ECHT Minicourse

What is the telescope conjecture?

Lecture 3
From algebra to geometry

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Bousfield localization
Finite localization
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The structure of $\langle BP\rangle$
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Introduction

We have seen that the Adams-Novikov E_2-term can be filtered in such a way that the hth subquotient displays v_h-periodic families, which is related to formal group laws of height h.

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This raised the question of whether this is an algebraic artifice or the reflection of a similar filtration of the stable homotopy category itself.

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Recall the chromatic short exact sequence for each $h \geq 0$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^h & \longrightarrow & M^h & \longrightarrow & N^{h+1} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & BP_*/(p^\infty, \ldots, v_{h-1}^\infty) & & v_h^{-1}N^h & & BP_*/(p^\infty, \ldots, v_h^\infty). & & \end{array}$$

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If there were a cofiber sequence of spectra having these comodules as their BP-homology, we would be in business.

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We are looking for spectra N_h with $BP_*N_h = N^h$, M_h with $BP_*M_h = M^h$, and a map $N_h \to M_h$ inducing the homomorphism $N^h \to M^h$, for all $h \geq 0$.

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We start with $N_0 = \mathbb{S}$ and $M_0 = \mathbb{S}\mathbb{Q}$, the rationalization of $\mathbb{S}$.

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We start with $N_0 = \mathbb{S}$ and $M_0 = \mathbb{S}\mathbb{Q}$, the rationalization of $\mathbb{S}$. This gives $N_1 = \mathbb{S}\mathbb{Q}/\mathbb{Z}_{(p)}$, the $\mathbb{Q}/\mathbb{Z}_{(p)}$ Moore spectrum.

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The functor we need to get from N_h to M_h for $h > 0$ is Bousfield localization.

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Bousfield localization

Suppose we have a model category $\mathcal{C}$, such as that of spaces or spectra.

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Suppose we have a model category $\mathcal{C}$, such as that of spaces or spectra. We want to alter the model structure in the following way.

- Enlarge the collection of weak equivalences in some way, and keep the same collection of cofibrations.

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Suppose we have a model category $\mathcal{C}$, such as that of spaces or spectra. We want to alter the model structure in the following way.

- Enlarge the collection of weak equivalences in some way, and keep the same collection of cofibrations.
- Since more of the cofibrations are trivial (meaning they are weak equivalences), there are fewer fibrations, since they must satisfy the right lifting property with respect to any trivial cofibration,

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- This could lead to a new fibrant replacement functor L.

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- This could lead to a new fibrant replacement functor L. It assigns to each object X in $\mathcal{C}$ a fibrant object LX, its localization.

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It is not obvious that this new “model structure” satisfies all of Quillen’s axiom.

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It is not obvious that this new "model structure" satisfies all of Quillen's axiom. The sticking point is the requirement that each map can be factored as a (redefined) trivial cofibration followed by a (redefined) fibration.

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A 2003 theorem of Phil Hirschhorn says that it can be done for any model category satisfying certain mild technical conditions,

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A 2003 theorem of Phil Hirschhorn says that it can be done for any model category satisfying certain mild technical conditions, which are met by the categories of spaces and of spectra.

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One way to enlarge the collection of weak equivalences in the category of spaces or of spectra is to require they they induce isomorphisms of homotopy groups only up to dimension n.

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Another to way to enlarge the collection of weak equivalences in the category of spaces or of spectra is to require they they induce isomorphisms in some generalized homology theory represented by a spectrum E.

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In the category of spectra

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In the category of spectra

- A spectrum Y is E-local iff for each X with $E_* X = 0$ (meaning that $E \otimes X$ is contractible),

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- A spectrum Y is E-local iff for each X with $E_* X = 0$ (meaning that $E \otimes X$ is contractible), the function spectrum $F(X, Y)$ is contractible.

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- Any map from X to an E-local spectrum Y factors uniquely (up to homotopy) through $L_E X$.

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- Any map from X to an E-local spectrum Y factors uniquely (up to homotopy) through $L_E X$.
- The map $X \to L_E X$ extends uniquely through any E_*-equivalance $X \to X'$.

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Bousfield localization (continued)

Two examples :

- Let $E = \mathbb{SQ} = H\mathbb{Q}$, the rational sphere spectrum, which is also the rational Eilenberg-Mac Lane spectrum.

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$$\operatorname*{holim}_{j} H\mathbb{Z}/p^j \cong H\mathbb{Z}_p$$

is not rationally acyclic even though each $H\mathbb{Z}/p^j$ is.

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Bousfield localization (continued)

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- Let $E = \mathbb{S}/p$, the mod p Moore spectrum.

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- Let $E = \mathbb{S}/p$, the mod p Moore spectrum. Then L_E is p-adic completion,

$$L_E X = \widehat{X}_p := \operatorname*{holim}_j X/p^j .$$

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The functor L_E^{fin} has formal properties similar to those of L_E,

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- The map $X \to L_E^{\mathrm{fin}}X$ extends uniquely through any E_*-equivalance $X \to X'$.

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Lurie's analog of Bousfield localization

[Lur09, Proposition 5.5.4.15] is statement about an ∞-categorical analog of Bousfield localization.

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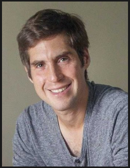

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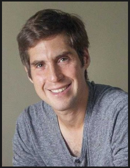

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Presentable means that $\mathcal{C}$ has small colimits and every object is a colimit of small objects.

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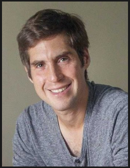

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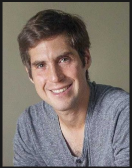

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In [Lur09, Definition 5.5.4.1] an object Z is said to be S-local if each morphism $s : X \to Y$ in S induces a weak equivalence $\mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$.

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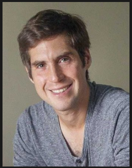

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Let $\overline{S}$ be the set of all S-equivalences.

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Let $\overline{S}$ be the set of all S-equivalences. It can be explicitly constructed from S.

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(i) For each object $X \in \mathcal{C}$, there exists an S-equivalence $s : X \to X'$ where X' is S-local.

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Let $\overline{S}$ be the set of all S-equivalences. It can be explicitly constructed from S. Let $\mathcal{C}'$ be the full subcategory of S-local objects. Then

(i) For each object $X \in \mathcal{C}$, there exists an S-equivalence $s : X \to X'$ where X' is S-local.

(ii) The ∞-category $\mathcal{C}'$ is presentable.

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Lurie's analog of Bousfield localization (continued)

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(iii) The inclusion functor $I : \mathcal{C}' \to \mathcal{C}$ has a left adjoint L.

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(ii) The ∞-category $\mathcal{C}'$ is presentable.

(iii) The inclusion functor $I : \mathcal{C}' \to \mathcal{C}$ has a left adjoint L. The composition IL (which need not be either a left or right adjoint) is the analog of Bousfield's fibrant replacement functor in model category theory.

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The following language of Bousfield is convenient for us. Two spectra E and E' are Bousfield equivalent, which we denote by $E \sim E'$, if their localization functors L_E and $L_{E'}$ are the same.

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We denote the equivalence class of E by $\langle E \rangle$. The wedge and smash product operations $\oplus$ and $\otimes$ of spectra induce corresponding operations on Bousfield classes.

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The complement of a class $\langle E \rangle$ is a class $\langle E \rangle^c$ such that $\langle E \rangle \oplus \langle E \rangle^c = \langle \mathbb{S} \rangle$ and $\langle E \rangle \otimes \langle E \rangle^c = \langle * \rangle$.

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Bousfield equivalence (continued)

The following was proved in [Rav84, Lemma 1.34].

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Bousfield equivalence (continued)

The following was proved in [Rav84, Lemma 1.34].

Proposition

For a self-map $v : \Sigma^d X \to X$, let C_v denotes its cofiber, and $v^{-1}X$ the telescope (meaning homotopy colimit) obtained by iterating v.

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The following was proved in [Rav84, Lemma 1.34].

Proposition

For a self-map $v : \Sigma^d X \to X$*, let* C_v *denotes its cofiber, and* $v^{-1}X$ *the telescope (meaning homotopy colimit) obtained by iterating* v*. Then*

$$\langle v^{-1}X\rangle \oplus \langle C_v\rangle = \langle X\rangle \qquad \text{and} \qquad \langle v^{-1}X\rangle \otimes \langle C_v\rangle = \langle * \rangle.$$

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In particular, $P(0) = BP$, and $k(0) = BP\langle 0\rangle = H_{(p)}$, the Eilenberg-Mac Lane spectrum for $\mathbb{Z}_{(p)}$.

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Each of these three admits a self map inducing multiplication by v_h in homotopy.

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The same goes for BP itself, the telescope being $v_h^{-1} BP$.

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The last of these is Morava K-theory.

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$$\pi_* E_h \cong W(\mathbb{F}_{p^h})[\![u_1, \dots u_{h-1}]\!][u^{\pm 1}] \qquad \text{with } |u| = -2 \text{ and } |u_i| = 0,$$

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where $v_h \mapsto u^{1-p^h}$ and $v_i \mapsto u_i u^{1-p^i}$

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$$\pi_* E_h \cong W(\mathbb{F}_{p^h})[\![u_1, \ldots u_{h-1}]\!][u^{\pm 1}] \qquad \text{with } |u| = -2 \text{ and } |u_i| = 0,$$

where $v_h \mapsto u^{1-p^h}$ and $v_i \mapsto u_i u^{1-p^i}$ under a map $E(h) \to E_h$.

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$$E(h) := v_h^{-1} BP\langle h \rangle \qquad B(h) := v_h^{-1} P(h) \qquad K(h) := v_h^{-1} k(h)$$

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E_h is an $\mathbb{E}_\infty$-ring spectrum by a theorem of Goerss, Hopkins and Miller.

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1. $\langle B(h) \rangle = \langle K(h) \rangle$.

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You might wonder (as I did) if $\langle BP\rangle = \langle \mathbb{S}_{(p)}\rangle$.

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You might wonder (as I did) if $\langle BP \rangle = \langle \mathbb{S}_{(p)} \rangle$. This is far from the case. There is a countable sequence of proper Bousfield inqualities between the two, as explained in [Rav84, §3].

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We say a spectrum E has **height h** if $K(n)_*E = 0$ iff $n > h$.

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Mitchell

Ausoni

Rognes

Hahn

Wilson

The chromatic filtration

The two most widely studied localization functors are $L_{K(h)}$ and $L_h := L_{E(h)}$.

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The two most widely studied localization functors are $L_{K(h)}$ and $L_h := L_{E(h)}$. Since $\langle E(h)\rangle > \langle E(h-1)\rangle$, there is a natural transformation $L_h \to L_{h-1}$,

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The chromatic filtration of π_*X is given by the kernels of the maps to π_*L_hX.

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The tower is known to converge, meaning that X is the homotopy limit of the diagram, if

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- X is a p-local finite spectrum, by a 1992 theorem of Mike Hopkins and myself.
- X is connective and p-local, and BP_*X has finite homological dimension as a BP_*-module, by a 2016 theorem of Toby Barthel.

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We know how to compute BP_*L_hX in terms of BP_*X.

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We know how to compute BP_*L_hX in terms of BP_*X. In particular when $v_{h-1}^{-1}BP_*X = 0$, we know that

$$BP_*L_hX = v_h^{-1}BP_*X.$$

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This condition is met for $X = N_h$, the inductively constructed spectrum with

$$BP_*N_h = N^h = BP_*/(p^\infty, \ldots, v_{h-1}^\infty).$$

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$$BP_* N_h = N^h = BP_* / (p^\infty, \ldots, v_{h-1}^\infty).$$

This means we can define M_h to be $L_h N_h$,

The chromatic filtration (continued)

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This condition is met for $X = N_h$, the inductively constructed spectrum with

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This means we can define M_h to be L_hN_h, so we have the desired geometric realization of the chromatic resolution.

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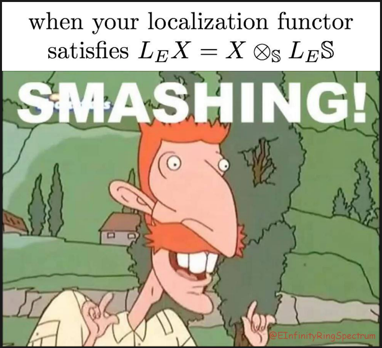

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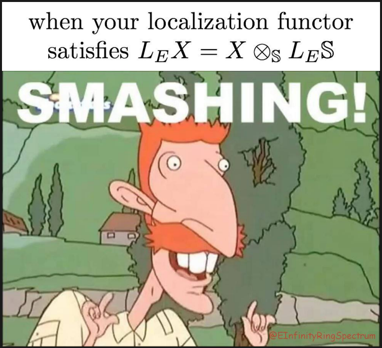

A smashing localization functor preserves homotopy colimits.

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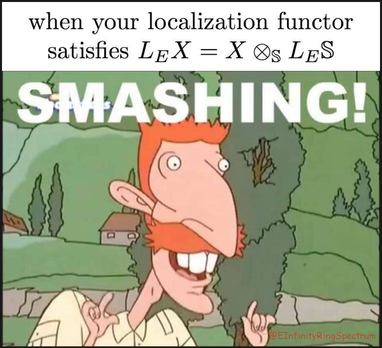

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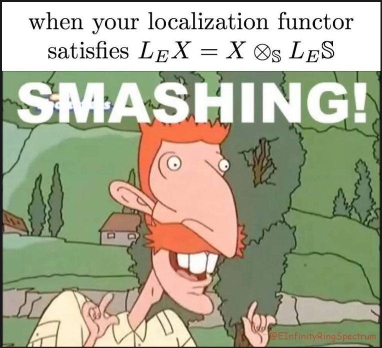

A smashing localization functor preserves homotopy colimits. A 1992 theorem of Hopkins and myself says that each L_h is smashing. This is not true of $L_{K(h)}$.

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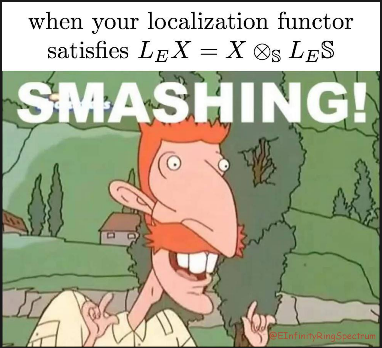

A smashing localization functor preserves homotopy colimits. A 1992 theorem of Hopkins and myself says that each L_h is smashing. This is not true of $L_{K(h)}$. Miller showed that L_h^{fin} is also smashing.

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THANK YOU!

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Finite localization
Lurie's analog of Bousfield localization
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The structure of $\langle BP \rangle$
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References

References

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