MATH 467, PRELIMINARY EXAMINATION

ALEX IOSEVICH

Please write your solutions in the blue books that are provided. If you choose to use a major theorem, state it and label it clearly, but do not prove it unless otherwise indicated.

The first five problems constitute the preliminary examination in complex analysis. The sixth problem is required only for those enrolled in Math 467 this semester.

Problem #1: State (do not prove) the Residue Theorem and use it to evaluate the following integrals. Justify all the calculations, including any estimates.

$$i) \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx.$$

$$ii) \int_{0}^{2\pi} \frac{d\theta}{2+\cos(\theta)}.$$

Problem #2:

Let $f(z)$ be an analytic function on $\mathbb{C}$ such that its values are contained in the open upper half-plane. Prove that f is constant.

Problem #3:

Let f be an analytic function in the open right half-plane $\{x \in \mathbb{C} : Re(z) > 0\}$. Suppose that $|f(z)| < 1$ for all z in the domain of f, and $f(1) = 0$. Find the largest possible value of $|f(2)|$.

Problem #4:

i) State Rouche's theorem.

ii) State the Fundamental Theorem of Algebra.

iii) Use Rouche's theorem to prove the Fundamental Theorem of Algebra.

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Warning: If you prove the Fundamental Theorem of Algebra without using Rouche's theorem, you **will not** get any credit.

Problem #5: Let f be a function analytic in $|z| < 1+\epsilon$ for some $\epsilon > 0$. Then for every **monic** polynomial p,

$$|f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})p(e^{it})|dt.$$

THE PRELIMINARY EXAMINATION ENDS HERE! THE REMAINING PROBLEM IS FOR THOSE ENROLLED IN MATH 467 THIS SEMESTER!

Problem #6:

i) Show that if f is analytic on an open set containing the closure of the disk of radius R centered at a, then

$$|f(a)|^2 \leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R \left|f(a+re^{i\theta})\right|^2 r dr d\theta.$$

ii) Let G be a region and let M be a fixed positive constant. Let $\mathcal{F}$ be a family of all functions in $H(G)$ (all holomorphic functions on G) such that

$$\int\int_G |f(z)|^2 dxdy \leq M.$$

Prove that $\mathcal{F}$ is normal.