Homotopy 2023
In Celebration of Paul Goerss

Northwestern University

# Hiking in the Alps: $C_p$-fixed points of Lubin-Tate spectra

Doug Ravenel
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Fortunately Mark Behrens took some careful notes for us.



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For any closed subgroup $H \subseteq \mathbb{G}_n$, one also has a homotopy fixed point spectrum $E_n^{hH}$ under $S^0_{K(n)}$. $\mathbb{G}_n$ is known to have a subgroup of order $p$ when $p - 1$ divides $n$.



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For any closed subgroup $H \subseteq \mathbb{G}_n$, one also has a homotopy fixed point spectrum $E_n^{hH}$ under $S^0_{K(n)}$. $\mathbb{G}_n$ is known to have a subgroup of order $p$ when $p-1$ divides $n$. Our goal is to study $E_{(p-1)f}^{hC_p}$ for positive integers $f$.



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- The power series variables $u_i$ each have degree 0.
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- The symbol ^ at the end denotes completion with respect to the maximal ideal $I_n = (p, u_1, \ldots u_{n-1})$.

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In short, we start with a graded polynomial local ring, invert each of its specified generators, and then complete at its graded maximal ideal. We will come back to this later.



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# *Properties of $E_n$ and $\mathbb{G}_n$ (continued)*

The extended Morava stabilizer group $\mathbb{G}_n$ is related to the automorphism group $\mathbb{S}_n$ of the Honda height $n$ formal group law $F_n$ over $\mathbb{F}_{p^n}$.



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**Theorem**

*$\mathrm{End}(F_n)$ is the algebra obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.*



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This algebra is a free module over $W$ of rank $n$, and hence a free module over $\mathbb{Z}_p$ of rank $n^2$. An element of the form

$$e = e_0 + e_1 F + \cdots + e_{n-1} F^{n-1} \qquad \text{with } e_i \in W$$



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is invertible if $e_0$ is a unit in $W$. They form a group under multiplication. This is the automorphism group $\mathrm{Aut}(F_n)$ of $F_n$, commonly known as the $n$th Morava stabilizer group $\mathbb{S}_n$.



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The main tool for computing the homotopy groups of the homotopy fixed point spectrum of $E^{hG}$ for a group $G$ acting on a spectrum $E$ is the homotopy fixed point spectral sequence

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# Group cohomology (continued)

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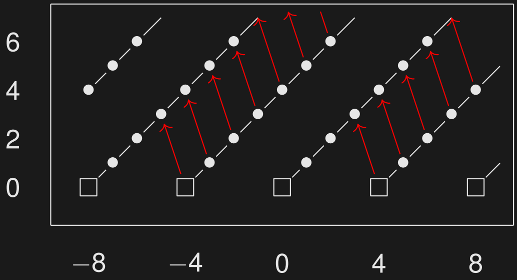
It follows that the $E_2$-term of the homotopy fixed point spectral sequence is

$$E_2^{s,t} = H^s(C_2; \pi_t E_2) = \begin{cases} \mathbb{Z}_2 & \text{for } s = 0 \text{ and } t \text{ divisible by } 4 \\ 0 & \text{for } s = 0 \text{ and } t \equiv 2 \bmod 4 \\ \mathbb{Z}/2 & \text{for } s > 0,\ t \text{ even,} \\ & \qquad \text{and } s \equiv t \bmod 2 \\ 0 & \text{otherwise.} \end{cases}$$



# The homotopy fixed point spectral sequence for $\pi_* KO$



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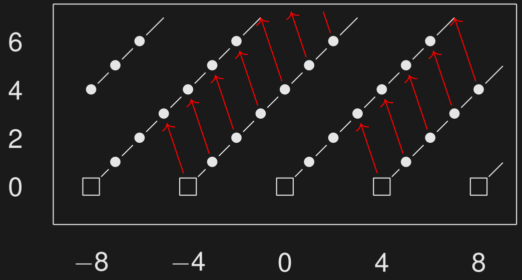


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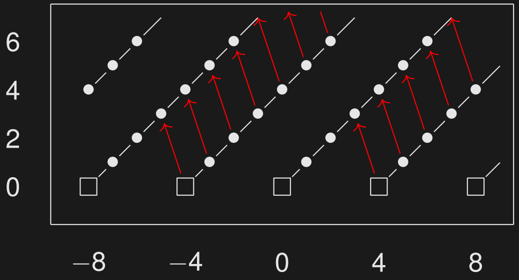


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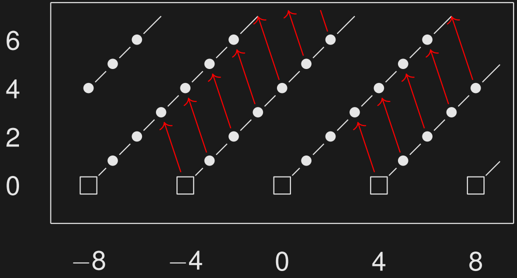


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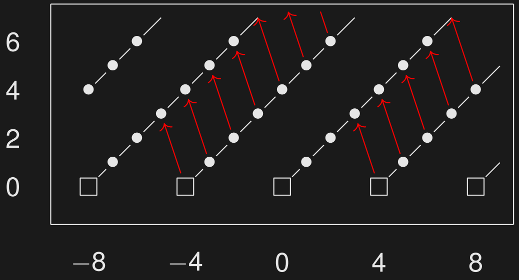


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The indicated $d_3$s can be established by equivariant methods, or by the requirement that the spectral sequence must converge to the known value of $\pi_* KO$.

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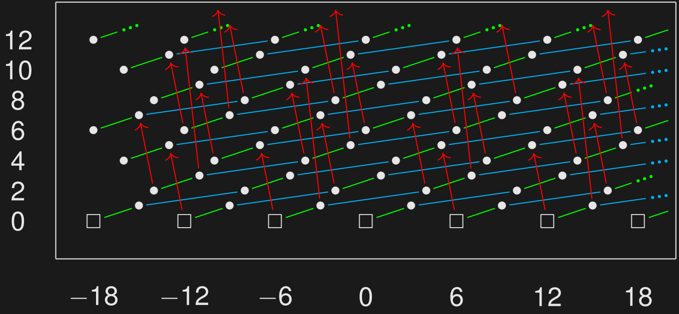

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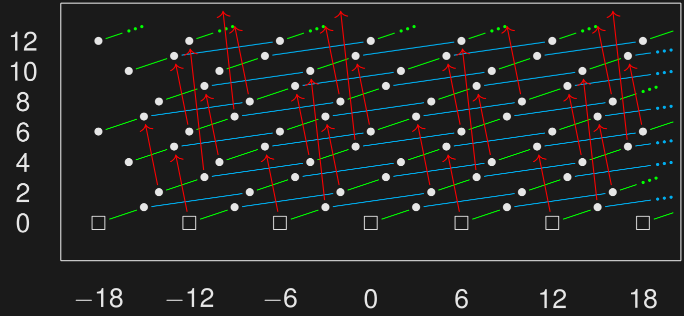


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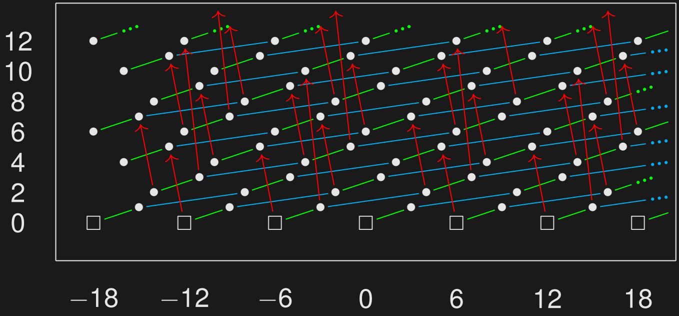


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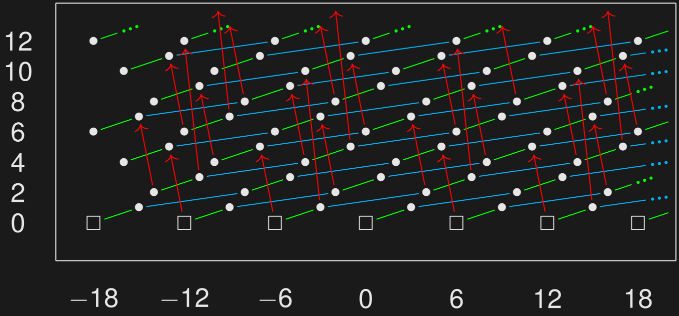


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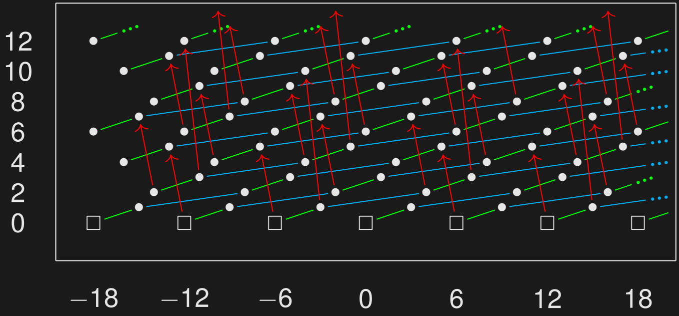


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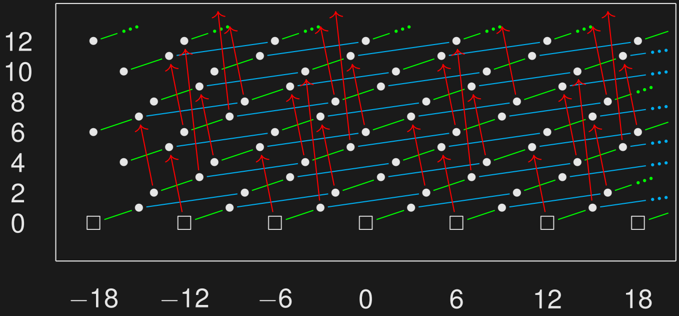




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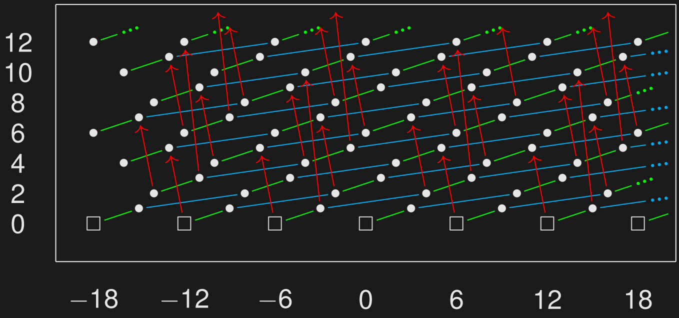


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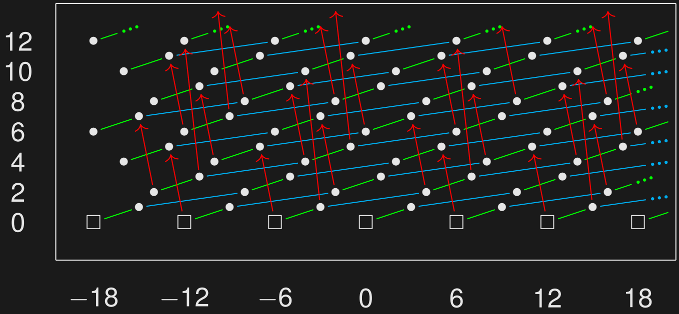


This pattern of differentials is 18-periodic. A comparable homotopy fixed point spectral sequence for *TMF* is 72-periodic.

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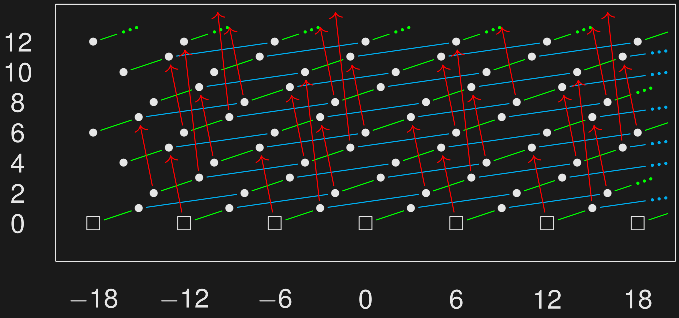


This pattern of differentials is 18-periodic. A comparable homotopy fixed point spectral sequence for *TMF* is 72-periodic. The picture above can be "spread out" by enlarging the group $C_3$ by adjoining the fourth roots of unity in $W$.

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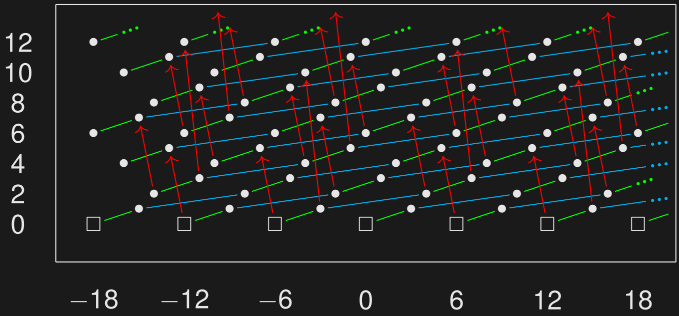


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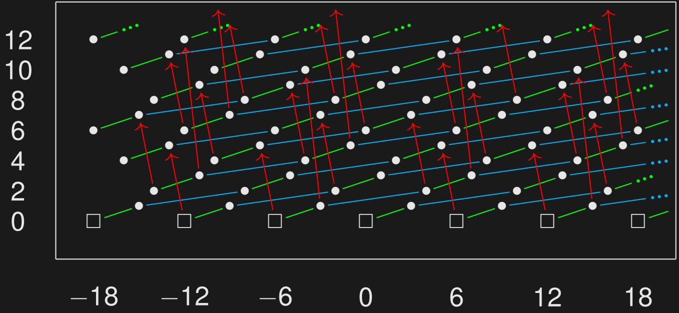


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# Some group theory

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# *TMF at $p = 3$ (continued)*

This is the homotopy fixed point spectral sequence for $E_2^{hG_{24}}$, which is $TMF_{K(2)}$, also known as $EO_3$.



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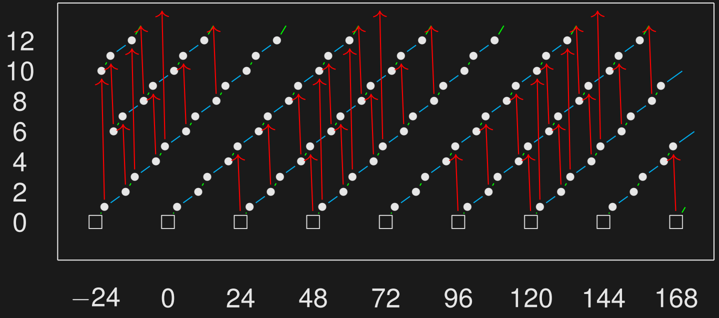

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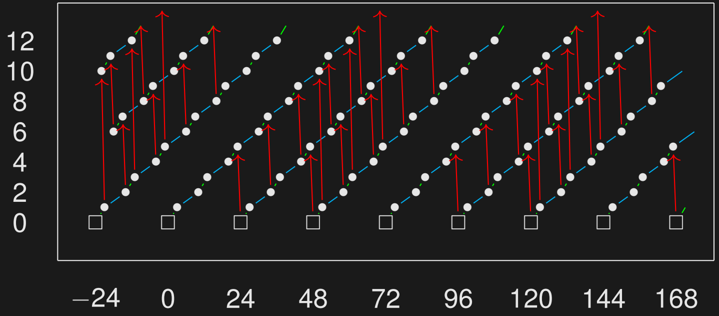


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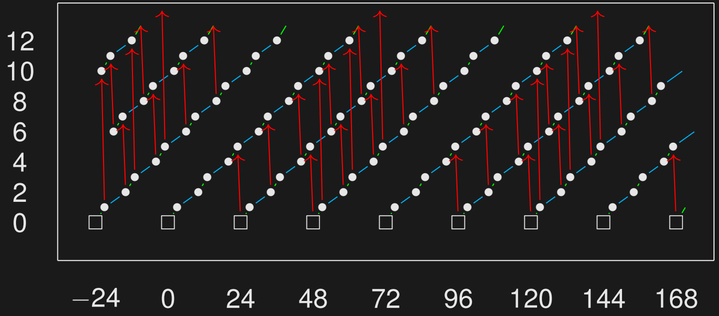


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| $x$ | $\beta_1$ | $\beta_{3/3}$ | $\beta_4$ | $\beta_{6/3}$ | $\beta_{9,9}$, $\beta_7$ | $\beta_{12/3}$ | $\beta_{13}$ |
|---|---|---|---|---|---|---|---|
| $\|x\|$ | 10 | 34 | 58 | 82 | 106 | 130 | 154 |



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Here are the dimensions of these elements for small primes.

| $p$ | $\lvert\alpha_1\rvert$ | $\lvert\beta_1\rvert$ | $\lvert\Delta\rvert$ |
|---|---|---|---|
| 3 | 3 | 10 | 24 |
| 5 | 7 | 38 | 160 |
| 7 | 11 | 82 | 504 |



# Larger primes (continued)

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Then there are differentials

$$d_{2p-1}\Delta = \alpha_1\beta_1^{p-1} \quad \text{and} \quad d_{2(p-1)^2+1}(\alpha_1\Delta^{p-1}) = \beta_1^{(p-1)^2+1}.$$



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From the Adams-Novikov $E_2$-term for the sphere spectrum we have

$$\theta_j := \beta_{p^{j-1}/p^{j-1}} \mapsto \beta_1 \Delta^{(p^{j-1}-1)/(p-1)} \qquad \text{for all } j \geq 1,$$



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and for $p = 5$ only, we have

$$\gamma_3 \mapsto \alpha_1\beta_1\Delta^4 \quad \text{in dimension 685.}$$



# *THANK YOU*

## *and have a wonderful retirement, Paul!*