ECHT Minicourse

What is the telescope conjecture?

Lecture 3
From algebra to geometry

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1 Introduction

Introduction

We have seen that the Adams-Novikov E_2-term can be filtered in such a way that the hth subquotient displays v_h-periodic families, which is related to formal group laws of height h.

This raised the question of whether this is an algebraic artifice or the reflection of a similar filtration of the stable homotopy category itself.

Recall the chromatic short exact sequence for each $h \geq 0$

$$\begin{tikzcd} 0 \arrow[r] & N^h \arrow[r] \arrow[d, equal] & M^h \arrow[r] \arrow[d, equal] & N^{h+1} \arrow[r] \arrow[d, equal] & 0 \\ & BP_*/(p^\infty, \ldots, v_{h-1}^\infty) & v_h^{-1}N^h & BP_*/(p^\infty, \ldots, v_h^\infty). & \end{tikzcd}$$

If there were a cofiber sequence of spectra having these comodules as their BP-homology, we would be in business.

Intrroduction (continued)

$$\begin{tikzcd} 0 \arrow[r] & N^h \arrow[r] \arrow[d, equal] & M^h \arrow[r] \arrow[d, equal] & N^{h+1} \arrow[r] \arrow[d, equal] & 0 \\ & BP_*/(p^\infty, \ldots, v_{h-1}^\infty) & v_h^{-1}N^h & BP_*/(p^\infty, \ldots, v_h^\infty). & \end{tikzcd}$$

We are looking for spectra N_h with $BP_*N_h = N^h$, M_h with $BP_*M_h = M^h$, and a map $N_h \to M_h$ inducing the homomorphism $N^h \to M^h$, for all $h \geq 0$. We will construct them by induction on h.

We start with $N_0 = \mathbb{S}$ and $M_0 = \mathbb{S}\mathbb{Q}$, the rationalization of $\mathbb{S}$. This gives $N_1 = \mathbb{S}\mathbb{Q}/\mathbb{Z}_{(p)}$, the $\mathbb{Q}/\mathbb{Z}_{(p)}$ Moore spectrum.

The functor we need to get from N_h to M_h for $h > 0$ is Bousfield localization. Pete Bousfield constructed it for the categories of spaces and spectra in 1975 and 1978, using model category methods, just in time for us!

2 Bousfield localization

Bousfield localization

Suppose we have a model category $\mathscr{C}$, such as that of spaces or spectra. We want to alter the model structure in the following way.

- Enlarge the collection of weak equivalences in some way, and keep the same collection of cofibrations.
- Since more of the cofibrations are trivial (meaning they are weak equivalences), there are fewer fibrations, since they must satisfy the right lifting property with respect to any trivial cofibration, of which there are more now than there were before.
- This could lead to a new fibrant replacement functor L. It assigns to each object X in $\mathscr{C}$ a fibrant object LX, its localization.

Bousfield localization (continued)

It is not obvious that this new "model structure" satisfies all of Quillen's axiom. The sticking point is the requirement that each map can be factored as a (redefined) trivial cofibration followed by a (redefined) fibration. Bousfield needed some delicate set theoretic arguments to prove it for the categories of spaces and of spectra.

A 2003 theorem of Phil Hirschhorn says that it can be done for any model category satisfying certain mild technical conditions, which are met by the categories of spaces and of spectra.

Bousfield localization (continued)

One way to enlarge the collection of weak equivalences in the category of spaces or of spectra is to require they they induce isomorphisms of homotopy groups only up to dimension n. Then the fibrant objects are those spaces or spectra with no homotopy above dimension n, and the fibrant replacement functor is the nth Postnikov section.

Another to way to enlarge the collection of weak equivalences in the category of spaces or of spectra is to require they they induce isomorphisms in some generalized homology theory represented by a spectrum E. In that case we denote the fibrant replace functor by L_E, and we refer to fibrant objects as E-local spectra.

Bousfield localization (continued)

In the category of spectra

- A spectrum Y is E-local iff for each X with $E_*X = 0$ (meaning that $E \otimes X$ is contractible), the function spectrum $F(X,Y)$ is contractible. Since the functor $F(X,-)$ preserves limits, this means that any limit of E-local spectra is E-local. It does not mean that L_E preverves limits, as we will see below.
- Any map from X to an E-local spectrum Y factors uniquely (up to homotopy) through $L_E X$.
- The map $X \to L_E X$ extends uniquely through any E_*-equivalance $X \to X'$.

Bousfield localization (continued)

Two examples :

- Let $E = \mathbb{S}\mathbb{Q} = H\mathbb{Q}$, the rational sphere spectrum, which is also the rational Eilenberg-Mac Lane spectrum. The functor L_E is rationalization, $L_E X = X \otimes H\mathbb{Q}$, which preserves homotopy colimits. The spectrum
$$\operatorname*{holim}_j H\mathbb{Z}/p^j \cong H\mathbb{Z}_p$$
is not rationally acyclic even though each $H\mathbb{Z}/p^j$ is. Hence the limit of rationally acyclic spectra need not be rationally acyclic, so L_E is not homotopy limit preserving.
- Let $E = \mathbb{S}/p$, the mod p Moore spectrum. Then L_E is p-adic completion,
$$L_E X = \widehat{X}_p := \operatorname*{holim}_j X/p^j.$$
It preserves homotopy limits but not homotopy colimits.

3 Finite localization

Finite localization

In the early 70's Adams suggested (in a lecture at the University of Chicago) defining $L_E X$ as the cofiber of the map $C_E X \to X$, where $C_E X$ is the colimit of all E_*-acyclic spectra mapping to X. Bousfield observed (in real time) that this colimit is not defined because the collection of such spectra need not be a set.

Bousfield later proved that it is enough to define $C_E X$ to be the colimit of all E_*-acyclic CW spectra with cardinality bounded by that of $\pi_* E$ mapping to X. This collection of spectra is a set, so the problem is solved.

In any case one could also consider the colimit $C_E^{\mathrm{fin}} X$ of all finite E_*-acyclic CW spectra mapping to X, and define $L_E^{\mathrm{fin}} X$ to be the cofiber of the map $C_E^{\mathrm{fin}} X \to X$. This is the finite E-localization of X. Such functors are studied by Miller in [Mil92] and by Bousfield in [Bou01].

Finite localization (continued)

In any case one could also consider the colimit $C_E^{\mathrm{fin}} X$ of all finite E_*-acyclic CW spectra mapping to X, and define $L_E^{\mathrm{fin}} X$ to be the cofiber of the map $C_E^{\mathrm{fin}} X \to X$. This is the finite E-localization of X. Such functors are studied by Miller in [Mil92] and by Bousfield in [Bou01].

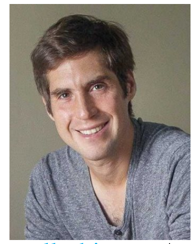
The functor L_E^{fin} has formal properties similar to those of L_E, to which it admits a natural transformation induced by that from C_E^{fin} to C_E as defined by Bousfield.

- We say a spectrum Y is finitely E-local iff for each finite X with $E_*X = 0$, the function spectrum $F(X,Y)$ is contractible. Since the functor $F(X,-)$ preserves limits, this means that any limit of finitely E-local spectra is finitely E-local.

- Any map from X to a finitely E-local spectrum Y factors uniquely (up to homotopy) through $L_E^{\text{fin}}X$.
- The map $X \to L_E^{\text{fin}}X$ extends uniquely through any E_*-equivalance $X \to X'$.

4 Lurie's analog of Bousfield localization

Lurie's analog of Bousfield localization



[Lur09, Proposition 5.5.4.15] is statement about an ∞-categorical analog of Bousfield localization. The input is a presentable ∞-category $\mathscr{C}$ with a set of morphisms S that are meant to be made into weak equivalences.

Presentable means that $\mathscr{C}$ has small colimits and every object is a colimit of small objects. An object is small if the mapping space from it to each filtered colimit is equivalent to the colimit of the mapping spaces.

In [Lur09, Definition 5.5.4.1] an object Z is said to be S-local if each morphism $s : X \to Y$ in S induces a weak equivalence $\mathscr{C}(Y,Z) \to \mathscr{C}(X,Z)$. A morphism $f : A \to B$ is an S-equivalence if it induces a weak equivalence $\mathscr{C}(B,Z) \to \mathscr{C}(A,Z)$ for each S-local object Z.

Lurie's analog of Bousfield localization (continued)

Let $\overline{S}$ be the set of all S-equivalences. It can be explicitly constructed from S. Let $\mathscr{C}'$ be the full subcategory of S-local objects. Then

(i) For each object $X \in \mathscr{C}$, there exists an S-equivalence $s : X \to X'$ where X' is S-local.

(ii) The ∞-category $\mathscr{C}'$ is presentable.

(iii) The inclusion functor $I : \mathscr{C}' \to \mathscr{C}$ has a left adjoint L. The composition IL (which need not be either a left or right adjoint) is the analog of Bousfield's fibrant replacement functor in model category theory.

5 Bousfield equivalence

Bousfield equivalence

The following language of Bousfield is convenient for us. Two spectra E and E' are Bousfield equivalent, which we denote by $E \sim E'$, if their localization functors L_E and $L_{E'}$ are the same. Equivalently $E \sim E'$ means that $E \otimes X = *$ iff $E' \otimes X = *$.

We denote the equivalence class of E by $\langle E \rangle$. The wedge and smash product operations $\oplus$ and $\otimes$ of spectra induce corresponding operations on Bousfield classes.

These classes are partially ordered by saying $\langle E \rangle \geq \langle E' \rangle$ if $E \otimes X = *$ implies $E' \otimes X = *$. This means the maximal equivalence class is that of the sphere spectrum $\mathbb{S}$, and the minimal one is that of a point $*$.

The complement of a class $\langle E \rangle$ is a class $\langle E \rangle^c$ such that $\langle E \rangle \oplus \langle E \rangle^c = \langle \mathbb{S} \rangle$ and $\langle E \rangle \otimes \langle E \rangle^c = \langle * \rangle$. Most classes do not have complements.

Bousfield equivalence (continued)

The following was proved in [Rav84, Lemma 1.34].

Proposition. *For a self-map $v : \Sigma^d X \to X$, let C_v denotes its cofiber, and $v^{-1}X$ the telescope (meaning homotopy colimit) obtained by iterating v. Then*

$$\langle v^{-1}X\rangle \oplus \langle C_v\rangle = \langle X\rangle \qquad \textit{and} \qquad \langle v^{-1}X\rangle \otimes \langle C_v\rangle = \langle *\rangle.$$

6 The structure of $\langle BP\rangle$

The structure of $\langle BP\rangle$

For each $h \geq 0$, there are BP-module spectra, the circus animals,

$$\begin{aligned} BP\langle h\rangle \text{ with } \pi_* BP\langle h\rangle &= BP_*/(v_{h+1}, v_{h+2}, \dots), \\ P(h) \text{ with } \pi_* P(h) &= BP_*/(p, v_1, \dots v_{h-1}), \\ \text{and} \qquad k(h) \text{ with } \pi_* k(h) &= BP_*/(p, v_1, \dots v_{h-1}, v_{h+1}, v_{h+2}, \dots). \end{aligned}$$

In particular, $P(0) = BP$, and $k(0) = BP\langle 0\rangle = H_{(p)}$, the Eilenberg-Mac Lane spectrum for $\mathbb{Z}_{(p)}$. H/p will denote the mod p Eilenberg-Mac Lane spectrum.

Each of these three admits a self map inducing multiplication by v_h in homotopy. In each case we can iterate the map to form a telescope, and we denote

$$E(h) := v_h^{-1} BP\langle h\rangle, \qquad B(h) := v_h^{-1} P(h), \qquad \text{and } K(h) := v_h^{-1} k(h).$$

The same goes for BP itself, the telescope being $v_h^{-1} BP$.

The structure of $\langle BP\rangle$ (continued)

$$E(h) := v_h^{-1} BP\langle h\rangle \qquad B(h) := v_h^{-1} P(h) \qquad K(h) := v_h^{-1} k(h)$$

The last of these is Morava K-theory. $E(0) = K(0) = H\mathbb{Q}$, the rational Eilenberg-Mac Lane spectrum. $BP\langle 1\rangle$ and $E(1)$, are the Adams summands of connective and periodic complex K-theory localized at p.

$E(h)$ is the Johnson-Wilson spectrum, not to be confused with the Morava spectrum E_h, which has the same Bousfield class. While $\pi_* E(h) \cong \mathbb{Z}_{(p)}[v_1, \dots v_{h-1}, v_h^{\pm 1}]$,

$$\pi_* E_h \cong W(\mathbb{F}_{p^h}) u_1, \dots u_{h-1}[u^{\pm 1}] \qquad \text{with } |u| = -2 \text{ and } |u_i| = 0,$$

where $v_h \mapsto u^{1-p^h}$ and $v_i \mapsto u_i u^{1-p^i}$ under a map $E(h) \to E_h$.

E_h is an $\mathbb{E}_\infty$-ring spectrum by a theorem of Goerss, Hopkins and Miller.



The structure of $\langle BP\rangle$ (continued)

The following was proved in [Rav84, Theorem 2.1].

$\langle BP\rangle$ Structure Theorem. *1.* $\langle B(h)\rangle = \langle K(h)\rangle$.

2. $\langle v_h^{-1} BP\rangle = \langle E(h)\rangle$.

3. $\langle P(h)\rangle = \langle K(h)\rangle \oplus \langle P(h+1)\rangle$.

4. $\langle E(h)\rangle = \bigoplus_{0\le i\le h} K(i)$.

5. $\langle BP\langle h\rangle\rangle = \langle E(h)\rangle \oplus \langle H/p\rangle$.

6. $\langle K(m)\rangle \otimes \langle K(n)\rangle = \langle *\rangle$ *for* $m \neq n$.

7. $\langle K(h)\rangle \otimes \langle H/p\rangle = \langle *\rangle$.

You might wonder (as I did) if $\langle BP\rangle = \langle \mathbb{S}_{(p)}\rangle$. This is far from the case. There is a countable sequence of proper Bousfield inqualities between the two, as explained in [Rav84, §3].

The structure of $\langle BP\rangle$ (continued)

We say a spectrum E has height h if $K(n)_*E = 0$ iff $n > h$. Thus $BP\langle h\rangle$ and $E(h)$ each have height h. The red shift conjecture of Christian Ausoni and John Rognes (2006) says that if a ring spectrum R has height h, then its algebraic K-theory $K(R)$ has height $h+1$. They proved this for $BP\langle 1\rangle$ in 2002, and Steve Mitchell had proved it in 1990 for HA for any discrete ring A. In 2022 Jeremy Hahn and Dylan Wilson proved this for $R = BP\langle h\rangle$, the first known example for all heights.


Mitchell


Ausoni


Rognes


Hahn


Wilson

7 The chromatic filtration

The chromatic filtration

The two most widely studied localization functors are $L_{K(h)}$ and $L_h := L_{E(h)}$. Since $\langle E(h)\rangle > \langle E(h-1)\rangle$, there is a natural transformation $L_h \to L_{h-1}$, leading to the chromatic tower

$$X \longrightarrow \cdots \longrightarrow L_3X \longrightarrow L_2X \longrightarrow L_1X \longrightarrow L_0X.$$

The chromatic filtration of π_*X is given by the kernels of the maps to π_*L_hX.

The tower is known to converge, meaning that X is the homotopy limit of the diagram, if

- X is a p-local finite spectrum, by a 1992 theorem of Mike Hopkins and myself.
- X is connective and p-local, and BP_*X has finite homological dimension as a BP_*-module, by a 2016 theorem of Toby Barthel.

The chromatic filtration (continued)

We know how to compute BP_*L_hX in terms of BP_*X. In particular when $v_{h-1}^{-1}BP_*X = 0$, we know that

$$BP_*L_hX = v_h^{-1}BP_*X.$$

This condition is met for $X = N_h$, the inductively constructed spectrum with

$$BP_*N_h = N^h = BP_*/(p^\infty, \ldots, v_{h-1}^\infty).$$

This means we can define M_h to be L_hN_h, so we have the desired geometric realization of the chromatic resolution.

8 The smash product theorem

The smash product theorem

when your localization functor satisfies $L_E X = X \otimes_{\mathbb{S}} L_E \mathbb{S}$

A smashing localization functor preserves homotopy colimits. A 1992 theorem of Hopkins and myself says that each L_h is smashing. This is not true of $L_{K(h)}$. Miller showed that L_h^{fin} is also smashing.

References

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