The Transatlantic Transchromatic
Homotopy Theory Conference II
Andy Baker 70
31 July – 4 August, 2023

Universität Regensburg

What is an ∞-category?

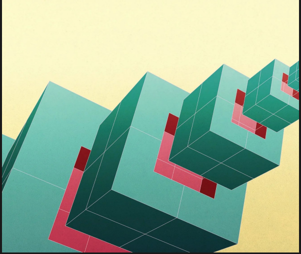

Image appearing in an article by Emily Riehl

in *Scientific American*, October 2021

Doug Ravenel
University of Rochester

4 August, 2023

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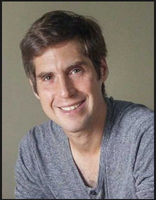

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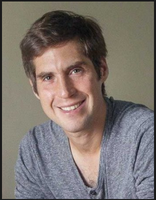

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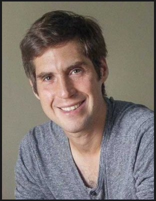

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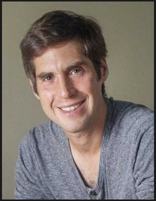

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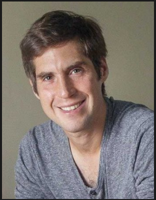

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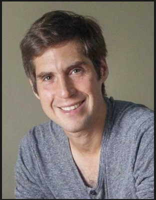

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- ∞-categories (that are not ordinary categories) will be written in lilac.

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The inner faces and horns are those for which $0 < i < n$.

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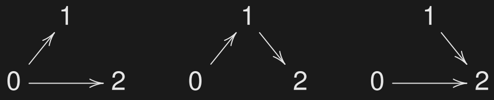

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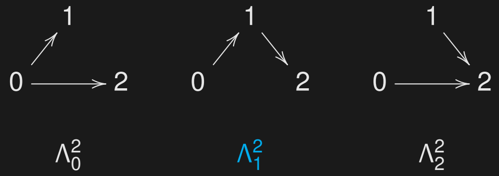

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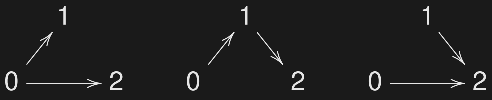

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A Kan complex is a simplicial set X for which every map from a horn $\Lambda^n_i \to X$ extends to Δ^n.

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$$\left\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0 \text{ and } \sum x_i = 1\right\}.$$

The geometric realization $|X|$ of a simplicial set X is the colimit of the Top-valued functor

$$[k] \mapsto X_k \times \Delta^k_{\text{top}}.$$

This space turns out to be the union of geometric realizations of the nondegenerate topological simplices of X, meaning ones not in the image of any degeneracy map. In particular, $|\Delta^n| = \Delta^n_{\text{top}} \approx D^n$, $|\partial\Delta^n| \approx S^{n-1}$,

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Review of simplicial sets (continued)

Given simplicial sets X and Y, one can define a simplicial set $X \times Y$ in which

$$(X \times Y)_n = \coprod_{0 \le i \le n} X_i \times Y_{n-i} \quad \text{and} \quad |X \times Y| = |X| \times |Y|.$$

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Hence $\mathcal{S}\mathrm{et}_\Delta$ is enriched over itself.

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and define NC_n to be the set of functors from $[n]$ to C.

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This simplicial set has the following property: Any simplicial map $\Lambda_i^n \to NC$ for $0 < i < n$ extends uniquely to Δ^n.

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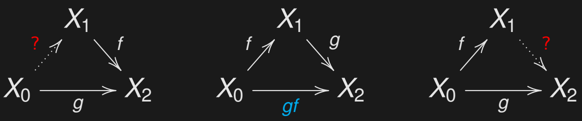

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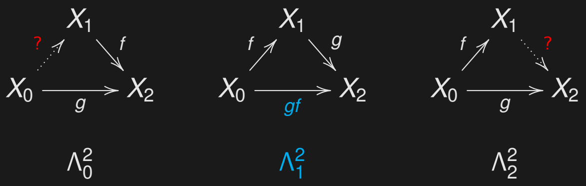

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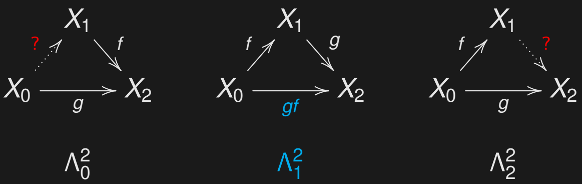

It is known that the category C is determined by its nerve, and that any simplicial set with property above is the nerve of some small category.

A small category is thus equivalent to a simplicial set (its nerve) in which each map from an inner horn Λ^n_i extends uniquely to a map from Δ^n.

The main definition

Definition

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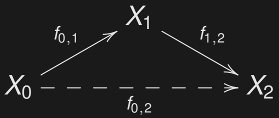

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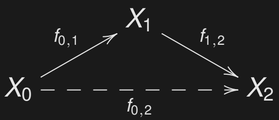

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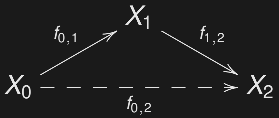

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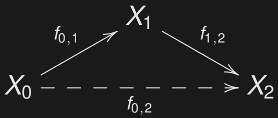

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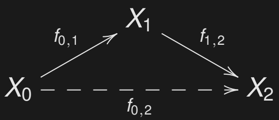

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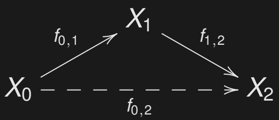

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To a topological space X we can associate an ∞-category X (also known as $\operatorname{Sing} X$, the singular simplicial set of X) in which X_n is the set of continuous maps $|\Delta^n| \to X$. X is also a Kan complex since a map $|\Lambda_i^n| \to X$, for any horn Λ_i^n, extends to $|\Delta^n|$.

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Such an ∞-category is called an ∞-groupoid because all morphisms, i.e., paths in X, are invertible up to homotopy.

The ∞-category of topological spaces

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There is another construction called the homotopy coherent nerve whose definition [HTT, Definition 1.1.5.5] baffled me for several years.

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The set of 2-simplices is more interesting. In the subcategory $N\mathbf{Top}$ (the ordinary nerve), it is the set of commutative diagrams of the form

$$\begin{array}{ccccc} & & X_1 & & \\ & \overset{f_{0,1}}{\nearrow} & & \overset{f_{1,2}}{\searrow} & \\ X_0 & & \xrightarrow[f_{1,2}f_{0,1}]{} & & X_2. \end{array}$$

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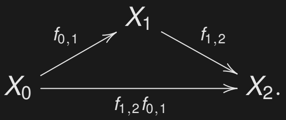
The top two edges can be viewed as a map $\Lambda^1_2 \to N\mathbf{Top}$, with the full diagram being its unique extension to Δ^2.

The ∞-category of topological spaces (continued)

$N\mathbf{Top}_2$ is the set of commutative diagrams of the form



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The set of 2-simplices $\mathcal{S}_2$ consists of similar diagrams in which the bottom arrow is replaced by any map $f_{0,2}$ homotopic to $f_{1,2}f_{0,1}$, with the homotopy $h_{0,2}$ being part of the datum.

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The ∞-category of topological spaces (continued)

$$\xymatrix{ & X_1 \ar[dr]^{f_{1,2}} \ar@{}[d]|{\Downarrow h_{0,2}} & \\ X_0 \ar[ur]^{f_{0,1}} \ar[rr]_{f_{0,2}} & & X_2 . }$$

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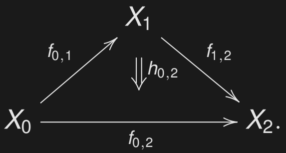
with certain properties. It is adjoint to a path (which we denote by the same symbol)

$$\begin{aligned} I &\xrightarrow{h_{0,2}} \mathbf{Top}(X_0, X_2) \\ 0 &\longmapsto f_{1,2}f_{0,1} \\ 1 &\longmapsto f_{0,2} \end{aligned}$$

The ∞-category of topological spaces (continued)



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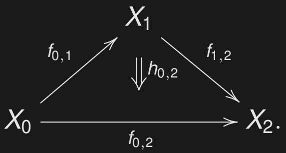

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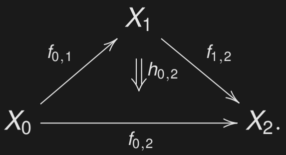

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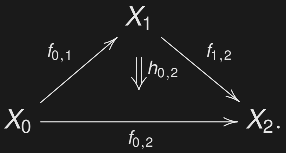

As in the ordinary case, the top two edges of the diagram can be viewed as a map $\Lambda_1^2 \to \mathcal{S}$. Now there is an extension of it to Δ^2 for each path $h_{0,2}$ in $\mathbf{Top}(X_0, X_2)$ starting at the point $f_{1,2}f_{0,1}$. The space of such paths is contractible.

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The following diagram shows four 2-simplices with their homotopies.

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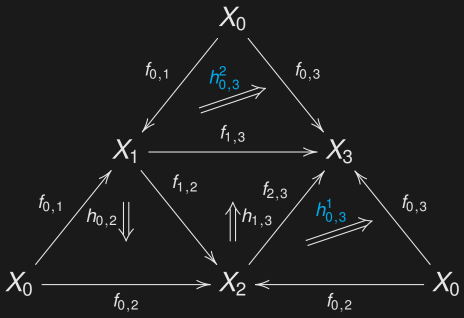

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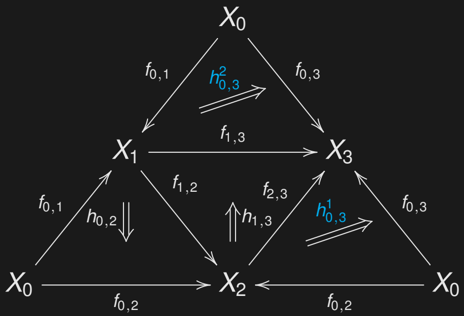

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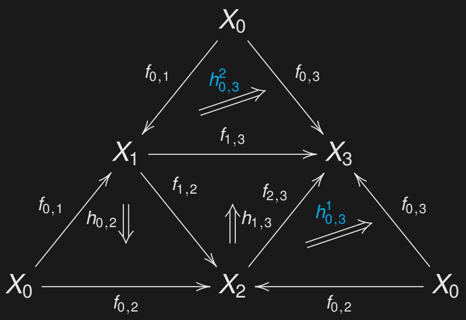

This is the boundary of a 3-simplex in $\mathcal{S}$ iff there is a certain double homotopy adjoint to a map $h_{0,3} : I^2 \to \mathbf{Top}(X_0, X_3)$ shown on the next slide.

The set of 3-simplices in $\mathcal{S}$ (continued)

The diagram on the previous is the boundary of a 3-simplex in $\mathcal{S}$ iff there a map $h_{0,3} : I^2 \to \mathbf{Top}(X_0, X_3)$ of the form

$$\begin{array}{ccc} f_{2,3}f_{1,2}f_{0,1} & \xrightarrow{f_{2,3}h_{0,2}} & f_{2,3}f_{0,2} \\ {\scriptstyle h_{1,3}f_{0,1}}\downarrow & & \downarrow{\scriptstyle h^1_{0,3}} \\ f_{1,3}f_{0,1} & \xrightarrow[h^2_{0,3}]{} & f_{0,3} \end{array}$$

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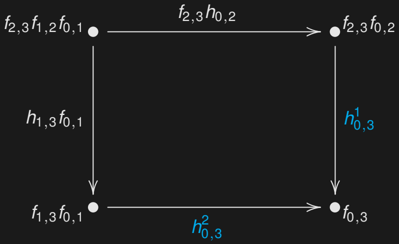

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What is an ∞-category?

Doug Ravenel

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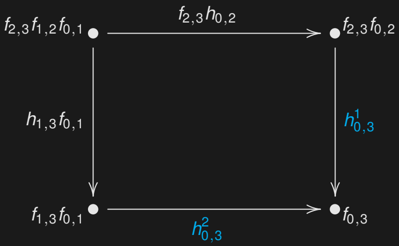

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For each 4-simplex, the additional datum is a map $h_{0,4} : I^3 \to \mathbf{Top}(X_0, X_4)$ of the form

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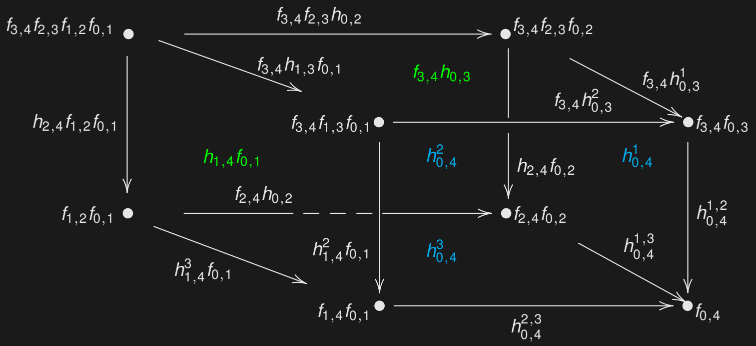

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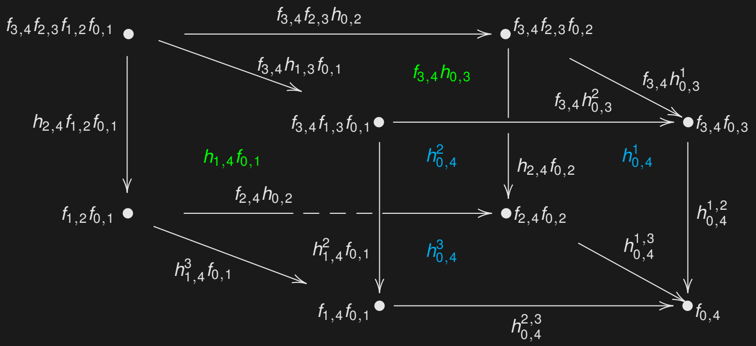

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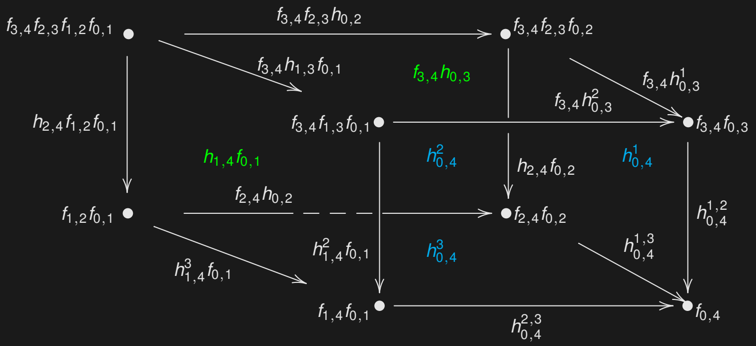
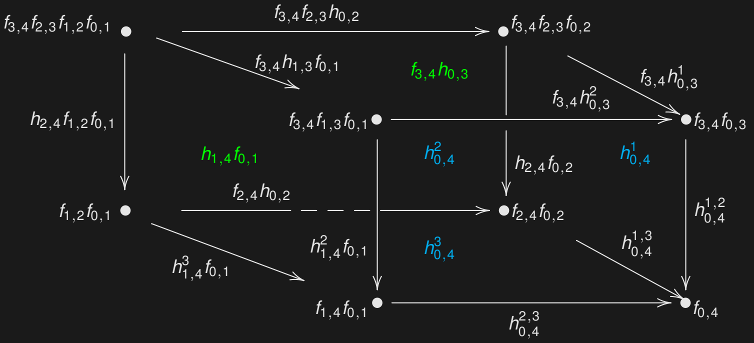


The restriction of $h_{0,4}$ to the left and top faces are the composite double homotopies indicated in green. The restrictions to the three faces abuting $f_{0,4}$ are double homotopies indicated in blue.

The set of 4-simplices in $\mathcal{S}$ (continued)



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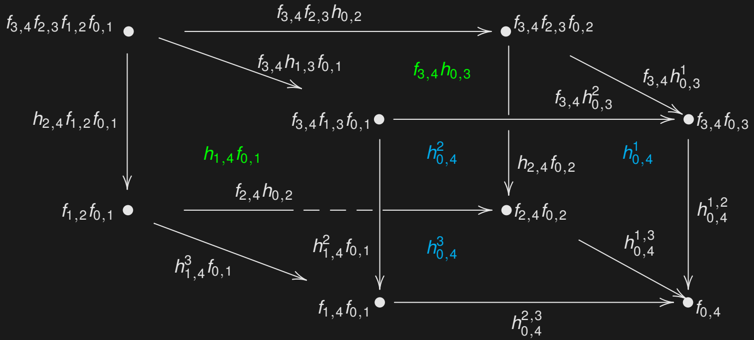

The restriction of $h_{0,4}$ to the back face (not labeled) is the composite

$$\begin{array}{c} I \times I \xrightarrow{h_{2,4} \times h_{0,2}} \mathrm{Top}(X_2, X_4) \times \mathrm{Top}(X_0, X_2) \\ \big\downarrow \mathrm{comp} \\ \mathrm{Top}(X_0, X_4). \end{array}$$

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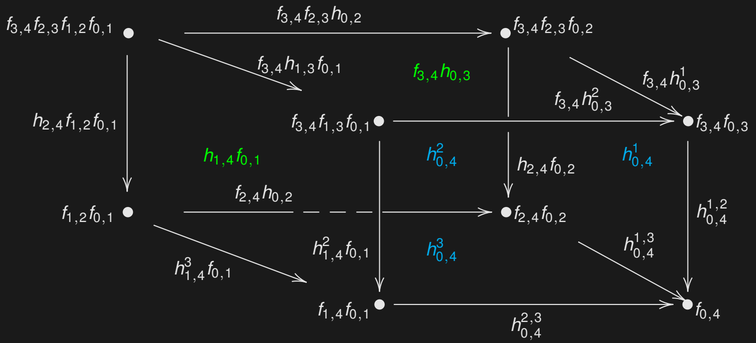

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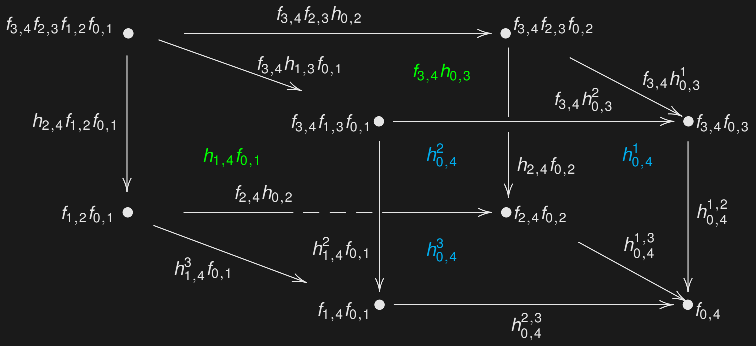

The five labeled faces are associated with the five 3-dimensional faces of the corresponding 4-simplex in $\mathcal{S}$.

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- for $n > 0$, there is an $(n+1)$-simplex for each sequence of spaces and continuous maps

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More details can be found in [HTT, 4.2.4].

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When we enlarge the class of weak equivalences (in the category of spaces or spectra) to those maps inducing an isomorphism in Morava E-theory (or Morava K-theory) for a fixed prime p and height n, this fibrant replacement functor is the L_n (or $L_{K(n)}$) of chromatic homotopy theory.

Bousfield localization in ∞-categories (continued)

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3. The inclusion functor $\mathcal{C}' \subseteq \mathcal{C}$ has a left adjoint L. This is the analog of Bousfield's fibrant replacement functor in model category theory.

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To unpack this definition, note that a vertex in this homotopy limit (meaning an object in the ∞-category Sp) consists of a sequence of pointed spaces $X_0, X_1, X_2, \ldots,$

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To unpack this definition, note that a vertex in this homotopy limit (meaning an object in the ∞-category Sp) consists of a sequence of pointed spaces $X_0, X_1, X_2, \ldots$, along with weak equivalences $X_i \to \Omega X_{i+1}$.

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An ∞-category $\mathcal{C}$ is stable if

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Thank you
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Happy Birthday Andy!

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