Algebra 2 Prelims (AKA Math 437 final): May 6, 2022

1a) Let $\begin{smallmatrix} K \\ | \\ E \\ | \\ k \end{smallmatrix}$ be an extension of fields where K is algebraic over E and E is algebraic over k. Prove that K must be algebraic over k. *You may use facts about towers of finite extensions without proof if you want.*

b) Let $f(x)=x^3+ax+b\in k[x]$ where $\mathrm{char}(k)\neq 2,3$, and let $\Delta=-4a^3-27b^2$ be the discriminant of $f(x)$ *(so that $\Delta=\delta^2$ where $\delta=(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)$ for α_1, α_2, α_3 the roots of $f(x)$).* Explain how one can use Δ to determine the Galois group of $f(x)$ over k when $f(x)$ is *irreducible* over k, giving a complete proof to justify your answer. Does your argument go through in characteristic 2? Why or why not?

2) Let k be a field of characteristic $p\neq 0$. Prove each of the following.

a) Let K be a cyclic extension of k of degree p. Then $K=k(\alpha)$ for some $\alpha\in K$ that is a root of a polynomial x^p-x-c for some $c\in k$. *[Hint: You might want to start by explaining why $Tr(-1)=0$ where Tr represents the Trace from K to k, and then applying the additive form of Hilbert's Theorem 90, which you can use without proof.]*

b) Conversely, for any $c\in k$, the polynomial x^p-x-c either has one root in k, in which case, all its roots are in k, or it is irreducible. Moreover, in the latter case, $k(\alpha)$ is Galois and cyclic of degree p over k.

3,4,5) For purposes of the prelims, I will count your best 3 of the following 4 problems *(A, B, C, or D, some of which have multiple parts, all of which were homework questions or are very related to ones that were).* For purposes of the class final, all 4 will count, but you do not have to complete all 4 problems, as the scores will be curved.

Ai) Suppose $\mathrm{char}(K)=p$, and let $a\in K$. If a has no p^{th} root in K, then $x^{p^n}-a$ is irreducible for all integers $n\geq 1$.

ii) Suppose $\mathrm{char}(K)=p$. Let α be algebraic over K. Show that α is separable over K if and only if $K(\alpha)=K(\alpha^{p^n})$ for all positive integers n.

B) Let E be an algebraic extension of k such that every nonconstant polynomial $f(x)\in k[x]$ has at least one root in E. Prove that E is algebraically closed.

Ci) Let G be a finite cyclic group. Prove there exists a Galois extension of $\mathbb{Q}$ with Galois group G.

ii) Prove the same result if G is a finite abelian *(not necessarily cyclic)* group.

Di) Let K/F be an extension of finite fields. Show that the norm N_F^K is surjective *(as a map from K^* to F^*). [Hint: As one step of your argument, explain why Hilbert's Theorem 90, which you can use without proof, can be applied to K/F.]*

ii) A polynomial is called reciprocal if whenever α is a root, so is $\frac{1}{\alpha}$. Suppose that $f(x)$ is a polynomial with coefficients in a subfield k of the real numbers, and $f(x)$ is irreducible over k. Suppose, moreover, that $f(x)$ has a non-real complex root of absolute value 1. Show that $f(x)$ is reciprocal of even degree.