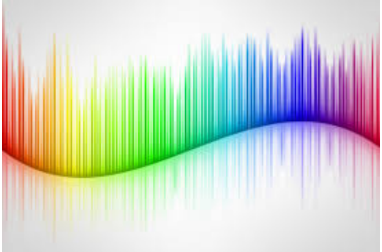
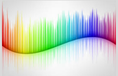
What is a G-spectrum?

Lehigh University Geometry and Topology Conference

May 22, 2015



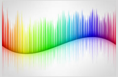
Mike Hill
University of Virginia
Mike Hopkins
Harvard University
Doug Ravenel
University of Rochester



Mike Hill
Mike Hopkins
Doug Ravenel

Introduction

Algebraic topologists have been studying spectra for over 50 years



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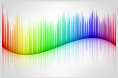

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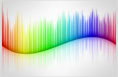

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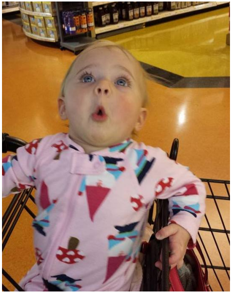

This is a peculiar state of affairs!

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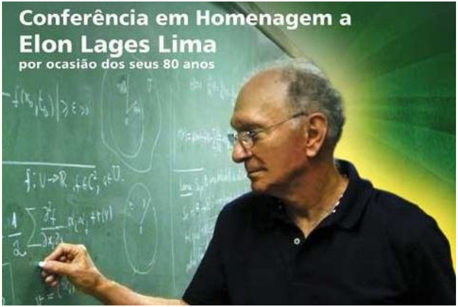
Introduction (continued)



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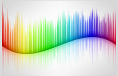

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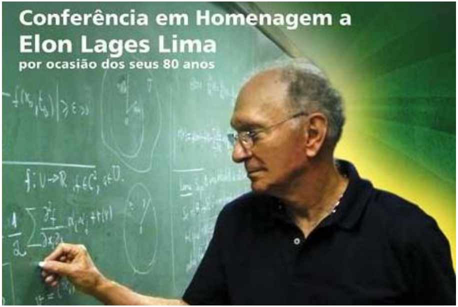

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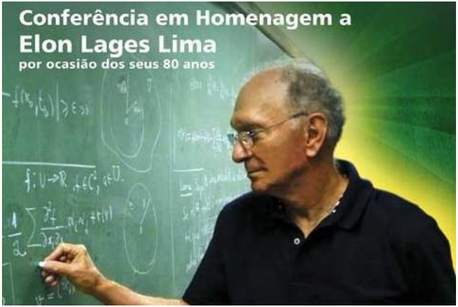

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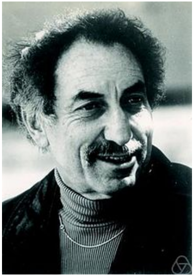

Ed Spanier
1921-1996

Introduction (continued)

George Whitehead
1918-2004

GENERALIZED HOMOLOGY THEORIES(1)

BY
GEORGE W. WHITEHEAD

Introduction (continued)

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is commutative.

4. **Spectra**(2). A *spectrum* **E** is a sequence(3) $\{E_n | n \in Z\}$ of spaces together with a sequence of maps

$$\epsilon_n : SE_n \to E_{n+1}.$$

If **E**, **E**$'$ are spectra, a map $f: \mathbf{E} \to \mathbf{E}'$ is a sequence of maps

$$f_n : E_n \to E_n'$$

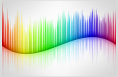
such that the diagrams

$$\begin{array}{ccc} SE_n & \xrightarrow{\epsilon_n} & E_{n+1} \\ Sf_n \downarrow & & \downarrow f_{n+1} \\ SE_n' & \xrightarrow{\epsilon_n'} & E_{n+1}' \end{array}$$

(3) By a *sequence* we shall always mean a function on *all* the integers.

Introduction (continued)

This definition was adequate for many calculations over the next 20 years.



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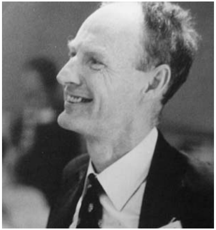

Frank Adams
1930-1989

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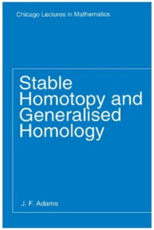

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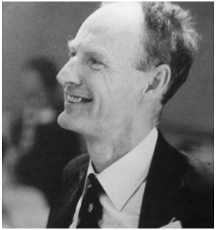

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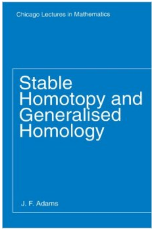

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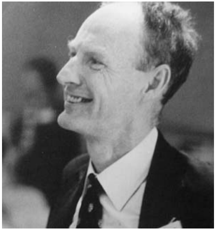

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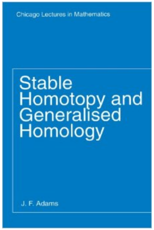

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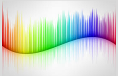

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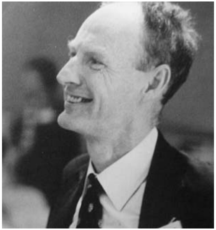

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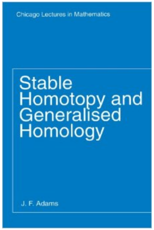

The definition led to a lot of technical problems especially in connection with smash products. The definition we use today is more categorical.

Introduction (continued)

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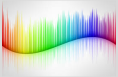
- up to homotopy



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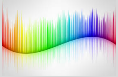

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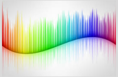

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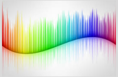

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- chromatic
- Mackey functor
- slice spectral sequence

Some categorical notions: Enrichment, I

In a (locally small) category $\mathcal{C}$,

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Some categorical notions: Enrichment, I

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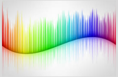

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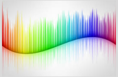

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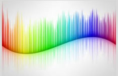

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We say that both of these categories are enriched over themselves.

Some categorical notions: Enrichment, I (continued)

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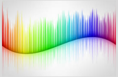

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Let G be a finite group. There are two categories whose objects are pointed G-spaces, where the base point is always fixed by G, because there are two types of morphisms to consider.

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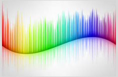
(i) Let $\mathcal{T}^G$ denote the category of pointed G-spaces and equivariant continuous pointed maps.



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Some categorical notions: Enrichment, I (continued)

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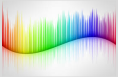

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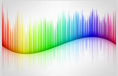

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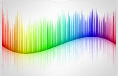

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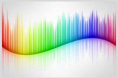

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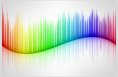

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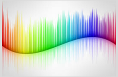
Symmetric monoidal categories

A symmetric monoidal category is a category $\mathcal{V}$ equipped with a map $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$



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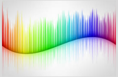

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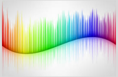

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The monoidal structure is closed if the functor $A \otimes (\cdot)$ has a right adjoint $(\cdot)^A$, the internal Hom with $\mathcal{V}(1, X^A) = \mathcal{V}(A, X)$.

Symmetric monoidal categories (continued)

Here are some familiar examples:

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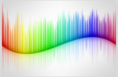

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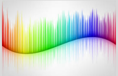
Enrichment II

The following definitions were first published by Eilenberg-Kelly in 1966.

Sammy Eilenberg
1913-1998

Max Kelly
1930-2007



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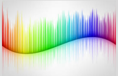

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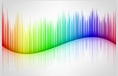
For each object X in $\mathcal{C}$ we have a morphism $1 \to \mathcal{C}(X, X)$ in $\mathcal{V}_0$



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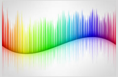
Enrichment II (continued)

A $\mathcal{V}$-category $\mathcal{C}$ is underlain by an ordinary category $\mathcal{C}_0$ having the same objects as $\mathcal{C}$



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Enrichment II (continued)

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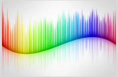
A functor $F : \mathcal{C} \to \mathcal{D}$ between $\mathcal{V}$-categories



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A functor $F : \mathcal{C} \to \mathcal{D}$ between $\mathcal{V}$-categories consists of a function $F : ob(\mathcal{C}) \to ob(\mathcal{D})$,

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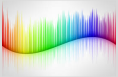

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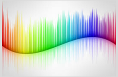

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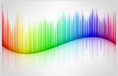

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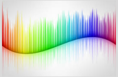

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The definition of a G-spectrum (continued)

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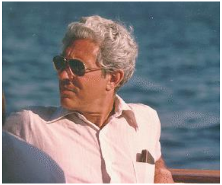
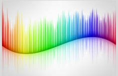


Peter May



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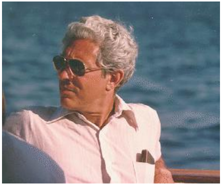

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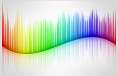

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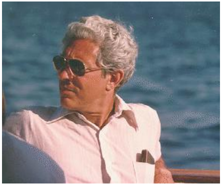

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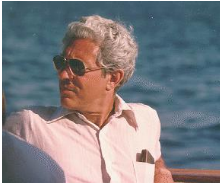
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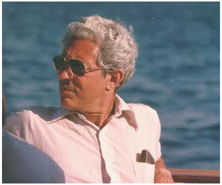
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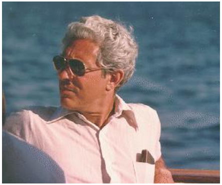

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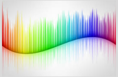

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First we examine the indexing spaces $\mathscr{J}_G(V, W)$.

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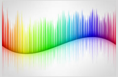

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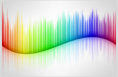

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- When $dim(V) > dim(W)$, the embedding space $O(V, W)$ is empty, so $\mathscr{J}_G(V, W) = *$.

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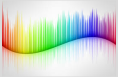

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An orthogonal G-spectrum E is a functor $\mathscr{J}_G \to \mathcal{T}_G$. We will denote its value on V by E_V.

This definition requires some unpacking!

First we examine the indexing spaces $\mathscr{J}_G(V, W)$.

- When $dim(V) > dim(W)$, the embedding space $O(V, W)$ is empty, so $\mathscr{J}_G(V, W) = *$.
- When $dim(V) = dim(W)$, the vector bundle is 0-dimensional, so $\mathscr{J}_G(V, W) = O(V, W)_+$, the orthogonal group (equipped with a G-action) with a disjoint base point.
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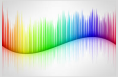

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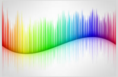

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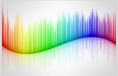

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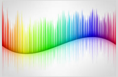

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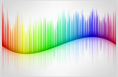

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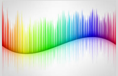

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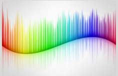

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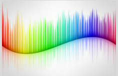

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Smash products with spaces and the sphere spectrum

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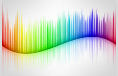

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For a pointed G-space X, the suspension spectrum $\Sigma^\infty X$ is $S^{-0} \wedge X$.

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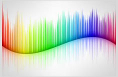

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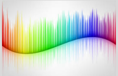
$$\mathcal{S}_G(\Sigma^\infty X, E) = \mathcal{S}_G(S^{-0} \wedge X, E) = \mathcal{T}_G(X, \Omega^\infty E),$$

so the functors $\Sigma^\infty : \mathcal{T}_G \to \mathcal{S}_G$ and $\Omega^\infty : \mathcal{S}_G \to \mathcal{T}_G$ are adjoint.



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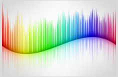

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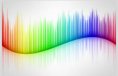

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Nevertheless, the categorical equivalence is useful for certain definitions.

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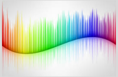

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Change of group (continued)

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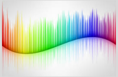
In proving the Kervaire invariant theorem we used this for $H = C_2$, $G = C_8$ and $E = MU_{\mathbf{R}}$.



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The tautological presentation and smash product

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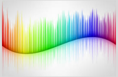

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Jean Dieudonné, Imogene Kelly, Max Kelly, Odette Dieudonné,
Brian Day, Margery Street and Ross Street
at a restaurant in Sydney in 1972

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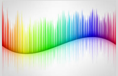

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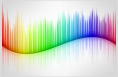

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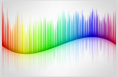

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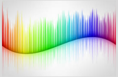

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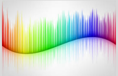
This can lead to a lot of technical problems!

Quillen model category structures

A way out of this difficulty is to define a Quillen model category structure on $\mathcal{S}_G$ and related categories.

Dan Quillen
1940-2011



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Quillen model category structures

A way out of this difficulty is to define a Quillen model category structure on $\mathcal{S}_G$ and related categories. This leads to two special collections of G-spectra, the fibrant and cofibrant ones.

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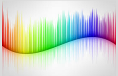

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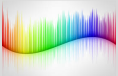

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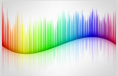

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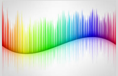

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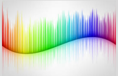

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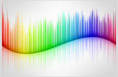

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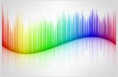

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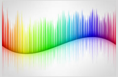

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Pete Bousfield

This observation (in the nonequivariant case) is due to Bousfield-Friedlander in a 1978 paper.

Eric Friedlander

The positive complete model category structure on $\mathcal{S}_G$

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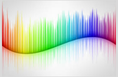

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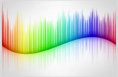

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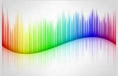

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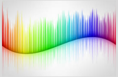

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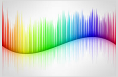

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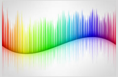

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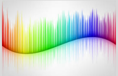

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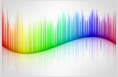

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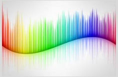

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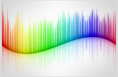

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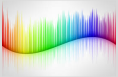

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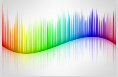

A counterexample: why we need genuine G-spectra

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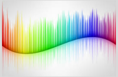

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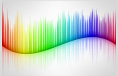

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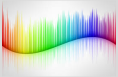

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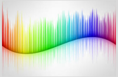

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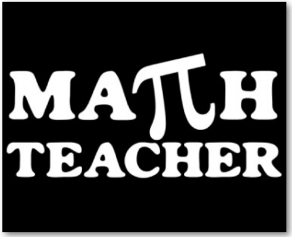

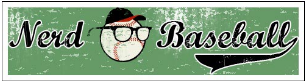

Happy Birthday Don!

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