MATH 403 (PROBABILITY), SPRING PRELIM

You must designate 5 of the 7 problems to be your **prelim** problems. Otherwise, by default, I will take the first 5 problems you turn in to be your prelim problems.

Problem 1 (20 points). Let $\{X(t)\colon t \in \mathbb{R}\}$ be a collection of jointly Gaussian mean zero random variables; i.e., $\forall t_1, t_2, \ldots, t_n \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R}^n)$ we have

$$\mathbb{P}\left((X(t_1), \ldots, X(t_n)) \in A\right) = \int_A \frac{1}{\sqrt{2\pi \det(\Sigma)}} \exp\left(-x^T \Sigma^{-1} x\right) dx$$

where

$$\Sigma_{ij} = \mathbb{E}[X(t_i)X(t_j)] =: R(t_i, t_j).$$

Σ_{ij} is called the covariance matrix, and $R(s,t)$ is called the covariance function.

(1) If $\mathbb{E}[X(t)^2] < \infty \ \forall \ t$, show that $|R(s,t)| < \infty \ \forall \ s, t$.
(2) If $(X(r), X(r+t)) \stackrel{d}{=} (X(0), X(t)) \ \ \forall \ r \in \mathbb{R}$, show that $R(s,t) = R(0, t-s)$. Such Gaussian processes are called **stationary**; i.e., their joint distribution is unchanged under translation.
(3) Show that $g(s) = \mathbb{E}[X(0)X(s)]$ is a positive definite function.
(4) Show that $R(0,t)$ is a constant multiple of the characteristic function of a random variable.

Problem 2 (10 points). Let $X_1, X_2, \ldots$ be an i.i.d. sequence of mean 0 finite positive variance random variables and let $S_n = X_1 + \cdots + X_n$.

(1) Use the central limit theorem and Kolmogorov's zero-one law to conclude that $\overline{\lim} \frac{S_n}{\sqrt{n}} = \infty$ almost surely.
(2) Use an argument by contradiction to show that $S_n/\sqrt{n}$ does not converge in probability.

Problem 3 (10 points). Suppose $\mu_n \Rightarrow \mu$ weakly where $\{\mu_n\}_{n=1}^{\infty}$ and μ are measures on $\mathbb{R}$. Let $\beta_n(A) = \mathbb{P}(g(X_n) \in A)$ for all Borel sets A, where g is some continuous function that is possibly unbounded. β is defined similarly. Show that $\beta_n \Rightarrow \beta$ weakly.

Problem 4 (10 points). Let X be a random variable in L^p where $0 < p < 1$.

(1) First show that

$$\int_{-\infty}^{\infty} \frac{1 - \cos(tx)}{|t|^{p+1}} dt = C_p |x|^p$$

where C_p is some constant that you must explicitly identify.
(2) Then, using Fubini, express $\mathbb{E}[|X|^p]$ in terms of the characteristic function $\phi(t)$ of X.

Problem 5 (10 points). Suppose S_n is Cauchy in probability; i.e., for every $\epsilon, \delta > 0$, there is an N such that for all $n, m > N$,

$$\mathbb{P}(|S_n - S_m| > \delta) < \epsilon$$

Show that there is a random variable S such that $S_n \to S$ in probability.

Problem 6 (10 points)**.** Let X, Y be two independent random variables with density functions $f, g \in L^2$. Show that $X + Y$ has a density that is in L^2 and that this density is continuous. *Hint:* Compute $\mathbb{P}(X + Y \le t)$ and differentiate it using the fundamental theorem of calculus: the fundamental theorem of calculus says that if $F(t) = \int_0^t f(s)ds$, then $F'(t) = f(t)$ if f is a continuous function.

Problem 7 (10 points)**.** Consider the following procedure: We begin with 1 red ball and 1 blue ball in an urn. At each turn, you put your hand into the urn and extract a ball. Then **two** balls of the same color of the ball you extracted are put into the urn. For example, if a red ball is drawn on the first turn, prior to the second drawing, there will be 2 red balls and 1 blue ball in the urn.

Let X_n be the fraction of red balls after the n^{th} draw.

(1) Show that X_n is a martingale; i.e.,
$$\mathbb{E}[X_n | \sigma(X_1, \ldots, X_{n-1}] = X_{n-1}$$
(2) Compute $\mathbb{P}((n + 1)X_n = k)$; i.e., compute the probability mass function of the number of red balls after n draws.
(3) Does there exist an X such that X_n converges weakly (in the sense of distributions) to X?