{i,j}$.

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Poster by
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- The height ∞ orbit is the linear subspace

$$\bigcap_{n>0} V_n.$$

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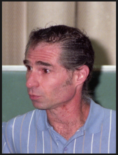
3. Smith-Toda complexes

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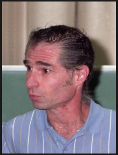

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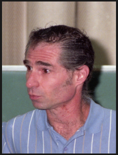


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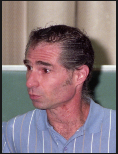


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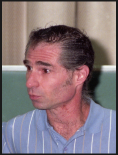


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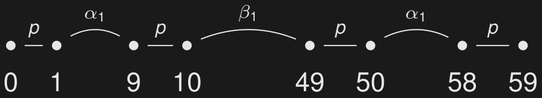
Cell diagram for $V(2)$ at $p = 5$, where $|v_1| = 8$ and $|v_2| = 48$:



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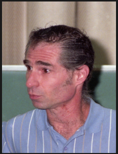


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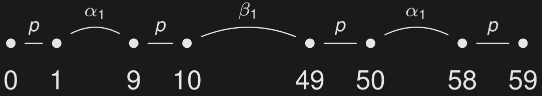

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Three developments in the early 70s (continued)

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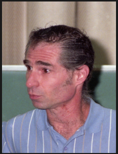

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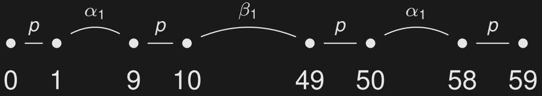

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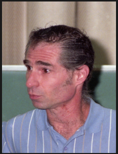


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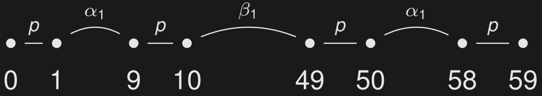

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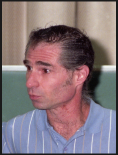

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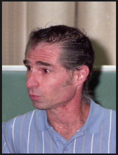

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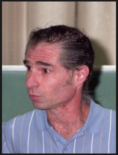

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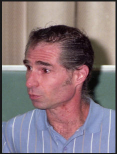

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SLMath Hot Topics
Life after the Telescope Conjecture
Why the Telescope Conjecture?

Doug Ravenel

Early 70s
Morava K-theory
Morava's vision
Smith-Toda complexes

Chromatic homotopy theory
Algebraic patterns
The chromatic filtration
Enter the telescope conjecture
The Hopkins-Smith periodicity theorem

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α_t is the composite

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Algebraic patterns

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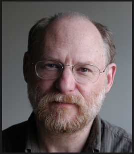

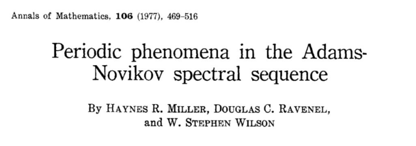

Annals of Mathematics, **106** (1977), 469–516

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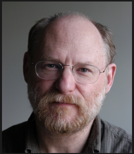

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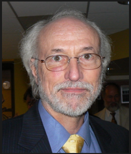

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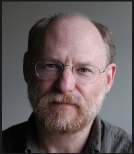

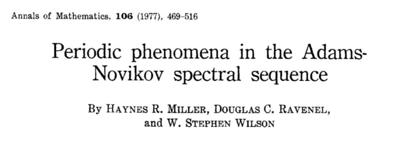

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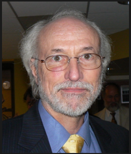

It organizes things into layers so that in the nth layer everything is v_n-periodic. The structure of this nth layer is controlled by the cohomology of the nth Morava stabilizer group $\mathbb{G}_n$.

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Let X be a p-local type n finite spectrum, meaning that $K(n)_* X \neq 0$ and $K(m)_* X = 0$ for $m < n$.

The Hopkins-Smith periodicity theorem

$$\Sigma^{|v_n|}V(n-1) \xrightarrow{w_n} V(n-1) \longrightarrow V(n)$$

Can we generalize this to $n > 3$? Not exactly. To this day, nobody has constructed $V(4)$ at any prime, and in 2010 Lee Nave showed that $V((p+1)/2)$ does not exist.

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$$w : \Sigma^d X \to X \qquad \text{where } K(n)_* w \text{ is an isomorphism.}$$

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Again the telescope conjecture equates the geometrically appealing telescope $w^{-1}X$ with the computationally accessible Bousfield localization $L_{K(n)}X$.

The telescope conjecture: Historical note

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San Francisco earthquake of October 17, 1989

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Jeremy, Tomer, myself, Ishan and Robert
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THANK YOU!

Scratch paper

More scratch paper

Even more scratch paper