Topology Prelim Fall 2020.

March 9, 2023

There are 5 questions in this exam each consisting of 3 subquestions. The best 8 out of 15 subquestions will be used as your course final score for MATH440. If you are a graduate student then this final also counts as a preliminary exam. For purposes of the prelim exam, a question is considered substantially correct if you get 2 out of 3 subquestions essentially correct. To get a prelim **Ph.D. pass** you need to get at least 3 questions substantially correct. To get a prelim **masters pass** you need to get at least 2 questions substantially correct.

1. (a) A subset S of a group G is called a generating set if every element $x \in G$ can be written as a finite product of elements in S and their inverses. Let G be a connected topological group and U be an open neighborhood of the identity element $e \in G$. Show that U is a generating set for G. (Hint: Find a symmetric open neighborhood V of e within U and consider $\cup_{n=1}^{\infty} V^n$ where V^n is the set of all n-fold products of elements from V.)
(b) Let U be an open connected subset of $\mathbb{R}^n$ in the standard Euclidean topology. Show that for any two points $z_0, z_1 \in U$, there exists a continuous path in U from z_0 to z_1 which consists of a finite concatenation of straight line segment paths, each parallel to one of the standard axis of $\mathbb{R}^n$.
(c) Let $GL_n(\mathbb{C})$ denote the complex general linear group topologized as a subspace of $Mat_n(\mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$. Show that $GL_n(\mathbb{C})$ is path connected.

2. (a) Let X be an Alexandrov topological space i.e., a topological space where arbitrary intersections of open sets are open. Show that X is locally path-connected. (Hint: First show that a topological space Y with a point y_0 such that the only open neighborhood of y_0 is Y itself, must be path-connected.)

(b) Give an example of a space X and a subset $A \subseteq X$ with $\alpha \in \bar{A}$, the closure of A in X, but where there is no sequence in A converging to α in X. Prove your example works.
(c) Describe the Klein Bottle as a quotient space of $[0, 2] \times [0, 1]$. Conduct a heuristic cut-and-paste analysis with this quotient model by cutting it along a suitable circle to relate the Klein Bottle to a process involving two Mobius bands. State the final relation clearly and justify it via suitable heuristic pictures.

3. (a) Let (X, d) be a compact metric space and $f : X \to X$ an isometry (i.e. $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that $f : X \to X$ is a homeomorphism (don't forget to prove it has to be onto!). Also give an example of a metric space (Y, D) and isometry $g : Y \to Y$ where g is not onto.
(b) Let $O(n) = \{A \in Mat_n(\mathbb{R}) | A^T A = I\}$ be the orthogonal group, topologized as a subspace of $Mat_n(\mathbb{R}) = \mathbb{R}^{n^2}$. Show that $O(n)$ is a compact smooth manifold. (Hint: Use the regular value theorem and the map $F : Mat_n(\mathbb{R}) \to Sym_n(\mathbb{R})$ given by $F(A) = A^T A$, here $Sym_n(\mathbb{R})$ denotes the vector space of symmetric $n \times n$ matrices.)
(c) Let (X, d) be a compact metric space and $f : X \to X$ a contraction, i.e. a function where there exists $\delta \in [0, 1)$ such that $d(f(x), f(x')) \leq \delta d(x, x')$ for all $x, x' \in X$. Show that f has a unique fixed point in X.

For questions 4 and 5, in addition to the standard basic facts of general topology, you may use all the following tools of differential topology freely as well as their basic properties: local immersion/submersion/inverse function theorems, regular value theorems, Sard's theorem, partitions of unity, ϵ-neighborhood theorem, tangent and normal bundles, mod-2 intersection numbers, degrees and winding numbers.

4. (a) Let $0 < a < b$ and let $B_d(0, R)$ denote the Euclidean ball of radius R about the origin in $\mathbb{R}^n$. Outline the construction of a smooth "bump" function $F : \mathbb{R}^n \to [0, 1]$ which is 1 on the ball $B_d(0, a)$ but 0 outside the ball $B_d(0, b)$. You should indicate all the main steps in the process but don't have to prove the justifying derivative computations but just state them.
(b) Let p be a homogeneous polynomial of degree $m \geq 1$ in n-variables. This means that $p(tx_1, \ldots, tx_n) = t^m p(x_1, \ldots, x_n)$ for all $t \in \mathbb{R}$. Prove Euler's Identity $\hat{x} \cdot \nabla p(\hat{x}) = mp(\hat{x})$ and then prove that $X_a = p^{-1}(a)$ is a smooth

codimension 1 submanifold of $\mathbb{R}^n$ for all $a > 0$. Finally prove that X_a is diffeomorphic to X_b for any $a, b > 0$.
(c) In this question you may use that metrizable spaces are normal/T_4 without proof. Use partitions of unity to prove the

Smooth Urysohn Lemma: Let M be a smooth manifold and A and B disjoint closed subsets in M. Then there exists a smooth function $F : M \to [0, 1]$ such that $F = 1$ on A and $F = 0$ on B.

5. (a) Let S^k denote the standard k-dimensional sphere (the set of unit vectors in $\mathbb{R}^{k+1}$). Prove, using Sard's theorem, that every smooth map $f : S^k \to S^n$ with $k < n$ is smoothly homotopic to a constant map.
(b) State the conditions of intersection theory i.e., the conditions on

$$f : X \to Y$$

and $Z \subseteq Y$ such that the mod-2 intersection number $I_2(f, Z)$ is defined. State the homotopy invariance property of these intersection numbers. Use this to find a map $f : S^1 \to T^2$ where T^2 is the 2-dimensional torus such that $f : S^1 \to T^2$ is not smoothly homotopic to a constant map. Then explain why this shows that S^2 is not diffeomorphic to T^2. (For this last part you may use the result of 5(a) freely even if you did not answer that subquestion. You may also describe f and compute intersection numbers $I_2(f, Z)$ pictorially though the rest of your answer should be more completely written out).
(c) State the conditions under which the mod-2 degree of a map $f : X \to Y$ can be defined. State the homotopy invariance and boundary map properties of this deg_2. Use deg_2 to prove that the following complex equation:

$$z^9 + \cos(|z|^2)(10z^8 + 3z^5 + \sin(|z|^2) + 3) = 0$$

has a solution $z \in \mathbb{C}$.