Math 453: Differentiable Manifolds

Final Exam

May 14, 2021

NAME (please print legibly): ____________________

Your University ID Number: ____________________

- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers. Please sign the pledge below.

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
TOTAL	50	

1. (10 points)

(a) Define the following terms:

- Lie Bracket of vector fields.
- Lie Derivative of a form.
- F-related vector fields.

(b) On $\mathbb{R}^2$ compute:

- $\left[y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right]$
- $\mathcal{L}_X\omega$, where $\omega = dx + xdy$ and $X = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

(c) Let $F : N \to M$ be a smooth map of manifolds, and let $X \in \mathfrak{X}(M)$ and $\bar{X} \in \mathfrak{X}(N)$ be F-related. Show that the local flows $\bar{\varphi}$ and φ of $\bar{X}$ and X, respectively, satisfy:

$$F \circ \bar{\varphi} = \varphi \circ F$$

2. (10 points)

(a) Define the following terms:

1.Germ of a function.

2.Point derivation.

3.Boundary of a manifold with boundary.

(b) Explain why the *boundary* of a manifold with boundary is well-defined.

(c) Let M be a manifold with boundary, and let $p \in \partial M$. Show that the dimension of T_pM is equal to the dimension of M.

3. (10 points)

(a) Define the following terms

- •Orientablility of a manifold in terms of atlases.
- •Orientability in terms terms of forms.
- •Regular value of a function.

(b) Suppose M is a manifold such that $M = U \cup V$ and $U \cap V$ is connected, where U and V are coordinate domains. Show that M is orientable

(c) Recall that if $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function and 0 is a regular value, then $S = f^{-1}(0)$ is submanifold of $\mathbb{R}^n$. Show that it is orientable.

4. (10 points)

(a) Define the following terms.

- Submanifold.
- Embedding.
- Immersion.

(b) Let $M \subset \mathbb{R}^3$ be the subset of Euclidean three-space defined by the zeroes of the function $f(x, y) = x^2 + y^2 - z$. Show that M is a sub-manifold of $\mathbb{R}^3$.

(c) Show that a $1 - 1$ immersion of a compact manifold is an embedding.

5. (10 points)

(a) Define the following terms:

- Define the exterior derivative of a form, either with respect to a basis or in coordinates.
- Partition of unity subordinate to a covering.
- Pullback of a form.

(b) Compute the Exterior Derivative of the form $\omega = x^2 dy + yzdx + 3dz$.

(c) Explain why the definition of integration of a form over an oriented manifold does not depend on the choice of a locally finite covering and subordinate partition of unity.

(d) State Stokes Theorem and prove it for the unit square in $\mathbb{R}^2$ oriented by $dx \wedge dy$.