Simple Zeros of Zeta Functions

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In this note we present an easy proof that $\zeta(s)$, the Riemann zeta-function, has infinitely many non-real simple zeros. We also state some theorems on simple zeros of the Dedekind zeta-function of a quadratic extension of the rationals.

It is known that a positive proportion of the zeros of $\zeta(s)$ are simple. This follows from Levinson's work on zeros on the critical line as observed by Selberg and, independently, Heath-Brown. In light of the difficulty of Levinson's method it seems desirable to have an easy proof that $\zeta(s)$ has infinitely many simple zeros. For this reason and to motivate the technique of proof of our other theorems we give this proof.

The facts that we use regarding the zeta-function are as follows:

(1) $\zeta(s) = \chi(s)\ \zeta(1-s),\ \chi(s) \ll T^{1/2-\sigma}\quad (|\sigma| \ll 1,\ |t| \ll T);$

(2) $\zeta(s),\ \zeta'(s) \ll T^{1/3(1-\sigma)}\quad (\sigma \geq 1/2,\ |t| \ll T,\ |s-1| \gg 1);$

(3) $\frac{\chi'}{\chi}(s) = -\log\frac{t}{2\pi} + O(\frac{1}{1+|t|})\quad (|\sigma| \ll 1,\ |s| \gg 1);$

(4) $N(T) = \#\{\ \rho = \beta + i\gamma:\ \zeta(\rho) = 0,\ 0 < \gamma \ll T\} \ll T\log T;$

(5) $N(3/4,\ T) = \#\{\ \rho = \beta + i\gamma\ :\ \zeta(\rho) = 0,\ \sigma \geq 3/4,\ |\gamma| \ll T\ \} \ll T^{2/3}$,

(6) $\exists\ T'$ s.t. $|T - T'| \ll 1$ and $\frac{\zeta'}{\zeta}(\sigma + i\,T') \ll \log^2 T\ (|\sigma| \ll 1);$

(7) if $a_n \ll_\varepsilon n^\varepsilon$ for all $\varepsilon > 0$, then for $c > 1$ and all $\varepsilon > 0$,

$$\frac{1}{2\pi i}\int_{c+i}^{c+iT} \chi(1-s)\ \Sigma\, a_n\, n^{-s}\, ds = \sum_{n \leq T/2\pi} a_n + O_\varepsilon(T^{c-1/2+\varepsilon}).$$

The first six of these assertions may be found in Titchmarsh [5] while (7) is in Gonek [4].

Our proof will show that if

$$N^{*}(T) = \# \{\rho = \beta + i\gamma : \zeta(\rho) = 0,\ \zeta'(\rho) \neq 0,\ 0 < \gamma \leq T\},$$

then

$$N^{*}(T) \gg T^{1/3}.$$

By using stronger density estimates and mean value theorems for Dirichlet polynomials at well-spaced points we can obtain $N^{*}(T) \gg T^{4/5+\delta}$ for some small $\delta > 0$.

The starting point of the proof is Cauchy's inequality:

$$\Big|\sum_{0<\gamma\leq T} \zeta'(\rho)\Big|^2 \leq \Big|\sum_{0<\gamma\leq T} 1\Big| \sum_{0<\gamma\leq T} |\zeta'(\rho)|^2 .$$

The second sum here is $N^{*}(T)$ so it suffices to bound the first sum from below and the third sum from above. (We shall obtain an asymptotic formula for the first sum.)

We begin by estimating the third sum, say S_3. Then by the symmetry of the zeros about the 1/2-line and the identity

$$\zeta'(\rho) = -\chi(\rho)\ \zeta'(1-\rho)$$

which follows from (1) we obtain

$$S_3 = \sum_{\substack{0<\gamma\leq T \\ \beta\geq 1/2}} \left(|\zeta'(\rho)|^2 + |\zeta'(1-\rho)|^2 \right)$$

$$= \sum_{\substack{0\leq\gamma<T \\ \beta\geq 1/2}} |\zeta'(\rho)|^2\ (1+ |\chi(1-\rho)|^2).$$

By (1) and (2) we obtain

$$S_3 \ll \sum_{\substack{0<\gamma\leq T \\ \beta\geq 1/2}} T^{(4/3)\beta-1/3} \ll \sum_{\substack{0\leq\gamma<T \\ 1/2\leq\beta\leq 3/4}} T^{2/3} + \sum_{\substack{0<\gamma\leq T \\ 3/4\leq\beta<1}} T$$

whence by (4) and (5),

$$S_3 \ll T^{2/3} N(T) + T\, N(3/4,T) \ll T^{5/3} \log T.$$

Next we evaluate the first sum, say S_1. Let T' be as in (6). Then by Cauchy's theorem and (1) and (2),

$$S_1 = \frac{1}{2\pi i} \int_{C} \frac{\zeta'}{\zeta}(s)\, ds + O_\varepsilon(T^{1/2 + \varepsilon})$$

where C is the positively oriented rectangular contour with vertices $c+i$, $c+iT'$, $1-c+iT'$, $1-c+i$ $(c=1 + (\log T)^{-1})$. The integral along the "bottom" edge is $\ll 1$; the integral along the "top" edge is $\ll_\varepsilon T^{1/2+\varepsilon}$ by (1), (2), and (6). Along the "right" side the integral is

$$= \frac{1}{2\pi} \sum_{m,n} \frac{\Lambda(m) \log n}{m^c n^c} \int_1^{T'} (mn)^{-it}\, dt$$

$$\ll \sum_{m,n} \frac{\Lambda(m) \log n}{m^c n^c \log(mn)} \ll \frac{\zeta'}{\zeta}(c)\, \zeta'(c) \ll (\log T)^3.$$

Finally, the integral along the left side is

$$= \frac{1}{2\pi i} \int_{1-c+iT'}^{1-c+i} \frac{\zeta'}{\zeta}(s)\, \zeta'(s)\, ds = \frac{-1}{2\pi i} \int_{c-iT'}^{c-i} \frac{\zeta'}{\zeta}(1-s)\, \zeta'(1-s)\, ds = -\bar{I}$$

where

$$I = \frac{1}{2\pi i} \int_{c+i}^{c+iT'} \frac{\zeta'}{\zeta}(1-s)\, \zeta'(1-s)\, ds.$$

From (1) it follows that

$$I = \frac{1}{2\pi i} \int_{c+i}^{c+iT'} \left(\frac{\chi'}{\chi}(s) - \frac{\zeta'}{\zeta}(s)\right) \chi(1-s) \left(-\zeta'(s) + \frac{\chi'}{\chi}(s)\, \zeta(s)\right) ds$$

$$= \frac{1}{2\pi i} \int_{c+i}^{c+iT'} \chi(1-s) \left(\frac{\zeta'}{\zeta}(s)\, \zeta'(s) - 2\, \frac{\chi'}{\chi}(s) + \left(\frac{\chi'}{\chi}(s)\right)^2 \zeta(s)\right) ds$$

$$= J_1 - 2 J_2 + J_3$$

say.

By (7) and the prime number theorem,

$$J_1 = \sum_{mn \le \frac{T}{2\pi}} \Lambda(m) \log n + O_\varepsilon(T^{1/2 + \varepsilon})$$

$$= \frac{T}{4\pi} \log^2 T + O(T \log T).$$

Now let

$$K_2(u) = \frac{1}{2\pi i} \int_{c+i}^{c+iu} \chi(1-s) \zeta'(s)\, ds$$

and

$$K_3(u) = \frac{1}{2\pi i} \int_{c+i}^{c+iu} \chi(1-s) \zeta(s)\, ds.$$

Then for $1 \le u \ll T$ it follows from (7) that

$$K_2(u) = \frac{-u}{2\pi} \log u + O(u)$$

and

$$K_3(u) = \frac{u}{2\pi} + O_\varepsilon(u^{1/2 + \varepsilon}).$$

Now by (1), (2), (3) and integration by parts we obtain

$$J_2 = \frac{-1}{2\pi i} \int_{c+i}^{c+iT'} \log \frac{t}{2\pi} \chi(1-s) \zeta'(s)\, ds + O_\varepsilon(T^{1/2 + \varepsilon})$$

$$= - \int_1^{T'} \log \frac{t}{2\pi}\, dK_2(t) + O_\varepsilon(T^{1/2 + \varepsilon})$$

$$= -\log \frac{t}{2\pi} K_2(t) \Big|_1^{T'} + \int_1^{T'} \frac{K_2(t)}{t} dt + O_\varepsilon(T^{1/2+\varepsilon})$$

$$= \frac{T}{2\pi} \log^2 T + O(T \log T).$$

Similarly, we find that

$$J_3 = \frac{T}{2\pi} \log^2 T + O(T \log T).$$

Thus,

$$I = \frac{-T}{4\pi} \log^2 T + O(T \log T)$$

which leads to

$$S_1 = \frac{T}{4\pi} \log^2 T + O(T \log T).$$

Finally,

$$N^*(T) \geq \frac{|S_1|^2}{S_3} \gg \frac{T^2 \log^4 T}{T^{5/3} \log T} \gg T^{1/3} .$$

An elaboration of this method leads to a result on simple zeros of the Dedekind zeta-function of a quadratic extension k of the rationals. Let $\zeta_k(s)$ denote such a zeta-function and

$$N_k^*(T) = \#\{\rho_k = \beta_k + i\gamma_k : \zeta_k(\rho_k) = 0,\ \zeta_k'(\rho_k) \neq 0,\ 0 < \gamma_k < T\}.$$

Then we can prove [2]

$$N_k^*(T) \gg T^{6/11} .$$

If we assume the Riemann-Hypothesis, then S_3 can be evaluated asymptotically (see Gonek [4]). In this case we obtain $N^*(T) \gg T$ (since $S_3 \ll T(\log T)^4$). To obtain $N^*(T) \gg N(T)$ we sum $\zeta'(\rho)\,B(\rho)$ instead of $\zeta'(\rho)$ where $B(s)$ is a function which makes $\zeta' B$ normal in terms of moments. Our choice is

$$B(s) = \sum_{n \leq y} \frac{b(n)}{n^s}$$

with $y = T^{1/2 - \epsilon}$ and $b(n) = \mu(n) P(\frac{\log y/n}{\log y})$ where μ is the Möbius function and P is a polynomial satisfying $P(0) = 0$, $P(1) = 1$. By the calculus of variations, $P(x) = \frac{-x}{2} + \frac{3}{2} x^2$ is the optimal choice of P and this leads to (on RH)

$$N^*(T) \geq (\frac{19}{27} + o(1)) N(T)$$

(see [1]). Of course, the proof we have given in this paper does not even begin to address the obstacles which must be submounted to obtain this result.

Finally, we mention that our last result can be used in conjunction with a theorem on common zeros of L-functions to derive (on RH) that

$$N_k^*(T) \geq (\frac{1}{54} + o(1)) N_k(T)$$

where

$$N_k(T) = \# \{\rho_k : \zeta_k(\rho_k) = 0, 0 < \gamma_k < T\}$$

(see [3]). This result does not seem to be accessible by other known methods.

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