A NOTE ON TURÁN'S METHOD

S. M. Gonek

1. STATEMENT OF RESULTS

Let $s_\nu = \sum_{n=1}^{N} b_n z_n^\nu$, where the b_n and the z_n are complex numbers, and $\nu = 0, 1, 2, \ldots$. A question of current interest is the size of $|s_\nu|$ in the special case that $b_n > 0$ and $|z_n| = 1$ for all n. In this connection, Leenman and Tijdeman [4] have shown that

$$\max_{1 \le \nu \le 2N} |s_\nu| \ge \frac{s_0}{2\sqrt{N}}. \tag{1}$$

On the other hand, by Dirichlet's theorem on uniform approximation there is a ν, $1 \le \nu \le 6^N$, such that $\left\| \nu \dfrac{\arg z_n}{2\pi} \right\| \le \dfrac{1}{6}$ for $1 \le n \le N$. For this ν we have $\operatorname{Re} z_n^\nu = \cos(\nu \arg z_n) \ge \cos \dfrac{\pi}{3} = \dfrac{1}{2}$, so that

$$\max_{1 \le \nu \le 6^N} |s_\nu| \ge \frac{s_0}{2}. \tag{2}$$

It is easy to see that we may have $s_\nu = 0$ for $1 \le \nu \le N - 1$ (take $z_n = e(n/N)$, $b_n = 1$ for $1 \le n \le N$), so the range of ν in (1) is essentially as short as one may consider. Furthermore, $|s_\nu| \le s_0$ for all ν, so we cannot hope to improve on (2) by more than a constant if we consider longer ranges of ν. Thus the two estimates (1) and (2) represent the extreme situations. In what follows we obtain (1) and (2) by a unified method which gives good lower bounds for ranges $1 \le \nu \le K$ of intermediate length as well.

THEOREM 1. *Let $s_\nu = \sum_{n=1}^{N} b_n z_n^\nu$, where $b_n > 0$ and $|z_n| = 1$ for all n. For a given r, $r = 1, 2, 3, \ldots$, we have*

$$\max_{1 \le \nu \le 2\binom{N+r-1}{r}} |s_\nu| \ge s_0 \left(2 \binom{N+r-1}{r} \right)^{-1/2r}. \tag{3}$$

From this we deduce

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COROLLARY 1. *In the above notation, if $r \le N$ then*

(4) $$\max_{1\le\nu\le(12N/r)^r} |s_\nu| \ge s_0 \sqrt{\frac{r}{12N}}\,.$$

If $r = 1$ in (4) then we have (1); if $r = N$ then we obtain (2); we ignore constants here.

By Cauchy's inequality we see that $s_0 \le N^{1/2}\left(\sum_{n=1}^{N} b_n^2\right)^{1/2}$, so one might wish to sharpen (4) by showing that

(5) $$\max_{1\le\nu\le(12N/r)^r} |s_\nu| \ge cr^{1/2}\left(\sum_{n=1}^{N} b_n^2\right)^{1/2}.$$

This coincides with (4) in the important special case $b_n = 1$, $1 \le n \le N$, and is true when $r = 1$ (see (8) of the Lemma below), but (5) is false in general. For example, if $b_n = n^{-2}$ then $|s_\nu| < \zeta(2)$ for all ν, while the right-hand side of (5) grows with r.

As a special case of Corollary 1, we have the following result of H. L. Montgomery [5; p. 22].

COROLLARY 2. (Montgomery). *Let $s_\nu = \sum_{n=1}^{N} z_n^\nu$, where $|z_n| = 1$ for all n. Let r be an integer, $1 \le r \le N$. Then*

(6) $$\max_{1\le\nu\le(12N/r)^r} |s_\nu| \ge \sqrt{\frac{rN}{12}}\,.$$

To get an idea of the strength of the bounds in (4) and (6) we may compare (6) with a result of Erdös and Rényi [2]. They showed that there exist z_n with $|z_n| = 1$ for $1 \le n \le N$, such that $s_\nu = \sum_{n=1}^{N} z_n^\nu$ satisfies $|s_\nu| \le (6N\log(\nu+1))^{1/2}$ for $\nu = 1, 2, 3, \ldots$. For these z_n it follows that the left-hand side of (6) is less than or equal to $\sqrt{12rN\log\frac{12N}{r}}$.

We may also apply our method to improve a recent result of G. Halász [3] for long ν ranges. Halász's inequality, of which we consider only a special instance, is a long range version of Turán's first main theorem [6; Satz VII]. Let $s_\nu = \sum_{n=1}^{N} b_n z_n^\nu$, where the b_n and z_n are complex, and $|z_n| \ge 1$ for all n. If $K \ge N$, then

(7) $$\max_{1\le\nu\le K} |s_\nu| \ge \frac{|s_0|}{K^{1/2}e^{4N^2/K}}.$$

For $K = N$, the lower bound in (7) is greater than or equal to $|s_0|/e^{5N}$, which is roughly the estimate given by Turán's first main theorem. If $K = 4N^2$, the lower bound is substantially increased to $|s_0|/(2eN)$. But we cannot improve on this (except for the constant) by considering larger K. We remedy this with the following theorem and its corollary.

THEOREM 2. *Let* $s_\nu = \sum_{n=1}^{N} b_n z_n^\nu$, *where the* b_n *and* z_n *are complex, and* $|z_n| \geq 1$ *for* $1 \leq n \leq N$. *For a given r,* $r = 1, 2, 3, \ldots$, *we have*

$$\max_{1 \leq \nu \leq 4\binom{N+r-1}{r}^2} |s_\nu| \geq |s_0| \left(2e \binom{N+r-1}{r} \right)^{-1/r}.$$

COROLLARY 3. *In the notation of Theorem 2, if* $r \leq N$, *then*

$$\max_{1 \leq \nu \leq (12N/r)^{2r}} |s_\nu| \geq |s_0| \frac{r}{30N}.$$

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2. PROOFS

We could obtain Theorem 1 with a slightly weaker lower bound in (3) from (1). Instead we derive it from the following Lemma which is more general than (1) but no more difficult to prove. This result is implicitly contained in a paper of Cassels [1] (also see Montgomery [5; p. 20]).

LEMMA (Cassels). *Let* s_ν *be as in Theorem* 1. *Then*

$$\max_{1 \leq \nu \leq K} |s_\nu|^2 \geq \frac{K-N}{K} \sum_{n=1}^{N} b_n^2. \tag{8}$$

Alternatively,

$$\max_{1 \leq \nu \leq K} |s_\nu|^2 \geq \frac{K-N}{KN} s_0^2. \tag{9}$$

Proof. Let $z_n = e(\theta_n)$. Then

$$\begin{aligned}\frac{K}{2} \max_{1 \leq \nu \leq K} |s_\nu|^2 &\geq \sum_{\nu=1}^{K} \left(1 - \frac{\nu}{K+1}\right) |s_\nu|^2 \\ &= \sum_{n=1}^{N} \sum_{m=1}^{N} b_n b_m \sum_{\nu=1}^{K} \left(1 - \frac{\nu}{K+1}\right) \cos 2\pi\nu (\theta_n - \theta_m).\end{aligned}$$

The sum over ν is equal to $K/2$ if $\theta_n = \theta_m$, and is always greater than or equal to $-1/2$, by the nonnegativity of Féjer's kernel. Thus we find that

$$\frac{K}{2}\max_{1\le\nu\le K}|s_\nu|^2 \ge \frac{K}{2}\sum_{n=1}^{N} b_n^2 - \frac{1}{2}\sum_{n=1}^{N}\sum_{\substack{m=1\\ n\ne m}}^{N} b_n b_m \ge \frac{K}{2}\sum_{n=1}^{N} b_n^2 - \frac{1}{2}s_0^2,$$

that is, $\max_{1\le\nu\le K}|s_\nu|^2 \ge \sum_{n=1}^{N} b_n^2 - \frac{1}{K}s_0^2$. By Cauchy's inequality $s_0^2 \le N\sum_{n=1}^{N} b_n^2$, so the above gives $\max_{1\le\nu\le K}|s_\nu|^2 \ge \frac{K-N}{K}\sum_{n=1}^{N} b_n^2$, which is (8). We obtain (9) by applying Cauchy's inequality to this.

Proof of Theorem 1. We apply (9) of the Lemma with $K = 2\binom{N+r-1}{r}$ to the sum

$$t_\nu = s_\nu^r = \sum_{\substack{e_1+\dots+e_N=r\\ e_i\ge 0}} \frac{r!}{e_1!\dots e_N!} b_1^{e_1}\dots b_N^{e_N} e(\nu(e_1\theta_1+\dots+e_N\theta_N))$$

$$= \sum_{\vec{e}} b(\vec{e})z(\vec{e})^\nu,$$

say. Note that for each $\vec{e}$, we have $b(\vec{e}) > 0$ and $|z(\vec{e})| = 1$, and that the sum has $\sum_{\substack{e_1+\dots+e_N=r\\ e_i\ge 0}} 1 = \binom{N+r-1}{r}$ terms. Thus

$$\max_{1\le\nu\le 2\binom{N+r-1}{r}} |t_\nu|^2 \ge t_0^2\left(2\binom{N+r-1}{r}\right)^{-1}.$$

Replacing t_ν by s_ν^r and t_0 by s_0^r, and then taking $2r^{\text{th}}$ roots gives Theorem 1.

Proof of Corollary 1. Since $k! \ge (k/e)^k$ for all $k \ge 1$, we find that

$$\binom{l}{k} \le \frac{l^k}{k!} \le \left(\frac{el}{k}\right)^k.$$

Thus

$$2\binom{N+r-1}{r} \le 2\left(\frac{e(N+r)}{r}\right)^r \le \left(\frac{6(N+r)}{r}\right)^r \le \left(\frac{12N}{r}\right)^r,$$

as $r \le N$. Hence $\left(2\binom{N+r-1}{r}\right)^{-1/2r} \ge \left(\frac{r}{12N}\right)^{1/2}$. The result now follows easily from Theorem 1.

Proof of Corollary 2. Replace s_0 by N in Corollary 1.

Proof of Theorem 2. The proof is analogous to the proof of Theorem 1 but is based on (7) rather than the Lemma; we take $K = 4\binom{N+r-1}{r}^2$.

Proof of Corollary 3. Similar to the proof of Corollary 1.

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Department of Mathematics
Temple University
Philadelphia, Pennsylvania 19122