ECHT Minicourse

What is the telescope conjecture?
Lecture 2
Morava's vision and the chromatic spectral sequence

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Recollections

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$$F_L^f(x, y) := f^{-1} F_L(f(x), f(y)).$$

This formal group law is induced by a ring automorphism $\theta_f : L \to L$.

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$$0 \longrightarrow \Sigma^{|v_{h-1}|} BP_*/I_{h-1} \xrightarrow{v_{h-1}} BP_*/I_{h-1} \longrightarrow BP_*/I_h \longrightarrow 0,$$

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- The height ∞ orbit is the linear subspace
$$\bigcap_{h>0} V_h.$$

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Here we describe the endomorphism ring and automorphism group of a height h formal group law over a field K of characteristic p containing $\mathbb{F}_{p^h}$.

We need some notation.

- $W := W(\mathbb{F}_{p^h})$ denotes the Witt ring for $\mathbb{F}_{p^h}$. It is the extension of the p-adic integers $\mathbb{Z}_p$ obtained by adjoining the $(p^h - 1)$th roots of unity. It is a complete local ring with residue field $\mathbb{F}_{p^h}$ and an extension of $\mathbb{Z}_p$ of degree h. It has an automorphism σ that lifts the Frobenius automorphism (pth power map) in the residue field. We denote the image of $w \in W$ under σ by w^σ.
- End_h denotes the $\mathbb{Z}_p$-algebra obtained from W by adjoining an indeterminate S with $Sw = w^\sigma S$ for $w \in W$ and setting $S^h = p$. We will see that it is the endomorphism ring of our formal group law.

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$$\log(x) = \sum_{k \geq 0} \frac{x^{p^{kh}}}{p^k} = x + \frac{x^{p^h}}{p} + \frac{x^{p^{2h}}}{p^2} + \cdots.$$

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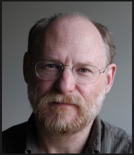

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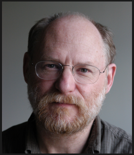

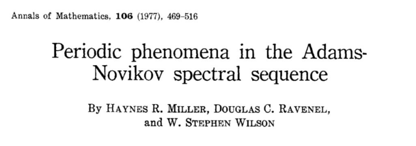

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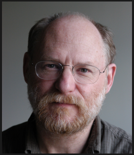

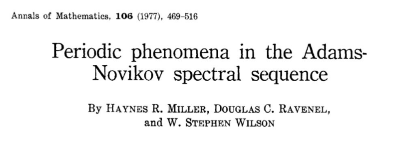

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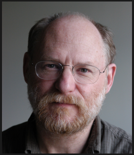

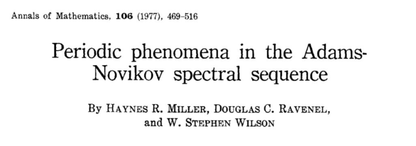

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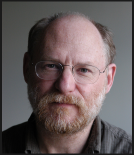

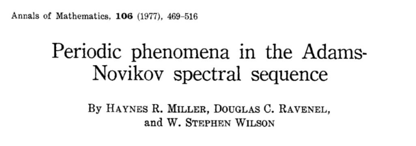
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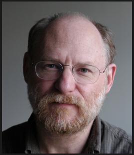

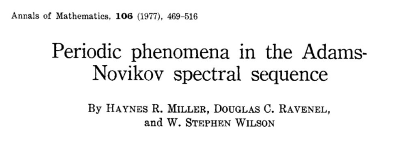

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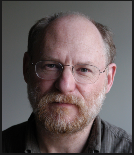

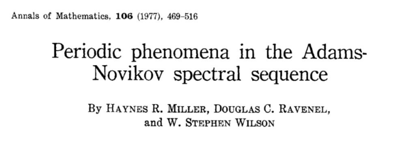

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$$E_2^{h,s} = \mathrm{Ext}^s\left(M^h\right) \implies \mathrm{Ext}^{s+h}\left(BP_*\right)$$

Roughly speaking, its hth column, $\mathrm{Ext}\left(M^h\right)$, displays v_h-periodic phenomena.

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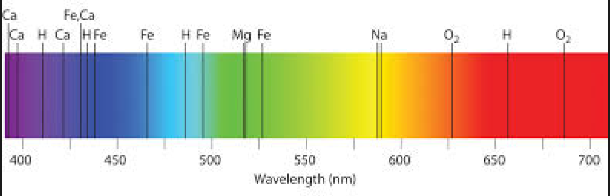
Roughly speaking, its hth column, $\mathrm{Ext}\left(M^h\right)$, displays v_h-periodic phenomena. This decomposition of the Adams-Novikov E_2-term into its various frequencies is our reason for the use of the word chromatic.



The chromatic spectral sequence (continued)

We will construct a long exact sequence of BP_*BP-comodules of the form

$$0 \longrightarrow BP_* \longrightarrow M^0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow M^3 \longrightarrow \ldots$$

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We will do so by splicing together the chromatic short exact sequences

$$\begin{aligned} 0 \longrightarrow N^0 := BP_* &\longrightarrow M^0 \longrightarrow N^1 \longrightarrow 0, \\ 0 \longrightarrow N^1 &\longrightarrow M^1 \longrightarrow N^2 \longrightarrow 0, \\ 0 \longrightarrow N^2 &\longrightarrow M^2 \longrightarrow N^3 \longrightarrow 0, \end{aligned}$$

and so on.

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$$N^1 = BP_*/p^\infty := \operatorname*{colim}_i BP_*/p^i.$$

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Our first chromatic short exact sequence is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^0 & \longrightarrow & M^0 & \longrightarrow & N^1 & \longrightarrow & 0, \\ & & \| & & \| & & \| & & \\ & & BP_* & & p^{-1}BP_* & & BP_*/(p^\infty) & & \end{array}$$

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We want the next one to be

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^1 & \longrightarrow & M^1 & \longrightarrow & N^2 & \longrightarrow & 0, \\ & & \| & & \| & & \| & & \\ & & BP_*/(p^\infty) & & v_1^{-1}BP_*/(p^\infty) & & BP_*/(p^\infty, v_1^\infty), & & \end{array}$$

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We want a short exact sequence of comodules

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$$\eta_R(v_1^k) = (v_1 + pt_1)^k = \sum_{i \geq 0} \binom{k}{i} p^i v_1^{k-i} t_1^i.$$

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When $k < 0$, this sum is infinite and therefore does not lie in $v_1^{-1}BP_*BP$.

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The chromatic spectral sequence (continued)

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$$0 \longrightarrow N^1 \longrightarrow M^1 \longrightarrow N^2 \longrightarrow 0,$$

$$N^1 = BP_*/(p^\infty), \quad M^1 = v_1^{-1}BP_*/(p^\infty), \quad N^2 = BP_*/(p^\infty, v_1^\infty),$$

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When $k < 0$, this sum is infinite and therefore does not lie in $v_1^{-1}BP_*BP$. This means that $v_1^{-1}BP_*$ is not a comodule. We claim that $v_1^{-1}BP_*/p^\infty$ is one nevertheless.

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$$\eta_R(v_1^k) = (v_1 + pt_1)^k = \sum_{i \geq 0} \binom{k}{i} p^i v_1^{k-i} t_1^i.$$

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Each element in BP_*/p^∞ can be written as a fraction of the form

$$\frac{x}{p^j} \qquad \text{where } j > 0 \text{ and } x \in BP_* \text{ is not divisible by } p.$$

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$$\eta_R\left(\frac{v_1^k x}{p^j}\right) = \sum_{0 \leq i < j} \binom{k}{i} \frac{v_1^{k-i} t_1^i \eta_R(x)}{p^{j-i}} .$$

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This sum is finite for all k, unlike the previous one, so $v_1^{-1} BP_*/p^\infty$ is a comodule as claimed.

The chromatic spectral sequence (continued)

Thus we have our second chromatic short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^1 & \longrightarrow & M^1 & \longrightarrow & N^2 & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & BP_*/(p^\infty) & & v_1^{-1}BP_*/(p^\infty) & & BP_*/(p^\infty, v_1^\infty). & & \end{array}$$

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In a similar manner we can work by induction on h and construct

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$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^h & \longrightarrow & M^h & \longrightarrow & N^{h+1} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & BP_*/(p^\infty, \ldots, v_{h-1}^\infty) & & v_h^{-1}N^h & & BP_*/(p^\infty, \ldots, v_h^\infty). & & \end{array}$$

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Splicing these together for all h gives the desired long exact sequence,

$$0 \longrightarrow BP_* \longrightarrow M^0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow M^3 \longrightarrow M^4 \longrightarrow \cdots.$$

The chromatic spectral sequence (continued)

Recall that the change-of-ring-isomorphism gives us a handle on $\mathrm{Ext}\left(v_h^{-1}BP_*/I_h\right)$.

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This leads to a Bockstein spectral sequence of the form

Doug Ravenel

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This leads to a Bockstein spectral sequence of the form

$$\mathrm{Ext}\left(M_1^0\right) \otimes P(a_0) \Longrightarrow \mathrm{Ext}\left(M^1\right)$$

$$x \otimes a_0^j \rightsquigarrow \frac{x}{p^{j+1}}$$

The chromatic spectral sequence (continued)

For $h = 2$ we have two short exact sequences

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and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0^2 & \longrightarrow & \Sigma^{|v_1|}M_1^1 & \xrightarrow{v_1} & M_1^1 & \longrightarrow & 0. \\ & & \| & & & & & & \\ & & v_2^{-1}BP_*/(p, v_1) & & \dfrac{x}{pv_1^{i+1}} & \longmapsto & \dfrac{x}{pv_1^i} & & \\ & & & & & & & & \\ & & x & \longmapsto & \dfrac{x}{pv_1} & & & & \end{array}$$

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Each one leads to a Bockstein spectral sequence, making the desired $\operatorname{Ext}(M^2)$ two steps removed from the known $\operatorname{Ext}\left(v_2^{-1}BP_*/(p, v_1)\right)$.

The chromatic spectral sequence (continued)

More generally we have a short exact sequence of comodules

$$0 \longrightarrow \Sigma^{|v_i|} M_{i+1}^{h-i-1} \longrightarrow \Sigma^{|v_i|} M_i^{h-i} \xrightarrow{v_i} M_i^{h-i} \longrightarrow 0$$

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$$\operatorname{Ext}\left(M_{i+1}^{h-i-1}\right) \otimes P(a_i) \Longrightarrow \operatorname{Ext}\left(M_i^{h-i}\right)$$

$$\frac{x}{p v_1 \cdots v_{i-1} v_i v_{i+1}^{j_{i+1}} \cdots v_{h-1}^{j_{h-1}}} \otimes a_i^j \rightsquigarrow \frac{x}{p v_1 \cdots v_{i-1} v_i^{j+1} \ldots v_{h-1}^{j_{h-1}}}.$$

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This makes $\mathrm{Ext}\left(M^h\right)$ *h* steps removed from the cohomology of $\mathbb{S}_h$.

The chromatic spectral sequence (continued)

Computations with these Bockstein spectral sequence can be quite delicate.

The chromatic spectral sequence (continued)

Computations with these Bockstein spectral sequence can be quite delicate. Nearly all of them published since 1977 have been due to Katsumi Shimomura and various coauthors.

THANK YOU!