Real Analysis Preliminary Exam, August 2022

1. Let $(X, \mathcal{S}, \mu)$ be a measure space such that $\mu(X) < \infty$. Prove that if $\mathcal{A}$ is a family of disjoint sets in $\mathcal{S}$ such that $\mu(A) > 0$ for all $A \in \mathcal{A}$, then $\mathcal{A}$ is a countable set.

2. Let $(X, \mathcal{S}, \mu)$ be a measure space and consider $(f_n)_n \subset L^1(X)$ to be a sequence of functions converging pointwise a.e. to $f \in L^1(X)$. Show that

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0$$

if and only if

$$\lim_{n\to\infty} \int_X |f_n| \, d\mu = \int_X |f| \, d\mu.$$

3. If $h : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function, then its associated Hardy-Littlewood maximal function $h^* : \mathbb{R} \to [0, \infty]$ is defined by

$$h^*(b) = \sup_{t>0} \frac{1}{2t} \int_{[b-t,b+t]} |h(x)| \, d\lambda(x), \qquad (\forall)\, b \in \mathbb{R}.$$

Prove that

$$\lambda\left(\{b \in \mathbb{R};\ h^*(b) = \infty\}\right) = 0$$

for all $h \in L^1(\mathbb{R})$.

4. Let λ denote the Lebesgue measure on $[0, 1]$.

i) Show that

$$\begin{aligned}
\int_{[0,1]} \int_{[0,1]} \frac{x^2 - y^2}{(x^2 + y^2)^2} \, d\lambda(y)\, d\lambda(x) \;&=\; \frac{\pi}{4}, \\
\int_{[0,1]} \int_{[0,1]} \frac{x^2 - y^2}{(x^2 + y^2)^2} \, d\lambda(x)\, d\lambda(y) \;&=\; -\frac{\pi}{4}.
\end{aligned}$$

ii) Argue why the previous two equalities violate neither Tonelli's theorem nor Fubini's theorem.

5. Let $f : [0, 1] \to \mathbb{R}$ be a Lebesgue measurable function. Prove that

$$\lim_{p\to\infty} \|f\|_{L^p([0,1])} \;=\; \|f\|_{L^\infty([0,1])}.$$