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# Mean values of the Riemann zeta-function and its derivatives

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## § 1. Introduction and statement of results

In 1918 Hardy and Littlewood [2] proved that as $T\to\infty$

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^2\,dt \sim T\log T. \tag{1}$$

In 1928 Ingham [3] generalized this considerably by showing that as $T\to\infty$

$$\int_1^T \zeta^{(\mu)}(\tfrac{1}{2}+it)\zeta^{(\nu)}(\tfrac{1}{2}-it)\,dt \sim \frac{T}{\mu+\nu+1}(\log T)^{\mu+\nu+1}, \tag{2}$$

wher $\zeta^{(\mu)}(s)$ denotes the $\mu^{\text{th}}$ derivative of $\zeta(s)$ $(=\zeta^{(0)}(s))$. Since $\zeta^{(\mu)}(\tfrac{1}{2}-it) = \overline{\zeta^{(\mu)}(\tfrac{1}{2}+it)}$, it follows in particular that

$$\int_1^T |\zeta^{(\mu)}(\tfrac{1}{2}+it)|^2\,dt \sim \frac{T}{2\mu+1}(\log T)^{2\mu+1}, \tag{3}$$

which gives (1) when $\mu=0$. Our object in this paper is to prove some new types of mean value formulae which are, when the Riemann hypothesis is assumed, discrete analogues of (1)–(3).

We denote the non-trivial zeros of $\zeta(s)$ by $\rho=\beta+i\gamma$ and we set $L=\frac{1}{2\pi}\log\frac{T}{2\pi}$. Our main result is the following

**Theorem.** *If $T$ is sufficiently large and $\alpha$ is any real number satisfying $|\alpha|\leqq\frac{1}{2}L$, then*

$$\sum_{1\leqq\gamma\leqq T}\zeta^{(\mu)}(\rho+i\alpha L^{-1})\zeta^{(\nu)}(1-\rho-i\alpha L^{-1})$$
$$=(-1)^{\mu+\nu}\left(\frac{1}{\mu+\nu+1}-H(\mu,\nu,2\pi\alpha)-H(\nu,\mu,-2\pi\alpha)\right)\frac{T}{2\pi}(\log T)^{\mu+\nu+2}$$
$$+O(T(\log T)^{\mu+\nu+1}), \tag{4}$$

*where*

$$H(\mu, \nu, 2\pi\alpha) = \mu! \sum_{l=0}^{\infty} \frac{(2\pi\alpha i)^l}{(l+\mu+1)!(l+\mu+\nu+2)}.$$

*The constant implicit in the O-term is independent of α.*

As an immediate consequence we have

**Corollary 1.** *Suppose that the Riemann hypothesis is true. If T, α, and H are as in the Theorem, then*

$$\sum_{1\leqq\gamma\leqq T} \zeta^{(\mu)}(\tfrac{1}{2}+i(\gamma+\alpha L^{-1}))\zeta^{(\nu)}(\tfrac{1}{2}-i(\gamma+\alpha L^{-1}))$$
$$=(-1)^{\mu+\nu}\left(\frac{1}{\mu+\nu+1} - H(\mu,\nu,2\pi\alpha) - H(\nu,\mu,-2\pi\alpha)\right)\frac{T}{2\pi}(\log T)^{\mu+\nu+2}$$
$$+O(T(\log T)^{\mu+\nu+1}. \tag{5}$$

*In particular,*

$$\sum_{1\leqq\gamma\leqq T} |\zeta^{(\mu)}(\tfrac{1}{2}+i(\gamma+\alpha L^{-1}))|^2$$
$$=\left(\frac{1}{2\mu+1} - H(\mu,\mu,2\pi\alpha) - H(\mu,\mu,-2\pi\alpha)\right)\frac{T}{2\pi}(\log T)^{2\mu+2}$$
$$+O(T(\log T)^{2\mu+1}). \tag{6}$$

*The constants implicit in the O-terms are independent of α.*

In (5) and (6) we have discrete analogues of (2) and (3). Note that there are $\sim \frac{T}{2\pi}\log T$ terms in the sum in (6) and that the right-hand side of (6) is

$$\sim \frac{\mu^2}{(2\mu+1)(\mu+1)^2}\cdot\frac{T}{2\pi}(\log T)^{2\mu+2}$$

when $\alpha=0$ and $\mu\geqq 1$. Thus, comparing (3) and (6), we see that on the Riemann hypothesis the average of $|\zeta^{(\mu)}(\frac{1}{2}+i\gamma)|^2$ over those zeros with $1\leqq\gamma\leqq T$ is smaller by a factor of $\left(\frac{\mu}{\mu+1}\right)^2$ than the average of $|\zeta^{(\mu)}(\frac{1}{2}+it)|^2$ over all points with $1\leqq t\leqq T$.

The case $\mu=0$ of (6) is a discrete analogue (1) and is of interest in its own right so we state it as

**Corollary 2.** *Assume the Riemann hypothesis is true. If T is sufficiently large and α is a real number such that* $|\alpha|\leqq L/2$, *then*

$$\sum_{1\leqq\gamma\leqq T} |\zeta(\tfrac{1}{2}+i(\gamma+\alpha L^{-1}))|^2 = \left(1-\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}\log^2 T + O(T\log T). \tag{7}$$

*The constant implicit in the O-term is independent of α.*

J. Mueller (see [6] and [7]) has recently found an interesting application of Corollary 2. Denote by $0 \leqq \gamma_1 \leqq \gamma_2 \leqq \dots$ the imaginary parts of the zeros of $\zeta(s)$ in the upper half-plane and set

$$\lambda = \limsup_n (\gamma_n - \gamma_{n-1}) \frac{\log \gamma_n}{2\pi},$$

$$\mu = \liminf_n (\gamma_n - \gamma_{n-1}) \frac{\log \gamma_n}{2\pi}.$$

A. Selberg [8] has remarked that $\mu < 1$ and $\lambda > 1$, and H. Montgomery [5], assuming the Riemann hypothesis, has shown that $\mu \leqq 0.68$. Mueller's result is

**Corollary 3.** *If the Riemann hypothesis is true* $\lambda \geqq 1.9$.

As the proof of this is brief, we give it in Sect. 6.

This paper is, in revised form, a part of my doctoral thesis written under the direction of Professor H.L. Montgomery. He suggested the problem which led to this investigation and made many helpful comments concerning the form and content of my thesis. It gives me gerat pleasure to express my gratitude to him here. I would also like to thank Professors D.J. Lewis, G. Piranian, and J. Ullman for carefully reading my thesis, and the referee for his very helpful suggestions.

## § 2. Some formulae and estimates

Before we develop the basic idea of the proof of the Theorem, it will be useful to set down certain formulae and estimates.

Throughout this paper $s = \sigma + it$ denotes a complex variable.

Let

$$\chi(1-s) = \pi^{\frac{1}{2}-s} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}, \tag{8}$$

$\Gamma(s)$ being the gamma-function. The unsymmetric form of the functional equation for $\zeta(s)$ *is*

$$\zeta(1-s) = \chi(1-s)\zeta(s). \tag{9}$$

We also require the symmetric form of the functional equation. Set

$$\xi(s) = \tfrac{1}{2} s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{10}$$

Then

$$\xi(s) = \xi(1-s). \tag{11}$$

The function $\xi(s)$ is entire of order one and its only zeros are the non-trivial zeros of $\zeta(s)$.

We write Stirling's formula for $\Gamma(s)$ in the form

$$\log \Gamma(s)=(s-\tfrac{1}{2})\log s-s+\tfrac{1}{2}\log 2\pi+O\left(\frac{1}{|s|}\right). \tag{12}$$

This is valid for $|s|\geqq\frac{1}{2}$ and $|\arg s|<\pi-\delta$, where $\delta>0$ is arbitrary but fixed (see Whittaker and Watson [10; Chaps. 12, 13]). Using this, it is not difficult to show that

$$\chi(1-s)=e^{-\frac{\pi i}{4}}\left(\frac{t}{2\pi}\right)^{\sigma-\frac{1}{2}}\exp[it\log t/2\pi e]\left(1+O\left(\frac{1}{t}\right)\right) \tag{13}$$

for $\sigma$ fixed and $t\geqq 1$, say.

Euler's psi-function is defined by

$$\psi(s)=\frac{\Gamma'(s)}{\Gamma(s)}. \tag{14}$$

When $|\arg s|<\pi-\delta$ and $|s|\geqq\frac{1}{2}$, we have

$$\psi(s)=\log s+O\left(\frac{1}{|s|}\right). \tag{15}$$

This may be derived from (12) by means of Cauchy's estimate for analytic functions.

As is well known, in each interval $(n, n+1)$ $(n=2, 3, \ldots)$ we can select a number $T_n$ such that if $\gamma$ is the ordinate of any zero of $\zeta(s)$, then $|T_n-\gamma| \gg \frac{1}{\log T_n}$. In this way we obtain a sequence $\mathscr{T}$ which will be fixed throughout this paper.

Recall that if $T$ is large and does not coincide with the ordinate of any zero of $\zeta(s)$, then

$$\frac{\zeta'}{\zeta}(\sigma+iT)=\sum_{|\gamma-T|<1}\frac{1}{s-\rho}+O(\log T)$$

uniformly for $-1\leqq\sigma\leqq 2$ (cf. Davenport [1; p. 99]). There are $\ll\log T$ terms in this sum and if $T\in\mathscr{T}$, each term is $\ll\log T$. Thus, for each large $T\in\mathscr{T}$ and uniformly for $-1\leqq\sigma\leqq 2$,

$$\frac{\zeta'}{\zeta}(\sigma+iT)\ll\log^2 T. \tag{16}$$

By logarithmic differentiation of (10) we have

$$\frac{\xi'}{\xi}(s)=\frac{\zeta'}{\zeta}(s)+\tfrac{1}{2}\psi(s/2)-\tfrac{1}{2}\log\pi+\frac{2s-1}{s(s-1)} \tag{17}$$

We deduce from this, (15), and (16) that

$$\frac{\xi'}{\xi}(\sigma+iT) \ll \log^2 T \tag{18}$$

for all large $T\in\mathcal{T}$, uniformly for $-1\leqq\sigma\leqq 2$.

Similarly, we may combine the estimate

$$\frac{\zeta'}{\zeta}(\sigma+it) \ll 1 \qquad (\sigma\geqq a>1)$$

with (15) and (17) to obtain

$$\frac{\xi'}{\xi}(\sigma+it) \ll \log 2|t| \tag{19}$$

for $\sigma\geqq a>1$ and $|t|\geqq 1$, say.

Finally, we need the estimates

$$\zeta^{(\nu)}(\sigma+it) \ll \begin{cases} |t|^{\frac{1}{2}-\sigma+\varepsilon} & \text{if } \sigma\leqq 0 \\ |t|^{\frac{1}{2}(1-\sigma)+\varepsilon} & \text{if } 0\leqq\sigma\leqq 1 \\ |t|^{\varepsilon} & \text{if } \sigma\geqq 1, \end{cases} \tag{20}$$

where $\varepsilon>0$ is arbitrary, $|t|\geqq\frac{1}{2}$, and $\nu=0,1,2,\ldots$. These may be derived from the case $\nu=0$, $|t|\geqq\frac{1}{4}$ (for which see Titchmarsh [9; pp. 81–82]) by applying Cauchy's estimate for the derivatives of analytic functions to $\zeta(s)$ in a small disc centered at $s=\sigma+it$.

## § 3. Beginning of the proof

We can now begin the proof of the Theorem, although we will require a section of lemmas (Sect. 4 below) to complete it.

Let $1<a<2$ and let $R$ denote the closed rectangle in the complex plane with vertices at $a+i$, $a+iT$, $1-a+iT$, $1-a+i$, where $T$ is large. We define

$$I=I(\mu,\nu,\delta)=\frac{1}{2\pi i}\int_{\partial R}\frac{\xi'}{\xi}(s)\zeta^{(\mu)}(s+i\delta)\zeta^{(\nu)}(1-s-i\delta)\,ds,$$

where $\partial R$ is the boundary of $R$ and the integral is taken in the counterclockwise sense. Also, we assume $\delta$ is real and $|\delta|\leqq\frac{1}{2}$. By the theory of residues

$$I(\mu,\nu,\delta)=\sum_{1\leqq\gamma\leqq T}\zeta^{(\mu)}(\rho+i\delta)\zeta^{(\nu)}(1-\rho-i\delta) \tag{21}$$

provided that no zero $\rho$ lies on $\partial R$. Since the ordinate of the first zero of $\zeta(s)$ above the real axis is $>14$ and no zeros lie on the vertical edges of $R$, we need only insure that $T$ is not the ordinate of a zero. This will be the case if $T\in\mathcal{T}$, the set constructed in Sec. 2. From now on we assume $T\in\mathcal{T}$; at the end of the proof this restriction will be removed.

To prove the Theorem we must estimate the integral $I(\mu, \nu, \delta)$. To do this we first split it into four parts corresponding to the four sides of $R$. We write

$$I(\mu, \nu, \delta) = \sum_{j=1}^{4} I_j(\mu, \nu, \delta),$$

where $I_1$ is the integral over $[a+i, a+iT)$, $I_2$ is over $[a+iT, 1-a+iT)$, $I_3$ is over $[1-a+iT, 1-a+i)$, and $I_4$ is over $[1-a+i, a+i)$. Since $|\delta| \leqq \frac{1}{2}$, the integral in $I_4$ is bounded, i.e. $I_4 \ll 1$. Next

$$\begin{aligned} I_2 &\ll \max_{1-a \leqq \sigma \leqq a} \left| \frac{\xi'}{\xi}(\sigma+iT)\zeta^{(\mu)}(\sigma+iT+i\delta)\zeta^{(\nu)}(1-\sigma-iT-i\delta) \right| \\ &\ll \log^2 T \max_{1-a \leqq \sigma \leqq a} |\zeta^{(\mu)}(\sigma+iT+i\delta)\zeta^{(\nu)}(1-\sigma-iT-i\delta)| \end{aligned}$$

by (18). The last line is

$$\leqq \log^2 T\Big(\max_{1-a \leqq \sigma \leqq 0} + \max_{0 \leqq \sigma \leqq 1} + \max_{1 \leqq \sigma \leqq a}\Big)|\zeta^{(\mu)}(\sigma+iT+i\delta)\zeta^{(\nu)}(1-\sigma-iT-i\delta)|,$$

which by (20) is

$$\leqq \log^2 T(T^{a-\frac{1}{2}+2\varepsilon} + T^{\frac{1}{2}+2\varepsilon} + T^{a-\frac{1}{2}+2\varepsilon}).$$

Since $a > 1$ and $\varepsilon > 0$ is arbitrary, we obtain

$$I_2 \ll T^{a-\frac{1}{2}+\varepsilon}.$$

This and the estimate for $I_4$ lead to

$$I(\mu, \nu, \delta) = I_1(\mu, \nu, \delta) + I_3(\mu, \nu, \delta) + O(T^{a-\frac{1}{2}+\varepsilon}). \tag{22}$$

We now consider $I_3$. The logarithmic derivative of (11) is

$$\frac{\xi'}{\xi}(s) = -\frac{\xi'}{\xi}(1-s).$$

Using this and the fact that both $\zeta^{(\lambda)}(s)$ and $\frac{\xi'}{\xi}(s)$ satisfy the reflection principle, we get

$$\begin{aligned} I_3(\mu, \nu, \delta) &= -\frac{1}{2\pi i} \int_{1-a+iT}^{1-a+i} \frac{\xi'}{\xi}(1-s)\zeta^{(\mu)}(s+i\delta)\zeta^{(\nu)}(1-s-i\delta)\,ds \\ &= \frac{1}{2\pi i} \int_1^T \frac{\xi'}{\xi}(a-it)\zeta^{(\mu)}(1-a+it+i\delta)\zeta^{(\nu)}(a-it-i\delta)\,i\,dt \\ &= \overline{\frac{1}{2\pi i} \int_1^T \frac{\xi'}{\xi}(a+it)\zeta^{(\mu)}(1-a-it-i\delta)\zeta^{(\nu)}(a+it+i\delta)\,i\,dt} \\ &= \overline{\frac{1}{2\pi i} \int_{a+i}^{a+iT} \frac{\xi'}{\xi}(s)\zeta^{(\mu)}(1-s-i\delta)\zeta^{(\nu)}(s+i\delta)\,ds} \\ &= \overline{I_1(\nu, \mu, \delta)}. \end{aligned}$$

This and (22) yield

$$I(\mu, \nu, \delta) = I_1(\mu, \nu, \delta) + \overline{I_1(\nu, \mu, \delta)} + O(T^{a-\frac{1}{2}+\varepsilon}). \tag{23}$$

Our problem is now reduced to estimating $I_1(\mu, \nu, \delta)$.

## § 3. Lemmas

Our first two lemmas are modified versions of Lemmas 3.2 and 3.3 of N. Levinson [4].

**Lemma 1.** *There is a small $c > 0$ such that*

$$\begin{aligned} I_0 &= \int_{r(1-c)}^{r(1+c)} \exp\left[it \log\left(\frac{t}{er}\right)\right] \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} dt \\ &= (2\pi)^{1-a} r^a e^{-ir+\pi i/4} + O(r^{a-\frac{1}{2}}) \end{aligned}$$

*for large $r$ and $a$ arbitrary but fixed.*

*Proof.* This result follows from the usual stationary phase techniques. If we set $t = r(1+x)$ then we can write

$$I_0 = (2\pi)^{\frac{1}{2}-a} e^{-ir} r^{a+\frac{1}{2}} I_1,$$

where

$$I_1 = \int_{-c}^{c} \exp(irf(x))(1+x)^{a-\frac{1}{2}} dx$$

with

$$f(x) = (1+x)\log(1+x) - (1+x).$$

Now $f(z) = \frac{z^2}{2} + \dots$ is holomorphic in a neighborhood of $z=0$ with only a double zero at $z=0$. Thus $u(z) = \sqrt{2f(z)} = z + \dots$ is holomorphic and $u'(z) \neq 0$ in a neighborhood of $z=0$. We make the change of variables $u(z) = u$ and obtain, if $c$ is sufficiently small,

$$I_1 = \int_{u(-c)}^{u(c)} e^{iru^2/2} g(u)\, du,$$

where $g(u) = 1 + \dots$ is holomorphic in a neighborhood of $u=0$. Now integration by parts yields

$$I_1 = \int_{u(-c)}^{u(c)} e^{iru^2/2} du + O\left(\frac{1}{r}\right),$$

and

$$\int_{u(-c)}^{u(c)} e^{iru^2/2} du = \left(\frac{2\pi}{r}\right)^{\frac{1}{2}} e^{\pi i/4} + O\left(\frac{1}{r}\right)$$

by the method of stationary phase; this proves Lemma 1.

**Lemma 2.** *For large A and* $A<r\leqq B\leqq 2A$

$$\int_A^B \exp\left[it\log\left(\frac{t}{re}\right)\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}dt=(2\pi)^{1-a}r^a e^{-ir+\pi i/4}+E(r,A,B), \qquad (24)$$

*where a is fixed and where*

$$E(r,A,B)=O(A^{a-\frac{1}{2}})+O\left(\frac{A^{a+\frac{1}{2}}}{|A-r|+A^{\frac{1}{2}}}\right)+O\left(\frac{B^{a+\frac{1}{2}}}{|B-r|+B^{\frac{1}{2}}}\right). \qquad (25)$$

*For* $r\leqq A$ *or* $r>B$,

$$\int_A^B \exp\left[it\log\left(\frac{t}{re}\right)\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}dt=E(r,A,B).$$

*Proof.* *If* $A<r\leqq A+A^{\frac{1}{2}}$ or if $B-B^{\frac{1}{2}}<r\leqq B$ the integral is $O(A^a)$ by Titchmarsh [9], Ch. IV, 4.5, Lemma 4.5. If instead $A+A^{\frac{1}{2}}<r<B-B^{\frac{1}{2}}$, we have

$$\int_A^B \exp\left[it\log\left(\frac{t}{re}\right)\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}dt=I_0+J_1+J_2,$$

where

$$J_1=\int_A^{r(1-c)} \exp\left[\left(it\log\left(\frac{t}{re}\right)\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}dt$$

and similarly for $J_2$. Now integration by parts shows that

$$|J_1|=O\left(A^{a-\frac{1}{2}}\Big/\left|\log\frac{r}{A}\right|\right)$$

and of course a similar estimate holds for $J_2$. Finally, if either $r<A-A^{\frac{1}{2}}$ or $r>B+B^{\frac{1}{2}}$, the required estimate follows by integration by parts. In view of Lemma 1, this completes the proof.

**Lemma 3.** *For* $m=0,1,2,\ldots$, *A large, and* $A<r\leqq B\leqq 2A$,

$$\int_A^B \exp\left[it\log\left(\frac{t}{re}\right)\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{t}{2\pi}\right)^m dt$$
$$=(2\pi)^{1-a}r^a e^{-ir+\pi i/4}\left(\log\frac{r}{2\pi}\right)^m+E(r,A,B)(\log A)^m,$$

*while for* $r\leqq A$ *or* $r>B$,

$$\int_A^B \exp\left[it\log\left(\frac{t}{re}\right)\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{t}{2\pi}\right)^m dt=E(r,A,B)(\log A)^m,$$

*where* $E(r,A,B)$ *is* (25).

*Proof.* The proof of Lemma 3 is easily obtained from Lemma 2 and integration by parts and therefore we omit it.

**Lemma 4.** *Let $E(r, A, B)$ be as in (25), where $A$ is large and $A<B\leqq 2A$. Assume $\{b_n\}_{n=1}^{\infty}$ is a sequence of complex numbers such that $b_n \ll n^{\varepsilon}$ for any $\varepsilon>0$. Then if $a>1$,*

$$\sum_{n=1}^{\infty} \frac{b_n}{n^a} E(2\pi n, A, B) \ll A^{a-\frac{1}{2}}.$$

*Proof.* Choose $\varepsilon$ so that $0<\varepsilon<a-1$. By (25)

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{b_n}{n^a} E(2\pi n, A, B) &\ll \sum_{n=1}^{\infty} n^{-a+\varepsilon} E(2\pi n, A, B)\\ &\ll A^{a-\frac{1}{2}} \sum_{n=1}^{\infty} n^{-a+\varepsilon} + A^{a+\frac{1}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{a-\varepsilon}(|A-2\pi n|+A^{\frac{1}{2}})}\\ &\quad + B^{a+\frac{1}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{a-\varepsilon}(|B-2\pi n|+B^{\frac{1}{2}})}.\end{aligned}$$

The proof of the lemma is completed by noting that

$$\sum_{n=1}^{\infty} n^{-a+\varepsilon} \ll 1$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^{a-\varepsilon}(|C-2\pi n|+C^{\frac{1}{2}})} \ll C^{-1};$$

indeed the last inequality is easily established by considering separately the ranges $|C-2\pi n|<C^{\frac{1}{2}}$ and $|C-2\pi n| \geqq C^{\frac{1}{2}}$.

**Lemma 5.** *Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that for any $\varepsilon>0$, $b_n \ll n^{\varepsilon}$. Let $a>1$ and let $m$ be a non-negative integer. Then for $T$ sufficiently large,*

$$\begin{aligned}&\frac{1}{2\pi} \int_1^T \left(\sum_{n=1}^{\infty} b_n n^{-a-it}\right) \chi(1-a-it) \left(\log \frac{t}{2\pi}\right)^m dt\\ &\quad = \sum_{1 \leqq n \leqq T/2\pi} b_n (\log n)^m + O(T^{a-\frac{1}{2}} (\log T)^m).\end{aligned} \tag{26}$$

*Proof.* By (13) we have

$$\begin{aligned}&\frac{1}{2\pi} \int_{T/2}^{T} \left(\sum_{n=1}^{\infty} b_n n^{-a-it}\right) \chi(1-a-it) \left(\log \frac{t}{2\pi}\right)^n dt\\ &= \frac{1}{2\pi} \int_{T/2}^{T} \left(\sum_{n=1}^{\infty} b_n n^{-a-it}\right) e^{-\pi i/4} \exp\left[it \log \frac{t}{2\pi e}\right] \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \left(\log \frac{t}{2\pi}\right)^m dt\\ &\quad + O\left(\int_{T/2}^{T} \left(\sum_{n=1}^{\infty} |b_n| n^{-a}\right) t^{a-\frac{3}{2}} (\log t)^m dt\right).\end{aligned} \tag{27}$$

Since $b_n \ll n^\varepsilon$, $\sum_{n=1}^{\infty} |b_n| n^{-a} \ll 1$ for $a>1$. The error term is therefore

$$\ll T^{a-\frac{1}{2}} (\log T)^m. \tag{28}$$

To treat the main term on the right-hand side of (27) we write it as

$$\sum_{n=1}^{\infty} b_n n^{-a} e^{-\pi i/4} \left( \frac{1}{2\pi} \int_{T/2}^{T} \exp\left[ it \log \frac{t}{2\pi n e} \right] \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \left(\log \frac{t}{2\pi}\right)^m dt \right), \tag{29}$$

the inversion of summation and integration being justified by absolute convergence. Now the integral in (29) is of the form estimable by Lemma 3 with $A = \frac{T}{2}$, $B=T$, and $r=2\pi n$. Thus (29) is equal to

$$\sum_{T/4\pi < n \leqq T/2\pi} b_n (\log n)^m + \left(\log \frac{T}{2}\right)^m \sum_{n=1}^{\infty} b_n n^{-a} E\left(2\pi n, \frac{T}{2}, T\right)$$

for large $T$. By Lemma 4 the second term is

$$\ll T^{a-\frac{1}{2}} (\log T)^m.$$

Hence (29) is equal to

$$\sum_{T/4\pi < n \leqq T/2\pi} b_n (\log n)^m + O(T^{a-\frac{1}{2}} (\log T)^m).$$

Using this and (28) in (27) we obtain

$$\begin{aligned} &\frac{1}{2\pi} \int_{T/2}^{T} \left( \sum_{n=1}^{\infty} b_n n^{-a-it} \right) \chi(1-a-it) \left(\log \frac{t}{2\pi}\right)^m dt \\ &\quad = \sum_{T/4\pi < n \leqq T/2\pi} b_n (\log n)^m + O(T^{a-\frac{1}{2}} (\log T)^m) \end{aligned} \tag{30}$$

for $T \geqq T_0$, say. Now let $l$ be the unique integer such that $T_0 \leqq \frac{T}{2^l} < 2T_0$. Adding together the result of (30) for the ranges $\left[\frac{T}{2^j}, \frac{T}{2^{j-1}}\right]$ $(j=1, \ldots, l)$, we find that

$$\begin{aligned} &\frac{1}{2\pi} \int_{T/2^l}^{T} \left( \sum_{n=1}^{\infty} b_n n^{-a-it} \right) \chi(1-a-it) \left(\log \frac{t}{2\pi}\right)^m dt \\ &\quad = \sum_{T/2^{l+1}\pi < n \leqq T/2\pi} b_n (\log n)^m + O(T^{a-\frac{1}{2}} (\log T)^m). \end{aligned}$$

Noting that

$$\frac{1}{2\pi} \int_{1}^{T/2^l} \left( \sum_{n=1}^{\infty} b_n n^{-a-it} \right) \chi(1-a-it) \left(\log \frac{t}{2\pi}\right)^m dt \ll 1$$

and

$$\sum_{1 \leqq n \leqq T/2^{l+1}\pi} b_n (\log n)^m \ll 1,$$

we obtain the result.

**Lemma 6.** *For $\sigma$ fixed, $v \geqq 0$, and $|t| \geqq 1$ we have*

$$\chi^{(v)}(1-s) = \chi(1-s)\left(-\log\frac{|t|}{2\pi}\right)^v + O(|t|^{\sigma-\frac{3}{2}}(\log|t|)^{v-1}). \tag{31}$$

*Proof.* We proceed by induction on $v$. The case $v=0$ is obviously true. Now suppose the lemma proved for $v=0, \ldots, \mu-1$. We differentiate the identity

$$\chi'(1-s) = \chi(1\text{-}s)\cdot\frac{\chi'}{\chi}(1-s)$$

and obtain

$$\chi^{(\mu)}(1-s) = \sum_{v=0}^{\mu-1}\binom{\mu-1}{v}\chi^{(v)}(1-s)\left(\frac{\chi'}{\chi}\right)^{(\mu-v-1)}(1-s). \tag{32}$$

We have

$$\frac{\chi'}{\chi}(1-s) = \log\pi - \tfrac{1}{2}\psi\left(\frac{s}{2}\right) - \tfrac{1}{2}\psi\left(\frac{1-s}{2}\right)$$

and by (15) and Cauchy's estimate for the derivatives of an analytic function applied to a small disc centered at $s$, we find that

$$\frac{\chi'}{\chi}(1-s) = -\log\frac{|t|}{2\pi} + O\left(\frac{1}{|t|}\right), \tag{33}$$

$$\left(\frac{\chi'}{\chi}\right)^{(v)}(1-s) = O\left(\frac{1}{|t|}\right) \quad \text{for } v \geqq 1. \tag{34}$$

Also

$$\chi(1-s) = O(|t|^{\sigma-\frac{1}{2}}) \quad \text{for } |t| \geqq 1. \tag{35}$$

The required result now follows from (32)-(35) and the induction hypothesis.

**Lemma 7.** *Let* $\zeta^{(\mu)}(s)\zeta^{(v)}(s) = \sum_{n=1}^{\infty}\frac{A_n(\mu, v)}{n^s}$ $(\mathrm{Re}\, s > 1)$, *where* $\mu$, $v \geqq 0$. *Then for* $x \geqq 1$,

$$\sum_{n \leqq x} A_n(\mu, v) = (-1)^{\mu+v}\frac{\mu!\,v!}{(\mu+v+1)!}x(\log x)^{\mu+v+1} + O(x(\log x)^{\mu+v}). \tag{40}$$

*Proof.* Lemma 7 is a simple exercise but we give a proof for completeness. We have

$$\begin{aligned}(-1)^{\mu+v}\sum_{n \leqq x} A_n(\mu, v) &= \sum_{dr \leqq x}(\log d)^{\mu}(\log r)^{v}\\ &= \Big(\sum_{d \leqq \sqrt{x}}\sum_{r \leqq x/d} + \sum_{r \leqq \sqrt{x}}\sum_{d \leqq x/r} - \sum_{d \leqq \sqrt{x}}\sum_{r \leqq \sqrt{x}}\Big)(\log d)^{\mu}(\log r)^{v}.\end{aligned}$$

Since

$$\sum_{d\leqq z}(\log d)^a = z(\log z)^a + O(z(\log z)^{a-1}),$$

our sum is

$$\sum_{d\leqq\sqrt{x}}(\log d)^{\mu}\frac{x}{d}\left(\log\frac{x}{d}\right)^{\nu} + \sum_{r\leqq\sqrt{x}}(\log r)^{\nu}\frac{x}{r}\left(\log\frac{x}{r}\right)^{\mu} + O(x(\log x)^{\mu+\nu}).$$

Now we can replace the last two sums by integrals, again introducing a remainder term $O(x(\log x)^{\mu+\nu})$, and we have to deal with

$$x\int_1^{\sqrt{x}}(\log t)^{\mu}\left(\log\frac{x}{t}\right)^{\nu}\frac{dt}{t} + x\int_1^{\sqrt{x}}(\log t)^{\nu}\left(\log\frac{x}{t}\right)^{\mu}\frac{dt}{t}$$
$$= x\int_1^{x}(\log t)^{\mu}\left(\log\frac{x}{t}\right)^{\nu}\frac{dt}{t}.$$

If we make the change of variable $t=\exp(u\log x)$, we see that

$$\int_1^{x}(\log t)^{\mu}\left(\log\frac{x}{t}\right)^{\nu}\frac{dt}{t} = (\log x)^{\mu+\nu+1}\int_0^1 u^{\mu}(1-u)^{\nu}du$$
$$= \frac{\mu!\,\nu!}{(\mu+\nu+1)!}(\log x)^{\mu+\nu+1}$$

by the well known beta integral, and our lemma follows.

**Lemma 8.** *Let* $\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\frac{\zeta'}{\zeta}(s-i\delta) = \sum_{n=1}^{\infty}\frac{B_n(\mu,\nu,\delta)}{n^s}$ $(\sigma>1)$, *where* $\mu,\nu\geqq 0$ *and* $\delta$ *is a real number. Then for* $x\geqq 1$

$$\sum_{n\leqq x}B_n(\mu,\nu,\delta) = (-1)^{\mu+\nu+1}\mu!\,\nu!\,x(\log x)^{\mu+\nu+2}\sum_{l=0}^{\infty}\frac{(i\delta\log x)^l}{(l+\mu+\nu+2)!}$$
$$+ O(x(\log x)^{\mu+\nu+1}).$$

*Proof.* We write $\zeta^{(\mu)}(s)\zeta^{(\nu)}(s) = \sum_{n=1}^{\infty}\frac{A_n(\mu,\nu)}{n^s}$ as in Lemma 7 and $\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$.

Then

$$\sum_{n\leqq x}B_n(\mu,\nu,\delta) = -\sum_{n\leqq x}\sum_{d\mid n}\Lambda(d)d^{i\delta}A_{n/d}(\mu,\nu)$$
$$= -\sum_{d\leqq x}\Lambda(d)\,d^{i\delta}\sum_{e\leqq x/d}A_e(\mu,\nu).$$

Using Lemma 7 to estimate the inner sum, we find that

$$\sum_{n\leqq x}B_n(\mu,\nu,\delta) = (-1)^{\mu+\nu+1}\frac{\mu!\,\nu!}{(\mu+\nu+1)!}x\sum_{d\leqq x}\frac{\Lambda(d)}{d^{1-i\delta}}\left(\log\frac{x}{d}\right)^{\mu+\nu+1}$$
$$+ O\left(x\sum_{d\leqq x}\frac{\Lambda(d)}{d}\left(\log\frac{x}{d}\right)^{\mu+\nu}\right),$$

or

$$\sum_{n\leqq x} B_n(\mu, \nu, \delta)=(-1)^{\mu+\nu+1}\frac{\mu!\nu!}{(\mu+\nu+1)!}xL_{\mu+\nu+1}(x,\delta)+O(xL_{\mu+\nu}(x,0)), \quad (36)$$

where

$$L_\kappa(x,\delta)=\sum_{d\leqq x}\frac{\Lambda(d)}{d^{1-i\delta}}\left(\log\frac{x}{d}\right)^\kappa.$$

To estimate $L_\kappa$ let $\psi(x)=\sum_{d\leqq x}\Lambda(d)$ and write

$$L_\kappa(x,\delta)=\int_1^x\frac{\left(\log\frac{x}{u}\right)^\kappa}{u^{1-i\delta}}\,d\psi(u).$$

By the prime number theorem with remainder, $\psi(x)=x+E(x)$, where $E(x)\ll x\exp(-c\sqrt{\log x})$ for some fixed $c>0$ and $x\geqq 1$. Thus

$$L_\kappa(x,\delta)=\int_1^x\frac{\left(\log\frac{x}{u}\right)^\kappa}{u^{1-i\delta}}\,du+\int_1^x\frac{\left(\log\frac{x}{u}\right)^\kappa}{u^{1-i\delta}}\,dE(u).$$

The second integral on the right is

$$\begin{aligned}&=E(1)(\log x)^\kappa+\int_1^x\frac{E(u)}{u^2}\left(\kappa\left(\log\frac{u}{u}\right)^{\kappa-1}+(1-i\delta)\left(\log\frac{x}{u}\right)^\kappa\right)du\\&\ll(\log x)^\kappa+(\log x)^\kappa\int_1^x\exp(-c\sqrt{\log u})\frac{du}{u}\\&\ll(\log x)^\kappa.\end{aligned}$$

In the first integral we replace $\left(\log\frac{x}{u}\right)^\kappa$ by $\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}(-1)^\lambda(\log x)^{\kappa-\lambda}(\log u)^\lambda$, $u^{i\delta}$ by $\sum_{l=0}^{\infty}\frac{(i\delta\log u)^l}{l!}$, and change the order of summation and integration to obtain

$$\begin{aligned}\int_1^x\frac{\left(\log\frac{x}{u}\right)^\kappa}{u^{1-i\delta}}\,du&=\sum_{l=0}^{\infty}\frac{(i\delta)^l}{l!}\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}(-1)^\lambda(\log x)^{\kappa-\lambda}\int_1^x(\log u)^{\lambda+l}\frac{du}{u}\\&=\sum_{l=0}^{\infty}\frac{(i\delta)^l}{l!}(\log x)^{\kappa+l+1}\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}\frac{(-1)^\lambda}{l+\lambda+1}.\end{aligned}$$

The innermost sum equals

$$\int_0^1 x^l(1-x)^\kappa\,dx=\frac{l!\kappa!}{(l+\kappa+1)!},$$

so the entire expression is equal to

$$\kappa!(\log x)^{\kappa+1}\sum_{l=0}^{\infty}\frac{(i\delta\log x)^l}{(l+\kappa+1)!}.$$

This gives

$$L_\kappa(x,\delta)=\kappa!(\log x)^{\kappa+1}\sum_{l=0}^{\infty}\frac{(i\delta\log x)^l}{(l+\kappa+1)!}+O((\log x)^\kappa).$$

Using this in (36) we easily find that

$$\sum_{n\leqq x}B_n(\mu,\nu,\delta)=(-1)^{\mu+\nu+1}\mu!\,\nu!\,x(\log x)^{\mu+\nu+2}\sum_{l=0}^{\infty}\frac{(i\delta\log x)^l}{(l+\mu+\nu+2)!}$$
$$+O(x(\log x)^{\mu+\nu+1}).$$

This proves the lemma.

**Lemma 9.** *Suppose that for a fixed* $\lambda\geqq 1$,

$$\sum_{n\leqq x}a_n=x(\log x)^\lambda+O(x(\log x)^{\lambda-1})\qquad(x\geqq 2).$$

*Then if* $\kappa\geqq 1$ *is fixed,*

$$\sum_{n\leqq x}a_n(\log n)^\kappa=x(\log x)^{\kappa+\lambda}+O(x(\log x)^{\kappa+\lambda-1})\qquad(x\geqq 2).$$

*Proof.* Trivial, by partial summation.

**Lemma 10.** *Let* $\lambda,\nu$ *be integers with* $\lambda\geqq 1$ *and* $\nu\geqq 0$. *Then*

$$\sum_{\kappa=0}^{\nu}(-1)^\kappa\binom{\nu}{\kappa}\frac{\kappa!}{(\lambda+\kappa)!}=\frac{1}{(\nu+\lambda)(\lambda-1)!}.$$

*Proof.* The sum equals

$$\sum_{\kappa=0}^{\nu}(-1)^\kappa\frac{\nu!}{(\nu-\kappa)!(\lambda+\kappa)!}=\frac{\nu!}{(\nu+\lambda)!}\sum_{\kappa=0}^{\nu}(-1)^\kappa\binom{\nu+\lambda}{\nu-\kappa}$$
$$=\frac{(-1)^\nu\nu!}{(\nu+\lambda)!}\sum_{\kappa=0}^{\nu}(-1)^\kappa\binom{\nu+\lambda}{\kappa}.$$

The last sum is the coefficient of $x^\nu$ in $(1-x)^{\nu+\lambda}(1-x)^{-1}$ and is therefore equal to the coefficient of $x^\nu$ in $(1-x)^{\nu+\lambda-1}$, i.e. $(-1)^\nu\binom{\nu+\lambda-1}{\nu}$. So the above is

$$=\frac{\nu!}{(\nu+\lambda)!}\binom{\nu+\lambda-1}{\nu}=\frac{(\nu+\lambda-1)!}{(\nu+\lambda)!(\lambda-1)!}=\frac{1}{(\nu+\lambda)(\lambda-1)!}.$$

## § 5. Completion of the proof

We are now in a position to estimate the integral $I_1(\mu,\nu,\delta)$ and thereby to complete the proof of the Theorem. By (21) and (23) we have

$$\sum_{1 \leqq \gamma \leqq T} \zeta^{(\mu)}(\rho+i\delta)\zeta^{(\nu)}(1-\rho-i\delta)=I_1(\mu,\nu,\delta)+I_1(\nu,\mu,\delta)+O(T^{a-\frac{1}{2}+\varepsilon}), \tag{37}$$

where $1<a<2$, $|\delta|\leqq\frac{1}{2}$, $T\in\mathscr{T}$, and

$$I_1(\mu,\nu,\delta)=\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\xi'}{\xi}(s)\zeta^{(\mu)}(s+i\delta)\zeta^{(\nu)}(1-s-i\delta)\,ds.$$

A simple change of variable gives

$$I_1(\mu,\nu,\delta)=\frac{1}{2\pi i}\int_{a+i(1+\delta)}^{a+i(T+\delta)}\frac{\xi'}{\xi}(s-i\delta)\zeta^{(\mu)}(s)\zeta^{(\nu)}(1-s)\,ds.$$

Now for a fixed $a>1$, the integrand is bounded over the interval $[a+i, a+i(1+\delta)]$. Also, by (19) and (20), the part of the integral along $[a+iT, a+i(T+\delta)]$ is

$$\ll \log T\cdot T^{\varepsilon/3}\cdot T^{a-\frac{1}{2}+\varepsilon/3}\ll T^{a-\frac{1}{2}+\varepsilon}.$$

Thus

$$I_1(\mu,\nu,\delta)=\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\xi'}{\xi}(s-i\delta)\zeta^{(\mu)}(s)\zeta^{(\nu)}(1-s)\,ds+O(T^{a-\frac{1}{2}+\varepsilon}).$$

Taking the $\nu$th derivative of (9) according to Leibniz's rule, we find that

$$\zeta^{(\nu)}(1-s)=\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}(-1)^{\kappa}\zeta^{(\kappa)}(s)\chi^{(\nu-\kappa)}(1-s).$$

Hence

$$I_1(\mu,\nu,\delta)=\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}(-1)^{\kappa}I_{1\kappa}(\mu,\nu,\delta)+O(T^{a-\frac{1}{2}+\varepsilon}), \tag{38}$$

where

$$I_{1\kappa}(\mu,\nu,\delta)=\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\xi'}{\xi}(s-i\delta)\zeta^{(\mu)}(s)\zeta^{(\kappa)}(s)\chi^{(\nu-\kappa)}(1-s)\,ds.$$

By Lemma 6, (19), and (20) it is not difficult to see that

$$I_{1\kappa}(\mu,\nu,\delta)=\frac{(-1)^{\nu-\kappa}}{2\pi}\int_1^T\frac{\xi'}{\xi}(a+it-i\delta)\zeta^{(\mu)}(a+it)\zeta^{(\kappa)}(a+it)$$
$$\cdot\chi(1-a-it)\left(\log\frac{t}{2\pi}\right)^{\nu-\kappa}dt+O(T^{a-\frac{1}{2}+\varepsilon}).$$

By (15) and (17)

$$\frac{\xi'}{\xi}(a+it-i\delta)=\frac{\zeta'}{\zeta}(a+it-i\delta)+\tfrac{1}{2}\log\frac{t}{2\pi}+\frac{\pi i}{4}+O\left(\frac{1}{t}\right)$$

for $t\geqq 1$, say. Hence

$$
\begin{aligned}
&I_{1\kappa}(\mu, \nu, \delta)\\
&=\frac{(-1)^{\nu-\kappa}}{2\pi}\int_1^T \frac{\zeta'}{\zeta}(a+it-i\delta)\zeta^{(\mu)}(a+it)\zeta^{(\kappa)}(a+it)\chi(1-a-it)\left(\log\frac{t}{2\pi}\right)^{\nu-\kappa} dt\\
&\quad+\frac{(-1)^{\nu-\kappa}}{2\pi}\int_1^T \tfrac{1}{2}\zeta^{(\mu)}(a+it)\zeta^{(\kappa)}(a+it)\chi(1-a-it)\left(\log\frac{t}{2\pi}\right)^{\nu-\kappa+1} dt\\
&\quad+\frac{(-1)^{\nu-\kappa}}{2\pi}\int_1^T \frac{\pi i}{4}\zeta^{(\mu)}(a+it)\zeta^{(\kappa)}(a+it)\chi(1-a-it)\left(\log\frac{t}{2\pi}\right)^{\nu-\kappa} dt\\
&\quad+O\left(\int_1^T |\zeta^{(\mu)}(a+it)\zeta^{(\kappa)}(a+it)\chi(1-a-it)|(\log t)^{\nu-\kappa}\frac{dt}{t}\right)\\
&\quad+O(T^{a-\frac{1}{2}+\varepsilon}).
\end{aligned}
$$

The next-to-last error term is $O(T^{a-\frac{1}{2}+\varepsilon})$ by (13) and (20) so we may write

$$I_{1\kappa}(\mu, \nu, \delta)=(-1)^{\nu-\kappa}(I_{1\kappa 1}+I_{1\kappa 2}+I_{1\kappa 3})+O(T^{a-\frac{1}{2}+\varepsilon}). \tag{39}$$

To treat $I_{1\kappa 1}$ write

$$\frac{\zeta'}{\zeta}(s-i\delta)\zeta^{(\mu)}(s)\zeta^{(\kappa)}(s)=\sum_{n=1}^{\infty}\frac{B_n(\mu, \kappa, \delta)}{n^s} \quad (\sigma>1)$$

as in Lemma 8. Then

$$I_{1\kappa 1}=\frac{1}{2\pi}\int_1^T\left(\sum_{n=1}^{\infty} B_n(\mu, \kappa, \delta)n^{-a-it}\right)\chi(1-a-it)\left(\log\frac{t}{2\pi}\right)^{\nu-\kappa} dt.$$

Since the $B_n$'s are easily seen to be $\ll n^{\varepsilon}$ for any $\varepsilon>0$, we have by Lemma 5 that for $T$ sufficiently large

$$I_{1\kappa 1}=\sum_{n\leqq T/2\pi} B_n(\mu, \kappa, \delta)(\log n)^{\nu-\kappa}+O(T^{a-\frac{1}{2}}(\log T)^{\nu-\kappa}).$$

It now follows from Lemmas 8 and 9 that

$$
\begin{aligned}
I_{1\kappa 1}&=(-1)^{\mu+\kappa+1}\mu!\,\kappa!\left(\sum_{l=0}^{\infty}\frac{\left(i\delta\log\frac{T}{2\pi}\right)^l}{(l+\mu+\kappa+2)!}\right)\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{\mu+\nu+2}\\
&\quad+O(T(\log T)^{\mu+\nu+1})+O(T^{a-\frac{1}{2}}(\log T)^{\nu-\kappa})
\end{aligned} \tag{40}
$$

for all large $T$.

Next, writing

$$\zeta^{(\mu)}(s)\zeta^{(\kappa)}(s)=\sum_{n=1}^{\infty}\frac{A_n(\mu, \kappa)}{n^s} \quad (\sigma>1)$$

as in Lemma 7, we have

$$I_{1\kappa 2}=\frac{1}{2\pi}\int_1^T\left(\tfrac{1}{2}\sum_{n=1}^{\infty} A_n(\mu, \kappa)n^{-a-it}\right)\chi(1-a-it)\left(\log\frac{t}{2\pi}\right)^{\nu-\kappa+1}.$$

The $A_n$'s are $\ll n^{\varepsilon}$ for any $\varepsilon>0$, hence by Lemma 5

$$I_{1\kappa 2}=\tfrac{1}{2}\sum_{n\leqq T/2\pi} A_n(\mu,\kappa)(\log n)^{\nu-\kappa+1}+O(T^{a-\frac{1}{2}}(\log T)^{\nu-\kappa+1})$$

for sufficiently large $T$. By Lemmas 7 and 9 we then find that

$$\begin{aligned} I_{1\kappa 2}=&\frac{(-1)^{\mu+\kappa}\mu!\kappa!}{2(\mu+\kappa+1)!}\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{\mu+\nu+2}+O(T(\log T)^{\mu+\nu+1})\\ &+O(T^{a-\frac{1}{2}}(\log T)^{\nu-\kappa+1}). \end{aligned} \tag{41}$$

The treatment of $I_{1\kappa 3}$ is analogous to that of $I_{1\kappa 2}$ and leads to

$$\begin{aligned} I_{1\kappa 3}=&\frac{(-1)^{\mu+\kappa}\mu!\kappa!}{(\mu+\kappa+1)!}\frac{\pi i}{4}\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{\mu+\nu+1}+O(T(\log T)^{\mu+\nu})\\ &+O(T^{a-\frac{1}{2}}(\log T)^{\nu-\kappa}), \end{aligned} \tag{42}$$

for all large $T$.

Combining (39)-(42), we see that

$$\begin{aligned} I_{1\kappa}(\mu,\nu,\delta)=&(-1)^{\mu+\nu}\mu!\kappa!\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{\mu+\nu+2}\left\{\frac{1}{2(\mu+\kappa+1)!}-\sum_{l=0}^{\infty}\frac{\left(i\delta\log\frac{T}{2\pi}\right)^l}{(l+\mu+\kappa+2)!}\right\}\\ &+O(T(\log T)^{\mu+\nu+1})+O(T^{a-\frac{1}{2}+\varepsilon}). \end{aligned}$$

Hence, by (38) and Lemma 10,

$$\begin{aligned} I_1(\mu,\nu,\delta)=&(-1)^{\mu+\nu}\mu!\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{\mu+\nu+2}\left\{\frac{1}{2}\sum_{\kappa=0}^{\nu}(-1)^{\kappa}\binom{\nu}{\kappa}\frac{\kappa!}{(\mu+\kappa+1)!}\right.\\ &\left.-\sum_{l=0}^{\infty}\left(i\delta\log\frac{T}{2\pi}\right)^l\sum_{\kappa=0}^{\nu}(-1)^{\kappa}\binom{\nu}{\kappa}\frac{\kappa!}{(l+\mu+\kappa+2)!}\right\}\\ &+O(T(\log T)^{\mu+\nu+1})+O(T^{a-\frac{1}{2}+\varepsilon})\\ =&(-1)^{\mu+\nu}\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{\mu+\nu+2}\left\{\frac{1}{2(\mu+\nu+1)}-\sum_{l=0}^{\infty}\frac{\mu!\left(i\delta\log\frac{T}{2\pi}\right)^l}{(l+\mu+1)!(l+\mu+\nu+2)}\right\}\\ &+O(T(\log T)^{\mu+\nu+1})+O(T^{a-\frac{1}{2}+\varepsilon}). \end{aligned}$$

It follows from this and (37) (with $a=\frac{5}{4}$, say) that

$$\begin{aligned} \sum_{1\leqq\gamma\leqq T}\zeta^{(\mu)}(\rho+i\delta)\zeta^{(\nu)}(1-\rho-i\delta)=&(-1)^{\mu+\nu}\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{\mu+\nu+2}\\ &\cdot\left\{\frac{1}{\mu+\nu+1}-H\left(\mu,\nu,\delta\log\frac{T}{2\pi}\right)-H\left(\nu,\mu,-\delta\log\frac{T}{2\pi}\right)\right\}\\ &+O(T(\log T)^{\mu+\nu+1}), \end{aligned}$$

where $T$ is a sufficiently large element of $\mathcal{T}$, $|\delta| \leqq \frac{1}{2}$, and

$$H(\mu, \nu, c) = \mu! \sum_{l=0}^{\infty} \frac{(ic)^l}{(l+\mu+1)!\,(l+\mu+\nu+2)}.$$

Taking $\delta = \alpha L^{-1}$, where $L = \frac{1}{2\pi} \log \frac{T}{2\pi}$ and $\alpha$ is a real number satisfying $|\alpha| \leqq L/2$, we obtain

$$\sum_{1 \leqq \gamma \leqq T} \zeta^{(\mu)}(\rho + i\alpha L^{-1})\zeta^{(\nu)}(1-\rho-i\alpha L^{-1}) = (-1)^{\mu+\nu} \frac{T}{2\pi}\left(\log \frac{T}{2\pi}\right)^{\mu+\nu+2}$$
$$\cdot\left\{\frac{1}{\mu+\nu+1} - H(\mu, \nu, 2\pi\alpha) - H(\nu, \mu, -2\pi\alpha)\right\} + O(T(\log T)^{\mu+\nu+1}). \quad (43)$$

This is clearly equivalent to (4) when $T \in \mathcal{T}$. To remove the restriction on $T$ note that increasing $T$ by a bounded amount introduces $O(\log T)$ terms into the sum in (43), and by (20) these are no larger than $O(T^{\frac{1}{2}+\varepsilon})$. Moreover, the right-hand side of (43) changes by at most $O((\log T)^{\mu+\nu+2})$. Since these errors are smaller than the $O$-term in (43), (43) holds for all large $T$ within $O(1)$ of an element of $\mathcal{T}$, that is, for all large $T$. The proof of the Theorem is now complete.

## § 6. Proof of corollary 3

Assume the Riemann hypothesis is true and let $0 < \gamma_1 \leqq \gamma_2 \leqq \ldots$ denote the ordinates of the zeros of $\zeta(s)$ in the upper half-plane. Integrating both sides of (7) with respect to $\alpha$ over the interval $[-\beta/2, \beta/2]$, we have

$$\sum_{1 \leqq \gamma_n \leqq T} \int_{\gamma_n - \beta/2L}^{\gamma_n + \beta/2L} |\zeta(\tfrac{1}{2}+it)|^2\, dt \sim F(\beta)\, T \log T.$$

where

$$F(\beta) = \int_{-\beta/2}^{\beta/2} 1 - \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha.$$

Now if we choose

$$\beta > \lambda = \limsup_{n} \frac{(\gamma_n - \gamma_{n-1})2\pi}{\log \gamma_n}$$

it is clear that the left-hand side above will be greater than

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^2\, dt,$$

which is $\sim T \log T$ by (1); that is, $F(\beta) > 1$. But a machine calculation shows that $F(1.9) = 0.997\ldots$. Hence $\lambda \geqq 1.9$.

The same argument could of course be based on a comparison of (3) and (6) with $\mu > 0$. But as $\mu$ increases this seems to lead to progressively worse lower bounds for $\lambda$.

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